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RESEARCH ARTICLE

Birationally rigid Fano-Mori fibre spaces

Aleksandr Pukhlikov

Department of Mathematical Sciences, The University of Liverpool; E-mail: pukh@liverpool.ac.uk.

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Abstract

In this paper, we prove the birational rigidity of Fano-Mori fibre spaces $\pi\colon V \to S$, every fibre of which is a Fano complete intersection of index 1 and codimension $k \geqslant 3$ in the projective space $\mathbb{P}^{M+k}$ for M sufficiently high, satisfying certain natural conditions of general position, in the assumption that the fibre space V/S is sufficiently twisted over the base. The dimension of the base S is bounded from above by a constant, depending only on the dimension M of the fibre (as the dimension of the fibre M grows, this constant grows as $\frac{1}{2}M^2$).

Introduction

0.1. Fano complete intersections

In the present paper, we study the birational geometry of algebraic varieties, fibred into Fano complete intersections of codimension $k \geqslant 3$ (fibrations into Fano hypersurfaces were studied in [1], into Fano complete intersections of codimension 2 in [2]). We start with a description of fibres of these fibre spaces. Let us fix an integer $k \geqslant 3$ and set

$$\varepsilon(k) = \min\left\{a \in \mathbb{Z} \,\middle|\, a \geqslant 1, \left(1 + \frac{1}{k}\right)^a \geqslant 2\right\}.$$

Now let us fix $M \in \mathbb{Z}$, satisfying the inequality

$$M \geqslant 10k^2 + 8k + 2\varepsilon(k) + 3. \tag{1}$$

The right-hand side of that inequality denote by the symbol $\rho(k)$. Let

$$\underline{d} = (d_1, \dots, d_k)$$

be an ordered tuple of integers,

$$2 \leqslant d_1 \leqslant d_2 \leqslant \dots \leqslant d_k,$$

satisfying the equality

$$d_1 + \dots + d_k = M + k.$$

Fano varieties, considered in this paper, are complete intersections of type $\underline{d}$ in the complex projective space $\mathbb{P}^{M+k}$. More precisely, let the symbol $\mathcal{P}_{a,N}$ stand for the space of homogeneous polynomials of degree $a \in \mathbb{Z}_+$ in $N \geqslant 1$ variables. Set

$$\mathcal{P} = \prod_{i=1}^{k} \mathcal{P}_{d_i, M+k+1}$$

to be the space of all tuples

$$\underline{f} = (f_1, \dots, f_k)$$

of homogeneous polynomials of degree $d_1, \dots, d_k$ on $\mathbb{P}^{M+k}$. If for $\underline{f} \in \mathcal{P}$ the scheme of common zeros of the polynomials $f_1, \dots, f_k$ is an irreducible reduced factorial variety $F = F(\underline{f})$ of dimension M with terminal singularities, then F is a primitive Fano variety:

$$\operatorname{Pic} F = \mathbb{Z} H_F, \quad K_F = -H_F,$$

where H_F is the class of a hyperplane section of the variety F (the Lefschetz theorem). Assuming that this is the case, let us give the following definition.

Definition 0.1. The variety F is *divisorially canonical*, if for every effective divisor $D \sim nH_F$, the pair $(F, \frac{1}{n}D)$ is canonical – that is, for every exceptional divisor E over F, the inequality

$$\operatorname{ord}_E D \leqslant n \cdot a(E)$$

holds, where $a(E)$ is the discrepancy of E with respect to F.

Below is the first main result of the present paper.

Theorem 0.1. *There exist a Zariski open subset $\mathcal{F} \subset \mathcal{P}$, such that for every tuple $\underline{f} \in \mathcal{F}$, the scheme of common zeros of the tuple $\underline{f}$ is an irreducible reduced factorial divisorially canonical variety $F(\underline{f})$ of dimension M with terminal singularities, and the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$ satisfies the inequality*

$$\operatorname{codim}((\mathcal{P} \setminus \mathcal{F}) \subset \mathcal{P}) \geqslant M - k + 5 + \binom{M - \rho(k) + 2}{2}.$$

(Thus, for a fixed k and growing M, the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$ grows as $\frac{1}{2}M^2$.)

It is convenient to express the property of divisorial canonicity in terms of the *global canonical threshold* of the variety F.

Recall that for a Fano variety X with the Picard number 1 and terminal $\mathbb{Q}$-factorial singularities, its global canonical threshold $\operatorname{ct}(X)$ is the supremum of $\lambda \in \mathbb{Q}_+$ such that for every effective divisor $D \sim -nK_X$ (here, $n \in \mathbb{Q}_+$), the pair $\left(X, \frac{\lambda}{n}D\right)$ is canonical. Therefore, Theorem 0.1 claims that for every $\underline{f} \in \mathcal{F}$, the inequality $\operatorname{ct}(F(\underline{f})) \geqslant 1$ holds.

If in the definition of the global canonical threshold instead of 'for every effective divisor $D \sim -nK_X$', we put 'for a general divisor D in any linear system $\Sigma \subset |-nK_X|$ with no fixed components', we get the definition of the *mobile canonical threshold* $\operatorname{mct}(X)$; obviously, $\operatorname{mct}(X) \geqslant \operatorname{ct}(X)$. The inequality $\operatorname{mct}(X) \geqslant 1$ is equivalent to the birational superrigidity of the Fano variety X; see [3]. If in the definition of the global canonical threshold the property of the pair $(X, \frac{\lambda}{n}D)$ to be canonical, we replace by the log-canonicity of that pair, we get the definition of the *global log-canonical threshold* $\operatorname{lct}(X)$; again, $\operatorname{lct}(X) \geqslant \operatorname{ct}(X)$.

For simplicity, we write $F \in \mathcal{F}$ instead of $F = F(\underline{f})$ for $\underline{f} \in \mathcal{F}$.

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0.2. Fano-Mori fibre spaces

By a *Fano-Mori fibre space* we mean a surjective morphism of projective varieties

$$\pi\colon V \to S,$$

where $\dim V \geqslant 3 + \dim S$, the base S is non-singular and rationally connected, and the following conditions are satisfied:

(FM1) every scheme fibre $F_s = \pi^{-1}(s)$, $s \in S$ is an irreducible reduced factorial Fano variety with terminal singularities and the Picard group $\operatorname{Pic} F_s \cong \mathbb{Z}$,

(FM2) the variety V itself is factorial and has at most terminal singularities,

(FM3) the equality

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S$$

holds.

So Fano-Mori fibre spaces are Mori fibre spaces with additional very good properties.

Definition 0.2. A Fano-Mori fibre space $\pi\colon V \to S$ is *stable with respect to fibre-wise birational modifications*, if for every birational morphism $\sigma_S\colon S^+ \to S$, where S^+ is a non-singular projective variety, the morphism

$$\pi_+\colon V^+ = V \times_S S^+ \to S^+$$

is a Fano-Mori fibre space.

We will consider birational maps $\chi\colon V \dashrightarrow V'$, where V is the total space of a Fano-Mori fibre space and V' is the total space of a fibre space $\pi'\colon V' \to S'$ which belongs to one of the two classes:

(1) *rationally connected fibre spaces*; that is, V' and S' are non-singular and projective and the base S' and a fibre of general position $(\pi')^{-1}(s')$ are rationally connected,

(2) *Mori fibre spaces*, where V' and S' are projective and the variety V' has $\mathbb{Q}$-factorial terminal singularities.

For a birational map $\chi\colon V \dashrightarrow V'$, where V'/S' is a rationally connected fibre space, we want to answer the question: is it fibre-wise – that is, is there a rational dominant map $\beta\colon S \dashrightarrow S'$, making the diagram

$$\begin{array}{ccc} V & \stackrel{\chi}{\dashrightarrow} & V' \\ \pi\downarrow & & \downarrow\pi' \\ S & \stackrel{\beta}{\dashrightarrow} & S' \end{array} \tag{2}$$

a commutative one – that is, $\pi' \circ \chi = \beta \circ \pi$?

For a birational map $\chi\colon V \dashrightarrow V'$, where V'/S' is a Mori fibre space with the additional properties (2) (only such Mori fibre spaces are considered in this paper), we want to answer the question: is there a *birational* map $\beta\colon S \dashrightarrow S'$, for which the diagram (2) is commutative? If the answer to this question is always affirmative (that is, it is affirmative for every fibre space from the class (2)), then the fibre space V/S is *birationally rigid*.

Now let us state the second main result of the present paper.

Theorem 0.2. *Assume that a Fano-Mori fibre space $\pi\colon V \to S$ is stable with respect to fibre-wise birational modifications, and moreover,*

(i) *for every point $s \in S$, the fibre F_s satisfies the inequalities* $\operatorname{lct}(F_s) \geqslant 1$ *and* $\operatorname{mct}(F_s) \geqslant 1$,

(ii) *(the K-condition) every mobile (that is, with no fixed components) linear system on V is a subsystem of a complete linear system $|-nK_V + \pi^*Y|$, where Y is a pseudoeffective class on S,*

(iii) *for every family $\overline{\mathcal{C}}$ of irreducible curves on S, sweeping out a dense subset of the base S, and $\overline{C} \in \overline{\mathcal{C}}$, no positive multiple of the class*

$$-(K_V \cdot \pi^{-1}(\overline{C})) - F \in A^{\dim S} V,$$

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where A^iV is the numerical Chow group of classes of cycles of codimension i on V and F – the class of a fibre of the projection π – is represented by an effective cycle on V.

Then for every rationally connected fibre space V'/S', every birational map $\chi\colon V \dashrightarrow V'$ (if such maps exist) is fibre-wise, and the fibre space V/S itself is birationally rigid.

By what was said in Subsection 0.1, the assumption (i) can be replaced by the single inequality $\mathrm{ct}(F_s) \geqslant 1$ for every $s \in S$; that is, it is sufficient to assume that every fibre of the fibre space V/S is a divisorially canonical variety.

As we will see from the proof of Theorem 0.2, instead of the conditions (ii) and (iii), it is sufficient to require that for every family $\overline{\mathcal{C}}$ of irreducible curves on S, sweeping out a dense subset, and $\overline{C} \in \overline{\mathcal{C}}$ the class

$$-N(K_V \cdot \pi^{-1}(\overline{C})) - F \in A^{\dim S}V,$$

is not represented by an effective cycle on V for any $N \geqslant 1$. The last condition is especially easy to verify: it is enough to have a numerically effective π-ample class H_V on V, satisfying the inequality

$$\left(K_V \cdot \pi^{-1}(\overline{C}) \cdot H_V^{\dim V - \dim S}\right) \geqslant 0 \tag{3}$$

for every dense family $\overline{\mathcal{C}} \ni \overline{C}$.

0.3. An explicit construction of a fibre space

Now let us construct a large class of Fano-Mori fibre spaces, satisfying the conditions of Theorem 0.2. Let S be a non-singular projective rationally connected positive-dimensional variety and $\pi_X\colon X \to S$ a locally trivial fibration with the fibre $\mathbb{P}^{M+k}$, where k and M are the same as in Subsection 0.1. We say that the subvariety $V \subset X$ of codimension k is a *fibration into complete intersections of type $\underline{d}$*, if the base S can be covered by Zariski open subsets U, over which the fibration π_X is trivial, $\pi_X^{-1}(U) \cong U \times \mathbb{P}^{M+k}$, and for every U, there is a regular map

$$\Phi_U : U \to \mathcal{P},$$

such that $V \cap \pi_X^{-1}(U)$ in the sense of the above-mentioned trivialization is the scheme of common zeros of a tuple

$$\underline{f}(s) = \Phi_U(s) = (f_1(x_*, s), \dots, f_k(x_*, s)),$$

where x_* are homogeneous coordinates on $\mathbb{P}^{M+k}$ and s runs through U.

Below (in §1), it will be clear that the open subset $\mathcal{F}$ from Theorem 0.1 is invariant under the action of the group $\mathrm{Aut}\,\mathbb{P}^{M+k}$. For that reason, the following definition makes sense.

Definition 0.3. A fibration $V \subset X$ into complete intersections of type $\underline{d}$ is a *$\mathcal{F}$-fibration*, if for any trivialization of the bundle π_X over an open set $U \subset S$, we have $\Phi_U(U) \subset \mathcal{F}$.

Obviously, if the inequality

$$\dim S \leqslant M - k + 4 + \binom{M - \rho(k) + 2}{2} \tag{4}$$

holds, then we may assume that V is a $\mathcal{F}$-fibration. Set $\pi = \pi_X|_V$. Now from Theorems 0.1 and 0.2 it is easy to obtain the third main result of the present paper.

Theorem 0.3. *Any $\mathcal{F}$-fibration $\pi\colon V \to S$ constructed above is a Fano-Mori fibre space. If the conditions (ii) and (iii) of Theorem 0.2 hold, then for every rationally connected fibre space V'/S', every birational map $\chi\colon V \dashrightarrow V'$ is fibre-wise, and the fibre space V/S itself is birationally rigid.*

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Example 0.1. Let H_X be a numerically effective divisorial class on X, the restriction of which onto the fibre $\pi_X^{-1}(s) \cong \mathbb{P}^{M+k}$ is the class of a hyperplane. Let $\Delta_1,\ldots, \Delta_k$ be very ample classes on the base S. Let us construct a $\mathcal{F}$-fibration V/S as a complete intersection of k general divisors

$$V = G_1 \cap \cdots \cap G_k,$$

where $G_i \in |d_i H_X + \pi_X^* \Delta_i|$. Let us find out, when V/S satisfies the conditions (ii) and (iii) of Theorem 0.2. Write

$$K_X = -(M + k + 1)H_X + \pi_X^* \Delta_X.$$

Then we get

$$K_V = \left(-H_X + \pi_X^*\left(\Delta_X + \sum_{i=1}^{k} \Delta_i\right)\right)\bigg|_V .$$

It is easy to check that the inequality (3) in this case takes the form of the estimate

$$\left(\left(\Delta_X + \sum_{i=1}^{k}\left(1 - \frac{1}{d_i}\right)\Delta_i\right) \cdot \overline{C}\right) \geqslant \left(H_X^{M+k+1} \cdot \pi_X^{-1}(\overline{C})\right),$$

where for the class H_V, we took $H_X|_V$. This inequality must be satisfied for every dense family $\overline{\mathcal{C}} \ni \overline{C}$.

Let us consider a very particular case, when $X = \mathbb{P}^m \times \mathbb{P}^{M+k}$ and G_i are divisors of bi-degree (m_i, d_i), $i = 1, \ldots, k$. Taking for H_X the pullback on X of the class of a hyperplane in $\mathbb{P}^{M+k}$, we get that the last inequality is equivalent to the numerical inequality

$$\sum_{i=1}^{k}\left(1 - \frac{1}{d_i}\right)m_i \geqslant m + 1. \tag{5}$$

If it is satisfied and the dimension $m = \dim S$ satisfies the inequality (4), then the intersection $V = G_1 \cap \cdots \cap G_k$ of general (in the sense of Zariski topology) divisors of bi-degree $(m_1, d_1), \ldots, (m_k, d_k)$, fibred over $S = \mathbb{P}^m$, is a birationally rigid Fano-Mori fibre space, and every birational map of V onto the total space of a rationally connected fibre space is fibre-wise. The inequality (5) shows that this claim holds for almost all tuples $(m_1, \ldots, m_k) \in \mathbb{Z}_+^k$ (except for finitely many of them) – that is, for almost all families of Fano-Mori fibre spaces – obtained by means of this construction. Note that the condition (5) is lose to a criterial one: if

$$m_1 + \cdots + m_k \leqslant m,$$

then the projection of V onto $\mathbb{P}^{M+k}$ defines on V a structure of a Fano-Mori fibre space (and a rationally connected fibre space), which is 'transversal' to the original structure $\pi\colon V \longrightarrow S$ (and is not fibre-wise), so that in this case, V/S is not birationally rigid.

0.4. The structure of the paper

The paper is organized in the following way. In §1, we produce the explicit local conditions defining the open subset $\mathcal{F} \subset \mathcal{P}$. The proof of divisorial canonicity of a variety $F \in \mathcal{F}$ (that is, of the inequality $\mathrm{ct}(F) \geqslant 1$) is reduced in §1 to a number of technical facts that will be shown in the subsequent sections (§§3–7). In §2, we show Theorem 0.2.

The proof of Theorem 0.1 consists of several pieces. The fact that the local conditions for the singularities that a variety $F \in \mathcal{F}$ can have (which are *multi-quadratic singularities*; see Subsection 1.2)

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guarantee that the variety F is factorial and its singularities are terminal is proven in §4, where we give a general definition of multi-quadratic singularities and study their properties. The estimate for the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$ (which is very important for constructing families of Fano-Mori fibre spaces, satisfying the assumptions of Theorem 0.2) is shown in §8. However, the main (and the hardest) part of the proof of Theorem 0.1 is to show that a variety $F \in \mathcal{F}$ is divisorially canonical. We assume that for some effective divisor $D \sim nH_F$, the pair $(F, \frac{1}{n}D)$ is not canonical; that is, for some exceptional divisor E over F, the inequality

$$\mathrm{ord}_E\, D > n \cdot a(E)$$

holds. Now we have to show that this assumption leads to a contradiction. In Subsections 1.3–1.6 it is shown how (using the inequalities for the multiplicity of subvarieties of the variety F at a given point, proven in §7) to obtain a contradiction in the case when a point of general position $o \in B$, where B is the centre of E on F, either is non-singular on F or is a quadratic singularity. The hardest task is to obtain a contradiction when the point o is a multi-quadratic singularity of the variety F. A plan of solving this problem is given in Subsection 1.7, where we introduce the concept of a *working triple* and describe the procedure of constructing a sequence of subvarieties of the variety F, in which each subvariety is a hyperplane section of the previous one and the last subvariety delivers the desired contradiction.

This program is realized in §3, where we study the properties of working triples; however, a number of key technical facts is only stated there – their proof is put off for a greater clarity of exposition. These key facts are shown in §§5, 6 (and the proof makes use of the facts on linear subspaces on complete intersections of quadrics, proven in Subsection 4.5).

Finally, in §7, we prove the estimates for the multiplicities of certain subvarieties of the variety F at given points in terms of the degrees of these subvarieties in $\mathbb{P}^{M+k}$. Here we use the well-known technique of hypertangent divisors. For the purposes of our proof of Theorem 0.1, we have to somewhat modify this technique.

0.5. General remarks

The birational rigidity of Fano-Mori fibre spaces over a positive-dimensional base was one of the most important topics in birational geometry in the past 40 years. For its history and place in the context of the modern birational geometry of rationally connected varieties, see [2, Subsection 0.4]. Here we just mention a few recent papers in the areas that are close to the direction, to which the present paper belongs.

These areas are the birational rigidity, explicit birational geometry of Mori fibre spaces (including the studies of their groups of birational automorphisms and, wider, Sarkisov links), the rationality problem, computing and estimating the global canonical thresholds and, related to these problems, the theory of K-stability.

In the papers [4, 5, 6], important results on the birational rigidity and rigidity-type results for fibrations over $\mathbb{P}^1$ were obtained. The paper [7] links the Sarkisov program with the problem of estimating the canonical threshold of certain divisors on Fano varieties. The papers [8, 9] prove the stable non-rationality of very general conic bundles and fibrations into del Pezzo surfaces, respectively, over a higher-dimensional base. The problem of stable rationality for hypersurfaces of various bi-degrees in the products of projective spaces (see Example 0.1 above) is considered in [10]. The theory of K-stability, which is on the border of birational geometry, is investigated in many papers (especially in the recent past) – in particular, see [11, 12, 13, 14]; we mentioned the papers that are the closest to the birational rigidity-type problems. Finally, there was a lot of development recently in the direction of applying the theory of Sarkisov links and relations between them to the study of the groups of birational automorphisms of such varieties that have a very large this group; see, for instance, [15, 16].

Getting back to the topic of this paper, we note that its immediate predecessor is [2], however, that paper investigates the non-canonical singularities, the centre of which is contained in the set of bi-quadratic points of the variety (from the technical viewpoint, this is the hardest part of the proof of divisorial canonicity), using the secant varieties of subvarieties of codimension 2 on an intersection of two quadrics. It is not possible to apply this approach to subvarieties of higher codimension on an intersection of $k \geqslant 3$ quadrics, and the present paper is based on a completely different construction (which applies to the bi-quadratic singularities, considered in [2], as well).

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1. Fano complete intersections

In this section, we describe the local conditions defining the open subset $\mathcal{F} \subset \mathcal{P}$ (Subsections 1.2 and 1.4). For a complete intersection $F \in \mathcal{F}$, the proof of its divisorial canonicity is reduced to a number of technical claims, which will be shown later. A more detailed plan of the proof of Theorem 0.1 is given in Subsection 1.1.

1.1. A plan of the proof of Theorem 0.1

In order to prove Theorem 0.1, one has to give an explicit definition of the open set $\mathcal{F} \subset \mathcal{P}$. This definition consists of two groups of conditions, which should be satisfied by the polynomials $f_1, \ldots, f_k$ at every point $o \in \mathbb{P}^{M+k}$ at which they all vanish. The first group of conditions is about the singularities of the complete intersection $F(\underline{f})$: they can be quadratic or multi-quadratic of a rank bounded from below. The corresponding definitions and facts are given in Subsection 1.2. Assuming that the conditions of the first group are satisfied, we get that the scheme of common zeros of the polynomials $f_1, \ldots, f_k$ is an irreducible reduced factorial variety $F = F(\underline{f}) \subset \mathbb{P}^{M+k}$ with terminal singularities, and so $\operatorname{Pic} F = \mathbb{Z}H_F$ and $K_F = -H_F$, so that the question, Is it divisorially canonical?, makes sense.

Assuming that F is not divisorially canonical, let us fix an effective divisor $D_F \sim n(D_F)H_F$, where $n(D_F) \geqslant 1$, such that the pair

$$\left(F, \frac{1}{n(D_F)} D_F\right)$$

is not canonical; that is, there is an exceptional divisor E over F, satisfying the Noether-Fano inequality:

$$\operatorname{ord}_E D_F > n(D_F)\, a(E).$$

We have to show that the existence of such a divisor leads to a contradiction. Let $B \subset F$ be the centre of the exceptional divisor E on F. The information about the singularities of the varieties F makes it possible to easily exclude the option when $\operatorname{codim}(B \subset F) = 2$. This is done in Subsection 1.3.

After that, in Subsection 1.4, we produce the second group of local conditions for the tuple of polynomials $\underline{f} \in \mathcal{F}$: now they are the regularity conditions. Assuming that they are satisfied at every point $o \in F$, we exclude the option $B \not\subset \operatorname{Sing} F$ in Subsection 1.5, and in Subsection 1.6, the option that the point $o \in B$ of general position is a quadratic singularity of F. In Subsection 1.7 we describe the procedure of excluding the multi-quadratic case, when the point $o \in B$ of general position is a multi-quadratic singularity of the type 2^l, $l \in \{2, \ldots, k\}$. This is the hardest part of the work, which is completed in the subsequent sections.

1.2. Multi-quadratic singularities

Let $o \in \mathbb{P}^{M+k}$ be a point at which $f_1, \ldots, f_k$ all vanish. Let us consider a system of affine coordinates $z_* = (z_1, \ldots, z_{M+k})$ with the origin at the point o on an affine chart $\mathbb{A}^{M+k} \subset \mathbb{P}^{M+k}$, containing that point. Write down

$$\begin{array}{lllll} f_1 = f_{1,1} + f_{1,2} + & \dots & +f_{1,d_1}, & & \\ f_2 = f_{2,1} + f_{2,2} + & \dots & & +f_{2,d_2}, & \\ & \dots & & & \\ f_k = f_{k,1} + f_{k,2} + & \dots & & & +f_{k,d_k}, \end{array}$$

where we use the same symbols f_i for the non-homogeneous polynomials in z_*, corresponding to the original polynomials f_i, and $f_{i,a}$ is a homogeneous polynomial of degree a in z_*. Obviously, if the linear forms $f_{1,1}, \dots, f_{k,1}$ are linearly independent, then in a neighborhood of the point o, the scheme of common zeros of the polynomials $f_1, \dots, f_k$ is a non-singular complete intersection of codimension k. In order to give the definition of a multi-quadratic singularity, we will need the concept of the rank of a tuple of quadratic forms.

Definition 1.1 [2]. *The rank of the tuple of quadratic forms* $q_1, \dots, q_l$ in N variables is the number

$$\mathrm{rk}(q_1, \dots, q_l) = \min\{\mathrm{rk}(\lambda_1 q_1 + \dots + \lambda_l q_l) \mid (\lambda_1, \dots, \lambda_l) \neq (0, \dots, 0)\}.$$

Obviously, $\mathrm{rk}(q_1, \dots, q_l) \leqslant N$. For that reason, in the sequel, the inequality $\mathrm{rk}(q_*) \geqslant a$ means implicitly that the forms q_i depend on a sufficient ($\geqslant a$) number of variables.

Take $l \in \{1, 2, \dots, k\}$.

Definition 1.2. The tuple $\underline{f}$ has at the point o a *multi-quadratic singularity of type* 2^l of rank a, if the following conditions are satisfied:

- $\dim\langle f_{1,1}, \dots, f_{k,1}\rangle = k - l$ (and in order to simplify the notations, we assume that the forms

$$f_{l+1}, \dots, f_k$$

  are linearly independent),
- the rank of the tuple of quadratic forms

$$f^*_{i,2} = f_{i,2} - \sum_{j=l+1}^{k} \lambda_{i,j} f_{j,2},$$

  $i = 1, \dots, l$, where $\lambda_{i,j} \in \mathbb{C}$ are defined by the equalities

$$f_{i,1} = \sum_{j=l+1}^{k} \lambda_{i,j} f_{j,1},$$

  is equal to the number a.

Now the first condition, defining the subset $\mathcal{F} \subset \mathcal{P}$, is stated in the following way.

(MQ1) For every point $o \in \mathbb{P}^{M+k}$, such that

$$f_1(o) = \dots = f_k(o) = 0,$$

either the linear forms $f_{1,1}, \dots, f_{k,1}$ are linearly independent or $\underline{f}$ has at the point o a multi-quadratic singularity of type 2^l, where $l \in \{1, 2, \dots, k\}$, of rank

$$\geqslant 2l + 4k + 2\varepsilon(k) - 1.$$

Theorem 1.1. *Assume that* $\underline{f}$ *satisfies the condition (MQ1). Then the scheme of common zeros of the polynomials* $f_1, \dots, f_k$ *is an irreducible reduced factorial variety F=F(*$\underline{f}$*) – a complete intersection of codimension k with terminal singularities, and, moreover,*

$$\mathrm{codim}(\mathrm{Sing}\, F \subset F) \geqslant 4k + 2\varepsilon(k).$$

Proof is given in §4 (Subsections 4.1-4.3).

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Assume that $\underline{f}$ satisfies the condition (MQ1). For a point $o \in F = F(\underline{f})$, the symbol T_oF stands for the subspace $\{f_{1,1} = \cdots = f_{k,1} = 0\} \subset \mathbb{C}^{M+k}$. For the proof of Theorem 0.1, we will need one more property of the tuple $\underline{f}$, which we include in the definition of the subset $\mathcal{F}$.

(MQ2) For any point $o \in F$, which is a multi-quadratic of type 2^l, where $l \geqslant 2$, the rank of the tuple of quadratic forms

$$f_{1,2}|_{T_oF}, \dots, f_{k,2}|_{T_oF}$$

is at least $10k^2 + 8k + 2\varepsilon(k) + 5$.

The condition (MQ2) for multi-quadratic points of type 2^l with $l \geqslant 2$ implies the condition (MQ1) because the rank of a quadratic form, restricted to a hyperplane, drops at most by 2; however, for the convenience of references, we state the conditions (MQ1) and (MQ2) independently of each other. These conditions are used in the proof of Theorem 0.1 in different ways.

So every tuple $\underline{f} \in \mathcal{F}$ satisfies (MQ1) and (MQ2).

1.3. Subvarieties of codimension 2

Following the plan, given in Subsection 1.1, let us fix an effective divisor $D_F \sim n(D_F)H_F$, $n(D_F) \geqslant 1$, such that the pair $(F, \frac{1}{n(D_F)}D_F)$ is not canonical. By the symbol

$$\mathrm{CS}\left(F, \frac{1}{n(D_F)}D_F\right),$$

we denote the union of the centres on F of all exceptional divisors over F, satisfying the Noether-Fano inequality (that is to say, of all non-canonical singularities of that pair). This is a closed subset of F. Let B be an irreducible component of maximal dimension of that set.

Proposition 1.1. *The following inequality holds:* $\mathrm{codim}(B \subset F) \geqslant 3$.

Proof. Assume the converse: $\mathrm{codim}(B \subset F) = 2$. Then $B \not\subset \mathrm{Sing}\, F$. Moreover, let $P \subset \mathbb{P}^{M+k}$ be a general linear subspace of dimension $2k + 2$. Theorem 1.1 implies that $P \cap \mathrm{Sing}\, F = \emptyset$, so that $F \cap P$ is a non-singular complete intersection of type $\underline{d}$ in $P \cong \mathbb{P}^{2k+2}$. Furthermore, the pair

$$\left(F \cap P, \frac{1}{n(D_F)}D_F|_{F\cap P}\right)$$

is not canonical, and the irreducible subvariety $B \cap P$ is an irreducible component of maximal dimension of the set

$$\mathrm{CS}\left(F \cap P, \frac{1}{n(D_F)}D_F|_{F\cap P}\right),$$

so that (as $F \cap P$ is non-singular)

$$\mathrm{mult}_{B\cap P}\, D_F|_{F\cap P} > n(D_F).$$

However, $D_F|_{F\cap P} \sim n(D_F)H_{F\cap P}$ (where $H_{F\cap P}$ is the class of a hyperplane section of $F \cap P$), so that by [17, Proposition 3.6] or [18], we get a contradiction, proving the proposition. Q.E.D. □

1.4. Regularity conditions

In order to continue the proof of Theorem 0.1, we need a second group of conditions defining the set $\mathcal{F}$. Let $o \in F$ be a point. We use the notations of Subsection 1.2. By the symbol T_oF we denote the *linear* tangent space

$$\{f_{1,1} = \cdots = f_{k,1} = 0\} \subset \mathbb{C}^{M+k},$$

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and by the symbol $\mathbb{P}(T_oF)$ its projectivization. Let $\mathcal{S} = (h_1, \dots, h_M)$ be the sequence of homogeneous polynomials

$$f_{i,j}|_{\mathbb{P}(T_oF)},$$

where $j \geqslant 2$, placed in the lexicographic order: (i_1, j_1) precedes (i_2, j_2), if $j_1 < j_2$ or $j_1 = j_2$, but $i_1 < i_2$. By the symbol $\mathcal{S}[-m]$ denote the sequence $\mathcal{S}$ with the last m members removed. Finally, the symbol $\mathcal{S}[-m]|_\Pi$ stands for the restriction of that sequence (that is, the restriction of each its member) onto a linear subspace $\Pi \subset \mathbb{P}(T_oF)$. The regularity conditions depend on the type of the singularity $o \in F$.

First, let the point $o \in F$ be non-singular, so that $\mathbb{P}(T_oF) \cong \mathbb{P}^{M-1}$. In that case, the regularity condition is stated in the following way.

(R1) The sequence

$$\mathcal{S}[-(k + \varepsilon(k) + 3)]|_\Pi$$

is regular for every subspace $\Pi \subset \mathbb{P}(T_oF)$ of codimension $k + \varepsilon(k) - 1$.

The condition (R1) is assumed for every non-singular point $o \in F$. It implies the following key fact.

Theorem 1.2. *Let $P \subset \mathbb{P}^{M+k}$ be an arbitrary linear subspace of codimension $\leqslant k + \varepsilon(k) - 1$. Then for every non-singular point $o \in F \cap P$ and every prime divisor $Y \sim n(Y)H_{F\cap P}$ on $F \cap P$, the inequality*

$$\operatorname{mult}_o Y \leqslant 2n(Y)$$

holds.

Proof is given in §7 (Subsections 7.1, 7.2).

Now let $o \in F$ be a quadratic singularity (this case corresponds to the value $l = 1$ in Definition 1.2). Here $\mathbb{P}(T_oF) \cong \mathbb{P}^M$. In this case, the regularity condition is stated as follows.

(R2) The sequence

$$\mathcal{S}[-4]|_\Pi$$

is regular for every hyperplane $\Pi \subset \mathbb{P}(T_oF)$.

The condition (R2) is assumed for every quadratic singular point $o \in F$ and implies the following key fact.

Theorem 1.3. *Let $o \in F$ be a quadratic singularity and $W \ni o$ the section of F by a hyperplane that is not tangent to F at the point o, and $Y \sim n(Y)H_W$ a prime divisor on W. Then the following inequality holds:*

$$\operatorname{mult}_o Y \leqslant 4n(Y).$$

Proof is given in §7 (Subsection 7.3).

(The symbol H_W stands for the class of a hyperplane section of the variety W; the linear form, defining the hyperplane that cuts out W, is not a linear combination of the forms $f_{1,1}, \dots, f_{k,1}$.)

Now let $o \in F$ be a multi-quadratic point of type 2^l, where $l \in \{2, \dots, k\}$. Here we will need two regularity conditions. In the first of them, the symbol T_oF means the projective closure of the linear subspace

$$\{f_{1,1} = \cdots = f_{k,1} = 0\} \subset \mathbb{C}^{M+k}$$

in $\mathbb{P}^{M+k}$.

(R3.1) For every subspace $P \subset T_oF$ of codimension $\varepsilon(k)$, containing the point o, the scheme of common zeros of the polynomials

$$f_1|_P, \dots, f_k|_P, \quad f_{i,2}|_P \quad \text{for all} \quad i : d_i \geqslant 3$$

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is an irreducible reduced subvariety of codimension $k + k_{\geqslant 3}$ in P, where

$$k_{\geqslant 3} = \sharp\{i = 1, \dots, k \mid d_i \geqslant 3\}.$$

Note that in the condition (R3.1), the homogeneous polynomials $f_{i,2}$ in the *affine* coordinates z_* are considered as quadratic forms in *homogeneous* coordinates on $\mathbb{P}^{M+k}$.

In the second regularity condition for multi-quadratic points, the symbol T_oF means a linear subspace in $\mathbb{C}^{M+k}$.

(R3.2) For every linear subspace $\Pi \subset \mathbb{P}(T_oF)$ of codimension $\varepsilon(k)$, the sequence

$$\mathcal{S}[-m^*]|_{\Pi}$$

is regular, where $m^* = \max\{\varepsilon(k) + 4 - l, 0\}$.

The conditions (R3.1) and (R3.2) are assumed for every multi-quadratic singular point $o \in F$. They imply the following key inequality. In Theorem 1.4, stated below, the symbol T_oF stands for the projective closure of the embedded tangent space – that is, a linear subspace in $\mathbb{P}^{M+k}$, containing the point o.

Theorem 1.4. *Let $P \subset T_oF$ be an arbitrary linear subspace of codimension $\leqslant \varepsilon(k)$ and $Y \ni o$ a prime divisor on $F \cap P$, $Y \sim n(Y)H_{F\cap P}$. Then the following inequality holds:*

$$\operatorname{mult}_o Y \leqslant \frac{3}{2} \cdot 2^k n(Y).$$

Proof is given in §7 (Subsections 7.3).

(The symbol $H_{F\cap P}$ stands for the class of a hyperplane section of the variety $F \cap P$; we will show below – see §4 – that $F \cap P$ is an irreducible factorial complete intersection.)

Summing up, let us give a complete definition of the subset $\mathcal{F} \subset \mathcal{P}$: it consists of the tuples $\underline{f}$, satisfying the conditions (MQ1,2), the condition (R1) at every non-singular point $o \in F(\underline{f})$, the condition (R2) at every quadratic point $o \in F(\underline{f})$ and the conditions (R3.1,2) at every multi-quadratic point $o \in F(\underline{f})$.

The inequality for the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$, given in Theorem 0.1, is shown in §8.

1.5. Exclusion of the non-singular case

We carry on with the proof of divisorial canonicity of the variety $F \in \mathcal{F}$. In the notations of Subsection 1.3, assume that the point of general position $o \in B$ is a non-singular point of F. We know (Proposition 1.1) that $\operatorname{codim}(B \subset F) \geqslant 3$. Consider a general subspace $P \ni o$ of dimension $k + 3$. Then $F \cap P$ is a non-singular three-dimensional variety, and the point o is a connected component of the set

$$\mathrm{CS}\left(F \cap P, \frac{1}{n(D_F)} D_F|_{F\cap P}\right)$$

(if $\operatorname{codim}(B \subset F) \geqslant 4$, then CS can be replaced, by inversion of adjunction, by LCS); that is, outside the point o in a neighborhood of that point, the pair

$$\left(F \cap P, \frac{1}{n(D_F)} D_F|_{F\cap P}\right) \tag{6}$$

is canonical. It is well known (see [19, Proposition 3] or [20, Chapter 7, Proposition 2.3]), and it follows from here that either the inequality

$$\operatorname{mult}_o D_F > 2n(D_F)$$

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holds or on the exceptional divisor $E \cong \mathbb{P}^{M-1}$ of the blow-up $F^+ \to F$ of the point o, there is a hyperplane $\Theta \subset E$ (uniquely determined by the pair (6)), such that the inequality

$$\mathrm{mult}_o D_F + \mathrm{mult}_\Theta D_F^+ > 2n(D_F)$$

holds, where D_F^+ is the strict transform of D_F on F^+.

The first option is impossible as it contradicts Theorem 1.2. In the second case, denote by the symbol $|H - \Theta|$ the projectively k-dimensional linear system of hyperplane sections of F, a general element of which $W \ni o$ is non-singular at the point o and satisfies the equality

$$W^+ \cap E = \Theta.$$

The restriction $D_W = (D_F \circ W)$ is an effective divisor on W, and $n(D_W) = n(D_F)$ and the inequality

$$\mathrm{mult}_o D_W \geqslant \mathrm{mult}_o D_F + \mathrm{mult}_\Theta D_F^+ > 2n(D_W)$$

holds, which again contradicts Theorem 1.2. We have shown the following fact.

Proposition 1.2. *The subvariety B is contained in the singular locus of F: $B \subset \mathrm{Sing}\, F$.*

1.6. Exclusion of the quadratic case

Again, let $o \in B$ be a point of general position.

Proposition 1.3. *The point o is a multi-quadratic singularity of type 2^l, $l \geqslant 2$.*

Proof. Assume the converse: the point o is a quadratic singularity of F. Let $P \ni o$ be a general $(k+3)$-dimensional linear subspace in $\mathbb{P}^{M+k}$. By the condition (MQ1) and Theorem 1.1, the intersection $F \cap P$ is a three-dimensional variety with the unique singular point o, which is a non-degenerate quadratic singularity. This intersection can be constructed in two steps: first, we consider the intersection $F \cap P'$ with a general linear subspace $P' \subset \mathbb{P}^{M+k}$, $P' \ni o$ of dimension

$$k + \mathrm{codim}(\mathrm{Sing}\, F \subset F),$$

and after that the intersection with a general subspace $P \subset P'$, $P \ni o$ of dimension $(k+3)$. Now we get the following: the pair

$$\left(F \cap P, \frac{1}{n(D_F)} D_F|_{F \cap P}\right)$$

is not log-canonical, but canonical outside the point o. Let us consider the blow-up

$$\varphi_P \colon P^+ \to P$$

of the point o with the exceptional divisor $\mathbb{E}_P \cong \mathbb{P}^{k+2}$ and let $(F \cap P)^+ \subset P^+$ be the strict transform of $F \cap P$ on P^+, so that $(F \cap P)^+ \to F \cap P$ is the blow-up of the quadratic singularity o with the exceptional divisor $E_P = (F \cap P)^+ \cap \mathbb{E}_P$, which is a non-singular two-dimensional quadric in the three-dimensional subspace $\langle E_P \rangle \subset \mathbb{E}_P$. Obviously, $a(E_P, F \cap P) = 1$, so that, writing down

$$D_P = D_F|_{F \cap P} \sim n(D_F) H_{F \cap P}$$

and $D_P^+ \sim n(D_F) H_{F \cap P} - \nu E_P$ (the strict transform of D_P on $(F \cap P)^+$), we obtain two options:

- either $\nu > 2n(D_F)$, so that E_P is a non-log-canonical singularity of the pair $(F \cap P, \frac{1}{n(D_F)} D_P)$,
- or $n(D_F) < \nu \leqslant 2n(D_F)$, and then the closed set

$$\mathrm{LCS}\left(\left(F \cap P, \frac{1}{n(D_F)} D_P\right), (F \cap P)^+\right)$$

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– the union of the centres of all non-log-canonical singularities of the original pair $(F \cap P, \frac{1}{n(D_F)} D_P)$ on $(F \cap P)^+$ – is a connected closed subset of the non-singular quadric E_P, which can be either a (possibly reducible) connected curve $C_P \subset E_P$ or a point $x_P \in E_P$. □

(It is well known – see, for instance, [20, Chapter 2, Proposition 3.7] – that the inequality $\nu \leqslant n(D_F)$ is impossible.) In the case $\nu > 2n(D_F)$, we get

$$\mathrm{mult}_o D_P = \mathrm{mult}_o D_F > 4n(D_F),$$

which contradicts Theorem 1.3, so that this case is impossible. Coming back to the original variety F, let us consider the blow-ups $\varphi_{\mathbb{P}} \colon (\mathbb{P}^{M+k})^+ \to \mathbb{P}^{M+k}$ and $\varphi \colon F^+ \to F$ of the point o, where F^+ is identified with the strict transform of F on $(\mathbb{P}^{M+k})^+$, with the exceptional divisors $\mathbb{E}$ and E, respectively, so that $E = F^+ \cap \mathbb{E}$ is a quadratic hypersurface E in the subspace $\langle E \rangle \subset \mathbb{E}$ of codimension $(k+1)$. By the condition for the rank (MQ1), the case of a point $x_P \in E_P$ is impossible: in that case, the quadric E would contain a linear subspace of codimension 2 (with respect to E), which cannot happen. Now, arguing word for word as in [2, Subsection 3.2] and using [2, Theorem 3.1], we get that on the quadric E, there is a hyperplane section $\Lambda \subset E$, such that

$$\nu + \mathrm{mult}_\Lambda D_F^+ > 2n(D_F).$$

Taking the linear system $|H_F - \Lambda|$ (of the projective dimension $(k-1)$) of hyperplane sections of the variety F, a general divisor $W \in |H_F - \Lambda|$ in which contains the point o and its strict transform W^+ cuts out Λ on E (that is, $W^+ \cap E = \Lambda$), we set $D_W = (D_F \circ W)$ and obtain the inequality

$$\mathrm{mult}_o D_W = 2(\nu + \mathrm{mult}_\Lambda D_F^+) > 4n(D_F) = 4n(D_W),$$

which contradicts Theorem 1.3. This completes the proof of Proposition 1.3.

1.7. Exclusion of the multi-quadratic case

This is the hardest and the longest part of our work. Fix a point $o \in B$ of general position, which by what was proven is a multi-quadratic singularity of type 2^l, satisfying the conditions (MQ1,2). The pair $(F, \frac{1}{n(D_F)} D_F)$ has a non-canonical singularity, the centre B of which is a component of the maximal dimension of the set $\mathrm{CS}(F, \frac{1}{n(D_F)} D_F)$, so that in a neighborhood of the point o, this pair is canonical outside B. We will show that this is impossible. This will be done in a few steps, and now we describe the scheme of the proof and state the key intermediate claims.

Definition 1.3. A pair $[X, o]$, where

$$X \subset \mathbb{P}(X) = \mathbb{P}^{N(X)}$$

is an irreducible reduced factorial complete intersection of type $\underline{d}$ in the projective space $\mathbb{P}(X)$, $\dim X = N(X) - k \geqslant 3$, and $o \in X$ is a point, is called a *complete intersection with a marked point* or, for brevity, a *marked complete intersection of level* (l_X, c_X)), where l_X, c_X are positive integers, satisfying the inequalities

$$2 \leqslant l_X \leqslant k \quad \text{and} \quad c_X \geqslant l_X + 4,$$

if the following conditions are satisfied:

(MC1) the inequality

$$\mathrm{codim}(\mathrm{Sing}\, X \subset X) \geqslant c_X$$

holds,

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(MC2) the point $o \in X$ is a multi-quadratic singularity of type 2^{l_X}, the rank of which satisfies the inequality

$$\mathrm{rk}(o \in X) \geqslant 2l_X + c_X - 1,$$

(MC3) the non-singular part $X \setminus \mathrm{Sing}\, X$ of the variety X satisfies the condition of divisorial canonicity,

$$\mathrm{ct}(X \setminus \mathrm{Sing}\, X) \geqslant 1;$$

that is, for every effective divisor $A \sim aH_X$, we have $\mathrm{CS}(X, \frac{1}{a}A) \subset \mathrm{Sing}\, X$.

The non-singular set of integers

$$I_X = [k + l_X + 3, k + c_X - 1] \cap \mathbb{Z}$$

is called the *admissible set* of the marked complete intersection $[X, o]$.

Remark 1.1. (i) Since $X \subset \mathbb{P}(X)$ is a complete intersection, the factoriality of the variety X follows from Grothendieck's theorem [21] by the condition (MC1). For that reason, $\mathrm{Pic}\, X = \mathbb{Z}H_X$, where H_X is the class of a hyperplane section.

(ii) By (MC1), for every $m \leqslant k + c_X - 1$ and a general subspace $P \ni o$ of dimension m in $\mathbb{P}(X)$, the point o is the only singularity of the variety $X \cap P$.

(iii) Let $\mathbb{P}(X)^+ \to \mathbb{P}(X)$ be the blow-up of the point o with the exceptional divisor $\mathbb{E}_X \cong \mathbb{P}^{N(X)-1}$. The strict transform $X^+ \subset \mathbb{P}(X)^+$ is the result of blowing up the point o on X with the exceptional divisor $E_X = X^+ \cap \mathbb{E}_X$. Obviously, E_X is an irreducible reduced non-degenerate complete intersection of l_X quadrics in a linear subspace of codimension $(k - l_X)$ in $\mathbb{E}_X$ (this follows from (MC2); see Proposition 1.4).

Proposition 1.4. *The following inequality holds:*

$$\mathrm{codim}(\mathrm{Sing}\, E_X \subset E_X) \geqslant c_X.$$

Proof. See in §4 (Subsection 4.2; by the condition (MC2) the claim of the proposition follows from Proposition 4.2, (ii)). □

Remark 1.2. Proposition 1.4 implies the estimate

$$\mathrm{codim}(\mathrm{Sing}\, E_X \subset \mathbb{E}_X) \geqslant k + c_X.$$

Therefore, for every $m \leqslant k + c_X$ and a general subspace $P \ni o$ of dimension m in $\mathbb{P}(X)$, the strict transform $P^+ \subset \mathbb{P}(X)^+$ does not meet the set $\mathrm{Sing}\, E_X$, since $P^+ \cap \mathbb{E}_X$ is a general linear subspace of dimension $m - 1 \leqslant k + c_X - 1$ in $\mathbb{E}_X$. Therefore, for $m = k + c_X$ an isolated, and for $m \leqslant k + c_X - 1$ the unique singularity o of the variety $X \cap P$ is resolved by the blow-up of that point, and moreover, the exceptional divisor

$$E_{X \cap P} = P^+ \cap E_X$$

of that blow-up is a non-singular complete intersection of l_X quadrics in the linear subspace of codimension $(k - l_X)$ in $\mathbb{E}_{X \cap P} = P^+ \cap \mathbb{E}_X$. The discrepancy of that exceptional divisor is

$$a(E_{X \cap P}) = a(E_{X \cap P}, X \cap P) = m - 1 - k - l_X,$$

so that for $m = k + l_X + 3$, we have $a(E_{X \cap P}) = 2$. The meaning of the lower end of the admissible set is in that equality.

In the following definition, we use the notations of Remarks 1.1 and 1.2. We continue to consider a marked complete intersection $[X, o]$ of level (l_X, c_X).

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Definition 1.4. A triple (X, D, o), where $D \sim n(D)H_X$ is an effective divisor on X, $n(D) \geqslant 1$, is called a *working triple*, if for a general subspace $P \ni o$ of dimension $k + c_X - 1$ in $\mathbb{P}(X)$, the pair

$$\left(X \cap P, \frac{1}{n(D)} D|_{X\cap P}\right) \tag{7}$$

is not log-canonical at the point o.

Remark 1.3. Since the point o is the unique singularity of the variety $X \cap P$, and by (MC3) the pair (7) is canonical outside the point o, there is a non-log-canonical singularity of that pair, the centre of which on $X \cap P$ is precisely the point o. By inversion of adjunction, the same is true for a general subspace $P \ni o$ of dimension $m \leqslant k + c_X - 2$.

Let us introduce one more notation. For the strict transform D^+ of the divisor D on X^+, write

$$D^+ \sim n(D)H_X - \nu(D)E_X$$

(in order to simplify the notations, the pullback of the divisorial class H_X on X^+ is denoted by the same symbol H_X). Respectively, for a general subspace $P \ni o$ in $\mathbb{P}(X)$ of dimension $m \leqslant k + c_X - 1$, we have

$$D_P = D|_{X\cap P} \sim n(D)H_{X\cap P}$$

and

$$D_P^+ \sim n(D)H_{X\cap P} - \nu(D)E_{X\cap P},$$

where $H_{X\cap P} = H_X|_{X\cap P}$ is the class of a hyperplane section of the variety $X \cap P \subset P \cong \mathbb{P}^m$.

Proposition 1.5. *Assume that $c_X \geqslant 2l_X + 4$. Then the inequality $\nu(D) > n(D)$ holds.*

Proof is given in §3 (Subsection 3.2).

Let us come back to the task of excluding the multi-quadratic case. Recall that $F \in \mathcal{F}$, so that we can use the conditions (MQ1,2) and the statement of Theorem 1.4. We fix a point of general position $o \in B$, where B is an irreducible component of the maximal dimension of the closed set $\mathrm{CS}(F, \frac{1}{n(D_F)} D_F)$.

Proposition 1.6. *The pair $[F, o]$ is a marked complete intersection of level (l, c_F), where $c_F = 4k + 2\varepsilon(k)$, and (F, D_F, o) is a working triple.*

Proof is given in §3 (Subsection 3.1).

Assume now that $l \leqslant k - 1$. The symbol T_oF stands again for a subspace of codimension $(k - l)$ of the projective space $\mathbb{P}^{M+k}$. Set

$$T = F \cap T_oF.$$

This is subvariety of codimension $(k - l)$ in F and a complete intersection of type $\underline{d}$ in $\mathbb{P}(T) = T_oF$.

Remark 1.4. Let us state here two well-known facts which we will use many times in the sequel: when a quadratic form is restricted to a hyperplane, its rank either remains the same or drops by 1 or 2; when a complete intersection in the projective space is intersected with a hyperplane, the codimension of its singular locus either remains the same or drops by 1 or 2 (for a proof of the second claim, see [22] or [23]).

If $l = k$, then for uniformity of notations, we set $T = F$.

Proposition 1.7. *The pair $[T, o]$ is a marked complete intersection of level (k, c_T), where $c_T = 2k + 2\varepsilon(k) + 4$. There is an effective divisor $D_T \sim n(D_T)H_T$ on T, such that (T, D_T, o) is a working triple.*

Proof is given in §3 (Subsection 3.4).

Proposition 1.5 (taking into account Remark 1.4) implies that $\nu(D_T) > n(D_T)$. Now the main stage in the exclusion of the multi-quadratic case (and thus in the proof of Theorem 0.1) is given by the following claim.

Proposition 1.8. *There is a sequence of marked complete intersections*

$$[R_0 = T, o], \quad [R_1, o], \quad \dots, \quad [R_a, o],$$

where $a \leqslant \varepsilon(k)$ and $\mathbb{P}(R_{i+1})$ is a hyperplane in $\mathbb{P}(R_i)$, containing the point o, and of effective divisors $D_i \sim n(D_i)H_{R_i}$ on R_i, $n(D_i) \geqslant 1$, such that $D_0 = D_T$ and

$$(R_0, D_0, o), \quad (R_1, D_1, o), \quad \dots, \quad (R_a, D_a, o)$$

are working triples, and moreover, for every $i = 0, \dots, a-1$, the inequality

$$2 - \frac{\nu(D_{i+1})}{n(D_{i+1})} < \frac{1}{1+\frac{1}{k}}\left(2 - \frac{\nu(D_i)}{n(D_i)}\right)$$

holds and $\nu(D_a) > \frac{3}{2}n(D_a)$.

Proof is given in §3 (Subsection 3.5) and §5.

Now let us complete the exclusion of the multi-quadratic case. The variety R_a is a section of $T = F \cap T_oF$ by a subspace of codimension $\leqslant \varepsilon(k)$, containing the point o, and D_a is an effective divisor on R_a, satisfying the inequality

$$\operatorname{mult}_o D_a = 2^k\nu(D_a) > \frac{3}{2} \cdot 2^k n(D_a).$$

This contradicts Theorem 1.4.

The contradiction completes the proof of divisorial canonicity of the variety $F \in \mathcal{F}$.

2. Fano-Mori fibre spaces

In this section, we prove Theorem 0.2. In Subsection 2.1, we associate with a birational map $\chi\colon V \dashrightarrow V'$ a mobile linear system Σ on V and state the key Theorem 2.1 about this system. In Subsection 2.2, we construct a fibre-wise birational modification of the fibre space V/S for the system Σ. In Subsection 2.3, we consider a mobile algebraic family of irreducible curves $\mathcal{C}$ on V and use it to prove (in Subsection 2.4) Theorem 2.1, which implies the first claim of Theorem 0.2 (that χ is fibre-wise). In Subsection 2.5, we prove the birational rigidity of the fibre space V/S.

2.1. The mobile linear system Σ

Assume that the Fano-Mori fibre space $\pi\colon V \to S$ satisfies all conditions of Theorem 0.2. Fix a fibre space $\pi'\colon V' \to S'$ that belongs to one of the two classes: either the class of rationally connected fibre spaces (and then we say that the rationally connected case is being considered) or the class of Mori fibre spaces in the sense of Subsection 0.2 (and then we say that the case of a Mori fibre space is being considered). We will study both cases simultaneously.

In the rationally connected case, let $Y' \in \operatorname{Pic} S'$ be a very ample class. Set

$$\Sigma' = |(\pi')^*Y'| = |-mK'_V + (\pi')^*Y'|,$$

where $m = 0$. This is a mobile complete linear system on V' (it defines the morphism π').

In the case of a Mori fibre space, let

$$\Sigma' = |-mK'_V + (\pi')^*Y'|$$

be a complete linear system on V', where $m \geqslant 0$ and Y' is a very ample divisorial class on S', and moreover, for $m \geqslant 1$, the system Σ' is very ample.

In both cases, set

$$\Sigma = (\chi^{-1})_* \Sigma' \subset |-nK_V + \pi^* Y|$$

to be the strict transform of Σ' on V with respect to the birational map $\chi\colon V \dashrightarrow V'$. Note that if $m = 0$ and $n = 0$, then by construction of these linear systems, the map χ is fibre-wise.

Theorem 2.1. *The following inequality holds: $n \leqslant m$.*

Proof. Assume the converse: $n > m$. In particular, if $m = 0$, then χ is not fibre-wise. Let us show that this assumption leads to a contradiction. □

2.2. *A fibre-wise birational modification of the fibre space* V/S

Let $\sigma_S\colon S^+ \to S$ be a composition of blow-ups with non-singular centres,

$$S^+ = S_N \overset{\sigma_{S,N}}{\to} S_{N-1} \to \ldots \overset{\sigma_{S,1}}{\to} S_0 = S,$$

where $\sigma_{S,i+1}\colon S_{i+1} \to S_i$ blows up a non-singular subvariety $Z_{S,i} \subset S_i$. Set $V_i = V \times_S S_i$ and $\pi_i\colon V_i \to S_i$; by the assumption on the stability with respect to birational modifications of the base, V_i/S_i is a Fano-Mori fibre space. Obviously,

$$V_{i+1} = V_i \times_{S_i} S_{i+1}$$

is the result of the blow-up $\sigma_{i+1}\colon V_{i+1} \to V_i$ of the subvariety $Z_i = \pi_i^{-1}(Z_{S,i}) \subset V_i$. Therefore, we get the commutative diagram

$$\begin{array}{ccccccccccc} V^+ & = & V_N & \overset{\sigma_N}{\to} & \ldots \to & V_{i+1} & \overset{\sigma_{i+1}}{\to} & V_i & \to \ldots & \overset{\sigma_1}{\to} & V_0 & = & V \\ & & \downarrow & & \ldots & \downarrow & & \downarrow & \ldots & & \downarrow & & \\ S^+ & = & S_N & \overset{\sigma_{S,N}}{\to} & \ldots \to & S_{i+1} & \overset{\sigma_{S,i+1}}{\to} & S_i & \to \ldots & \overset{\sigma_{S,1}}{\to} & S_0 & = & S, \end{array}$$

where the vertical arrows $\pi\colon V_i \to S_i$ are Fano-Mori fibre spaces. The symbol Σ^i stands for the strict transform of the system Σ on V_i, $\Sigma^+ = \Sigma^N$. In these notations, let us consider a sequence of blow-ups $\sigma_{S,*}$ such that for every $i = 0, 1, \ldots, N-1$,

$$Z_i \subset \operatorname{Bs} \Sigma^i,$$

and the base set of the system Σ^+ contains entirely no fibre $\pi_+^{-1}(s_+)$, where $s_+ \in S^+$ and $\pi_+ = \pi_N$. (If this is true already for the original system Σ, then we set $\sigma_S = \mathrm{id}_S$, $S^+ = S$ and $V^+ = V$, and there is no need to make any blow-ups; but we will soon see that this case is impossible.)

By the assumptions on the fibre space V/S, the fibre $\pi_+^{-1}(s_+)$ is isomorphic to the fibre $F_s = \pi^{-1}(s)$ of the original fibre space, where $s = \sigma_S(s_+)$. Let $\mathcal{T}$ be the set of all prime σ_S-exceptional divisors on S^+. We get

$$\Sigma^+ \subset \left| -n\sigma^* K_V + \pi_+^* \left(\sigma_S^* Y - \sum_{T \in \mathcal{T}} b_T T \right) \right| =$$

$$= \left| -nK^+ + \pi_+^* \left(\sigma_S^* Y + \sum_{T \in \mathcal{T}} (na_T - b_T) T \right) \right|,$$

where $\sigma\colon V^+ \to V$ is the composition of the morphisms σ_i, $K^+ = K_{V^+}$, $b_T \geqslant 1$ and $a_T \geqslant 1$ for all $T \in \mathcal{T}$, $a_T = a(T, S)$ is the discrepancy of T with respect to S.

Let $\varphi\colon \widetilde{V} \to V^+$ be the resolution of singularities of the composite map $\chi_+ = \chi \circ \sigma\colon V^+ \dashrightarrow V'$, $\mathcal{E}$ the set of prime φ-exceptional divisors on $\widetilde{V}$ and $\psi = \chi \circ \sigma \circ \varphi\colon \widetilde{V} \to V'$ is a birational morphism.

Proposition 2.1. *For a general divisor $D^+ \in \Sigma^+$, the pair $(V^+, \frac{1}{n}D^+)$ is canonical.*

Proof. Assume that this is not the case. Then there is an exceptional divisor $E \in \mathcal{E}$, satisfying the Noether-Fano inequality

$$\mathrm{ord}_E\, D^+ = \mathrm{ord}_E\, \Sigma^+ > na(E, V^+)$$

(we write D^+, Σ^+ instead of $\varphi^* D^+, \varphi^*\Sigma^+$ for simplicity). Set $B = \varphi(E) \subset V^+$.

There are two options:

(1) $\pi_+(B) = S^+$,

(2) $\pi_+(B)$ is a proper irreducible closed subset S^+.

If (1) is the case, then the fibre $F = F_s$ of general position intersects B. The restriction

$$\Sigma^+_F = \Sigma^+|_F \subset |-nK_F|$$

is a mobile linear system, and moreover, the pair $(F, \frac{1}{n}D^+_F)$ is not canonical for $D^+_F = D^+|_F$. This contradicts the condition $\mathrm{mct}(F) \geqslant 1$.

Therefore, (2) is the case. Let $p \in B$ be a point of general position and $F = \pi_+^{-1}(\pi_+(p))$, so that $p \in F$. Since $F \not\subset \mathrm{Bs}\,\Sigma^+$, the restriction $D^+_F = D^+|_F$ is well defined (although the linear system Σ^+_F may have fixed components). By inversion of adjunction, the pair $(F, \frac{1}{n}D^+_F)$ is not log-canonical. This contradicts the condition $\mathrm{lct}(F) \geqslant 1$. Q.E.D. for the proposition. □

Denote by the symbol $\widetilde{\Sigma}$ the strict transform of the system Σ^+ on $\widetilde{V}$. Obviously,

$$\widetilde{\Sigma} = \psi^*\Sigma' = |-m\psi^* K' + \psi^*(\pi')^* Y'|, \tag{8}$$

where $K' = K_{V'}$; that is, $\widetilde{\Sigma}$ is a complete linear system. We have another presentation for this linear system:

$$\widetilde{\Sigma} = \left|\varphi^* D^+ - \sum_{E\in\mathcal{E}} b_E E\right| =$$

$$= \left|-n\widetilde{K} + \varphi^*\pi_+^*\left(\sigma_S^* Y + \sum_{T\in\mathcal{T}}(na_T - b_T)T\right) + \sum_{E\in\mathcal{E}}(na_E - b_E)E\right|, \tag{9}$$

where $\widetilde{K} = K_{\widetilde{V}}$, $D^+ \in \Sigma^+$ is a general divisor and $a_E = a(E, V^+)$ is the discrepancy.

2.3. The mobile system of curves

Take a family of irreducible curves $\mathcal{C}'$ on V', contracted by the projection π', sweeping out a Zariski dense subset of the variety V' and not meeting the set where the birational map ψ^{-1} is not determined. Assume that for a general pair of points p, q in a fibre of general position of the projection π', there is a curve $C' \in \mathcal{C}'$ containing the both points. In the rationally connected case, the curves of the family $\mathcal{C}'$ are rational (the existence of such family is shown in [24, Chapter II]); in the case of a Mori fibre space, we do not require this. For a curve $C' \in \mathcal{C}'$, set $\widetilde{C} = \psi^{-1}(C')$ (at every point of the curve C', the map ψ^{-1} is an isomorphism); thus, we get a family $\widetilde{\mathcal{C}}$ of irreducible curves on $\widetilde{V}$. Both in the rationally connected case and the case of a Mori fibre space, the inequality

$$(C' \cdot K') < 0$$

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holds, so that $(\widetilde{C}\cdot\widetilde{K})=(C'\cdot K')<0$. Furthermore,.

$$(\widetilde{C}\cdot\widetilde{D})=(C'\cdot D')=-m(C'\cdot K')\geqslant 0,$$

and $(\widetilde{C}\cdot\widetilde{D})=0$ if and only if $m=0$ (since obviously $(C'\cdot(\pi')^*Y')=0$).

Let $\mathcal{C}^+=\varphi_*\widetilde{\mathcal{C}}$ be the image of the family $\widetilde{\mathcal{C}}$ on V^+ and $\mathcal{C}=\sigma_*\mathcal{C}^+$ its image on V.

Proposition 2.2. *The curves $C\in\mathcal{C}$ are not contracted by the projection π.*

Proof. Assume the converse: $\pi(C)$ is a point on S. By the construction of the family $\mathcal{C}'$ this means that the map χ^{-1} is fibre-wise: there is a rational dominant map $\beta'\colon S'\dashrightarrow S$, such that the diagram

$$\begin{array}{ccc} V & \overset{\chi^{-1}}{\dashleftarrow} & V' \\ \downarrow & & \downarrow \\ S & \overset{\beta'}{\dashleftarrow} & S' \end{array}$$

is commutative, and moreover, $\dim S'>\dim S$ (otherwise, β' is birational and then χ is fibre-wise, contrary to our assumption). In that case, for a point $s\in S$ of general position, the fibre $F_s=\pi^{-1}(s)$ is birational to $(\pi')^{-1}(\beta')^{-1}(s)$. Here, $\dim(\beta')^{-1}(s)\geqslant 1$ and either the fibre $(\pi')^{-1}(s')$ for a point $s'\in(\beta')^{-1}(s)$ of general position is rationally connected or the anti-canonical class of the variety $(\pi')^{-1}(\beta')^{-1}(s)$ is π'-ample, and we get a contradiction with the condition $\mathrm{mct}(F_s)\geqslant 1$ (the fibre F_s is a birationally superrigid Fano variety). Q.E.D. for the proposition. □

For a general curve $C\in\mathcal{C}$, set

$$\pi_*C=d_C\overline{C},$$

where $d_C\geqslant 1$. Replacing, if necessary, the family $\mathcal{C}'$ by some open subfamily, we may assume that the integer d_C does not depend on C. For the corresponding curve $C^+\in\mathcal{C}^+$, we have $(\pi_+)_*C^+=d_C\overline{C}^+$, where $\overline{C}^+$ is the strict transform of the curve $\overline{C}$ on S^+.

2.4. *Proof of Theorem 2.1*

Recall that we assume that $n>m$. Using the two presentations (8) and (9) for the class of a divisor $\widetilde{D}\in\widetilde{\Sigma}$, we get

$$d_C\left(\overline{C}^+\cdot\left(\sigma_S^*Y+\sum_{T\in\mathcal{T}}(na_T-b_T)T\right)\right)+\sum_{E\in\mathcal{E}}(na_E-b_E)(\widetilde{C}\cdot E)=(n-m)(\widetilde{C}\cdot\widetilde{K})<0,$$

whence, taking into account the inequalities $b_E\leqslant na_E$ for all $E\in\mathcal{E}$ (Proposition 2.1), it follows that

$$\left(\overline{C}^+\cdot\left(\sigma_S^*Y+\sum_{T\in\mathcal{T}}(na_T-b_T)T\right)\right)<0.$$

However, by the K-condition (the assumption (ii) in Theorem 0.2), the class Y is pseudo-effective, so that

$$(\overline{C}^+\cdot\sigma_S^*Y)=(\overline{C}\cdot Y)\geqslant 0,$$

and $(\overline{C}^+\cdot T)\geqslant 0$ for all $T\in\mathcal{T}$, so that $\mathcal{T}\neq\emptyset$, and for some $T\in\mathcal{T}$, such that $(\overline{C}^+\cdot T)>0$, the inequality $b_T>na_T$ holds. Since $a_T\geqslant 1$ for all $T\in\mathcal{T}$, we conclude that

$$\left(\overline{C}^+\cdot\left(\sigma_S^*Y-\sum_{T\in\mathcal{T}}b_TT\right)\right)<-n\left(\overline{C}^+\cdot\sum_{T\in\mathcal{T}}a_TT\right)\leqslant -n.$$

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For a general curve $\overline{C}^+$, consider the algebraic cycle of the scheme-theoretic intersection

$$(D^+ \circ \pi_+^{-1}(\overline{C}^+)) = \left(\left(\sigma^* D - \pi_+^*\left(\sum_{T\in\mathcal{T}} b_T T\right)\right) \circ \pi_+^{-1}(\overline{C}^+)\right).$$

The numerical class of that effective cycle is

$$n(\sigma^*(-K_V)\cdot\pi_+^{-1}(\overline{C}^+)) + \left(\overline{C}^+ \cdot \left(\sigma_S^* Y - \sum_{T\in\mathcal{T}} b_T T\right)\right)F$$

(where F is the class of a fibre of the projection π_+) and the class of the effective cycle $\sigma_*(D^+ \circ \pi_+^{-1}(\overline{C}^+))$ in the numerical Chow group is

$$-n(K_V\cdot\pi^{-1}(\overline{C})) + bF,$$

where $b < -n$. This contradicts the condition (iii) of Theorem 0.2. The proof of Theorem 2.1 is complete. Therefore, in both cases (that of a rationally connected fibre space and of a Mori fibre space), the map χ is fibre-wise. The first claim of Theorem 0.2 (in the rationally connected case) is shown. It remains to prove the birational rigidity.

2.5. *Proof of birational rigidity*

Starting from this moment, we assume that V'/S' is a Mori fibre space and the birational map $\chi\colon V \dashrightarrow V'$ is fibre-wise; however, the corresponding map of the bases $\beta\colon S \dashrightarrow S'$ is not birational: $\dim S > \dim S'$ and the fibres $\beta^{-1}(s')$ for $s' \in S'$ are of positive dimension. We have to obtain a contradiction, showing that this case is impossible.

First of all, let us consider the fibre-wise modification of the fibre space V/S (Subsection 2.2). Now we will need a composition of blow-ups $\sigma_S\colon S^+ \to S$ with non-singular centres such that as in Subsection 2.2, none of the fibres of the Fano-Mori fibre space V^+/S^+ are contained in the base set $\operatorname{Bs}\Sigma^+$, and, in addition, σ_S resolves the singularities of the rational dominant map $\beta\colon S \dashrightarrow S'$; that is,

$$\beta_+ = \beta\circ\sigma_S\colon S^+ \to S'$$

is a morphism. (So that the inclusion $Z_i = \pi_i^{-1}(Z_{S,i}) \subset \operatorname{Bs}\Sigma^i$ – see Subsection 2.2 – no longer takes place for all $i = 0, \dots, N-1$.)

The fibre $\beta_+^{-1}(s')$ over a point $s' \in S'$ of general position is an irreducible non-singular subvariety of positive dimension. Set $G(s') = (\pi')^{-1}(s')$ and let $G^+(s')$ be the strict transform of $G(s')$ on V^+. Obviously,

$$G^+(s') = \pi_+^{-1}(\beta_+^{-1}(s'))$$

is a union of fibres of the projection π_+ over the points of the variety $\beta_+^{-1}(s')$.

Since $\pi'\colon V' \to S'$ is a Mori fibre space, we have the equality $\rho(V') = \rho(S') + 1$. Let $\mathcal{E}'$ be the set of all ψ-exceptional divisors $E' \in \mathcal{E}'$, satisfying the equality $\pi'(\psi(E')) = S'$. Furthermore, let $\mathcal{Z} \subset \operatorname{Pic}\widetilde{V}\otimes\mathbb{Q}$ be the subspace, generated by the subspace $\psi^*(\pi')^*\operatorname{Pic}S'\otimes\mathbb{Q}$ and the classes of all ψ-exceptional divisors on $\widetilde{V}$, the images of which on V' do not cover S'. Then the equality

$$\operatorname{Pic}\widetilde{V}\otimes\mathbb{Q} = \mathbb{Q}\widetilde{K} \oplus \left(\bigoplus_{E'\in\mathcal{E}'} \mathbb{Q}E' \oplus \mathcal{Z}\right)$$

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holds; in particular, the subspace in brackets is a hyperplane in $\mathrm{Pic}\,\widetilde{V}\otimes\mathbb{Q}$. Writing down the class $\widetilde{K}$ with respect to the morphisms φ and ψ, we get the equality

$$\varphi^*K^+ + \sum_{E\in\mathcal{E}} a_E^+ E = \psi^*K' + \sum_{E'\in\mathcal{E}'} a'(E')E' + Z_1, \tag{10}$$

where $Z_1\in\mathcal{Z}$ is some effective class, $a_E^+ = a(E,V^+)$ for φ-exceptional divisors $E\in\mathcal{E}$ and $a'(E') = a(E',V')$ for ψ-exceptional divisors $E'\in\mathcal{E}'$, covering S'. Here, all $a_E^+\geqslant 1$ and $a'(E')>0$. Using the ψ-presentation (8) and the φ-presentation (9) of the divisorial class $\widetilde{D}$ and expressing K' from the formula (10), we get the following equality in $\mathrm{Pic}\widetilde{V}\otimes\mathbb{Q}$:

$$(m-n)\varphi^*K^+ + \varphi^*\pi_+^*Y_+ + \sum_{E\in\mathcal{E}}(ma_E^+ - b_E)E = m\sum_{E'\in\mathcal{E}'} a'(E')E' + Z_2, \tag{11}$$

where $Y_+ = \sigma_S^*Y + \sum_{T\in\mathcal{T}}(na_T - b_T)T$ and $Z_2 = mZ_1 + \psi^*(\pi')^*Y'\in\mathcal{Z}$ is an effective class. Applying to both sides of (11) φ_* and restricting onto a fibre of general position of the projection π_+, we get that

$$(m-n)K^+|_{\pi_+^{-1}(s_+)}$$

is an effective class. Since $m\geqslant n$ and the fibre $\pi_+^{-1}(s_+)$ is a Fano variety, we conclude that $m=n$ and (11) turns into

$$\varphi^*\pi_+^*Y_+ + \sum_{E\in\mathcal{E}}(na_E^+ - b_E)E = n\sum_{E'\in\mathcal{E}'} a'(E')E' + Z_2. \tag{12}$$

By Proposition 2.1, we have $b_E\leqslant na_E^+$ for all $E\in\mathcal{E}$. Again, we apply φ_* and get that the class Y_+ is effective on S^+.

Now let us consider the defined above fibre $G=G(s')$ of general position of the morphism π' and its strict transforms $\widetilde{G}$ on $\widetilde{V}$ and G^+ on V^+ (the symbol s' for simplicity of notations is omitted). Obviously, for every $Z\in\mathcal{Z}$, we have $Z|_{\widetilde{G}}=0$. Furthermore, for any linear combination with non-negative coefficients,

$$\left.\left(\sum_{E'\in\mathcal{E}'} b'_{E'}E'\right)\right|_{\widetilde{G}}$$

is a fixed divisor on $\widetilde{G}$. Now let Δ be a very ample divisor on S^+. Then the restriction $\varphi^*\pi_+^*\Delta|_{\widetilde{G}}$ is mobile (recall that $\beta_+^{-1}(s')$ is a variety of positive dimension, so that $\Delta|_{\beta_+^{-1}(s')}$ is a mobile class). Therefore,

$$\varphi^*\pi_+^*\Delta\notin\bigoplus_{E'\in\mathcal{E}'}\mathbb{Q}E'\oplus\mathcal{Z},$$

whence we conclude that

$$\mathrm{Pic}\,\widetilde{V}\otimes\mathbb{Q} = \mathbb{Q}[\varphi^*\pi_+^*\Delta]\oplus\left(\bigoplus_{E'\in\mathcal{E}'}\mathbb{Q}E'\oplus\mathcal{Z}\right).$$

However, this cannot be the case. Let $F^+\subset G^+$ be a fibre of general position of the morphism π_+ and $\widetilde{F}\subset\widetilde{G}$ its strict transform on $\widetilde{V}$. Restricting (12) onto $\widetilde{F}$, we obtain the equality

$$\sum_{E\in\mathcal{E}}(na_E^+ - b_E)E|_{\widetilde{F}} = n\sum_{E'\in\mathcal{E}'} a'(E')E'|_{\widetilde{F}},$$

where on the right-hand side, it is a linear combination of all divisors $E'|_{\widetilde{F}}$, $E'\in\mathcal{E}'$, with *positive* coefficients (it is here that we use the assumption that the singularities of the variety V' are terminal;

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see Subsection 0.2), and on the left-hand side, it is a linear combination of φ-exceptional divisors $E|_{\widetilde{F}}$, $E \in \mathcal{E}$, with non-negative coefficients. Since by construction $\pi_+^*\Delta|_{F^+} = 0$, we have $\varphi^*\pi_+^*\Delta|_{\widetilde{F}} = 0$, whence it follows that the restriction of *every* divisorial class in $\mathrm{Pic}\,\widetilde{V} \otimes \mathbb{Q}$ onto $\widetilde{F}$ is fixed (is a linear combination of φ-exceptional divisors $E|_F$, $E \in \mathcal{E}$), which is impossible. This contradiction completes the proof of Theorem 0.2.

3. Hyperplane sections

This section is an immediate follow up of §1: we develop the technique of working triples and consider its first applications.

3.1. The working triple (F, D_F, o)

Let us prove Proposition 1.6. Proposition 1.2, shown in Subsection 1.5, implies the condition (MC3). Theorem 1.1 gives the condition (MC1) for $c_F = 4k + 2\varepsilon(k)$ (the inequality $c_F \geqslant l + 4$ is satisfied in the obvious way, since $l \leqslant k$). Finally, the condition (MQ1) gives precisely (MC2). Therefore, $[F, o]$ is indeed a marked complete intersection of level (l, c_F).

Consider a general subspace $P^\sharp \ni o$ of dimension $k + c_F$ in $\mathbb{P}^{M+k}$. The pair

$$\left(F \cap P^\sharp, \frac{1}{n(D_F)} D_F|_{F\cap P^\sharp}\right)$$

is not canonical. By (MC1), the singularities of the variety $F \cap P^\sharp$ are zero-dimensional, and moreover, $o \in \mathrm{Sing}\, F \cap P^\sharp$ and

$$\mathrm{CS}\left(F \cap P^\sharp, \frac{1}{n(D_F)} D_F|_{F\cap P^\sharp}\right) \subset \mathrm{Sing}(F \cap P^\sharp),$$

and the point o is an (isolated) centre of some non-canonical singularity of that pair. For a general subspace $P \ni o$ of dimension $k + c_F - 1$, take a general hyperplane in $P^\sharp$, containing the point o. By inversion of adjunction, we have the equalities

$$\{o\} = \mathrm{LCS}\left(F \cap P, \frac{1}{n(D_F)} D_F|_{F\cap P}\right) = \mathrm{CS}\left(F \cap P, \frac{1}{n(D_F)} D_F|_{F\cap P}\right),$$

and this is precisely (7). Q.E.D. for Proposition 1.6.

As we explained in Subsection 1.7, from now, our work is constructing a certain special sequence of working triples. This sequence starts with the working triple (F, D_F, o). In order to construct the sequence, we will need certain facts about working triples.

3.2. Multiplicity at the marked point

Let us prove Proposition 1.5. We use the notations of Subsection 1.7 and work with a working triple (X, D, o), where $[X, o]$ is a marked complete intersection. Assume that $\nu(D) \leqslant 2n(D)$ (otherwise, there is nothing to prove).

Since for a general subspace $P \ni o$ of dimension $m \in I_X$, the inequality $a(E_{X\cap P}) \geqslant 2$ holds (see Remark 1.2), the pair

$$\left((X \cap P)^+, \frac{1}{n(D)} D_P^+\right)$$

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is not log-canonical, and moreover,

$$\mathrm{LCS}\left((X\cap P)^+, \frac{1}{n(D)}D_P^+\right)\subset E_{X\cap P}.$$

Let $B(P)\subset E_{X\cap P}$ be the centre of some non-log-canonical singularity of that pair. Then the inequality

$$\mathrm{mult}_{B(P)}\, D_P^+ > n(D)$$

holds, and the more so,

$$\mathrm{mult}_{B(P)}\, D_P^+|_{E_{X\cap P}} > n(D).$$

Considering a general subspace $P^*\ni o$ of the minimal admissible dimension $k+l_X+3$ in I_X as a general subspace of codimension $\geqslant l_X$ in a general subspace $P\ni o$ of the maximal admissible dimension $k+c_X-1$ in I_X (recall that by assumption $c_X\geqslant 2l_X+4$), we see that the centre $B(P)$ of some non-log-canonical singularity is of dimension $\geqslant l_X$. However, $E_{X\cap P}$ is a non-singular complete intersection of l_X quadrics in the projective space of dimension l_X+c_X-2, and the divisor $D_P^+|_{E_{X\cap P}}$ is cut out on $E_{X\cap P}$ by a hypersurface of degree $\nu(D)$ in that projective space. Therefore, (for example, by [17, Proposition 3.6]), the inequality

$$\nu(D)\geqslant \mathrm{mult}_{B(P)}\, D_P^+|_{E_{X\cap P}}$$

holds. Therefore, $\nu(D)>n(D)$. Q.E.D. for Proposition 1.5.

3.3. Transversal hyperplane sections

We still work with an arbitrary working triple (X,D,o), where $[X,o]$ is a marked complete intersection of level (l_X,c_X).

Proposition 3.1. *Let $R\ni o$ be the section of the variety X by a hyperplane $\mathbb{P}(R)\subset\mathbb{P}(X)$, which is not tangent to X at the point o. Then $D\neq bR$ for $b\geqslant 1$. Moreover, if D contains R as a component – that is,*

$$D=D^*+bR,$$

where $b\geqslant 1$ – then (X,D^*,o) is a working triple.

Proof. If $c_X\geqslant 2l_X+4$, then the first claim (that D is not a multiple of R) follows immediately from Proposition 1.5: indeed, the hyperplane $\mathbb{P}(R)$ is not tangent to X at the point o; that is, for the strict transform R^+ on the blow-up of that point, we have

$$R^+\sim H_X-E_X,$$

so that the equality $D=bR$ implies that $n(D)=b=\nu(D)$, which contradicts Proposition 1.5. However, we will show now that the additional assumptions for the parameters l_X and c_X are not needed.

By Remark 1.4, the condition (MC1) for X implies the inequality

$$\mathrm{codim}(\mathrm{Sing}\, R\subset R)\geqslant c_X-2. \tag{13}$$

Since the hyperplane $\mathbb{P}(R)$ is not tangent to X at the point o, this point is a multi-quadratic singularity of the variety R of type 2^{l_X}, the rank of which (by Remark 1.4 and the condition (MC2)) satisfies the inequality

$$\mathrm{rk}(o\in R)\geqslant 2l_X+c_X-3.$$

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Consider a general linear subspace $P \ni o$ in $\mathbb{P}(X)$ of dimension $k + c_X - 2$. That dimension, generally speaking, does not belong to I_X, and only the inequality

$$a(E_{X\cap P}) \geqslant 1$$

holds. The variety $X \cap P$ has a unique singularity, the point o, and its strict transform $(X \cap P)^+$ and the exceptional divisor $E_{X\cap P}$ are non-singular.

The intersection $P \cap \mathbb{P}(R)$ is a general linear subspace of dimension $k + c_X - 3$ in $\mathbb{P}(R)$, containing the point o. For that reason, $R \cap P$ has a unique singularity, the point o, and moreover, the exceptional divisor

$$E_{R\cap P} = (R \cap P)^+ \cap \mathbb{E}_X = R^+ \cap E_{X\cap P}$$

is non-singular, and the map $(R \cap P)^+ \to R \cap P$ is the blow-up of the point o on $R \cap P$, which resolves the singularities of that variety. From here, taking into account that

$$\nu(R) = 1 \leqslant a(E_{X\cap P}),$$

it follows that the pair $(X \cap P, R \cap P)$ is canonical. By inversion of adjunction, we get that for every $m \in I_X$ and a general subspace $P^\sharp \ni o$ of dimension m, the pair $(X \cap P^\sharp, R \cap P^\sharp)$ is canonical. Therefore, $D \neq bR$, $b \geqslant 1$, and the first claim of the proposition is shown.

Assume now that $D = D^* + bR$, where $b \geqslant 1$. Then for a general subspace $P \ni o$ of dimension $k + c_X - 1$ in $\mathbb{P}(X)$, the pair $(X \cap P, \frac{1}{n(D)} D_P)$ is not log canonical at the point o. As we saw above, the pair $(X \cap P, R_P)$ is log-canonical (and even canonical). The condition of being log-canonical is linear, so we conclude that the pair

$$\left(X \cap P, \frac{1}{n(D^*)} D^*|_{X\cap P}\right)$$

is not log-canonical at the point o. Therefore, (X, D^*, o) is a working triple. Q.E.D. for the proposition. □

Theorem 3.1 (on the transversal hyperplane section). *Let $[X, o]$ be a marked complete intersection of level $(l_X = k, c_X)$, where $c_X \geqslant k + 6$, and (X, D, o) a working triple. Let $R \ni o$ be a hyperplane section, which is not a component of the divisor D. Assume that the inequality* $\mathrm{ct}(R \setminus \mathrm{Sing}\, R) \geqslant 1$ *holds. Then $(R, (D \circ R), o)$ is a working triple on the marked complete intersection $[R, o]$ of level $(l_R = k, c_R)$, where $c_R = c_X - 2$.*

Proof. First of all, let us check that $[R, o]$ is a marked complete intersection of level (k, c_R). The inequality $c_R \geqslant k + 4$ holds by assumption. Furthermore,

$$\mathrm{codim}(\mathrm{Sing}\, R \subset R) \geqslant c_X - 2 = c_R,$$

so that the condition (MC1) is satisfied. Furthermore, the point $o \in R$ is a multi-quadratic singularity, the rank of which satisfies the inequality

$$\mathrm{rk}(o \in R) \geqslant \mathrm{rk}(o \in X) - 2 \geqslant 2k + c_R - 1,$$

so that the condition (MC2) holds. The condition (MC3) holds by assumption. The bound for the codimension of the singular set $\mathrm{Sing}\, R$ guarantees that the complete intersection $R \subset \mathbb{P}(R)$ is irreducible, reduced and factorial. Therefore, $[R, o]$ is a marked complete intersection of level (k, c_R). Set

$$I_R = [2k + 3, k + c_R - 1].$$

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Obviously, $(D \circ R) \sim n(D \circ R)H_R = n(D)H_R$, where H_R is the class of a hyperplane section of R. It remains to check that for a general subspace $P \ni o$ of dimension $k + c_R - 1$ in $\mathbb{P}(R)$, the pair

$$\left(R \cap P, \frac{1}{n(D \circ R)}(D \circ R)|_{R\cap P}\right) \tag{14}$$

is not log-canonical at the point o. In order to do this, we present P as the intersection

$$P = P^\sharp \cap \mathbb{P}(R),$$

where $P^\sharp \ni o$ is a general subspace of dimension

$$k + c_R = k + c_X - 2$$

in $\mathbb{P}(X)$. As $k + c_R \in I_X$, the point o is the only singularity of the variety $X \cap P^\sharp$ (and the only singularity of the variety $R \cap P$), and

$$\{o\} = \mathrm{LCS}\left(X \cap P^\sharp, \frac{1}{n(D)}D|_{X\cap P^\sharp}\right).$$

The variety $R \cap P$ is the section of the variety $X \cap P^\sharp$ by the hyperplane $P = P^\sharp \cap \mathbb{P}(R)$, containing the point o, so that by inversion of adjunction the pair (14) is not log-canonical. At the same time, it is canonical outside the point o since the subspace $P \subset \mathbb{P}(R)$ is generic, the non-singular part $R\setminus \mathrm{Sing}\, R$ is divisorially canonical and the equality $\{o\} = \mathrm{Sing}(R \cap P)$ holds. Therefore, the pair (14) is not log-canonical precisely at the point o, which completes the proof of Theorem 3.1. □

3.4. Tangent hyperplane sections

Now let us consider a marked complete intersection $[X, o]$ of level (l_X, c_X), where $l_X \leqslant k - 1$. Let R be the section of the variety X by a hyperplane $\mathbb{P}(R) \subset \mathbb{P}(X)$, which is tangent to X at the point o. By the symbol $\mathbb{P}(R)^+$, denote the strict transform of the hyperplane $\mathbb{P}(R)$ on $\mathbb{P}(X)^+$ and set

$$\mathbb{E}_R = \mathbb{P}(R)^+ \cap \mathbb{E}_X.$$

Obviously, $\mathbb{E}_R \cong \mathbb{P}^{N(X)-2}$ is the exceptional divisor of the blow-up of the point o on the hyperplane $\mathbb{P}(R)$. Set also $E_R = R^+ \cap \mathbb{E}_X$. Obviously, $E_R \subset \mathbb{E}_R$, and

$$\mathrm{codim}(E_R \subset \mathbb{E}_R) = k.$$

Proposition 3.2. *Assume that $c_X \geqslant l_X + 5$ and the point $o \in R$ is a multi-quadratic singularity of type 2^{l_X+1}, and moreover, the inequality*

$$\mathrm{rk}(o \in R) \geqslant 2l_X + c_X - 2$$

holds. Then $D \neq bR$ for $b \geqslant 1$. Moreover, if the divisor D contains R as a component – that is,

$$D = D^* + bR,$$

where $b \geqslant 1$ – then (X, D^*, o) is a working triple.

Proof is completely similar to the proof of Proposition 3.1, but we give it in full details because there are some small points where the two arguments are different. The inequality (13) holds in this case again. Let us use the additional assumption about the singularity $o \in R$. Consider a general linear subspace $P \ni o$ in $\mathbb{P}(X)$ of dimension $k + l_X + 3$ (it is the minimal admissible dimension). We get the equality $a(E_{X\cap P}) = 2$. Obviously,

$$R^+ \sim H_X - 2E_X$$

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and, respectively, on $(X \cap P)^+$, we have

$$(R \cap P)^+ \sim H_{X\cap P} - 2E_{X\cap P}.$$

Arguing as in the transversal case, we note that the intersection $P \cap \mathbb{P}(R)$ is a general subspace of dimension $k + l_X + 2$ in $\mathbb{P}(R)$. Taking into account that by the inequality (13), the inequality

$$\operatorname{codim}(\operatorname{Sing} R \subset \mathbb{P}(R)) \geqslant k + c_X - 2$$

holds, and that by assumption $c_X \geqslant l_X + 5$, we see that $R \cap P$ has a unique singularity, the point o. Furthermore, by the assumption about the rank of the singular point $o \in R$, we get the inequality

$$\operatorname{codim}(\operatorname{Sing} E_R \subset E_R) \geqslant c_X - 3 \geqslant l_X + 2,$$

so that

$$\operatorname{codim}(\operatorname{Sing} E_R \subset \mathbb{E}_R) \geqslant k + l_X + 2.$$

The exceptional divisor $E_{R\cap P}$ is the section of the subvariety $E_R \subset \mathbb{E}_R$ by a general linear subspace of dimension $k + l_X + 1$, whence we conclude that the variety $E_{R\cap P}$ is non-singular. Thus, we have shown that the singularity $o \in R \cap P$ is resolved by one blow-up. Therefore, the pair

$$((X \cap P)^+, (R \cap P)^+)$$

is canonical, so that the pair

$$(X \cap P, R \cap P)$$

is canonical, too. We have shown that $D \neq bR$ for $b \geqslant 1$.

By inversion of adjunction for every $m \in I_X$ and a general subspace $P^\sharp \ni o$ of dimension m, the pair $(X \cap P^\sharp, R \cap P^\sharp)$ is canonical (recall that $n(R) = 1$). Repeating the arguments given in the transversal case (the proof of Proposition 3.1) word for word, we complete the proof of Proposition 3.2.

Remark 3.1. If for $l_X \leqslant k - 1$ the intersection $X \cap T_oX$ has the point o as a multi-quadratic singularity of type 2^k, the rank of which satisfies the inequality

$$\operatorname{rk}(o \in X \cap T_oX) \geqslant 2l_X + c_X - 2,$$

the assumption about the rank $\operatorname{rk}(o \in R)$ in the statement of Proposition 3.2 holds automatically for every tangent hyperplane at the point o.

Theorem 3.2 (on the tangent hyperplane section). *Let $[X, o]$ be a marked complete intersection of level (l_X, c_X), where $2 \leqslant l_X \leqslant k - 1$ and $c_X \geqslant l_X + 7$, and (X, D, o) a working triple. Let R be the section of X by a hyperplane which is tangent to X at the point o, and assume that R is not a component of the divisor D. Assume that the point $o \in R$ is a multi-quadratic singularity of type 2^{l_R}, where $l_R = l_X + 1$, the rank of which satisfies the inequality*

$$\operatorname{rk}(o \in R) \geqslant 2l_R + c_R - 1 = 2l_X + c_X - 1,$$

where $c_R = c_X - 2$, and also that the inequality $\operatorname{ct}(R \backslash \operatorname{Sing} R) \geqslant 1$ *holds. Then $(R, (D \circ R), o)$ is a working triple on the marked complete intersection $[R, o]$ of level (l_R, c_R).*

Proof is similar to the transversal case (Theorem 3.1), and we just emphasize the necessary modifications. The fact that $[R, o]$ is a marked complete intersection of level (l_R, c_R) is checked in the tangent case even easier than in the transversal one, because the assumption about the singularity $o \in R$ is among the assumptions of the theorem.

A general subspace $P \ni o$ of dimension $k + c_R - 1 = k + c_X - 3$ in $\mathbb{P}(R)$ is again presented as the intersection $P = P^\sharp \cap \mathbb{P}(R)$, where $P^\sharp \ni o$ is a general subspace of dimension $k + c_R \in I_X$ in $\mathbb{P}(X)$,

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and now, repeating the arguments in the transversal case and using inversion of adjunction, we get that $(R, (D \circ R), o)$ is a working triple. Q.E.D. for the theorem.

Proof of Proposition 1.7. We assume that $l \leqslant k-1$. Recall that the symbol T stands for the intersection $F \cap T_o F$; this is a subvariety of codimension $(k-l)$ in F. Let us construct a sequence of subvarieties

$$T_0 = F \supset T_1 \supset \cdots \supset T_{k-l} = T,$$

where T_{i+1} is the section of $T_i \ni o$ by some hyperplane $\mathbb{P}(T_{i+1}) = \langle T_{i+1} \rangle \ni o$, which is tangent to T_i at the point o. Theorem 1.1 implies that the inequality

$$c_F \geqslant l + 3(k-l) + 4$$

holds (the inequality of Theorem 1.1 for the codimension c_F is much stronger, but for the clarity of exposition, we give the weakest estimate that is sufficient for the proof of Proposition 1.7; this remark also applies to the estimate of the rank of the multi-quadratic singularity $o \in T$ below). Furthermore, the condition (MQ2) implies that $o \in T$ is a multi-quadratic singularity of type 2^k, and moreover, the inequality

$$\operatorname{rk}(o \in T) \geqslant 2k + c_F - 1 \tag{15}$$

holds. Finally, by Theorem 1.2 for every hyperplane section W of every subvariety T_i, $i = 0, 1, \ldots, k-l$, every non-singular point $p \in W$ and every prime divisor Y on W the inequality

$$\frac{\operatorname{mult}_p}{\deg} Y \leqslant \frac{2}{\deg F} \tag{16}$$

holds. Then for all $i = 0, 1, \ldots, k-l$, the pair $[T_i, o]$ is a marked complete intersection of level

$$(l_i = l + i, c_i = c_F - 2i).$$

Indeed, the inequality $c_i \geqslant l_i + 4$ is true by the definition of the numbers l_i, c_i, the condition (MC1) follows from Remark 1.4, the point $o \in T_i$ by construction is a multi-quadratic singularity of type 2^{l+1}, and moreover, by (15), we have

$$\operatorname{rk}(o \in T_i) \geqslant 2l + c_F - 1 = 2l_i + c_i - 1,$$

and, finally, repeating the proof of Proposition 1.1 and the arguments of Subsection 1.5 word for word, we get that by (16), the condition (MC3) holds. Therefore, $[T, o]$ is a marked complete intersection of level (k, c_{k-l}), where $c_{k-l} = c_F - 2(k-l) \geqslant k + 4$. Recall (Proposition 1.6) that $c_F = 4k + 2\varepsilon(k)$. Since $l \geqslant 2$, the inequality

$$c_{k-l} \geqslant c_T = 2k + 2\varepsilon(k) + 4$$

holds, so that $[T, o]$ is a marked complete intersection of level (k, c_T), as we claimed. □

It remains to construct the working triple (T, D_T, o). We will construct a sequence of working triples (T_i, D_i, o), where $i = 0, 1, \ldots, k-l$ and $D_0 = D_F$. Assume that (T_i, D_i, o) is already constructed and $i \leqslant k-l-1$. Let us check that all assumptions that allow us to apply Proposition 3.2 are satisfied.

Indeed, the fact that $i \leqslant k-l-1$ implies the inequality $c_i \geqslant l_i + 7$. The point $o \in T_{i+1}$ is a multi-quadratic singularity of type 2^{l_i+1}, the rank of which satisfies the inequality

$$\operatorname{rk}(o \in T_{i+1}) \geqslant 2l_{i+1} + c_{i+1} - 1 = 2l_i + c_i - 1$$

(see above). Applying Proposition 3.2, we remove T_{i+1} from the effective divisor D_i (if it is necessary) and obtain the working triple (T_i, D_i^*, o), where the effective divisor D_i^* does not contain T_{i+1} as a component.

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It is easy to see that we have all assumptions of Theorem 3.2. Set

$$D_{i+1} = (D_i^* \circ T_{i+1}).$$

Now (T_{i+1}, D_{i+1}, o) is a working triple. Proof of Proposition 1.7 is complete.

3.5. Plan of the proof of Proposition 1.8

Recall that by the condition (MQ2), the inequality

$$\mathrm{rk}(o \in T) \geqslant 10k^2 + 8k + 2\varepsilon(k) + 5$$

holds. The pair $[T, o]$ is a marked complete intersection of level (k, c_T), where $c_T = 2k + 2\varepsilon(k) + 4$. Let

$$R_0 = T, R_1, \dots, R_a,$$

where $a \leqslant \varepsilon(k)$, be an *arbitrary* sequence of subvarieties in T, where R_{i+1} is the section of R_i by the hyperplane $\mathbb{P}(R_{i+1})$ in $\mathbb{P}(R_i)$, containing the point o. Set $c_i = c_T - 2i$, where $i = 0, 1, \dots, a$.

Proposition 3.3. *The pair $[R_i, o]$ is a marked complete intersection of level (k, c_i).*

Proof. Since $a \leqslant \varepsilon(k)$, the inequality $c_i \geqslant k + 4$ holds in an obvious way (in fact, $c_i \geqslant 2k + 4$). The condition (MC1) holds by Remark 1.4. The condition (MC3) is obtained by repeating the proof of Proposition 1.1 and the arguments of Subsection 1.5 word for word, taking into account Theorem 1.2. Finally, again by Remark 1.4, the inequality

$$10k^2 + 8k + 2\varepsilon(k) + 4 \geqslant 2k + c_i + 2i - 1$$

implies the condition (MC2). Q.E.D. for the proposition. □

Now let us construct for every $i = 0, 1, \dots, a$ an effective divisor D_i on R_i in the same way as we did it in Subsection 3.4 in the proof of Proposition 1.7, applying instead of Proposition 3.2, its 'transversal' analog, Proposition 3.1, and Theorem 3.1 instead of Theorem 3.2. More precisely, if the effective divisor D_i, where $i \leqslant a - 1$, is already constructed, we remove from this divisor all components that are hyperplane sections (if there are such components), and obtain an effective divisor D_i^* that does not contain hyperplane sections as components, and such that (R_i, D_i^*, o) is a working triple (Proposition 3.1).

Proposition 3.4. *The following inequality holds:*

$$\frac{\nu(D_i^*)}{n(D_i^*)} \geqslant \frac{\nu(D_i)}{n(D_i)}.$$

Proof. It is sufficient to consider the case when D_i^* is obtained from D_i by removing one hyperplane section $Z \ni o$. Write down

$$D_i = D_i^* + bZ,$$

where $b \geqslant 1$. Since $c_i \geqslant 2k + 4$, we can apply Proposition 1.5: $\nu(D_i) > n(D_i)$. However, $\nu(Z) = n(Z) = 1$. Set $\nu(D_i) = \alpha n(D_i)$, where $\alpha > 1$. We get

$$\frac{\nu(D_i^*)}{n(D_i^*)} = \frac{\alpha n(D_i^*) + (\alpha - 1)b}{n(D_i^*)} > \alpha,$$

which proves the proposition. Q.E.D. □

(If we remove from D_i, a hyperplane section that does not contain the point o, the claim of Proposition 3.4 is trivial.)

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Now we apply Theorem 3.1, setting $D_{i+1} = (D_i^* \circ R_{i+1})$. This cycle of the scheme-theoretic intersection is well defined as an effective divisor on R_{i+1}, and moreover, (R_{i+1}, D_{i+1}, o) is a working triple and

$$\frac{\nu(D_{i+1})}{n(D_{i+1})} \geqslant \frac{\nu(D_i^*)}{n(D_i^*)} \geqslant \frac{\nu(D_i)}{n(D_i)}.$$

We emphasize that $R_1, \ldots, R_a$ is an arbitrary sequence of consecutive hyperplane sections. By Remark 1.4, for all $i = 0, 1, \ldots, a$, the inequality $c_i \geqslant 2k + 4$ holds, and the rank of the multi-quadratic singularity $o \in R_i$ of type 2^k is at least $10k^2 + 8k + 5$. By Theorem 1.4 and Proposition 1.5, we have the inequalities

$$n(D_i) < \nu(D_i) \leqslant \frac{3}{2} n(D_i).$$

Therefore, at every step of our construction, the assumptions of the following claim are satisfied.

Theorem 3.3 (on the special hyperplane section). *Let $[X, o]$ be a marked complete intersection of level (k, c_X), where $c_X \geqslant 2k + 4$ and the inequality*

$$\mathrm{rk}(o \in X) \geqslant 10k^2 + 8k + 5$$

holds. Let (X, D, o) be a working triple, where the effective divisor D does not contain hyperplane sections and satisfies the inequalities

$$n(D) < \nu(D) \leqslant \frac{3}{2} n(D).$$

Then there is a section $R \ni o$ of the variety X by a hyperplane $\mathbb{P}(R) = \langle R \rangle \subset \mathbb{P}(X) = \mathbb{P}^{N(X)}$, such that the effective divisor $D_R = (R \circ D)$ on R satisfies the inequality

$$2 - \frac{\nu(D_R)}{n(D_R)} < \frac{1}{1 + \frac{1}{k}} \left(2 - \frac{\nu(D)}{n(D)} \right).$$

Now by the definition of the integer $\varepsilon(k)$ and what was said above, Theorem 3.3 immediately implies Proposition 1.8.

Proof of Theorem 3.3 is given in §5.

4. Multi-quadratic singularities

In this section, we consider the properties of multi-quadratic singularities, the rank of which is bounded from below: they are factorial, stable with respect to blow-ups and terminal. In Subsection 4.5, we study linear subspaces on complete intersections of quadrics and the properties of projections from these subspaces.

4.1. The definition and the first properties

Let $\mathcal{X}$ be an (irreducible) algebraic variety, $o \in \mathcal{X}$ a point.

Definition 4.1. The point o is a *multi-quadratic singularity* of the variety $\mathcal{X}$ of type 2^l and rank $r \geqslant 1$, if in some neighborhood of this point, $\mathcal{X}$ can be realized as a subvariety of a non-singular $N = (\dim \mathcal{X} + l)$-dimensional variety $\mathcal{Y} \ni o$, and for some system $(u_1, \ldots, u_N)$ of local parameters on $\mathcal{Y}$ at the point o, the subvariety $\mathcal{X}$ is the scheme of common zeros of regular functions

$$\alpha_1, \ldots, \alpha_l \in \mathcal{O}_{o,\mathcal{Y}} \subset \mathbb{C}[[u_1, \ldots, u_N]],$$

which are represented by the formal power series

$$\alpha_i = \alpha_{i,2} + \alpha_{i,3} + \ldots,$$

where $\alpha_{i,j}(u_1,\dots,u_N)$ are homogeneous polynomials of degree j and

$$\mathrm{rk}(\alpha_{1,2},\dots,\alpha_{l,2})=r.$$

(Obviously, the order of the formal power series, representing α_i, and the rank of the tuple of quadratic forms $\alpha_{i,2}$ do not depend on the choice of the local parameters on $\mathcal{Y}$ at the point o.)

It is convenient to work in a more general context. Assume that in a neighborhood of the point o, the variety $\mathcal{X}$ is realized as a subvariety $\mathcal{X}\subset\mathcal{Z}$, where $\dim\mathcal{Z}=\dim\mathcal{X}+e=N(\mathcal{Z})$, and for a certain system of local parameters $(v_1,\dots,v_{N(\mathcal{Z})})$ on $\mathcal{Z}$ at the point o, the subvariety $\mathcal{X}$ is the scheme of common zeros of regular functions

$$\beta_1,\dots,\beta_e\in\mathcal{O}_{o,\mathcal{Z}}\subset\mathbb{C}[[v_*]],$$

which are represented by the formal power series

$$\beta_i=\beta_{i,1}+\beta_{i,2}+\dots,$$

where $\beta_{i,j}(v_*)$ are homogeneous polynomials of degree j. Assume that for some $l\in\{0,1,\dots,e\}$,

$$\dim\langle\beta_{1,1},\dots,\beta_{e,1}\rangle=e-l,$$

where we assume (for the convenience of notations) that the linear forms $\beta_{j,1}$ for $l+1\leqslant j\leqslant e$ are linearly independent, so that for $1\leqslant i\leqslant l$ and $l+1\leqslant j\leqslant e$, there are uniquely determined numbers $a_{i,j}$, such that

$$\beta_{i,1}=\sum_{j=l+1}^{e}a_{i,j}\beta_{j,1}.$$

Set $\mathcal{Y}=\{\beta_j=0\,|\,l+1\leqslant j\leqslant e\}$ and

$$\beta_i^*=\beta_i-\sum_{j=l+1}^{e}a_{i,j}\beta_j.$$

Then (in a neighborhood of the point o) the variety $\mathcal{Y}$ is non-singular, and $\mathcal{X}\subset\mathcal{Y}$ is realized as the scheme of common zeros of the regular functions β_i^*, $1\leqslant i\leqslant l$. Set

$$T_o\mathcal{Y}=T_o\mathcal{X}=\{\beta_{j,1}=0\,|\,l+1\leqslant j\leqslant e\}.$$

If

$$\mathrm{rk}\left(\beta^*_{i,2}|_{T_o\mathcal{X}}\,|\,1\leqslant i\leqslant l\right)=r,$$

then obviously $o\in\mathcal{X}$ is a multi-quadratic singularity of rank r.

The rank of the multi-quadratic point $o\in\mathcal{X}$ is denoted by the symbol $\mathrm{rk}(o\in\mathcal{X})$ or just $\mathrm{rk}(o)$, if it is clear which variety is meant. For uniformity of notations, we treat a non-singular point as a multi-quadratic one of type 2^0.

Proposition 4.1. *Assume that $o\in\mathcal{X}$ is a multi-quadratic singularity of type 2^l, where $l\geqslant 1$, and of rank $r\geqslant 2l$. Then in a neighborhood of the point o, every point $p\in\mathcal{X}$ is either non-singular or a multi-quadratic of type 2^b, where $b\in\{1,\dots,l\}$, of rank $\geqslant r-2(l-b)$.*

Proof. Using the notations for the embedding $\mathcal{X}\subset\mathcal{Z}$ introduced above, with $e=l$ (so that $\beta_{i,1}=0$ for all $i=1,\dots,l$) and setting $N(\mathcal{Z})=N$, consider an open set $U\subset\mathcal{Z}$, $U\ni o$, such that for every point $p\in U$, the 'shifted' functions

$$v_i^{(p)}=v_i-v_i(p),\quad i=1,\dots,N$$

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form a system of local parameters at the point p, and in the formal expansion

$$\beta_i = \beta_{i,0}^{(p)} + \beta_{i,1}^{(p)} + \beta_{i,2}^{(p)} + \dots$$

with respect to the system of parameters $v_*^{(p)}$, the quadratic components satisfy the inequality

$$\mathrm{rk}(\beta_{i,2}^{(p)} \mid 1 \leqslant i \leqslant l) \geqslant r.$$

If the point p is a common zero of $\beta_1, \dots, \beta_l$, then $\beta_{i,0}^{(p)} = 0$ for $1 \leqslant i \leqslant l$. Set

$$T_p\mathcal{X} = \{\beta_{i,1}^{(p)} = 0 \mid 1 \leqslant i \leqslant l\}$$

and assume (for the convenience of notations) that the forms $\beta_{i,1}^{(p)}$ for $b+1 \leqslant i \leqslant l$ are linearly independent, where

$$\dim\langle\beta_{i,1}^{(p)} \mid 1 \leqslant i \leqslant l\rangle = l - b.$$

Since $\mathrm{codim}(T_p\mathcal{X} \subset T_p\mathcal{Z}) = l - b$, by Remark 1.4, the inequality

$$\mathrm{rk}(\beta_{i,2}^{(p)}|_{T_p\mathcal{X}} \mid 1 \leqslant i \leqslant l) \geqslant r - 2(l - b)$$

holds. It is easy to see from the construction of the quadratic forms $\beta_{i,2}^{(p)*}$, $1 \leqslant i \leqslant b$ that every linear combination of these forms with coefficients $(\lambda_1, \dots, \lambda_b) \neq (0, \dots, 0)$ is a linear combination of the original forms $\beta_{i,2}^{(p)}$, $1 \leqslant i \leqslant l$, not all coefficients in which are equal to zero. Therefore, the point p is a multi-quadratic singularity of rank $\geqslant r - 2(l - b)$, as we claimed. Q.E.D. for the proposition. □

4.2. Complete intersections of quadrics

In the notations of Definition 4.1, let $\mathcal{Y}^+ \to \mathcal{Y}$ be the blow-up of the point o with the exceptional divisor $E_{\mathcal{Y}} \cong \mathbb{P}^{N-1}$ and $\mathcal{X}^+ \subset \mathcal{Y}^+$ the strict transform of $\mathcal{X}$ on $\mathcal{Y}^+$, so that $\mathcal{X}^+ \to \mathcal{X}$ is the blow-up of the point o on $\mathcal{X}$ with the exceptional divisor $E_{\mathcal{Y}}|_{\mathcal{X}^+} = E_{\mathcal{X}}$. Therefore, $E_{\mathcal{X}}$ is the scheme of common zeros of the quadratic forms $\alpha_{i,2}$, $i = 1, \dots, l$, on $E_{\mathcal{Y}} \cong \mathbb{P}^{N-1}$.

Let $q_1, \dots, q_l$ be quadratic forms on $\mathbb{P}^{N-1}$, where $N \geqslant l + 4$. By the symbol $q_{[1,l]}$, we denote the tuple $(q_1, \dots, q_l)$.

Proposition 4.2. (i) *Assume that the inequality*

$$\mathrm{rk}\, q_{[1,l]} \geqslant 2l + 3$$

holds. Then the scheme of common zeros of the forms $q_1, \dots, q_l$ is an irreducible non-degenerate factorial variety $Q \subset \mathbb{P}^{N-1}$ of codimension l – that is, a complete intersection of type 2^l.

(ii) *Assume that for some $e \geqslant 4$, the inequality*

$$\mathrm{rk}\, q_{[1,l]} \geqslant 2l + e - 1$$

holds. Then the following inequality is true:

$$\mathrm{codim}(\mathrm{Sing}\, Q \subset Q) \geqslant e.$$

Proof is given below in Subsection 4.4.

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Corollary 4.1. (i) *Assume that the rank of the tuple* $\alpha_{*,2} = (\alpha_{1,2}, \dots, \alpha_{l,2})$ *of quadratic forms satisfies the inequality*

$$\mathrm{rk}(\alpha_{*,2}) \geqslant 2l + 3.$$

Then in a neighborhood of the point o*, the scheme of common zeros of the regular functions* $\alpha_1, \dots, \alpha_l$ *is an irreducible reduced factorial subvariety* $\mathcal{X}$ *of codimension* l *in* $\mathcal{Y}$*.*

(ii) *Assume that for some* $e \geqslant 4$*, the inequality*

$$\mathrm{rk}(\alpha_{*,2}) \geqslant 2l + e - 1$$

holds. Then in a neighborhood of the point o*, the following inequality is true:*

$$\mathrm{codim}(\mathrm{Sing}\,\mathcal{X} \subset \mathcal{X}) \geqslant e.$$

Proof. Both claims obviously follow from Proposition 4.2, taking into account Grothendieck's theorem [21] on the factoriality of a complete intersection, the singular set of which is of codimension $\geqslant 4$.

Therefore, for $r \geqslant 2l + 3$, the assumption in Definition 4.1, that $\mathcal{X}$ is an irreducible variety, is unnecessary: in a neighborhood of the point o, the scheme of common zeros of the functions α_* is automatically irreducible and reduced, and moreover, it is a factorial variety. This proves all claims of Theorem 1.1, except for that the singularities of the variety F are terminal. □

4.3. Stability with respect to blow-ups

Let $\underline{r} = (r_1, r_2, \dots, r_k)$ be a tuple of integers, satisfying the inequalities $r_{i+1} \geqslant r_i + 2$ for $i = 1, \dots, k-1$, where $r_1 \geqslant 5$. Again, let $\mathcal{Y}$ be a non-singular N-dimensional variety, where $N \geqslant k + 3$, and $\mathcal{X} \subset \mathcal{Y}$ an (irreducible) subvariety of codimension k, every point $o \in \mathcal{X}$ of which is either non-singular or a multi-quadratic singularity of type 2^l, where $l \in \{1, \dots, k\}$, of rank $\geqslant r_l$. Somewhat abusing the terminology, we say in this case that $\mathcal{X}$ has *multi-quadratic singularities of type* $\underline{r}$.

Theorem 4.1. *In the assumptions above, let* $B \subset \mathcal{X}$ *be an irreducible subvariety of codimension* $\geqslant 2$*. Then there is an open subset* $U \subset \mathcal{X}$*, such that* $U \cap B \neq \emptyset$*,* $U \cap B$ *is non-singular and the blow-up*

$$\sigma_B \colon U_B \to U$$

along B *gives a quasi-projective variety* U_B *with multi-quadratic singularities of type* $\underline{r}$*.*

Proof. If a point of general position $o \in B$ is non-singular on $\mathcal{X}$, there is nothing to prove. If $o \in \mathcal{X}$ is a multi-quadratic singularity of type 2^l, where $l < k$, then a certain Zariski open subset $U \subset \mathcal{X}$, $U \ni o$ has multi-quadratic singularities of type $(r_1, \dots, r_l)$ (see Subsection 4.1), so that it is sufficient to consider the case when a point of general position $o \in B$ is a multi-quadratic point of type 2^k on $\mathcal{X}$. Passing over to an open subset, we may assume that the subvariety B is non-singular. Let $(u_1, \dots, u_N)$ be a system of local parameters at the point o, such that $B = \{u_1 = \dots = u_m = 0\}$. Since $B \subset \mathcal{X}$, the subvariety $\mathcal{X} \subset \mathcal{Y}$ is the scheme of common zeros of regular functions

$$\beta_1, \dots, \beta_k \in \mathcal{O}_{o,\mathcal{Y}} \subset \mathcal{O}_{o,B}[[u_1, \dots, u_m]],$$

where for all $i = 1, \dots, k$,

$$\beta_i = \beta_{i,2} + \beta_{i,3} + \dots,$$

where $\beta_{i,j}$ are homogeneous polynomials of degree j in $u_1, \dots, u_m$ with coefficients from $\mathcal{O}_{o,B}$. Again replacing $\mathcal{Y}$, if necessary, by an open subset, containing the point o, we have

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$$\beta_i \in \mathcal{O}(\mathcal{Y}) \subset \mathcal{O}(B)[[u_1, \dots, u_m]],$$

so that all coefficients of the forms $\beta_{i,j}$ are regular functions on B; in particular,

$$\beta_{i,2} = \sum_{1 \leqslant j_1 \leqslant j_2 \leqslant m} A_{j_1,j_2} u_{j_1} u_{j_2},$$

where $A_{j_1,j_2} \in \mathcal{O}(B)$. In terms of the embedding $\mathcal{O}_{o,\mathcal{Y}} \subset \mathbb{C}[[u_1, \dots, u_N]]$, we get the presentation

$$\beta_i = \overline{\beta}_{i,2} + \overline{\beta}_{i,3} + \dots,$$

where $\overline{\beta}_{i,j}$ is a homogeneous polynomial of degree j in u_*, and moreover, in the right-hand side, there are no monomials that do not contain the variables $u_1, \dots, u_m$, or that contain precisely one of them (in the power 1): every monomial in the right-hand side is divisible by some quadratic monomial in $u_1, \dots, u_m$. □

Let $\mathcal{Y}_B \to \mathcal{Y}$ be the blow-up of the subvariety B and $\mathcal{X}_B \subset \mathcal{Y}_B$ the strict transform of $\mathcal{X}$. Obviously, the morphism $\mathcal{X}_B \to \mathcal{X}$ is the blow-up of B on $\mathcal{X}$. The symbol E_B denotes the exceptional divisors of the blow-up of B on $\mathcal{Y}$. Since outside E_B the varieties $\mathcal{X}_B$ and $\mathcal{X}$ are isomorphic, it is sufficient to show that every point $p \in \mathcal{X}_B \cap E_B$ is either non-singular or a multi-quadratic singularity of the variety U_B of type 2^l, where $l \geqslant 1$, and of rank $\geqslant r_l$. We assume that the point p lies over the point $o \in U$ and is a singularity of the variety U_B.

By a linear change of local parameters $u_1, \dots, u_m$, we may ensure that at the point $p \in \mathcal{Y}_B$, there is a system of local parameters

$$(v_1, \dots, v_m, u_{m+1}, \dots, u_N)$$

linked to the original system of parameters u_* by the standard relations

$$u_1 = v_1, \ u_2 = v_1 v_2, \dots, \ u_m = v_1 v_m.$$

The local equation of the exceptional divisor E_B at the point p is $v_1 = 0$, and the subvariety $\mathcal{X}_B \subset \mathcal{Y}_B$ at that point is defined by the equations

$$\widetilde{\beta}_1, \dots, \widetilde{\beta}_k \in \mathcal{O}_{p,\mathcal{Y}_B} \subset \mathbb{C}[[v_1, \dots, v_m, u_{m+1}, \dots, u_N]].$$

Write down $\widetilde{\beta}_i = \widetilde{\beta}_{i,1} + \widetilde{\beta}_{1,2} + \dots$ and assume that for some $l \in \{1, \dots, k\}$, the linear forms $\widetilde{\beta}_{j,1}$, $l+1 \leqslant j \leqslant k$ are linearly independent, and moreover,

$$\dim\langle \widetilde{\beta}_{i,1} \mid 1 \leqslant i \leqslant k \rangle = k - l,$$

so that there are relations

$$\widetilde{\beta}_{i,1} = \sum_{j=l+1}^{k} a_{i,j} \widetilde{\beta}_{i,1},$$

$i = 1, \dots, l$. Replacing the original system of local equations $\beta_1, \dots, \beta_k$ by

$$\beta_i - \sum_{j=l+1}^{k} a_{i,j} \beta_j, \ i = 1, \dots, l, \quad \beta_{l+1}, \dots, \beta_k,$$

we may assume that the linear forms $\widetilde{\beta}_{i,1}$, $i = 1, \dots, l$ are identically zero. In that case, the following claim is true.

Lemma 4.1. *For $i = 1, \dots, l$ the quadratic forms $\overline{\beta}_{i,2}$ depend only on $u_2, \dots, u_m$ and*

$$\widetilde{\beta}_{i,2} = \overline{\beta}_{i,2}(v_2, \dots, v_m) + \beta^{\sharp}_{i,2},$$

where every monomial in the quadratic form $\beta^{\sharp}_{i,2}$ is divisible either by v_1, or by u_i, $m + 1 \leqslant i \leqslant N$.

Proof. This is obvious because every monomial in $\overline{\beta}_{i,j}$ is divisible by some quadratic monomial in $u_1, \dots, u_m$, and $\widetilde{\beta}_{i,1} \equiv 0$ for $i = 1, \dots, l$, and by the standard formulas, transforming regular functions under a blow-up. Q.E.D. for the lemma. □

The lemma gives us the inequality

$$\mathrm{rk}(\widetilde{\beta}_{i,2},\ 1 \leqslant i \leqslant l) \geqslant \mathrm{rk}(\overline{\beta}_{i,2},\ 1 \leqslant i \leqslant l) \geqslant r_k.$$

Setting $T_p\mathcal{X}_B = \{\widetilde{\beta}_{i,1} = 0 \,|\, l + 1 \leqslant j \leqslant k\}$ and using Remark 1.4, we get

$$\mathrm{rk}(\widetilde{\beta}_{i,2}|_{T_p\mathcal{X}_B},\ 1 \leqslant i \leqslant l) \geqslant r_k - 2(k - l) \geqslant r_l.$$

Therefore, $p \in \mathcal{X}_B$ is a multi-quadratic singularity of type 2^l and rank $\geqslant r_l$. Q.E.D. for Theorem 4.1.

Corollary 4.2. *Assume that $\mathcal{X}$ has multi-quadratic singularities of type $\underline{r}$, where $r_l \geqslant 3l + 1$ for all $l = 1, \dots, k$. Then the singularities of $\mathcal{X}$ are terminal.*

Proof. In the notations of the proof of Theorem 4.1, it is sufficient to show the inequality

$$a(\mathcal{X}_B \cap E_B, \mathcal{X}) \geqslant 1.$$

Assume that a point $o \in B$ of general position is a multi-quadratic singularity of type 2^l. From the claim (ii) of Corollary 4.1, we get the inequality

$$\mathrm{codim}(B \subset \mathcal{X}) \geqslant l + 2,$$

so that $\mathrm{codim}(B \subset \mathcal{Y}) \geqslant k + l + 2$, and for that reason,

$$a(E_B, \mathcal{Y}) \geqslant k + l + 1.$$

By the adjunction formula,

$$a(\mathcal{X}_B \cap E_B, \mathcal{X}) = a(E_B, \mathcal{Y}) - (k - l) - 2l,$$

which implies the required inequality. Q.E.D. for the corollary. □

This completes the proof of Theorem 1.1.

4.4. Singularities of complete intersections

Let us show Proposition 4.2. We will prove the claims (i) and (ii) simultaneously: by Grothendieck's theorem on parafactoriality [21, 25], the claim (ii) for $e = 4$ implies the factoriality of the variety Q.

We argue by induction on $l \geqslant 1$. For one quadric ($l = 1$), the claims (i) and (ii) are obvious. Since

$$\mathrm{rk}\, q_{[1,l-1]} \geqslant \mathrm{rk}\, q_{[1,l]},$$

we may assume that the claims (i) and (ii) are true for the tuple of quadratic forms $q_1, \dots, q_{l-1}$. In particular, the scheme of their common zeros Q_{l-1} is an irreducible reduced factorial complete

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intersection of type 2^{l-1} in $\mathbb{P}^{N-1}$, so that $\operatorname{Pic} Q_{l-1} = \mathbb{Z}H_{l-1}$, where H_{l-1} is the class of a hyperplane section: every effective divisor on Q_{l-1} is cut out on Q_{l-1} by a hypersurface in $\mathbb{P}^{N-1}$.

The scheme of common zeros of the quadratic forms $q_1, \dots, q_l$ is the divisor of zeros of the form q_l on the variety Q_{l-1}. This divisor is reducible or non-reduced if and only if there is a form q_l^* of rank $\leqslant 2$ such that

$$q_l - q_l^* \in \langle q_1, \dots, q_{l-1} \rangle,$$

and in that case, $\operatorname{rk} q_{[1,l]} \leqslant 2$, which contradicts the assumption. Therefore, Q is an irreducible reduced complete intersection. It is easy to see that $Q \subset \mathbb{P}^{N-1}$ is non-degenerate. Since

$$\operatorname{rk} q_{[1,l-1]} \geqslant 2(l-1) + (e+2) - 1$$

(for the claim (i) we set $e = 4$), we have

$$\operatorname{codim}(\operatorname{Sing} Q_{l-1} \subset Q_{l-1}) \geqslant e + 2,$$

so that

$$\operatorname{codim}((Q \cap \operatorname{Sing} Q_{l-1}) \subset Q) \geqslant e + 1.$$

It is easy to see that a point $p \in Q$, which is non-singular on Q_{l-1}, is singular on Q if and only if for some $\lambda_1, \dots, \lambda_{l-1}$, the quadric

$$q_l - \lambda_1 q_1 - \dots - \lambda_{l-1} q_{l-1} = 0$$

is singular at that point. Since the singular set of a quadric of rank r in $\mathbb{P}^{N-1}$ has dimension $N-1-r$, we conclude that the dimension of the set

$$\operatorname{Sing} Q \cap (Q_{l-1} \setminus \operatorname{Sing} Q_{l-1})$$

does not exceed $N - 1 - \operatorname{rk} q_{[1,l]} + (l-1)$, whence it follows that the codimension of that set with respect to Q is at least $\operatorname{rk} q_{[1,l]} - 2l + 1 \geqslant e$. Q.E.D. for Proposition 4.2.

4.5. Linear subspaces and projections

Now let us consider the questions that are naturally close to Proposition 4.2 and its proof. These questions are of key importance in the proof of Theorem 3.3 (which will be given in §5). Since in Theorem 3.3 the multi-quadratic singularity is of type 2^k, starting from this moment, we consider k quadratic forms $q_1, \dots, q_k$ in N variables (that is, on $\mathbb{P}^{N-1}$), and the tuple of them is denoted by the symbol $q_{[1,k]}$. The symbol Q, as above, stands for the complete intersection of these k quadrics $\{q_i = 0\}$ in $\mathbb{P}^{N-1}$.

Proposition 4.3. *Assume that for some $b \geqslant 0$, the inequality*

$$\operatorname{rk} q_{[1,k]} \geqslant 2(1+b)k + 3$$

holds. Then for every point $p \in Q \setminus \operatorname{Sing} Q$, there is a linear space $\Pi \subset \mathbb{P}^{N-1}$ of dimension b, such that $p \in \Pi \subset Q$, and moreover, $\Pi \cap \operatorname{Sing} Q = \emptyset$.

Proof contains the (obvious) construction of such linear subspaces. We argue by induction on b. If $b = 0$, then Π is the point p itself and there is nothing to prove. Assume that $b \geqslant 1$ and for $b - 1$ the claim of the Proposition is true.

Consider the linear subspace $T = T_p Q$ of codimension k in $\mathbb{P}^{N-1}$. Obviously, every linear space in $\mathbb{P}^{N-1}$ that contains the point p and is contained in Q is contained in T, too. Furthermore, $Q \cap T$ is defined by the quadratic forms $q_1|_T, \dots, q_k|_T$. Since $\operatorname{rk} q_{[1,k]}|_T \geqslant \operatorname{rk} q_{[1,k]} - 2k$, the inequality

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$$\operatorname{rk} q_{[1,k]}|_T \geqslant 2bk+3$$

holds, where every quadric $\{q_i|_T = 0\}$, $i = 1, \dots, k$, is by construction a cone with the vertex at p. Therefore, $Q \cap T$ is a cone with the vertex at the point p. Let $P \subset T$ be a hyperplane in T that does not contain the point p. Then the cone $Q \cap T$ is a cone with the base $Q \cap P$, where $Q \cap P$ is a complete intersection of the quadrics $\{q_i|_P = 0\}$, where, obviously,

$$\operatorname{rk} q_{[1,k]}|_P = \operatorname{rk} q_{[1,k]}|_T \geqslant 2(1+(b-1))k+3.$$

By the induction hypothesis, there is a linear subspace $\Pi^\sharp \subset P$ of dimension $(b-1)$, such that $\Pi^\sharp \subset Q \cap P$ and $\Pi^\sharp \cap \operatorname{Sing}(Q \cap P) = \emptyset$.

Furthermore, the set of singular points $\operatorname{Sing}(Q \cap T)$ is a cone with the vertex p, the base of which is $\operatorname{Sing}(Q \cap P)$, so that for the subspace $\Pi = \langle p, \Pi^\sharp \rangle$, which is a cone with the vertex p and the base $\Pi^\sharp$, we have $\Pi \cap \operatorname{Sing}(Q \cap T) = \{p\}$. Since $T \cap \operatorname{Sing} Q \subset \operatorname{Sing}(Q \cap T)$ and $p \notin \operatorname{Sing} Q$, we get $\Pi \cap \operatorname{Sing} Q = \emptyset$, which completes the proof of the proposition.

Proposition 4.4. *Let $b \geqslant \beta \geqslant 0$ be some integers. Assume that the inequality*

$$\operatorname{rk} q_{[1,k]} \geqslant 2k(b+\beta+1)+2\beta+3$$

holds. Then for every linear subspace $P \subset \mathbb{P}^{N-1}$ of codimension β and a general linear subspace $\Pi \subset Q$, $\Pi \cap \operatorname{Sing} = \emptyset$, of dimension b, the intersection $P \cap \Pi$ has codimension β in Π.

Proof. Again, we argue by induction on β; the case $\beta = 0$ is trivial. Only the equality $\Pi \cap \operatorname{Sing} Q = \emptyset$ for a general subspace $\Pi \subset Q$ of dimension $b \geqslant 0$ is needed, and it is true by Proposition 4.3. □

Let us show our claim in the assumption that it is true for $\beta - 1$.

First of all, note that

$$\operatorname{rk} q_{[1,k]}|_P \geqslant 2k(b+\beta+1)+3 > 2k+3,$$

so that by Proposition 4.2, the intersection $Q \cap P$ is an irreducible reduced complete intersection of type 2^k in P; in particular, a point of general position $p \in Q \cap P$ is non-singular. This means that

$$T_p(Q \cap P) = T_pQ \cap P$$

is of codimension k in P, so that T_pQ and P are in general position. The property to be in general position is an open property; therefore, for a point of general position $p \in Q$ (in particular, $p \notin P$), the linear subspaces T_pQ and P are in general position and their intersection $T_pQ \cap P$ is of codimension k in P and of codimension β in T_pQ.

Consider a general hypersurface Z in T_pQ, containing the subspace $T_pQ \cap P$ and not containing the point p. We have

$$\operatorname{rk} q_{[1,k]}|_Z \geqslant 2k(b+\beta)+2\beta+1 = 2k(b+(\beta-1)+1)+2(\beta-1))+3,$$

so that by the induction hypothesis for a general linear subspace $\Pi^\sharp \subset Q \cap Z$ of dimension $(b-1)$ that does not meet the set $\operatorname{Sing}(Q \cap Z)$, the intersection

$$(P \cap T_pQ) \cap \Pi^\sharp = P \cap \Pi^\sharp$$

is of codimension $\beta - 1 = \operatorname{codim}((P \cap T_pQ) \subset Z)$ with respect to $\Pi^\sharp$.

Then the linear space

$$\Pi = \langle p, \Pi^\sharp \rangle \subset T_pQ$$

of dimension b is contained in Q and does not meet the set $\mathrm{Sing}\, Q$ (see the proof of Proposition 4.3), and finally, the subspace

$$P \cap \Pi = P \cap T_p Q \cap \Pi = P \cap T_p Q \cap Z \cap \Pi = P \cap \Pi^\sharp$$

is of codimension β with respect to Π. Q.E.D. for the proposition.

Corollary 4.3. *In the assumptions of Proposition 4.4, where $\beta \geqslant k$, let $Y \subset Q$ be an irreducible subvariety of codimension $\beta - k$. Then the restriction onto Y of the projection*

$$\mathrm{pr}_\Pi : \mathbb{P}^{N-1} \dashrightarrow \mathbb{P}^{N-b-2}$$

from a general subspace $\Pi \subset Q$ of dimension b is dominant.

Proof. Let $p \in Y$ be a non-singular point. We apply Proposition 4.4 to the subspace $P = T_p Y \subset \mathbb{P}^{N-1}$ of codimension β. A general subspace $\Pi \subset Q$ of dimension $b \geqslant \beta$ does not contain the point p and is in general position with P, so that $\mathrm{pr}_\Pi |_P$ is regular in a neighborhood of the point p and its differential at the point p is an epimorphism. Therefore, $\mathrm{pr}_\Pi |_Y$ is regular at the point p, and its differential at that point is an epimorphism. Q.E.D. for the corollary. □

Note an important particular case.

Corollary 4.4. *Assume that $b \geqslant k$ and the inequality*

$$\mathrm{rk}\, q_{[1,k]} \geqslant 2k(b + k + 2) + 3$$

holds. Then the restriction of the projection pr_Π from a general subspace $\Pi \subset Q$ of dimension b onto Q is dominant, and its general fibre is a linear subspace of dimension $b + 1 - k$.

Proof. That it is dominant follows from the previous corollary, so that the dimension of a general fibre is $b + 1 - k$. Furthermore, pr_Π fibres $\mathbb{P}^{N-1}$ (more precisely, $\mathbb{P}^{N-1} \setminus \Pi$) into linear subspaces $\Pi^\sharp \supset \Pi$ of dimension $b + 1$. The centre Π of the projection is a hyperplane in $\Pi^\sharp$. Since $\Pi \subset Q$, the quadric $\{q_i|_{\Pi^\sharp} = 0\}$ is the union of two hyperplanes, one of which is Π. Now the claim of the corollary is obvious. □

Let $\Pi \subset Q$ be a linear subspace of dimension $b \geqslant k$, not meeting the set $\mathrm{Sing}\, Q$, and $\sigma : \widetilde{Q} \to Q$ and $\sigma_{\mathbb{P}} : \widetilde{\mathbb{P}^{N-1}} \to \mathbb{P}^{N-1}$ the blow-ups of Π on Q and $\mathbb{P}^{N-1}$, respectively, so that we can identify $\widetilde{Q}$ with the strict transform of Q on $\widetilde{\mathbb{P}^{N-1}}$. By the symbols E_Q and $E_{\mathbb{P}}$ we denote the exceptional divisors of these blow ups; we consider E_Q as a subvariety in $E_{\mathbb{P}}$. Let $\varphi : \widetilde{Q} \to \mathbb{P}^{N-b-2}$ and $\varphi_{\mathbb{P}} : \widetilde{\mathbb{P}^{N-1}} \to \mathbb{P}^{N-b-2}$ be the regularizations of the rational maps $\mathrm{pr}_\Pi |_Q$ and pr_Π, respectively. We have the natural identification $E_{\mathbb{P}} = \Pi \times \mathbb{P}^{N-b-2}$, where the map

$$\varphi_{\mathbb{P}}|_{E_{\mathbb{P}}} : E_{\mathbb{P}} \to \mathbb{P}^{N-b-2}$$

is the projection onto the second factor. In the assumptions of Corollary 4.4, the morphism φ is surjective, and for a point of general position $p \in \mathbb{P}^{N-b-2}$, the fibre $\varphi^{-1}(p)$ is a linear subspace of dimension $b + 1 - k$ in $\varphi_{\mathbb{P}}^{-1}(p) \cong \mathbb{P}^{b+1}$, which is not contained entirely in the hyperplane

$$\varphi_{\mathbb{P}}^{-1}(p) \cap E_{\mathbb{P}} = \left(\varphi_{\mathbb{P}}|_{E_{\mathbb{P}}}\right)^{-1}(p),$$

which identifies naturally with Π, and for that reason, $\varphi^{-1}(p) \cap E_{\mathbb{P}}$ identifies naturally with a subspace of dimension $b - k$ in Π (and a hyperplane in $\varphi^{-1}(p)$). However,

$$\varphi^{-1}(p) \cap E_{\mathbb{P}} = \varphi^{-1}(p) \cap E_Q = (\varphi|_{E_Q})^{-1}(p),$$

so that arguing by dimensions, we conclude that the restriction $\varphi|_{E_Q}$ is surjective.

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Proposition 4.5. *In the assumptions of Corollary 4.4, let $Y \subset \Pi$ be an irreducible closed subset, and assume that*

$$b \geqslant k + \operatorname{codim}(Y \subset \Pi).$$

Then the restriction $\varphi|_{\sigma^{-1}(Y)}$ is surjective, so that for a point of general position $p \in \mathbb{P}^{N-b-2}$, the intersection $\varphi^{-1}(p) \cap \sigma^{-1}(Y)$ is nonempty and each of its components is of codimension $\operatorname{codim}(Y \subset \Pi)$ *in the projective space $\varphi^{-1}(p) \cap E_{\mathbb{P}}$.*

Proof. Obviously,

$$\sigma^{-1}(Y) = \sigma_{\mathbb{P}}^{-1}(Y) \cap \widetilde{Q} = \sigma_{\mathbb{P}}^{-1}(Y) \cap E_Q.$$

Since $\varphi^{-1}(p) \subset \widetilde{Q}$, the equality

$$\varphi^{-1}(p) \cap \sigma^{-1}(Y) = \varphi^{-1}(p) \cap \sigma_{\mathbb{P}}^{-1}(Y)$$

holds, but $\sigma^{-1}(Y) = Y \times \mathbb{P}^{N-b-2}$ in terms of the direct decomposition of the exceptional divisor $E_{\mathbb{P}}$. Therefore, identifying the fibre of the projection $\varphi_{\mathbb{P}}|_{E_{\mathbb{P}}}$ with the projective space Π, we get that the intersection $\varphi^{-1}(p) \cap \sigma^{-1}(Y)$ identifies naturally with the intersection of Y and the linear subspace $\varphi^{-1}(p) \cap E_{\mathbb{P}}$ of dimension $b-k$ in Π. By our assumption, this intersection is nonempty, so that the morphism $\varphi|_{\sigma^{-1}(Y)}$ is surjective. Q.E.D. for the proposition. □

5. The special hyperplane section

In this section, we prove Theorem 3.3.

5.1. Start of the proof

We use the notations of Subsection 1.7 and the assumptions of Theorem 3.3. Recall that

$$I_X = [2k+3, k+c_X-1] \cap \mathbb{Z}$$

is the set of admissible dimensions for the working triple (X, D, o). Consider a general subspace $P \ni o$ in $\mathbb{P}(X)$ of the minimal admissible dimension $2k+3$. Since $a(E_{X\cap P}) = 2$ and $\nu(D) \leqslant \frac{3}{2}n(D) < 2n(D)$, we conclude that the pair

$$\left((X \cap P)^+, \frac{1}{n(D)} D|^+_{X\cap P}\right)$$

is not log-canonical, but canonical outside the exceptional divisor $E_{X\cap P}$. By the inequality $\nu(D) < 2n(D)$, we can apply the connectedness principle to this pair:

$$\operatorname{LCS}\left((X \cap P)^+, \frac{1}{n(D)} D|^+_{X\cap P}\right) \tag{17}$$

is a proper connected closed subset of the exceptional divisor $E_{X\cap P}$. There are the following options:

$(1)_P$ this subset contains a divisor,

$(2)_P$ some irreducible component of maximal dimension $B(P) \subset E_{X\cap P}$ in this set has a positive dimension and codimension $\geqslant 2$ in $E_{X\cap P}$,

$(3)_P$ this subset is a point.

Remark 5.1. In the case $(1)_P$, the divisor in the subset (17) is unique and is a hyperplane section of the variety $E_{X\cap P} \subset \mathbb{E}_{X\cap P}$, since $D|^+_{X\cap P}$ has along this subvariety the multiplicity $> n(D)$ (since it is the

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centre of some non-log-canonical singularity), whereas the restriction $D^+|_{E_{X\cap P}}$ is cut out on $E_{X\cap P}$ by a hypersurface of degree $\nu(D) < 2n(D)$.

Since $P \ni o$ is a subspace of general position, we go back to the original variety X and get that the pair $(X^+, \frac{1}{n(D)}D|^+)$ is not log-canonical, and moreover, for the centre $B \subset E_X$ of some non-log-canonical singularity of that pair, one of the three option takes place:
(1) B is a hyperplane section of $E_X \subset \mathbb{E}_X$,
(2) $\operatorname{codim}(B \subset E_X) \in \{2, \dots, k+1\}$,
(3) B is a linear subspace of codimension $2k+2$ in $\mathbb{E}_X$, which is contained in E_X.

Proposition 5.1. *The option (1) does not take place.*

Proof. Assume the converse: B is a hyperplane section of E_X. Let $R \subset X$, $R \ni o$ be the uniquely determined hyperplane section, such that $R^+ \cap E_X = B$ (in other words, $\mathbb{P}(R)^+ \cap \mathbb{E}_X$ is the hyperplane in $\mathbb{E}_X$ that cuts out B on E_X). Since $\operatorname{mult}_B D^+ > n(D)$, we get that for the effective divisor $D_R = (D \circ R)$ on R, the inequality

$$\nu(D_R) \geqslant \nu(D) + \operatorname{mult}_B D^+ > 2n(D) = 2n(D_R)$$

holds, which is impossible by Theorem 1.4. Q.E.D. for the proposition. □

Proposition 5.2. *The option (3) does not take place.*

Proof. Since $\operatorname{codim}(B \subset E_X) = k+2$, this is impossible by the Lefschetz theorem (in order to apply the Lefschetz theorem, it is sufficient to have the inequality $\operatorname{codim}(\operatorname{Sing} E_X \subset E_X) \geqslant 2k+6$, for which by Proposition 4.2 it is sufficient to have the inequality $\operatorname{rk}(o \in X) \geqslant 4k+5$; we have a much stronger condition for the rank of the singularity). Q.E.D. for the proposition. □

Therefore, the option (2) takes place. By construction (or arguing by dimension), $B \not\subset \operatorname{Sing} E_X$. Recall that there is a non-log-canonical singularity of the pair $(X^+, \frac{1}{n(D)}D^+)$, the centre of which is B.

Let $p \in B$ be a point of general position; in particular, $p \notin \operatorname{Sing} E_X$ and the more so, $p \notin \operatorname{Sing} X^+$. Applying inversion of adjunction in the word for word the same way as in [20, Chapter 7, Proposition 2.3] (that is, restricting D^+ onto a general non-singular surface, containing the point p), we get the alternative: either $\operatorname{mult}_B D^+ > 2n(D)$ or on the blow-up

$$\varphi_p \colon X^{(p)} \to X^+$$

of the point p with the exceptional divisor $E(p) \subset X^{(p)}$, $E(p) \cong \mathbb{P}^{N(X)-1}$, there is a hyperplane $\Theta(p) \subset E(p)$ in $E(p)$, satisfying the inequality

$$\operatorname{mult}_B D^+ + \operatorname{mult}_{\Theta(p)} D^{(p)} > 2n(D), \tag{18}$$

where $D^{(p)}$ is the strict transform of the divisor D^+ on $X^{(p)}$, and moreover, the hyperplane $\Theta(p)$ is uniquely determined by the pair $(X^+, \frac{1}{n(D)}D^+)$ and varies algebraically with the point $p \in B$.

The case when the inequality $\operatorname{mult}_B D^+ > 2n(D)$ holds is excluded (with simplifications) by the arguments, excluding the option $(2)_\Theta$, given below; see Subsection 5.3, Remark 5.2.

There are two options for the hyperplane $\Theta(p)$:

$(1)_\Theta$ $\Theta(p) \neq \mathbb{P}(T_p E_X)$ (where we identify $E(p)$ with the projectivization of the tangent space $T_p X^+$), so that $\Theta(p)$ intersects $\mathbb{P}(T_p E_X)$ by some hyperplane $\Theta_E(p)$,

$(2)_\Theta$ the hyperplanes $\Theta(p)$ and $\mathbb{P}(T_p E_X)$ in $E(p)$ are equal.

Below (see Subsection 5.3, Remark 5.2), we show that the option $(2)_\Theta$ does not take place: it implies that $E_X \subset D^+$, which is impossible; the same arguments exclude the inequality $\operatorname{mult}_B D^+ > 2n(D)$, too.

Therefore, we may assume that the option $(1)_\Theta$ takes place.

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5.2. The existence of the special hyperplane section

Adding the upper index (p) means the strict transform on $X^{(p)}$: we used this principle for the divisor D above and will use it for other subvarieties on X^+. Our aim is to prove the following claim.

Theorem 5.1. *There is a hyperplane section Λ of the exceptional divisor $E_X \subset \mathbb{E}_X$, containing B, satisfying the inequality*

$$\operatorname{mult}_\Lambda D^+ > \frac{2n(D) - \nu(D)}{k+1}.$$

Moreover, for a point of general position $p \in B$, the following equality holds:

$$\Lambda^{(p)} \cap E(p) = \Theta_E(p).$$

Proof. Let $L \subset E_X, L \ni p$ be a line in the projective space $\mathbb{E}_X$, such that $L \cap \operatorname{Sing} E_X = \emptyset$ and

$$L^{(p)} \cap E(p) \in \Theta_E(p).$$ □

Lemma 5.1. *The line L is contained in D^+.*

Proof. Assume the converse. Then $D^+|_L$ is an effective divisor on L of degree $\nu(D) \leqslant \frac{3}{2}n(D) < 2n(D)$. At the same time, the divisor $D^+|_L$ contains the point p with multiplicity $> 2n(D)$ due to the inequality (18). The contradiction proves the lemma. Q.E.D. □

Proposition 5.3. *The following inequality holds:*

$$\operatorname{mult}_L D^+ > \frac{2n(D) - \nu(D)}{k+1}.$$

Proof is given in §6.

Let us go back to the proof of Theorem 5.1.

We will construct the set $\Lambda \subset E_X$ explicitly and then prove that it is a hyperplane section. The exceptional divisor E_X is a complete intersection of k quadrics in $\mathbb{E}_X$:

$$E_X = \{q_1 = \ldots q_k = 0\},$$

using the notations of Subsection 4.5. Let $U_B \subset B$ be a nonempty Zariski open subset, where

$$U_B \cap \operatorname{Sing} E_X = \emptyset,$$

and for every point $p \in B$, the option $(1)_\Theta$ takes place. By the assumption on the rank of the multi-quadratic point $o \in X$ for $p \in U_B$, the set $E_X \cap T_p E_X$ (where $T_p E_X \subset \mathbb{E}_X$ is the embedded tangent space – that is, a linear subspace of codimension k in $\mathbb{E}_X$) is irreducible and reduced, and moreover, every hyperplane section of that set is also irreducible and reduced. Indeed, by Proposition 4.2, in order to have these properties, it is sufficient to have the inequality $\operatorname{rk} q_{[1,k]} \geqslant 4k+5$ because by Remark 1.4, it implies the inequality

$$\operatorname{rk} q_{[1,k]}|_{T_p E_X} \geqslant 2k+5,$$

and we can apply Proposition 4.2. Obviously, $E_X \cap T_p E_X$ is a cone with the vertex p, consisting of all lines $L \subset E_X$, $L \ni p$. The singular set of that cone is of codimension $\geqslant 6$ (Proposition 4.2), and so for a general line $L \ni p$,

$$L \cap \operatorname{Sing}(E_X \cap T_p E_X) = \{p\},$$

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so that $L \cap \operatorname{Sing} E_X = \emptyset$, and the same is true for every hyperplane section of the cone $E_X \cap T_p E_X$, containing the point p, since its singular set is of codimension $\geqslant 4$ (Remark 1.4).

Let $\mathcal{L}(p)$ be the union of all lines $L \subset E_X, L \ni p$, such that

$$L^{(p)} \cap E(p) \in \Theta_E(p).$$

Obviously, $\mathcal{L}(p)$ is the section of the cone $E_X \cap T_p E_X$ by some hyperplane, containing the point p (this hyperplane corresponds to the hyperplane $\Theta_E(p)$). As we have shown above, $\mathcal{L}(p)$ is an irreducible closed subset of codimension $k+1$ in E_X, and

$$\operatorname{mult}_{\mathcal{L}(p)} D^+ > \frac{2n(D) - \nu(D)}{k+1}.$$

Set

$$\Lambda = \overline{\bigcup_{p \in U_B} \mathcal{L}(p)}$$

(the overline means the closure). By what was said above, the inequality

$$\operatorname{mult}_\Lambda D^+ > \frac{2n(D) - \nu(D)}{k+1}$$

holds.

Theorem 5.2. *The subset $\Lambda \subset E_X$ is a hyperplane section of the variety $E_X \subset \mathbb{E}_X$.*

We will prove Theorem 5.2 in two steps: first, we will show that Λ is a prime divisor on E_X and then that this divisor is a hyperplane section. By construction, the set Λ is irreducible.

5.3. The set Λ is a divisor

By our assumption about the rank of the point $o \in X$ for $b = k+1$, the inequality

$$\operatorname{rk} q_{[1,k]} \geqslant 2k(b + 2k + 2) + 2(2k+1) + 3 \tag{19}$$

holds. By Corollary 4.3, for a general subspace $\Pi \subset E_X$ of dimension b, the restriction onto B of the projection

$$\operatorname{pr}_\Pi \colon \mathbb{P}^{N(X)-1} \dashrightarrow \mathbb{P}^{N(X)-b-2}$$

from the subspace Π is dominant. Let $s \in \mathbb{P}^{N(X)-b-2}$ be a point of general position. By the symbol $\langle \Pi, s \rangle$ denote the closure

$$\overline{\operatorname{pr}_\Pi^{-1}(s)} \subset \mathbb{P}^{N(X)-1}$$

(this is a $(\dim \Pi + 1)$-dimensional subspace) and set

$$E_X(\Pi, s) = E_X \cap \langle \Pi, s \rangle.$$

For the blow-ups $\sigma \colon \widetilde{E}_X \to E_X$ and $\sigma_\mathbb{P} \colon \widetilde{\mathbb{P}^{N(X)-1}} \to \mathbb{P}^{N(X)-1}$ of the subspace Π on E_X and $\mathbb{P}^{N(X)-1}$, respectively, let $\varphi \colon \widetilde{E}_X \to \mathbb{P}^{N(X)-b-2}$ and $\varphi_\mathbb{P} \colon \widetilde{\mathbb{P}^{N(X)-1}} \to \mathbb{P}^{N(X)-b-2}$ be the regularizations of the projections $\operatorname{pr}_\Pi|_{E_X}$ and pr_Π, respectively. Obviously, the fibre $\varphi_\mathbb{P}^{-1}(s)$ identifies naturally with $\langle \Pi, s \rangle$, and the fibre $\varphi^{-1}(s)$ with $E_X(\Pi, s)$. The fibre of the surjective morphism $\varphi|_{\sigma^{-1}(B)}$ over the point s we denote by the symbol $B(s)$; this is a possibly reducible closed subset in $\varphi_\mathbb{P}^{-1}(s)$, each irreducible component of

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which is of codimension $c_B = \mathrm{codim}(B \subset E_X)$ and is not contained entirely in the hyperplane Π (with respect to the identification $\varphi_{\mathbb{P}}^{-1}(s) = \langle \Pi, s \rangle$). Write down $B(s)$ as a union of irreducible components:

$$B(s) = \bigcup_{i \in I} B_i(s),$$

and let $p \in B_i(s)$ be a point of general position on one of them; in particular, $p \notin \Pi$, so that the projection pr_Π is regular at that point and $p \notin B_j(s)$ for $j \neq i$. We will consider the point p as a point of general position on B, which was introduced in Subsection 5.1, and use the notations for the blow-up φ_p of this point and for objects linked to this blow-up. Note that for $b = k + 1$, we have the inequality

$$\dim B(s) = \dim B_i(s) \geqslant 1.$$

The set of lines $L \subset E_X(\Pi, s)$, $L \ni p$, such that $L^{(p)} \cap E(p) \in \Theta(p)$, forms a hyperplane in $E_X(\Pi, s)$, which we denote by the symbol $\Lambda(\Pi, s, p)$. By construction, $\Lambda(\Pi, s, p) \subset \Lambda$.

Since any nontrivial algebraic family of hyperplanes in a projective space sweeps out that space and for a general point s we have $E_X(\Pi, s) \not\subset \Lambda$ (otherwise, $\Lambda = E_X$, which is impossible), we conclude that the hyperplane $\Lambda(\Pi, s, p)$ does not depend on the choice of a point of general position $p \in B_i(s)$, so that

$$\Lambda(\Pi, s, p) = \Lambda(\Pi, s, B_i(s))$$

is a hyperplane in $\varphi^{-1}(s) = E_X(\Pi, s)$, containing the component $B_i(s)$. Therefore, for a general point s, the intersection $\Lambda \cap E_X(\Pi, s)$ contains a divisor in $E_X(\Pi, s)$, whence we get that $\Lambda \subset E_X$ is a (prime) divisor on E_X, as we claimed. This divisor is cut out on E_X by a hypersurface of degree d_Λ in $\mathbb{E}_X$. It remains to show that $d_\Lambda = 1$.

Remark 5.2. We promised above that the option $(2)_\Theta$ does not take place. Indeed, if it does, then every line $L \ni p$ in $E_X(\Pi, s)$ is contained in Λ, so that $E_X(\Pi, s) \subset \Lambda$, and for that reason, $E_X \subset \Lambda$, which is absurd. In a similar way, if $\mathrm{mult}_B D^+ > \nu(D)$, then every line in $E_X(\Pi, s)$, meeting B, is contained in Λ, so that $E_X \subset \Lambda$, which is impossible. Therefore, the inequality

$$\mathrm{mult}_B D^+ \leqslant \nu(D)$$

holds.

5.4. The divisor Λ is a hyperplane section

Let us consider the intersection $\Lambda \cap E_X(\Pi, s)$ for a general point s in more details. This is a possibly reducible divisor, each component of which has multiplicity 1, containing at least one hyperplane. If in this divisor there are components of degree $\geqslant 2$, then the union of hyperplanes in $\Lambda(\Pi, s)$ gives a proper closed subset of $\Lambda_1(\Pi, s)$, which is also a divisor. Then

$$\overline{\bigcup_{s} \Lambda_1(\Pi, s)}$$

(the union is taken over a nonempty open subset in $\mathbb{P}^{N(X)-b-2}$) is a proper closed subset in Λ, which is of codimension 1 in E_X, which is impossible as Λ is a prime divisor. We conclude that $\Lambda(\Pi, s)$ is a union of precisely d_Λ distinct hyperplanes in $E_X(\Pi, s)$.

Assume that $d_\Lambda \geqslant 2$. By our assumptions about the rank $\mathrm{rk}(o \in X)$, the inequality (19) holds for $b = 3k$:

$$\mathrm{rk}\, q_{[1,k]} \geqslant 10k^2 + 8k + 5.$$

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Again, we apply Corollary 4.3, now to a general subspace $\Pi^* = E_X(\Pi, s)$ of dimension $b^* = b+1-k \geqslant 2k+1$. The subspace Π^* does not meet the set Sing E_X, and the restriction of the projection from Π^*

$$\mathrm{pr}_{\Pi^*}\colon \mathbb{P}^{N(X)-1} \dashrightarrow \mathbb{P}^{N(X)-b^*-2}$$

onto B is dominant. Let $s^* \in \mathbb{P}^{N(X)-b^*-2}$ be a point of general position. We use the notations introduced above and write $E_X(\Pi^*, s^*)$. For the blow-ups of the subspace Π^*, we use the symbols σ_{Π^*} and $\sigma_{\mathbb{P},\Pi^*}$, respectively, and for the regularized projections, the symbols φ_{Π^*} and $\varphi_{\mathbb{P},\Pi^*}$. The symbol $\langle \Pi^*, s^*\rangle$ has the same meaning as above. Set

$$E^* = \sigma_{\Pi^*}^{-1}(\Pi^*) \quad \text{and} \quad E^*_{\mathbb{P}} = \sigma_{\mathbb{P},\Pi^*}^{-1}(\Pi^*)$$

to be the exceptional divisors of the blow-up of Π^* on E_X and $\mathbb{P}^{N(X)-1}$. By the arguments immediately before the statement of Proposition 4.5, the map $\varphi_{\Pi^*}|_{E^*}$ is surjective, and by Proposition 4.5 (which applies since $b^* \geqslant k+1$), the intersection

$$\varphi_{\Pi^*}^{-1}(s^*) \cap \sigma_{\Pi^*}^{-1}(\Lambda \cap \Pi^*)$$

is nonempty, and each of its irreducible components is of codimension 1 in the projective space $\varphi_{\Pi^*}^{-1}(s^*) \cap E^*_{\mathbb{P}}$.

By what was shown above, $\Lambda \cap \Pi^*$ is a union of d_Λ distinct hyperplanes Λ^*_i, $i \in I$. In a similar way,

$$\Lambda \cap E_X(\Pi^*, s^*) = \sigma_{\Pi^*}^{-1}(\Lambda) \cap \varphi_{\Pi^*}^{-1}(s^*)$$

is the union of d_Λ distinct hyperplanes in $\varphi_{\Pi^*}^{-1}(s^*)$, none of which coincides with the hyperplane $\varphi_{\Pi^*}^{-1}(s^*) \cap E^*_{\mathbb{P}}$. Note that the strict transform of the divisor Λ with respect to the blow-up σ_{Π^*} is just its full inverse image $\sigma_{\Pi^*}^{-1}(\Lambda)$, since $\Lambda \not\subset \Pi^*$. Furthermore,

$$\sigma_{\Pi^*}^{-1}(\Lambda) \cap E^* = \sigma_{\Pi^*}^{-1}(\Lambda \cap \Pi^*) = \bigcup_{i \in I} \sigma_{\Pi^*}^{-1}(\Lambda^*_i),$$

and every intersection $\varphi_{\Pi^*}^{-1}(s^*) \cap \sigma_{\Pi^*}^{-1}(\Lambda^*_i)$ is a hyperplane in $\varphi_{\Pi^*}^{-1}(s^*) \cap E^*_{\mathbb{P}}$. It follows that each irreducible component of set $\Lambda \cap E_X(\Pi^*, s^*)$ intersects the hyperplane $\varphi_{\Pi^*}^{-1}(s^*) \cap E^*_{\mathbb{P}}$ by one of the hyperplanes $\sigma_{\Pi^*}^{-1}(\Lambda^*_i) \cap \varphi_{\Pi^*}^{-1}(s^*)$, $i \in I$. Thus, one can write down

$$\Lambda \cap E_X(\Pi^*, s^*) = \bigcup_{i \in I} \Lambda_i(\Pi^*, s^*),$$

where $\Lambda_i(\Pi^*, s^*)$ is a hyperplane in $E_X(\Pi^*, s^*)$, satisfying the equality

$$\Lambda_i(\Pi^*, s^*) \cap E^* = \sigma_{\Pi^*}^{-1}(\Lambda^*_i) \cap \varphi_{\Pi^*}^{-1}(s^*).$$

In other words, the choice of a component of the intersection $\Lambda \cap \Pi^*$ determines uniquely the component of the intersection of Λ with $\langle \Pi^*, s^*\rangle = \varphi_{\Pi^*}^{-1}(s^*) = E_X(\Pi^*, s^*)$ for a general point s^*. Now set

$$\Lambda_i = \sigma_{\Pi^*}\left(\overline{\bigcup_{s^*} (\Pi^*, s^*)}\right),$$

where the union is taken over a nonempty Zariski open subset of the projective space $\mathbb{P}^{N(X)-b^*-2}$. This is a prime divisor on E_X, and moreover, $\Lambda_i \subset \Lambda$, and for that reason, $\Lambda_i = \Lambda$, whence we conclude that all hyperplanes $\Lambda_i(\Pi^*, s^*)$ are the same, which is a contradiction with the assumption that $d_\Lambda \geqslant 2$.

Thus, $d_\Lambda = 1$ and Λ is a hyperplane section of $E_X \subset \mathbb{E}_X$. Q.E.D. for Theorem 5.2, and therefore for Theorem 5.1.

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5.5. The construction of a new working triple

Now we can complete the proof Theorem 3.3 and construct the new working triple (R, D_R, o). Let $R \ni o$ the section of X by the hyperplane $\mathbb{P}(R) = \langle R \rangle$, such that

$$R^+ \cap \mathbb{E}_X = R^+ \cap E_X = \Lambda$$

(in other words, the hyperplane $\mathbb{P}(R)^+ \cap \mathbb{E}_X$ cuts out Λ on E_X). Since R is not a component of the effective divisor D_X, the scheme-theoretic intersection $(R \circ D_X)$ is well defined, and we treat this intersection as an effective divisor on R. Set $D_R = (R \circ D_X)$ in that sense.

On $R^+ \subset X^+$ with the exceptional divisor

$$E_R = (R^+ \cap E_X) = \Lambda \subset \mathbb{E}_R = \mathbb{P}(R)^+ \cap \mathbb{E}_X,$$

we have the equivalence

$$D_R^+ \sim n(D_R)H_R - \nu(D_R)E_R,$$

where H_R is the class of a hyperplane section of R and

$$\nu(D_R) \geqslant \nu(D_X) + \mathrm{mult}_\Lambda D_X^+ > \nu(D_X) + \frac{2n(D_X) - \nu(D_X)}{k+1}.$$

Again, $[R, o]$ is a marked complete intersection, of level (k, c_R), where $c_R = c_X - 2$, (R, D_R, o) is a working triple, and the inequality

$$2n(D_R) - \nu(D_R) < \left(1 - \frac{1}{k+1}\right)(2n(D_X) - \nu(D_X))$$

holds (since $n(D_R) = n(D_X)$).

The procedure of constructing the special hyperplane section is complete. Q.E.D. for Theorem 3.3.

6. Multiplicity of a line

In this section, we prove Proposition 5.3.

6.1. Blowing up a point and a curve

Since we completed our study of working triples, the symbol X is now free and will mean an arbitrary non-singular quasi-projective variety of dimension $\geqslant 3$. Let $C \subset X$ be a non-singular projective curve, $p \in C$ a point. Furthermore, let

$$\sigma_C \colon X(C) \to X$$

be the blow-up of the curve C with the exceptional divisor E_C and $\sigma_C^{-1}(p) \cong \mathbb{P}^{\dim X - 2}$ the fibre over the point p. Let

$$\sigma \colon X(C, \sigma_C^{-1}(p)) \to X(C)$$

be the blow-up of that fibre with the exceptional divisor E and $E_C^{(p)}$ the strict transform of E_C on that blow-up.

However, consider the blow-up

$$\varphi_p \colon X(p) \to X$$

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of the point p with the exceptional divisor E_p and denote by the symbol $C(p)$ the strict transform of the curve C on $X(p)$. Finally, let

$$\varphi\colon X(p, C(p)) \to X(p)$$

be the blow-up of the curve $C(p)$, $E_{C(p)}$ the exceptional divisor of that blow-up and E_p^C the strict transform of E_p.

Proposition 6.1. *The identity map* id_X *extends to an isomorphism*

$$X(C, \sigma_C^{-1}(p)) \cong X(p, C(p)),$$

identifying the subvarieties E *and* E_p^C *and the subvarieties* E_C^p *and* $E_{C(p)}$.

Proof. This is a well-known fact, which can be checked by elementary computations in local parameters. Q.E.D. for the proposition. □

Taking into account the identifications above, we will use the notations E_p^C and $E_{C(p)}$, and forget about E and E_C^p. The variety $X(C, \sigma_C^{-1}(p))$ will be denoted by the symbol $\widetilde{X}$. Let D be an effective divisor on X. The symbols D^C and D^p stand for its strict transforms on $X(C)$ and $X(p)$, respectively, and the symbol $\widetilde{D}$ for its strict transform on $\widetilde{X}$. Set

$$\mu = \mathrm{mult}_C D \quad \text{and} \quad \mu_p = \mathrm{mult}_p D,$$

where, of course, $\mu_p \geqslant \mu$.

Lemma 6.1. *The following equality holds:*

$$\mathrm{mult}_{\sigma_C^{-1}(p)} D^C = \mu_p - \mu.$$

Proof. (This is a well-known fact, and we give a proof for the convenience of the reader, and also because a similar argument is used below.) We have the sequence of obvious equalities:

$$\sigma_C^* D = D^C + \mu E_C,$$

so that

$$\sigma^* \sigma_C^* D = \widetilde{D} + \mu E_{C(p)} + (\mu + \mathrm{mult}_{\sigma_{C(p)}^{-1}} D^C) E_p^C.$$

Considering the second sequence of blow-ups, we get

$$\varphi_p^* D = D^p + \mu_p E_p$$

and, respectively,

$$\varphi^* \varphi_p^* D = \widetilde{D} + \mu E_{C(p)} + \mu_p E_p^C.$$

Comparing the two presentations of the same effective divisor, we get the claim of the lemma.

6.2. *Blowing up two points and a curve*

In the notations of the previous subsection, let us consider the point

$$q = C(p) \cap E_p.$$

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Set $\mu_q = \mathrm{mult}_q\, D^p$. Obviously,

$$\mu_q \geqslant \mathrm{mult}_{C(p)}\, D^p = \mathrm{mult}_C\, D = \mu.$$

Let

$$\varphi_q \colon X(p,q) \to X(p)$$

be the blow-up of the point q with the exceptional divisor E_q and $C(p,q)$ the strict transform of the curve $C(p)$. Finally, let

$$\varphi_\sharp \colon X^\sharp \to X(p,q)$$

be the blow-up of the curve $C(p,q)$ with the exceptional divisor $E_{C(p,q)}$ and $E_q^\sharp$ the strict transform of E_q. Note that the curve $C(p)$ intersects E_p transversally, and therefore, $C(p,q)$ does not meet the strict transform E_p^q of the divisor $E_p \subset X(p)$ on $X(p,q)$.

Proposition 6.2. *The restriction of the divisor D^C onto the exceptional divisor E_C contains the fibre $\sigma_C^{-1}(p)$ with multiplicity at least $\mu_p + \mu_q - 2\mu$.*

Proof. Obviously, on $X^\sharp$ we have the equality

$$\varphi_\sharp^* \varphi_q^* \varphi_p^* D = D^\sharp + (\mu_q + \mu_p) E_q^\sharp + \mu_p E_p^q + \mu E_{C(p,q)},$$

where $D^\sharp$ is the strict transform of D on $X^\sharp$. However, using the constructions of Subsection 6.1, we see that $X^\sharp$ can be obtained as the blow-up of the curve $C(p)$ on $X(p)$ with the subsequent blowing up of the fibre of the exceptional divisor $E_{C(p)}$ over the point q or, applying the construction of Subsection 6.1 twice, as the blow-up of the curve C on X with the subsequent blowing up of the fibre $\sigma_C^{-1}(p)$ and then the blowing up of the subvariety

$$E_p^C \cap E_C^p.$$

In the last presentation, the three prime exceptional divisors are

$$E_C^\sharp = E_{C(p,q)}, \quad E_p^\sharp \quad \text{and} \quad E_q^\sharp.$$

We denote the blow-up $X^\sharp \to \widetilde{X}$ of the subvariety $E_p^C \cap E_C^p$, mentioned above, by the symbol $\sigma_\sharp$. Thus, we obtain the following commutative diagram of birational morphisms:

$$\begin{array}{ccccc} X(C) & \stackrel{\sigma}{\leftarrow} & \widetilde{X} & \stackrel{\sigma_\sharp}{\leftarrow} & X^\sharp \\ \downarrow & & \downarrow & & \downarrow \\ X & \stackrel{\varphi_p}{\leftarrow} & X(p) & \stackrel{\varphi_q}{\leftarrow} & X(p,q), \end{array}$$

where the vertical arrows (from the left to the right) are σ_C, φ and $\varphi_\sharp$, respectively. We have the equality

$$\sigma_\sharp^* \sigma^* \sigma_C^* D = \varphi_\sharp^* \varphi_q^* \varphi_p^* D.$$

This pullback can be written down as

$$D^\sharp + \mu E_C^\sharp + \mu_p E_p^\sharp + (\mu_p + \mu_q) E_q^\sharp,$$

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since $E_{C(p,q)} = E_C^\sharp$ and $E_q^\sharp$ is the exceptional divisor of the blow-up $\sigma_\sharp$. However, $D^C = \sigma_C^* D - \mu E_C$ and, besides,

$$\sigma_\sharp^* \sigma^* E_C = E_C^\sharp + E_p^\sharp + 2E_q^\sharp,$$

so that the exceptional divisor $E_q^\sharp$ comes into the pullback of the divisor D^C on $X^\sharp$ with multiplicity $(\mu_p + \mu_q - 2\mu)$. However, the blow-ups σ and $\sigma_\sharp$ do not change the divisor E_C, as they blow-up subvarieties of codimension 1 on the variety:

$$\sigma \circ \sigma_\sharp|_{E_C^\sharp} : E_C^\sharp \to E_C$$

is an isomorphism. Since the restriction $D^\sharp$ onto $E_C^\sharp$ is an effective divisor, it follows from here that the restriction of the divisor D^C onto E_C contains the fibre $\sigma^{-1}(p)$ (which is precisely the restriction of $E_q^\sharp$ onto $E_C^\sharp$ in terms of the isomorphism between $E_C^\sharp$ and E_C, discussed above) with multiplicity $\geqslant \mu_p + \mu_q - 2\mu$. Proof of Proposition 6.2 is complete. □

6.3. The multiplicity of an infinitely near line

Let us come back to the proof of Proposition 5.3. We will obtain its claim from a more general fact. Let $o \in \mathcal{X}$ be a germ of a multi-quadratic singularity of type 2^k, where $\mathcal{X} \subset \mathcal{Y}$, $\mathcal{Y}$ is non-singular, $\operatorname{codim}(\mathcal{X} \subset \mathcal{Y}) = k$ and the inequality

$$\operatorname{rk}(o \in \mathcal{X}) \geqslant 2k + 3$$

holds, so that $\operatorname{codim}(\operatorname{Sing} \mathcal{X} \subset \mathcal{X}) \geqslant 4$ and $\mathcal{X}$ is factorial. Let $\sigma_{\mathcal{Y}} : \mathcal{Y}^+ \to \mathcal{Y}$ be the blow-up of the point o with the exceptional divisor $E_{\mathcal{Y}}$, $\mathcal{X}^+ \subset \mathcal{Y}^+$ the strict transform, so that

$$\sigma = \sigma_{\mathcal{Y}}|_{\mathcal{X}^+} : \mathcal{X}^+ \to \mathcal{X}$$

is the blow-up of the point o on $\mathcal{X}$ with the exceptional divisor $E_{\mathcal{X}}$, which is a complete intersection of k quadrics in $E_{\mathcal{Y}} \cong \mathbb{P}^{\dim \mathcal{Y} - 1}$. By Proposition 4.2,

$$\operatorname{codim}(\operatorname{Sing} E_{\mathcal{X}} \subset E_{\mathcal{X}}) \geqslant 4.$$

Let $L \subset E_{\mathcal{X}}$ be a line, where $L \cap \operatorname{Sing} E_{\mathcal{X}} = \emptyset$ and $p \in L$ a point. Let us blow-up this point on $\mathcal{Y}^+$ and $\mathcal{X}^+$, respectively:

$$\sigma_{p,\mathcal{Y}} : \mathcal{Y}_p \to \mathcal{Y}^+ \quad \text{and} \quad \sigma_p : \mathcal{X}_p \to \mathcal{X}^+$$

are these blow-ups with the exceptional divisors $E_{p,\mathcal{Y}}$ and E_p. Set

$$q = L^{(p)} \cap E_p,$$

where $L^{(p)} \subset \mathcal{X}_p$ is the strict transform.

Let $D_{\mathcal{X}}$ be an effective divisor on $\mathcal{X}$. For its strict transform on $\mathcal{X}^+$, we have the equality

$$D_{\mathcal{X}}^+ = \sigma^* D_{\mathcal{X}} - \nu E_{\mathcal{X}}$$

for some $\nu \in \mathbb{Z}_+$. Furthermore, we denote the strict transform of $D_{\mathcal{X}}^+$ on $\mathcal{X}_p$ by the symbol $D_{\mathcal{X}}^{(p)}$ and set

$$\mu_p = \operatorname{mult}_p D_{\mathcal{X}}^+ \quad \text{and} \quad \mu_q = \operatorname{mult}_q D_{\mathcal{X}}^{(p)}.$$

Set also $\mu = \operatorname{mult}_L D_{\mathcal{X}}^+$; obviously, $\mu \leqslant \mu_p$.

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Theorem 6.1. *The following inequality holds:*

$$\mu \geqslant \frac{1}{k+1}(\mu_p + \mu_q - \nu).$$

Proof. Let $P \subset E_{\mathcal{Y}}$ be a general linear subspace of dimension $(k+2)$, containing the line L. □

Lemma 6.2. *The surface $S = P \cap \mathcal{X}^+ = P \cap E_{\mathcal{X}}$ is non-singular.*

Proof. We argue by induction on $\dim E_{\mathcal{X}} \geqslant 2$. If $\dim E_{\mathcal{X}} = 2$, then there is nothing to prove. Let $\dim E_{\mathcal{X}} \geqslant 3$. The hyperplanes in $E_{\mathcal{Y}}$, tangent to $E_{\mathcal{X}}$ at at least one point of the line L, form a k-dimensional family. The hyperplanes, containing the line L, form a family (a linear subspace) of codimension 2 in the dual projective space for $E_{\mathcal{Y}}$ – that is, of dimension $k + \dim E_{\mathcal{X}} - 2 \geqslant k+1$ – so that for a general hyperplane $R_{\mathcal{Y}} \supset L$ in $E_{\mathcal{Y}}$, we have the following: $E_{\mathcal{X}} \cap R_{\mathcal{Y}}$ is non-singular along L (and, of course, for the codimension of the singular set, we have the equality $\operatorname{codim}(\operatorname{Sing}(E_{\mathcal{X}} \cap R_{\mathcal{Y}}) \subset (E_{\mathcal{X}} \cap R_{\mathcal{Y}})) = \operatorname{codim}(\operatorname{Sing} E_{\mathcal{X}} \subset E_{\mathcal{X}})$. Applying the induction hypothesis, we complete the proof of the lemma. Q.E.D. □

Let $\mathcal{Z} \subset \mathcal{Y}$, $\mathcal{Z} \ni o$, be a general subvariety of dimension $(k+3)$, non-singular at the point o, such that $\mathcal{Z}^+ \cap E_{\mathcal{Y}} = P$, and

$$\mathcal{X}_P = \mathcal{X} \cap \mathcal{Z}$$

(the notation $\mathcal{X}_P$ is chosen for convenience: $\mathcal{X}_P$ is determined by $\mathcal{Z}$, not by P). Then $\mathcal{X}_P$ is a three-dimensional variety with the isolated multi-quadratic singularity $o \in \mathcal{X}_P$, and the blow-up of the point o resolves this singularity: the exceptional divisor $\mathcal{X}_P^+ \cap E_{\mathcal{Y}}$ is the non-singular surface S.

The restriction of the divisor $D_{\mathcal{X}}$ onto $\mathcal{X}_P$ is denoted by the symbol D_P, and its strict transform on $\mathcal{X}_P^+$ by the symbol D_P^+.

Lemma 6.3. *The normal sheaf $\mathcal{N}_{L/\mathcal{X}_P^+} \cong \mathcal{O}_L(-\alpha) \oplus \mathcal{O}_L(-\beta)$, where $\alpha + \beta = k$ and $\alpha \geqslant \beta \geqslant 1$.*

Proof. Since $P \cong \mathbb{P}^{k+2}$, by the adjunction formula $K_S = (k-3)H_S$, where H_S is the class of a hyperplane section of $S \subset P$, whence it follows that $(L^2)_S = 1 - k$. Furthermore, the surface S is the exceptional divisor of the blow-up of the point o on $\mathcal{X}_P$ ($S = \mathcal{X}_P^+ \cap E_{\mathcal{Y}}$), so that $\mathcal{O}_{\mathcal{X}_P^+}(S)|_L = \mathcal{O}_L(-1)$, and we have the exact sequence

$$0 \to \mathcal{N}_{L/S} \to \mathcal{N}_{L/\mathcal{X}_P^+} \to \mathcal{N}_{S/\mathcal{X}_P^+}|_L \to 0$$

or

$$0 \to \mathcal{O}_L(1-k) \to \mathcal{N}_{L/\mathcal{X}_P^+} \to \mathcal{O}(-1) \to 0.$$

From this, the claim of the lemma follows at once. Q.E.D. □

Let $\sigma_L \colon \mathcal{X}_{P,L} \to \mathcal{X}_P^+$ be the blow-up of the line L, and $E_{P,L} \subset \mathcal{X}_{P,L}$ the exceptional divisor. The lemma implies that $E_{P,L}$ is a ruled surface of type $\mathbb{F}_{\alpha-\beta}$ and its Picard group is $\mathbb{Z}s \oplus \mathbb{Z}f$, where f is the class of a fibre, s the class of the exceptional section, $s^2 = -(\alpha - \beta)$. Again, from the lemma shown above, it follows that

$$(E_{P,L}^3)_{\mathcal{X}_{P,L}} = (E_{P,L}|_{E_{P,L}}^2) = -\deg \mathcal{N}_{L/\mathcal{X}_P^+} = k,$$

so that

$$-E_{P,L}|_{P,L} = s + \frac{1}{2}(k + \alpha - \beta)f.$$

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Obviously (since the subspace P is general),

$$\mathrm{mult}_p D^+_P = \mu_p.$$

Let $\mathcal{X}^{(p)}_P \subset \mathcal{X}_p$ be the strict transform of $\mathcal{X}^+_P$ on $\mathcal{X}_p$. By construction, $q \in \mathcal{X}^{(p)}_P$. Setting $D^{(p)}_P = D^{(p)}_{\mathcal{X}}|_{\mathcal{X}^{(p)}_P}$, we obtain

$$\mu_q = \mathrm{mult}_q D^{(p)}_P.$$

Finally, let $D^{(L)}_P$ be the strict transform of the divisor D^+_P on $\mathcal{X}_{P,L}$. Obviously,

$$D^{(L)}_P = \sigma^*_L D^+_P - \mu E_{P,L},$$

so that, writing the pullback on $\mathcal{X}_{P,L}$ of the restriction $E_{\mathcal{X}}|_{\mathcal{X}^+_P}$ for simplicity as the restriction $E_{\mathcal{X}}|_{\mathcal{X}_{P,L}}$, we have

$$(-\nu E_{\mathcal{X}}|_{\mathcal{X}_{P,L}} - \mu E_{P,L})|_{E_{P,L}} \sim$$

$$\sim \nu f + \mu(s + \frac{1}{2}(k + \alpha - \beta) f) = \mu s + (\nu + \frac{1}{2}\mu(k + \alpha - \beta)) f.$$

By Proposition 6.2, this effective divisor contains the fibre $\sigma^{-1}_L(p)$ with multiplicity at least $\mu_p + \mu_q - 2\mu$, whence we get the inequality

$$\nu + \frac{1}{2}\mu(k + \alpha - \beta) \geqslant \mu_p + \mu_q - 2\mu,$$

which after easy transformations gives us that

$$\mu > \frac{2(\mu_p + \mu_q) - 2\nu}{k + (\alpha - \beta) + 4}.$$

The denominator of the right-hand side is maximal when $\alpha = k - 1$ and $\beta = 1$ and so

$$\mu > \frac{2(\mu_p + \mu_q) - 2\nu}{2k + 2} = \frac{(\mu_p + \mu_q) - \nu}{k + 1}.$$

Q.E.D. for the theorem.

Proposition 5.3 follows immediately from the theorem that we have just shown, taking into account the construction of the line L and the inequality (18).

7. Hypertangent divisors

In this section, we prove Theorems 1.2, 1.3 and 1.4.

7.1. Non-singular points. Tangent divisors

Let us start the proof of Theorem 1.2. Obviously, it is sufficient to consider the case when the subspace P is of maximal admissible codimension $k + \varepsilon(k) - 1$ in $\mathbb{P}^{M+k}$. Theorem 1.1 and Remark 1.4 imply that the inequality

$$\mathrm{codim}(\mathrm{Sing}(F \cap P) \subset (F \cap P)) \geqslant 2k + 2$$

holds. In particular, $F \cap P$ is a factorial complete intersection of codimension k in $\mathbb{P} \cong \mathbb{P}^{M-\varepsilon(k)+1}$. Moreover, by the Lefschetz theorem, applied to the section of the variety $F \cap P$ by a general linear subspace of dimension $3k+1$ in P (this section is a non-singular complete intersection of codimension k in $\mathbb{P}^{3k+1}$), we get that the section of $F \cap P$ by an arbitrary linear subspace of codimension $a \leqslant k$ is irreducible and reduced, since for the numerical Chow group, we have

$$A^a F \cap P = \mathbb{Z} H^a_{F \cap P},$$

where $H_{F\cap P}$ is the class of a hyperplane section.

Assume that Theorem 1.2 is not true and $\mathrm{mult}_o Y > 2n(Y)$. We will argue precisely as in [26, §2]; see also [20, Chapter 3, Section 2.1]. Let $T_1, \dots, T_k$ be the tangent hyperplane sections of $F \cap P$ at the point o (in the notations of Subsection 1.4 they are defined by the linear forms $f_{i,1}|_P$, $i = 1, \dots, k$). By what was said above, for each $i = 1, \dots, k$, the intersection $T_1 \cap \dots \cap T_i$ is of codimension i in $F \cap P$ and coincides with the scheme-theoretic intersection $(T_1 \circ \dots \circ T_i)$ and its multiplicity at the point o equals precisely 2^i, since the quadratic forms

$$f_{1,2}|_{T_o(F\cap P)}, \dots, f_{k,2}|_{T_o(F\cap P)}$$

satisfy the regularity condition. Now we argue as in [26, §2]. We set $Y_1 = Y$ and see that $Y_1 \neq T_1$ because $\mathrm{mult}_o T_1 = 2n(T_1) = 2$. We consider the cycle $(Y_1 \circ T_1)$ of the scheme-theoretic intersection and take for Y_2 the component of that cycle that has the maximal value of the ratio $\mathrm{mult}_o/\deg$. Assume that the subvariety Y_i of codimension i in $F \cap P$, satisfying the inequality

$$\mathrm{mult}_o Y_i > \frac{2^i}{\deg F} \deg Y_i$$

is already constructed, and $i \leqslant k-1$. Then

$$Y_i \neq T_1 \cap \dots \cap T_i.$$

However, by construction, Y_i is contained in the divisors $T_1, \dots, T_{i-1}$, so that $Y_i \not\subset T_i$, and the cycle of scheme-theoretic intersection $(Y_i \circ T_i)$ of codimension $i+1$ is well defined. For Y_{i+1}, we take the component of this cycle with the maximal value of the ratio $\mathrm{mult}_o/\deg$. Completing this process, we obtain an irreducible subvariety $Y_{k+1} \subset (F \cap P)$ of codimension $k+1$, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k+1} > \frac{2^{k+1}}{\deg F}.$$

7.2. Non-singular points. Hypertangent divisors

We continue the proof of Theorem 1.2. In the notations of Subsection 1.4 for each $j = 2, \dots, d_k - 1$, construct the hypertangent linear systems

$$\Lambda_j = \left| \sum_{i=1}^{k} \sum_{\alpha=1}^{\min\{j, d_i - 1\}} f_{i,[1,\alpha]} s_{i,j-\alpha} \right|_{F\cap P},$$

where $f_{i,[1,\alpha]} = f_{i,1} + \dots + f_{i,\alpha}$ is the left segment of the polynomial f_i of length α, the polynomials $s_{i,j-\alpha}$ are homogeneous polynomials of degree $j - \alpha$, running through the spaces $\mathcal{P}_{j-\alpha, M+k}$ independently of each other and the restriction onto $F \cap P$ means the restriction onto the affine part of that variety in $\mathbb{A}^{M+k}_{z_*}$ followed by the closure.

Let h_a, where $a \geqslant k+1$, be the a-th polynomial in the sequence $\mathcal{S}$. Then $h_a = f_{i,j}|_{\mathbb{P}(T_o F)}$ for some i and $j \geqslant 3$. Set $\mathcal{H}_a = \Lambda_{j-1}$. In this way, we obtain a sequence of linear systems $\mathcal{H}_{k+1}, \mathcal{H}_{k+2}, \dots, \mathcal{H}_M$, where the system Λ_j occurs, in the notations of [26, §2],

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$$w_j^+ = \sharp\{i, 1 \leqslant i \leqslant k \mid j \leqslant d_i - 1\}$$

times. By the symbol $\mathcal{H}[-m]$ we denote the space

$$\prod_{a=k+1}^{M-m} \mathcal{H}_a$$

of all tuples $(D_{k+1}, \dots, D_{M-m})$ of divisors, where $D_a \in \mathcal{H}_a$. For $a \in \{k+1, \dots, M\}$ set

$$\beta_a = \frac{j+1}{j},$$

if $\mathcal{H}_a = \Lambda_j$. The number β_a is called the *slope* of the divisor D_a. It is easy to see that

$$\prod_{a=k+1}^{M} \beta_a = \frac{d_1 \dots d_k}{2^k} = \frac{\deg F}{2^k}. \tag{20}$$

Set $m_* = k + \varepsilon(k) + 3$. Let

$$(D_{k+1}, \dots, D_{M-m_*}) \in \mathcal{H}[-m_*]$$

be a general tuple. The technique of hypertangent divisors, applied in precisely the same way as in [26, §2] or [20, Chapter 3, Subsection 2.2] – see also [2, Proposition 2.1] – gives the following claim.

Proposition 7.1. *There is a sequence of irreducible subvarieties*

$$Y_{k+1}, Y_{k+2}, \dots, Y_{M-m_*},$$

where Y_{k+1} *has been constructed above, such that* $\operatorname{codim}(Y_i \subset (F \cap P)) = i$, *the subvariety* Y_i *is not contained in the support of the divisor* D_{i+1} *for* $i \leqslant M - m_* - 1$, *the subvariety* Y_{i+1} *is an irreducible component of the effective cycle* $(Y_i \circ D_{i+1})$ *and the following inequality holds:*

$$\frac{\operatorname{mult}_o}{\deg} Y_{i+1} \geqslant \beta_{i+1} \frac{\operatorname{mult}_o}{\deg} Y_i.$$

There is no need to give a **proof** of that claim because it is identical to the arguments mentioned above. Note only that the key point in the construction of the sequence of subvarieties Y_i is the fact that Y_i is not contained in the support of a general divisor $D_{i+1} \in \mathcal{H}_{i+1}$, and this fact follows from the regularity condition (R1). Since $\dim(F \cap P) = M + 1 - k - \varepsilon(k)$, the subvariety $Y^* = Y_{M-m_*}$ is of dimension 4 and satisfies the inequality

$$\frac{\operatorname{mult}_o}{\deg} Y^* > \frac{\dfrac{2^{k+1}}{\deg F} \cdot \dfrac{\deg F}{2^k}}{\dfrac{3}{2} \cdot \displaystyle\prod_{a=M-m_*+1}^{M} \beta_a} = \frac{4}{3} \frac{1}{\displaystyle\prod_{a=M-m_*+1}^{M} \beta_a}.$$

(The number $\frac{3}{2}$ appears in the denominator because the hypertangent divisor D_{k+1} is skipped in the procedure of intersection, in the same way and for the same reason as in [26, §2], and its slope is $\frac{3}{2}$.) Now the inequality

$$\frac{4}{3} \geqslant \prod_{a=M-m_*+1}^{M} \beta_a, \tag{21}$$

shown below in Proposition 7.2, completes the proof of Theorem 1.2.

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Proposition 7.2. *Assume that for* $k = 3, 4, 5$*, the dimension* M *is, respectively, at least* 96, 160, 215*, and for* $k \geqslant 6$*, the inequality* $M \geqslant 8k^2 + 2k$ *holds. Then the inequality* (21) *is true.*

Proof. Using the obvious fact that the function $\frac{t+1}{t}$ is decreasing, it is easy to see that the right-hand side of the inequality (21) with k and M fixed attains the maximum when the degrees $d_1, \dots, d_k$ are equal or 'almost equal' in the following sense: let $M \equiv e \bmod k$ with $e \in \{0, 1, \dots, k-1\}$, then the 'almost equality' means that

$$d_1 = \dots = d_{k-e} = \frac{M-e}{k} + 1, \quad d_{k-e+1} = \dots = d_k = \frac{M-k}{k} + 2.$$

For $k \in \{3, \dots, 9\}$, the claim of the proposition can be checked for each case of almost equal degrees – that is, for each possible value of e – manually, computing $\varepsilon(k)$ explicitly. For $k \geqslant 10$, it is easy to see that $\varepsilon(k) \leqslant k-3$, so that $m_* \leqslant 2k$. Therefore (again considering the case of almost equal degrees), the right-hand side of (21) does not exceed the number

$$\left(\frac{\frac{M}{k}}{\frac{M}{k} - 2}\right)^k = \left(\frac{M}{M-2k}\right)^k,$$

from which we get that (21) is true if

$$M \geqslant 2k \frac{(1+\frac{1}{3})^{\frac{1}{k}}}{(1+\frac{1}{3})^{\frac{1}{k}} - 1}.$$

If in the numerator and denominator we replace $(1+\frac{1}{3})^{\frac{1}{k}}$ by the smaller number $1 + \frac{1}{4k}$, the right-hand side of the last inequality gets higher. This proves the proposition. Q.E.D. □

Note that for $M \geqslant \rho(k)$ (see the inequality (1) in Subsection 0.1), the assumptions of the previous proposition are satisfied. This completes the proof of Theorem 1.2.

7.3. *Quadratic points*

Let us show Theorem 1.3. Note first of all that if Y is a section of the variety W by a hyperplane that is tangent to W at the point o (that is, the equation of the hyperplane is a linear combination of the forms $f_{1,1}, \dots, f_{k,1}$, restricted onto the hyperplane $\mathbb{P}(W) \cong \mathbb{P}^{M+k-1}$), then

$$\mathrm{mult}_o Y = 4n(Y) = 4,$$

so that the claim of the theorem is optimal. Thus, we assume the converse: the inequality

$$\mathrm{mult}_o Y > 2n(Y)$$

holds. We argue as in the non-singular case (Subsection 7.1): let $T_1, \dots, T_{k-1}$ be the tangent hyperplane sections, given by $(k-1)$ independent forms taken from the set $\{f_{1,1}, \dots, f_{k,1}\}$. Since $\mathrm{codim}(\mathrm{Sing}\, F \subset F) \geqslant 2k+2$, all scheme-theoretic intersections $(T_1 \circ \dots \circ T_i)$, $1 \leqslant i \leqslant k-1$, are irreducible, reduced and coincide with the set-theoretic intersection $T_1 \cap \dots \cap T_i$, and moreover, by the condition (R2), the equality

$$\mathrm{mult}_o T_1 \cap \dots \cap T_i = 2^{i+1}$$

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holds. In particular, $\mathrm{mult}_o T_1 = 4n(T_1) = 4$, so that $Y \neq T_1$. Arguing as in Subsection 7.1, we construct a sequence of irreducible subvarieties $Y_1 = Y, Y_2, \dots, Y_k$, where $\mathrm{codim}(Y_i \subset W) = i$ and the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_k > \frac{2^{k+1}}{\deg F}$$

holds. Now the proof of Theorem 1.3 repeats the arguments of Subsection 7.2, where m_* is replaced by 4. Since $4 < m_*$, the inequality (21) guarantees the inequality which is obtained from (21) when m_* is replaced by 4. This completes the proof of Theorem 1.3.

7.4. Multi-quadratic points. Tangent divisors

We start the proof of Theorem 1.4, the structure of which is similar to the structure of the proof of Theorem 1.2. At first, we argue as in Subsection 7.1: it is sufficient to consider a linear subspace P in $T_o F$ of maximal admissible codimension $\varepsilon(k)$. Assume that the prime divisor Y on $F \cap P$ satisfies the inequality

$$\mathrm{mult}_o Y > \frac{3}{2} \cdot 2^k n(Y),$$

or the equivalent inequality

$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{3}{2} \cdot \frac{2^k}{\deg F},$$

and consider the second hypertangent linear system (which in this case plays the role of the tangent linear system)

$$\Lambda_2 = \left| \sum_{d_i \geqslant 3} s_{i,0} f_{i,2} \right|_{F \cap P},$$

where $s_{i,0} \in \mathbb{C}$ are constants, independent of each other. Instead of the Lefschetz theorem, we use the condition (R3.1): the system of equations $f_{i,2}|_{F\cap P} = 0$, where $d_i \geqslant 3$ defines an irreducible reduced subvariety of codimension $k + k_{\geqslant 3}$ in P, and by (R3.2), the multiplicity of that subvariety at the point o is precisely $2^k \cdot (\frac{3}{2})^{k_{\geqslant 3}}$. More precisely, for a general tuple $(D_{2,1}, \dots, D_{2,k_{\geqslant 3}})$ of divisors in the system Λ_2, the following claim is true: for each $i = 1, \dots, k_{\geqslant 3}$, the cycle $(D_{2,1} \circ \cdots \circ D_{2,i})$ of the scheme-theoretic intersection of the divisors $D_{2,1}, \dots, D_{2,i}$ is an irreducible reduced subvariety of codimension i in $F \cap P$, the multiplicity of which at the point o is $2^k \cdot (\frac{3}{2})^i$. Arguing as in Subsection 7.1, we construct a sequence $Y_1 = Y, Y_2, \dots, Y_{k_{\geqslant 3}}$ of irreducible subvarieties of codimension $\mathrm{codim}(Y_i \subset (F \cap P)) = i$, where Y_{i+1} is an irreducible component of the cycle $(Y_i \circ D_{2,i})$ with the maximal value of the ratio $\mathrm{mult}_o/\deg$. Therefore,

$$\frac{\mathrm{mult}_o}{\deg} Y_{k_{\geqslant 3}} > \left(\frac{3}{2}\right)^{k_{\geqslant 3}} \cdot \frac{2^k}{\deg F}.$$

It follows from here that $Y_{k_{\geqslant 3}} \neq D_{2,1} \cap \cdots \cap D_{2,k_{\geqslant 3}}$, but since by construction,

$$Y_{k_{\geqslant 3}} \subset D_{2,1} \cap \cdots \cap D_{2,k_{\geqslant 3}-1},$$

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we conclude that $Y_{k_{\geqslant 3}} \not\subset D_{2,k_{\geqslant 3}}$, so that the effective cycle $(Y_{k_{\geqslant 3}} \circ D_{2,k_{\geqslant 3}})$ of the scheme-theoretic intersection of these varieties is well defined and one of its components $Y_{k_{\geqslant 3}+1}$ satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k_{\geqslant 3}+1} > \left(\frac{3}{2}\right)^{k_{\geqslant 3}+1} \cdot \frac{2^k}{\deg F}.$$

7.5. Multi-quadratic points. Hypertangent divisors

Now we argue almost word for word as in Subsection 7.2: construct the hypertangent systems

$$\Lambda_j = \left| \sum_{\alpha=2}^{j} \sum_{d_i \geqslant \alpha+1} f_{i,[2,\alpha]} s_{i,j-\alpha} \right|_{F \cap P},$$

where $j = 3, \ldots, d_k - 1$ and all symbols have the same meaning as in Subsection 7.2. If h_a, where $a \geqslant k + k_{\geqslant 3} + 1$ is the a-th polynomial in the sequence $\mathcal{S}$, $h_a = f_{i,j}|_{\mathbb{P}(T_o F)}$, for some i and $j \geqslant 4$, then we set $\mathcal{H}_a = \Lambda_{j-1}$ and obtain the sequence of linear systems

$$\mathcal{H}_{k+k_{\geqslant 3}+1}, \quad \mathcal{H}_{k+k_{\geqslant 3}+2}, \quad \ldots, \quad \mathcal{H}_M.$$

By the symbol $\mathcal{H}[-m]$ we denote the space

$$\prod_{a=k+k_{\geqslant 3}+1}^{M-m} \mathcal{H}_a.$$

Instead of the equality (20), we get the equality

$$\prod_{a=k+k_{\geqslant 3}+1}^{M} \beta_a = \frac{\deg F}{2^k \left(\frac{3}{2}\right)^{k_{\geqslant 3}}}.$$

Let $(D_{k+k_{\geqslant 3}+1}, \ldots, D_{M-m^*}) \in \mathcal{H}[-m^*]$ be a general tuple. Now the technique of hypertangent divisors, applied in the word for word the same way as in Subsection 7.2, gives the following claim.

Proposition 7.3. *There is a sequence of irreducible subvarieties*

$$Y_{k_{\geqslant 3}+1}, Y_{k_{\geqslant 3}+2}, \ldots, Y_{M-k-m^*},$$

where $Y_{k_{\geqslant 3}+1}$ is constructed above, such that $\mathrm{codim}(Y_i \subset (F \cap P)) = i$, *the subvariety Y_i is not contained in the support of the divisor D_{k+i+1} for $i \leqslant M - m^* - 1$, the subvariety Y_{i+1} is an irreducible component of the effective cycle $(Y_i \circ D_{k+i+1})$ and the following inequality holds:*

$$\frac{\mathrm{mult}_o}{\deg} Y_{i+1} \geqslant \beta_{k+i+1} \frac{\mathrm{mult}_o}{\deg} Y_i.$$

Now since $\dim F \cap P = M - (k - l) - \varepsilon(k)$, by the definition of the number m^*, the last subvariety $Y^* = Y_{M-k-m^*}$ in that sequence is of dimension $\geqslant 4$ and satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y^* > \frac{\left(\frac{3}{2}\right)^{k_{\geqslant 3}+1} \cdot \frac{2^k}{\deg F} \cdot \frac{\deg F}{2^k \left(\frac{3}{2}\right)^{k_{\geqslant 3}}}}{\frac{4}{3} \prod\limits_{a=M-m_*+1}^{M} \beta_{k+a}} = \frac{9}{8} \frac{1}{\prod\limits_{a=M-m_*+1}^{M} \beta_{k+a}}.$$

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(The number $\frac{4}{3}$ appears in the denominator of the right-hand side because the hypertangent divisor $D_{k+k_{\geqslant 3}+1}$ is skipped in the process of constructing the sequence Y_*; see the similar remark above, before the inequality (21).) If $m^* = 0$, then the product in the denominator is assumed to be equal to 1. Now the inequality

$$\frac{9}{8} \geqslant \prod_{a=M-m^*+1}^{M} \beta_{k+a}, \tag{22}$$

shown below in Proposition 7.4, completes the proof of Theorem 1.4.

Proposition 7.4. *Assume that for $k \in \{3, \dots, 7\}$, the number M is at least the number shown in the corresponding column of the table*

k	3	4	5	6	7
$M \geqslant$	128	204	255	357	477

and for $k \geqslant 8$, the inequality $M \geqslant 9k^2 + k$ holds. Then the inequality (22) holds.

Proof. As in the non-singular case (the proof of Proposition 7.2), we see that the right-hand side of the inequality (22) for k and M fixed attains the maximum when the degrees d_i are equal or 'almost equal'. For $k \in \{3, \dots, 7\}$, the claim of the proposition is checked manually. For $k \geqslant 8$, we have $\varepsilon(k) \leqslant k - 2$, so that $m^* \leqslant k$. Therefore, (considering the case of equal or almost equal degrees) the right-hand side of the inequality (22) does not exceed the number

$$\left(\frac{M}{M-k}\right)^k,$$

which, in its turn, does not exceed $\frac{9}{8}$ for $M \geqslant 9k^2+k$, which is easy to check by elementary computations, similar to the proof Proposition 7.2. Q.E.D. □

8. The codimension of the complement

In this section, we show the estimate for the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$, given in Theorem 0.1.

8.1. Preliminary constructions

Set

$$\gamma = M - k + 5 + \binom{M - \rho(k) + 2}{2};$$

see Subsection 0.1. We consider γ as a function of M with $k \geqslant 3$ fixed, where $M \geqslant \rho(k)$. Let $o \in \mathbb{P}^{M+k}$ be an arbitrary point. The symbol $\mathcal{P}(o)$ stands for the linear subspace of the space $\mathcal{P}$, consisting of all tuples $\underline{f}$, vanishing at the point o: $\underline{f}(o) = (0, \dots, 0)$. Obviously, $\operatorname{codim}(\mathcal{P}(o) \subset \mathcal{P}) = k$. Fixing the point o, we use the notations of Subsections 1.2–1.4, considering the polynomials f_i as non-homogeneous polynomials in the affine coordinates z_*. By the symbols

$$\mathcal{B}_{MQ1}, \mathcal{B}_{MQ2}, \mathcal{B}_{R1}, \mathcal{B}_{R2}, \mathcal{B}_{R3.1}, \mathcal{B}_{R3.2}$$

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we denote the subsets of the subspace $\mathcal{P}(o)$, consisting of the tuples $\underline{f}$ that do not satisfy the conditions

$$(MQ1),\ (MQ2),\ (R1),\ (R2),\ (R3.1),\ (R3.2)$$

at the point o, respectively. Since the point o varies in $\mathbb{P}^{M+k}$, it is sufficient to show that the codimension of each of the six sets $\mathcal{B}_*$ in $\mathcal{P}(o)$ is at least $\gamma + M$.

Furthermore, for an arbitrary tuple

$$\underline{\xi} = (\xi_1, \dots, \xi_k)$$

of linear forms in z_*, the symbol $\mathcal{P}(o, \underline{\xi})$ denotes the affine subspace, consisting of the tuples $\underline{f}$, such that

$$f_{1,1} = \xi_1, \quad \dots, \quad f_{k,1} = \xi_k.$$

By the symbol $\dim \underline{\xi}$ denote the dimension

$$\dim\langle \xi_1, \dots, \xi_k\rangle,$$

so that $\mathcal{P}(o)$ is fibred into disjoint subsets

$$\mathcal{P}^{(i)}(o) = \bigcup_{\dim \underline{\xi} = i} \mathcal{P}(o, \underline{\xi}),$$

where $i = 0, 1, \dots, k$. Obviously, the equality

$$\mathrm{codim}(\mathcal{P}^{(i)}(o) \subset \mathcal{P}(o)) = (k-i)(M+k-i)$$

holds. In particular, $\mathcal{P}^{(k)}(o)$ consists of the tuples $\underline{f}$, such that the scheme of their common zeros is a non-singular subvariety of codimension k in a neighborhood of the point o. Set

$$\mathcal{B}_{R1}(\underline{\xi}) = \mathcal{B}_{R1} \cap \mathcal{P}(o, \underline{\xi}).$$

For the case of a non-singular point, it is sufficient to prove the inequality

$$\mathrm{codim}(\mathcal{B}_{R1}(\underline{\xi}) \subset \mathcal{P}(o, \underline{\xi})) \geqslant \gamma + M,$$

where $\dim \underline{\xi} = k$.

Furthermore, let $\mathcal{B}_{MQ1}(\underline{\xi}) = \mathcal{B}_{MQ1} \cap \mathcal{P}(o, \underline{\xi})$, where $\dim \underline{\xi} = i \leqslant k-1$, be the set of the tuples $\underline{f}$, such that the condition (MQ1) for $l = k - i$ is not satisfied, and $\mathcal{B}_{MQ2}(\underline{\xi}) = \mathcal{B}_{MQ2} \cap \mathcal{P}(o, \underline{\xi})$, where $\dim \underline{\xi} = i \leqslant k-2$, be the set of the tuples $\underline{f}$, such that the condition (MQ2) for $l = k - i$ is not satisfied.

In a similar way, we define the sets $\mathcal{B}_{R2}(\underline{\xi})$ for $\dim \underline{\xi} = k-1$ and $\mathcal{B}_{R3.1}(\underline{\xi}), \mathcal{B}_{R3.2}(\underline{\xi})$ for $\dim \underline{\xi} \leqslant k-2$.

Clearly, it is sufficient to prove that for $\dim \underline{\xi} = i$, the codimension of the set $\mathcal{B}_*(\underline{\xi})$ in $\mathcal{P}(o, \underline{\xi})$ is at least

$$\gamma + M - (k-i)(M+k-i).$$

In the conditions (R1), (R2) and (R3.2), we have also an arbitrary subspace $\Pi \subset \mathbb{P}(T_oF)$ of the corresponding codimension, and in the condition (R3.1) an arbitrary subspace P in the embedded tangent space $T_oF \subset \mathbb{P}^{M+k}$ of codimension $\varepsilon(k)$, containing the point o. For an arbitrary subspace $\Pi \subset \mathbb{P}(T_oF)$ of the corresponding codimension, let

$$\mathcal{B}_{R1}(\underline{\xi}, \Pi), \quad \mathcal{B}_{R2}(\underline{\xi}, \Pi), \quad \mathcal{B}_{R3.2}(\underline{\xi}, \Pi)$$

be the set of tuples $\underline{f} \in \mathcal{P}(o, \underline{\xi})$, such that the respective condition (R1), (R2) and (R3.2) is violated precisely for that subspace Π. In a similar way, we define the subset $\mathcal{B}_{R3.1}(\underline{\xi}, P)$. These definitions are

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meaningful because the tangent space T_oF is given by the fixed linear forms ξ_i and for that reason is fixed.

Since the subspace Π varies in a $(\dim \Pi + 1)\operatorname{codim}(\Pi \subset \mathbb{P}(T_oF))$-dimensional Grassmanian, the estimate for the codimension of the set $\mathcal{B}_*(\underline{\xi}, \Pi)$ in $\mathcal{P}(o, \underline{\xi})$ should be stronger than the estimate for the codimension of the set $\mathcal{B}_*(\underline{\xi})$ by that number. Similarly, $\overline{P}$ varies in a

$$\varepsilon(k)(\dim T_oF - \varepsilon(k))$$

-dimensional family, so that the estimate for the codimension of the set $\mathcal{B}_{R3.1}(\underline{\xi}, P)$ should be stronger than the estimate for $\mathcal{B}_{R3.1}(\underline{\xi})$ by that number.

Now everything is ready to consider each of the six subsets $\mathcal{B}_*$.

8.2. The conditions (MQ1) and (MQ2)

For a non-singular point $o \in F$, these conditions contain no restrictions, so we assume that $\dim \underline{\xi} \leqslant k-1$. It is well known that the closed subset of quadratic forms of rank $\leqslant r \leqslant N-1$ in the space $\mathcal{P}_{2,N}$ has the codimension

$$\binom{N-r+1}{2}.$$

From here, it is easy to see that the closed subset of tuples $(q_1, \dots, q_e) = q_{[1,e]}$ of quadratic forms in N variables, defined by the condition

$$\operatorname{rk} q_{[1,e]} \leqslant r,$$

is of codimension

$$\geqslant \binom{N-r+1}{2} - (e-1)$$

in the space $\mathcal{P}_{2,N}^{\times e}$. As we noted in Subsection 1.2 (after stating the condition (MQ2)), for $l \geqslant 2$, the condition (MQ2) is stronger than (MQ1); therefore, it is sufficient to estimate the codimension of the set $\mathcal{B}_{MQ2}$ (in the case of quadratic points, when $l = 1$, it is easy to check that the codimension of the set $\mathcal{B}_{MQ1}$ is higher than required). So we assume that $\dim \underline{\xi} = k - l \leqslant k-2$. The condition (MQ2) requires the rank of the tuple of quadratic forms $q_{[1,k]}$, where $q_i = f_{i,2}|_{T_oF}$, to be at least $\rho(k) + 2$; see (1) in Subsection 0.1. Taking into account the variation of the tuple $\underline{\xi}$, from what was said above, it is easy to obtain that the codimension of the set $\mathcal{B}_{MQ2} \cap \mathcal{P}^{k-l}(o) \geqslant$

$$-k + 1 + l(M+l) + \binom{M + l - \rho(k)}{2}.$$

The minimum of this expression is attained for $l = 2$, and it is easy to check that this minimum is precisely $\gamma + M$. Therefore, the codimension of the set $\mathcal{B}_{MQ2}$ is at least γ, and the codimension of the set $\mathcal{B}_{MQ1}$ for $l \geqslant 2$ is higher. For $l = 1$, the last codimension is also higher. This completes our consideration of the conditions (MQ1) and (MQ2).

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8.3. Regularity at the non-singular and quadratic points

Let us estimate the codimension of the set $\mathcal{B}_{R1}(\underline{\xi},\Pi)$ in the space $\mathcal{P}(o,\underline{\xi})$. Here, $\dim\underline{\xi}=k$ and $\Pi\subset\mathbb{P}(T_oF)$ is a subspace of codimension $k+\varepsilon(k)-1=m_*-4$. Let

$$\mathcal{G}(\underline{d})=\prod_{i=1}^{M-m_*}\mathcal{P}_{\deg h_i,\dim\Pi+1}$$

be the space, parameterizing all sequences $\mathcal{S}[-m_*]|_\Pi$. Since the polynomials h_i are distinct homogeneous components of the polynomials of the tuple $\underline{f}$, restricted onto the subspace Π, the codimension of the subset $\mathcal{B}_{R1}(\underline{\xi},\Pi)$ in $\mathcal{P}(o,\underline{\xi})$ is equal to the codimension of the subset $\mathcal{B}\subset\mathcal{G}(\underline{d})$, which consists of the sequences that do not satisfy the condition (R1).

Using the approach that was applied in [27, 20, 28] and many other papers, let us present $\mathcal{B}$ as a disjoint union

$$\mathcal{B}=\bigsqcup_{i=1}^{M-m_*}\mathcal{B}_i,$$

where $\mathcal{B}_i$ consists of sequences

$$(h_1|_\Pi,\dots,h_{M-m_*}|_\Pi),$$

such that the first $i-1$ polynomials form a regular sequence but h_i vanishes on one of the components of the set of their common zeros. The 'projection method' estimates the codimension of $\mathcal{B}_i$ in $\mathcal{G}(\underline{d})$ from below by the integer

$$\binom{\dim\Pi-i+1+\deg h_i}{\deg h_i}=\binom{\dim\Pi-i+1+\deg h_i}{\dim\Pi-i+1} \tag{23}$$

(we will use both presentations). It follows easily from here (see [28, §3]) that the worst estimate corresponds to the case of equal or 'almost equal' degrees d_i, described above. We will consider this case.

Thus, we need to estimate from below the minimum of $M-m_*$ integers (23). Here are the first $(k+1)$ of them:

$$\binom{\dim\Pi+2}{2},\ \binom{\dim\Pi+1}{2},\ \dots,\ \binom{\dim\Pi+3-k}{2},\ \binom{\dim\Pi+3-k}{3}.$$

We call the left-hand side of the equality (23) the presentation of type (I); the right-hand side is the presentation of type (II). Let us write down each of the numbers (23) in the form

$$\binom{A(i)}{B(i)},$$

where $A(i)\geqslant 2B(i)$, using the presentation of type (I) or of type (II).

At first (for the starting segment of the sequence), we use the presentation of type (I). It is easy to see that when we change i by $i+1$, we have one of the two options:

- either $\deg h_{i+1}=\deg h_i$, and then $A(i+1)=A(i)-1$ and $B(i+1)=B(i)$, so that

$$\binom{A(i+1)}{B(i+1)}<\binom{A(i)}{B(i)},$$

  and moreover, $C(i)=A(i)-2B(i)$ decreases: $C(i+1)=C(i)-1$,

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◦ or $\deg h_{i+1} = \deg h_i + 1$, and then $A(i+1) = A(i)$ and $B(i+1) = B(i)+1$, so that $C(i+1) = C(i) - 2$, and if $C(i+1) \geqslant 0$, then

$$\binom{A(i+1)}{B(i+1)} > \binom{A(i)}{B(i)}.$$

This is how it goes on until the 'equilibrium': $C(i_*) \geqslant 0$, but $C(i_* + 1) < 0$, and after that, we use the presentation of type (II).

Now when we change i by $(i+1)$, we have one of the two options:

◦ either $\deg h_{i+1} = \deg h_i$, and then $A(i+1) = A(i) - 1$ and $B(i+1) = B(i) - 1$, so that $C(i+1) = C(i)+1$ and

$$\binom{A(i+1)}{B(i+1)} < \binom{A(i)}{B(i)},$$

◦ or $\deg h_{i+1} = \deg h_i + 1$, and then $A(i+1) = A(i)$ and $B(i+1) = B(i) - 1$, so that $C(i+1) = C(i)+2$ and

$$\binom{A(i+1)}{B(i+1)} < \binom{A(i)}{B(i)}.$$

Therefore, after the 'equilibrium', our sequence is strictly decreasing. Moreover, if

$$\binom{A(i_1)}{B(i_1)} \quad \text{and} \quad \binom{A(i_2)}{B(i_2)}$$

are two numbers in our sequence, where $i_1 \leqslant i_*$ and $i_2 > i_*$ and $B(i_1) \geqslant B(i_2)$, then, obviously,

$$\binom{A(i_1)}{B(i_1)} > \binom{A(i_2)}{B(i_2)}.$$

Recall that the degrees d_i are equal or 'almost equal'.

Lemma 8.1. *For $M \geqslant 3k^2$, the following inequality holds: $i_* < M - m_*$.*

Proof. Elementary computations, using the equality $C(i+k) = C(i) - (k+1)$ if $C(i+k) \geqslant 0$. Q.E.D. for the lemma. □

Therefore, the 'equilibrium' is reached earlier than the sequence $h_i, \ldots, h_{M-m_*}$ comes to an end, so that there is a nonempty segment after the 'equilibrium'. By construction, $B(M - m_*) = 4$. By what was said above, the minimum of the numbers $\binom{A(i)}{B(i)}$ for $i = 1, \ldots, M - m_*$ is the minimum of the following three numbers:

$$\binom{\dim \Pi + 3 - k}{2}, \quad \binom{\dim \Pi + 4 - 2k}{3}, \quad \binom{\deg h_{M-m_*} + 4}{4}.$$

Lemma 8.2. *For $\dim \Pi \geqslant 3k + 1$, the following inequality holds:*

$$\binom{\dim \Pi + 4 - 2k}{3} > \binom{\dim \Pi + 3 - k}{2}.$$

Proof. Elementary computations. Q.E.D. □

Lemma 8.3. *For $M \geqslant 2\sqrt{3}k^2$, the following inequality holds:*

$$\binom{\deg h_{M-m_*} + 4}{4} > \binom{\dim \Pi + 3 - k}{2}.$$

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Proof. It is easy to check the inequalities

$$\frac{(M-2k)^2}{2} \geqslant \binom{\dim\Pi+3-k}{2}$$

and

$$\binom{\deg h_{M-m_*}+4}{4} \geqslant \frac{1}{24}\left(\frac{M}{k}+1\right)\left(\frac{M}{k}\right)\left(\frac{M}{k}-1\right)\left(\frac{M}{k}-2\right),$$

so that it is sufficient to show that for $M \geqslant 2\sqrt{3}k^2$, the inequality

$$\left(\frac{M^2}{k^2}-1\right)\left(\frac{M^2}{k^2}-2\frac{M}{k}\right) > 12(M-2k)^2$$

holds or, equivalently, $M(M^2-k^2) > 12k^4(M-2k)$. It is easy to check the last inequality, considering the cubic polynomial

$$t^3-(12k^4+k^2)t+24k^5$$

in the real variable t. Q.E.D. for the lemma. □

The work that was carried out above gives the inequality

$$\operatorname{codim}(\mathcal{B}\subset\mathcal{G}(\underline{d})) \geqslant \binom{M+3-2k-\varepsilon(k)}{2}.$$

From here, by elementary computations (taking into account the variation of the subspace Π, see Subsection 8.1), it is easy to obtain the required inequality $\operatorname{codim}(\mathcal{B}_{R1}\subset\mathcal{P}(o)) \geqslant \gamma+M$. This completes the proof in the case of smooth points.

It is easy to see that the methods used above give a stronger estimate for the codimension of the set $\mathcal{B}_{R2}$ because the dimension of the subspace Π is higher. The computations are completely similar to the computations given above for the case of a non-singular point. For that reason, we do not consider the case of a quadratic point, and we move on to estimating the codimension of the sets $\mathcal{B}_{R3.1}$ and $\mathcal{B}_{R3.2}$.

8.4. Regularity at the multi-quadratic points

Let us estimate the codimension of the set $\mathcal{B}_{R3.2}(\underline{\xi},\Pi)$, where $\Pi\subset\mathbb{P}(T_oF)$ is an arbitrary subspace of codimension $\varepsilon(k)$. Our arguments are completely similar to the arguments of Subsection 8.3 for a non-singular point and give a stronger estimate for the codimension. We just point out the necessary changes in the constructions of Subsection 8.3. Set

$$\mathcal{G}(\underline{d}) = \prod_{i=1}^{M-m^*}\mathcal{P}_{\deg h_i,\dim\Pi+1}.$$

Denote by the symbol $\mathcal{B}$ the subset in $\mathcal{G}(\underline{d})$, consisting of the sequences that do not satisfy the condition (R3.2). Again, we break $\mathcal{B}$ into subsets:

$$\mathcal{B} = \bigsqcup_{i=1}^{M-m^*}\mathcal{B}_i,$$

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where $\mathcal{B}_i$ has the same meaning as in Subsection 8.3 (but for the multi-quadratic point o). Again, the codimension of $\mathcal{B}_i$ in $\mathcal{G}(\underline{d})$ is bounded from below by the number (23), and for k and M fixed, the worst estimate corresponds to the case of equal or 'almost equal' degrees d_i.

Arguing precisely in the same way as in the non-singular case (Subsection 8.3), we see, since the dimension of the subspace Π is higher than in the non-singular case, that the claim of Lemma 8.1 is true. Note that if $m^* = 0$, then in the notations of Subsection 8.3, we have $B(M - m^*) \geqslant 4$. Thus, replacing $B(M - m^*)$ by 4, we get that $\operatorname{codim}(\mathcal{B} \subset \mathcal{G}(\underline{d}))$ is bounded from below by the least of the three numbers

$$\binom{\dim \Pi + 3 - k}{2}, \quad \binom{\dim \Pi + 4 - 2k}{3}, \quad \binom{\deg h_{M-m^*} + 4}{4}.$$

The claim of Lemma 8.2 is true since, as we noted above, $\dim \Pi$ in the multi-quadratic case is higher than in the non-singular case. Obviously, $m^* < m_*$, so that $\deg h_{M-m^*} \geqslant \deg h_{M-m_*}$, and the claim of Lemma 8.3 is also true. As a result, we get the inequality

$$\operatorname{codim}(\mathcal{B} \subset \mathcal{G}(\underline{d})) \geqslant \binom{M + 2 + l - k - \varepsilon(k)}{2},$$

where $\dim \underline{\xi} = k - l$. The minimum of the right-hand side is attained for $l = 2$, and it is easy to see that this minimum is significantly higher than in the non-singular case. It is easy to check, taking into account the variation of the subspace Π, that

$$\operatorname{codim}(B_{R3.2} \subset \mathcal{P}(o)) > \gamma + M.$$

This completes our consideration of the condition (R3.2) in the multi-quadratic case.

8.5. The condition (R3.1)

In order to estimate the codimension of the set $\mathcal{B}_{R3.1}(\underline{\xi}, P)$, we need the following known general fact. Take $e \geqslant 1$ and let $\underline{w} = (w_1, \dots, w_e) \in \mathbb{Z}^e$ be a tuple of integers, where $2 \leqslant w_1 \leqslant \dots \leqslant w_e$.

Set

$$\mathcal{P}(\underline{w}) = \prod_{i=1}^{e} \mathcal{P}_{w_i, N+1}$$

to be the space of tuples $\underline{g} = (g_1, \dots, g_e)$ of homogeneous polynomials in $N + 1$ variables, $\deg g_i = w_i$, which we consider as homogeneous polynomials on $\mathbb{P}^N$. Let

$$\mathcal{B}^*(\underline{w}) \subset \mathcal{P}(\underline{w})$$

be the set of tuples $\underline{g}$, such that the scheme of their common zeros is not an irreducible reduced subvariety of codimension e in $\mathbb{P}^N$.

Theorem 8.1. *The following inequality holds:*

$$\operatorname{codim}(\mathcal{B}^*(\underline{w}) \subset \mathcal{P}(\underline{w})) \geqslant \frac{1}{2}(N - e - 1)(N - e - 4) + 2.$$

Proof. This is Theorem 2.1 in [2]. □

Let us estimate the codimension of the set $\mathcal{B}_{R3.1}(\underline{\xi}, \Pi)$. In order to do this, consider in the projective space P a hypersurface $P^\sharp$ that does not contain the point o – for instance, the intersection of the hyperplane 'at infinity' with respect to the system of affine coordinates $(z_1, \dots, z_{M+k})$ with the subspace P. If the scheme of common zeros of the tuple of polynomials, consisting of

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$$f_1|_P, \ldots, f_k|_P$$

and the polynomials $f_{i,2}|_P$ for i such that $d_i \geqslant 3$ is not an irreducible reduced subvariety of codimension $k + k_{\geqslant 3}$ in P (that is, the condition (R3.1) is violated; see Subsection 1.4), then the scheme of common zeros of the set of polynomials

$$f_1|_{P^\sharp}, \ldots, f_k|_{P^\sharp}, f_{i,2}|_{P^\sharp} \quad \text{for} \quad d_i \geqslant 3, \tag{24}$$

respectively, is reducible, non-reduced or of codimension $< k + k_{\geqslant 3}$ in $P^\sharp$. However, for each i, such that $d_i \geqslant 3$, the homogeneous polynomials

$$f_i|_{P^\sharp} = f_{i,d_i}|_{P^\sharp} \quad \text{and} \quad f_{i,2}|_{P^\sharp}$$

on the projective space $P^\sharp$ are linear combinations of disjoint sets of monomials in f_i, so that the coefficients of those polynomials belong to disjoint subsets of coefficients of the polynomial f_i. Therefore (re-ordering the polynomials of the tuple (24) so that their degrees do not decrease), applying Theorem 8.1 to the tuple (24), we get that the codimension of the set $\mathcal{B}_{R3.1}(\underline{\xi}, P)$ is at least

$$\frac{1}{2}(\dim P^\sharp - k - k_{\geqslant 3} - 1)(\dim P^\sharp - k - k_{\geqslant 3} - 4) + 2,$$

where $\dim P^\sharp = M + l - \varepsilon(k) - 1$, $\dim \underline{\xi} = k - l$. It is easy to check by elementary computations that this estimate (with the correction due to the variation of the subspace P and the set of linear forms $\underline{\xi}$) is stronger than we need.

This completes the proof of the estimate for the codimension of the complement $\mathcal{P} \setminus \mathcal{F}$ in Theorem 0.1.

Note that (for the technique of estimating the codimension that we used) the estimate of Theorem 0.1 is optimal for the condition (MQ2); that requirement turns out to be the strongest.

Competing interest. The authors have no competing interest to declare.

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