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On complex oscillation, function-theoretic quantization of non-homogeneous periodic ordinary differential equations and special functions*

Yik-Man Chiang
Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
(machiang@ust.hk)

Kit-Wing Yu
Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
(kitwing@hotmail.com)

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New necessary and sufficient conditions are given for the quantization of a class of periodic second-order non-homogeneous ordinary differential equations in the complex plane. The problem is studied from the viewpoint of complex oscillation theory first developed in works by Bank and Laine and Gundersen and Steinbart. We show that, when a solution is complex non-oscillatory (finite exponent of convergence of zeros), then the solution, which can be written as special functions, must degenerate. This gives a necessary and sufficient condition when the Lommel function has finitely many zeros in every branch, and this is a type of quantization for the non-homogeneous differential equation. The degenerate solutions are polynomial/rational-type functions, which are of independent interest. In particular, this shows that complex non-oscillatory solutions of this class of differential equations are equivalent to the subnormal solutions considered in a recent paper by Chiang and Yu. In addition to the asymptotics of special functions, the other main idea that we apply in our proof is a classical result by Wright that gives precise asymptotic locations of large zeros of a functional equation.

1. Introduction and the main results

Let $A(z)$ be a transcendental entire function, and let $f(z)$ be an entire function solution of the differential equation

$$f'' + A(z)f = 0. \tag{1.1}$$

We use the $\sigma(f)$ and $\lambda(f)$ to denote the *order* and *exponent of convergence of zeros*, respectively, of an entire function $f(z)$. A solution is called *complex oscillatory* if $\lambda(f) = +\infty$ and is called *complex non-oscillatory* if $\lambda(f) < +\infty$. Interested

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readers are referred to [14, 16] for the notation and background relating to Nevanlinna theory, where this research originates. However, we shall not make use of Nevanlinna theory in the remainder of this paper. For earlier treatments of the various complex oscillation problems considered, we refer the intrusted reader to, for example, [1–4, 8, 9, 13, 15, 23]. We consider the complex oscillation problem of a class of non-homogeneous differential equations which includes the simple-looking equation

$$f'' + (\mathrm{e}^{2z} - \nu^2)f = \sigma \mathrm{e}^{(\mu+1)z} \tag{1.2}$$

as a special case, where $\mu, \nu, \sigma \in \boldsymbol{C}$ and $\sigma \neq 0$. It is well known that all solutions of the equation are entire [16].

In [10, theorem 1.2], several explicit solutions of (1.2) in terms of the sum of the Bessel functions of first and second kinds $J_\nu(\zeta)$ and $Y_\nu(\zeta)$, and the *Lommel function* $S_{\mu,\nu}(\zeta)$ (see the appendix), are given. In fact, the general solution of (1.2) can be written as

$$f(z) = AJ_\nu(\mathrm{e}^z) + BY_\nu(\mathrm{e}^z) + \sigma S_{\mu,\nu}(\mathrm{e}^z), \tag{1.3}$$

where $A, B \in \boldsymbol{C}$ and $S_{\mu,\nu}(\zeta)$ is a *particular integral* of the non-homogeneous Bessel differential equation

$$\zeta^2 y''(\zeta) + \zeta y'(\zeta) + (\zeta^2 - \nu^2)y(\zeta) = \zeta^{\mu+1}. \tag{1.4}$$

The functions $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ are two *linearly independent* solutions of the corresponding homogeneous Bessel differential equation of (1.4).

The Lommel function $S_{\mu,\nu}(\zeta)$ is a special function that plays important roles in numerous physical applications (see, for example, [19, 20, 26]) and was first studied by Lommel [18] (we refer the interested reader to [10] and the references therein for further discussion about the background and the applications of the Lommel function).

The authors' previous work [10, theorem 1.2] concerns the *subnormality* of the solutions of (1.2). We recall that an entire function $f(z)$ is called *subnormal* if either

$$\limsup_{r\to+\infty} \frac{\log\log M(r,f)}{r} = 0 \quad \text{or} \quad \limsup_{r\to+\infty} \frac{\log T(r,f)}{r} = 0 \tag{1.5}$$

holds, where $M(r,f) = \max_{|z|\leqslant r} |f(z)|$ denotes the usual maximum modulus of the entire function $f(z)$ and $T(r,f)$ is the Nevanlinna characteristics of $f(z)$. We have shown that solutions of (1.2) are subnormal, that is, if (1.5) holds, if and only if $(A, B) = (0, 0)$ and either $\mu + \nu = 2p + 1$ or $\mu - \nu = 2p + 1$ holds for a non-negative integer p in (1.3) and the subnormal solutions have the form given by formulae (1.6) and (1.7). In other words, *subnormal solutions and finite order solutions of* (1.2) *are equivalent* (see corollary 1.5). This provides a new *non-homogeneous function-theoretic quantization-type* result for equation (1.2), an explanation of which will be given in § 8.

Remark 1.1. The authors have also generalized the above results to the much more general equation (1.8) [10, theorem 1.4], and a number of interesting corollaries. For example, orders of growth of the entire functions $S_{\mu,\nu}(\mathrm{e}^z)$ and $\boldsymbol{H}_\nu(\mathrm{e}^z)$ were determined and *non-homogeneous function-theoretic quantization-type results* were also obtained (see [10, theorem 1.7, § 6] for details).

For the homogeneous differential equation (1.1), it is known that a solution $f(z)$ could have $\lambda(f) < +\infty$ but its growth is *not subnormal*, that is,

$$\limsup_{r\to+\infty} \log\log M(r,f)/r = +\infty.$$

Thus, one may ask what the relationship is between subnormality of solutions and its exponent of convergence of zeros of (1.2). We show that the finiteness of the two measures are equivalent.

Our main results are as follows.

THEOREM 1.2. *Let $f(z)$ be a solution of (1.2). Then $\lambda(f) < +\infty$ if and only if $A = B = 0$ in (1.3) and either $\mu + \nu = 2p + 1$ or $\mu - \nu = 2p + 1$ for a non-negative integer p and*

$$S_{\mu,\nu}(\zeta) = \zeta^{\mu-1}\left[\sum_{k=0}^{p} \frac{(-1)^k c_k}{\zeta^{2k}}\right], \tag{1.6}$$

where the coefficients c_k, $k = 0, 1, \dots, p$, are defined by

$$c_0 = 1 \quad \textit{and} \quad c_k = \prod_{m=1}^{k} [(\mu - 2m + 1)^2 - \nu^2]. \tag{1.7}$$

The above result is a special case of the following theorem 1.3. We first introduce a set of more general coefficients. Suppose that n is a positive integer and that A, B, L, M, N, σ, σ_i, μ_j and ν are complex numbers such that L and M are non-zero and at least one of σ_j, $j \in \{1, 2, \dots, n\}$, are non-zero.

THEOREM 1.3. *Let $f(z)$ be an entire solution to the differential equation*

$$f'' + 2Nf' + [L^2M^2\mathrm{e}^{2Mz} + (N^2 - \nu^2M^2)]f = \sum_{j=1}^{n} \sigma_j L^{\mu_j+1} M^2 \mathrm{e}^{[M(\mu_j+1)-N]z}. \tag{1.8}$$

Then $f(z)$ is given by

$$f(z) = \mathrm{e}^{-Nz}\left[AJ_\nu(L\mathrm{e}^{Mz}) + BY_\nu(L\mathrm{e}^{Mz}) + \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(L\mathrm{e}^{Mz})\right]. \tag{1.9}$$

Moreover, suppose that all the $\mathrm{Re}(\mu_j)$ are distinct. Then we have $\lambda(f) < +\infty$ if and only if $A = B = 0$ and, for each non-zero σ_j, we have either

$$\mu_j + \nu = 2p_j + 1 \quad \textit{or} \quad \mu_j - \nu = 2p_j + 1, \tag{1.10}$$

where p_j is a non-negative integer and

$$S_{\mu_j,\nu}(\zeta) = \zeta^{\mu_j-1}\left[\sum_{k=0}^{p_j} \frac{(-1)^k c_{k,j}}{\zeta^{2k}}\right], \tag{1.11}$$

where $c_{k,j}$, $k = 0, 1, \dots, p-1$, $j = 1, 2, \dots, n$, are defined by

$$c_{0,j} = 1 \quad \textit{and} \quad c_{k,j} = \prod_{m=1}^{k} [(\mu_j - 2m + 1)^2 - \nu^2]. \tag{1.12}$$

As an immediate consequence of theorems 1.2 or 1.3, we obtain the following result, which gives us information about the number of zeros of the Lommel function $S_{\mu,\nu}(\zeta)$ in the sense of Nevanlinna's value distribution theory.

Corollary 1.4. *Suppose that σ_j are complex constants such that at least one of σ_j is non-zero, where $j = 1, 2, \dots, n$. Suppose, furthermore, that $\mathrm{Re}(\mu_j)$ are distinct and that $S_{\mu_j,\nu}(\zeta)$ are Lommel functions of arbitrary branches given in lemma 3.1. Then each branch of the function*

$$F(\zeta) = \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(\zeta) \tag{1.13}$$

has finitely many zeros if and only if either $\mu_j + \nu = 2p_j + 1$ or $\mu_j - \nu = 2p_j + 1$ for non-negative integers p_j. In particular, the special case $n = 1$ implies that each branch of $S_{\mu,\nu}(\zeta)$ has finitely many zeros and must satisfy satisfy either $\mu + \nu = 2p + 1$ or $\mu - \nu = 2p + 1$ for a non-negative integer p.

The Lommel function $S_{\mu,\nu}(\zeta)$, as in the cases of many classical special functions, has, in general, infinitely many branches, that is, its covering manifold has infinitely many sheets. The values of the function in different branches are given by so-called *analytic continuation formulae*. Such analytic continuation formulae in their full generality, first derived by Chiang and Yu [10], are given in lemma 3.1.

Corollary 1.5. *Suppose that $f(z)$ is a solution of (1.2). Then we have $\lambda(f) < +\infty$ if and only if the solution $f(z)$ is subnormal.*

Remark 1.6. We note that, for all values of μ_j and ν, $J_\nu(L\mathrm{e}^{Mz})$, $Y_\nu(L\mathrm{e}^{Mz})$ and $S_{\mu_j,\nu}(L\mathrm{e}^{Mz})$ are entire functions in the complex z-plane. Hence, they are single-valued functions and so are *independent of the branches* of $S_{\mu_j,\nu}(\zeta)$.

The main idea of our argument in the proofs is based on the asymptotic expansions of special functions (Bessel and Lommel functions), the analytic continuation formulae for $S_{\mu,\nu}(\zeta)$ (these formulae were first discovered by the present authors, and they play a very important role in [10] and also in this paper), the asymptotic locations of the zero of the transcendental equation $z\mathrm{e}^z = a$ given in [29,30] and the application of Rouché's theorem on suitably chosen contours in the complex plane.

This paper is organized as follows. We introduce the Lommel transformation in § 2, which serves as a crucial step in our proof to transform equation (1.8) into the equation

$$\zeta^2 y''(\zeta) + \zeta y'(\zeta) + (\zeta^2 - \nu^2) y(\zeta) = \sum_{j=1}^{n} \sigma_j \zeta^{\mu_j+1}. \tag{1.14}$$

Since we need to consider the *different branches* of the function (2.1) in the proof of theorem 1.3, so the analytic continuation formulae for $S_{\mu,\nu}(\zeta)$ come into play at this stage. We quote these formulae (which were derived in [10]) in § 3 for easy reference. Besides, we need information about the zeros of the function $g(\zeta) = \hat{C}\mathrm{e}^{\mathrm{i}\zeta} + \hat{\sigma}\zeta^{\mu-(1/2)}$, where $\hat{C} \neq 0$ and $\mu \neq \frac{1}{2}$. It turns out that Wright has already investigated the precise locations of zeros of the equation $z\mathrm{e}^z = a$ in [29,30], where

$a \neq 0$. In fact, this problem is of considerable scientific interest (see, for example, [5, 28]). Since we need to modify Wright's method in the preliminary construction of one of the contours used in the proof of theorem 1.3, we shall give an outline of Wright's method in § 4. A detailed study of the zeros of the function $g(\zeta) = \hat{C}e^{i\zeta} + \hat{\sigma}\zeta^{\mu-(1/2)}$ will be given in § 5, followed by a proof of theorem 1.3 in § 6. A proof of corollary 1.4 is presented in § 7 and a discussion about the non-homogeneous function-theoretic quantization-type result will be given in § 8. The appendix contains all of the necessary information regarding the Bessel functions and the Lommel functions that are used in this paper.

2. The Lommel transformations

Lommel investigated transformations that involve Bessel equations [17] in 1868.[1] Our standard references are [27, § 4.31], [12, p. 13] and [10, § 2]. Lommel considered the transformation $\zeta = \alpha x^\beta$ and $y(\zeta) = x^\gamma u(x)$, where x and $u(x)$ are the new independent and dependent variables, respectively, $\alpha, \beta \in \boldsymbol{C} \setminus \{0\}$ and $\gamma \in \boldsymbol{C}$. We apply this transformation to equation (1.14) to obtain a second-order differential equation in $u(x)$ whose general solution is $x^{-\gamma} y(\alpha x^\beta)$. Following the idea in [10], we apply a further change of variable by $x = e^z$ and $f(z) = u(x)$ to that differential equation in $u(x)$, replacing α, β and γ by L, M and N, respectively. This process yields (1.8). As noted in § 1, the general solution of (1.4) is given by a combination of the Bessel functions of first and second kinds and the Lommel function $S_{\mu,\nu}(\zeta)$ [12, § 7.7.5], hence, the general solution to (1.14) is

$$y(\zeta) = AJ_\nu(\zeta) + BY_\nu(\zeta) + \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(\zeta) \tag{2.1}$$

$$= CH_\nu^{(1)}(\zeta) + DH_\nu^{(2)}(\zeta) + \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(\zeta), \tag{2.2}$$

where $C = \frac{1}{2}(A - iB)$ and $D = \frac{1}{2}(A + iB)$. It is easily seen that $A = B = 0$ if and only if $C = D = 0$. Thus, the general solution $f(z) = e^{-Nz} y(Le^{Mz})$ of (1.8) assumes the form

$$f(z) = e^{-Nz}\biggl[CH_\nu^{(1)}(Le^{Mz}) + DH_\nu^{(2)}(Le^{Mz}) + \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(Le^{Mz})\biggr]. \tag{2.3}$$

3. Analytic continuation formulae for the Lommel function

We first note that the Lommel functions $S_{\mu,\nu}(\zeta)$ have a rather complicated definition with respect to different subscripts μ and ν (in four different cases) even in the principal branch. In this section, we shall not repeat the description of its definition (interested readers are referred to [10, § 3.1] and the references therein). Here we only record the analytic continuation formulae of $S_{\mu,\nu}(\zeta)$. Proofs can be found in [10, §§ 3.2–3.5]. The asymptotic expansion and the linear independence property of $S_{\mu,\nu}(\zeta)$ will be given in the appendix.

[1] We mentioned that the same transformations were also considered independently by Pearson [27, p. 98] in 1880.

Let $\chi_\pm := \frac{1}{2}(\mu \pm \nu + 1)$. We define the constants

$$\left.\begin{aligned} K &:= 2^{\mu-1}\Gamma(\chi_+)\Gamma(\chi_-), \\ K_+ &:= K\mathrm{i}[1 + \mathrm{e}^{(-\mu+\nu)\pi\mathrm{i}}]\cos(\tfrac{1}{2}(\mu+\nu)\pi), \\ K'_\pm &:= \pi 2^{\nu-2}\mathrm{i}\mathrm{e}^{-m\nu\pi\mathrm{i}}\Gamma(\nu)[U_{m-1}(\cos\nu\pi)\mathrm{e}^{(m\pm1)\nu\pi\mathrm{i}} - m], \\ K''_\pm &:= -\tfrac{1}{4}(m\pi^2(m\pm1)), \end{aligned}\right\} \tag{3.1}$$

where Γ is the gamma function and where $U_m(\cos\nu\pi) := \sin m\nu\pi/\sin\nu\pi$ is the *Chebyschev polynomial of the second kind.*

When $\mu \pm \nu \neq 2p+1$ for any integer p, we have the following lemma.

Lemma 3.1 (Chiang and Yu [10, theorem 3.4]). *Let m be an integer.*

(i) *We have*

$$\begin{aligned} &S_{\mu,\nu}(\zeta\mathrm{e}^{-m\pi\mathrm{i}}) \\ &\quad = (-1)^m\mathrm{e}^{-m\mu\pi\mathrm{i}}S_{\mu,\nu}(\zeta) + K_+[P_m(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}})H_\nu^{(1)}(\zeta) \\ &\qquad\qquad + \mathrm{e}^{-\nu\pi\mathrm{i}}P_{m-1}(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}})H_\nu^{(2)}(\zeta)], \end{aligned} \tag{3.2}$$

where $P_m(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}})$ is a rational function of $\cos\nu\pi$ and $\mathrm{e}^{-\mu\pi\mathrm{i}}$ given by

$$\begin{aligned} &P_m(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}}) \\ &\quad = \frac{U_{m-1}(\cos\nu\pi) + \mathrm{e}^{-\mu\pi\mathrm{i}}U_m(\cos\nu\pi) + (-1)^{m+1}\mathrm{e}^{-(m+1)\mu\pi\mathrm{i}}}{[1+\mathrm{e}^{-(\mu+\nu)\pi\mathrm{i}}][1+\mathrm{e}^{-(\mu-\nu)\pi\mathrm{i}}]}. \end{aligned} \tag{3.3}$$

(ii) *Furthermore, the coefficients $P_m(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}})$ and $P_{m-1}(\cos\nu\pi,\mathrm{e}^{-\mu\pi\mathrm{i}})$ are not identically zero simultaneously for all μ,ν and all non-zero integers m.*

When either $\mu+\nu$ or $\mu-\nu$ is an odd negative integer $-2p-1$, where p is a non-negative integer, then we have another set of analytic continuation formulae which are given by the following lemma.

Lemma 3.2. (Chiang and Yu [10, lemmas 3.6, 3.8, 3.10]). *Let m be an integer. Then we have the following.*

(i) *If $-\nu \notin \{0,1,2,\dots\}$, then we have*

$$\begin{aligned} S_{\nu-2p-1,\nu}(\zeta\mathrm{e}^{-m\pi\mathrm{i}}) &= \mathrm{e}^{-m\nu\pi\mathrm{i}}S_{\nu-2p-1,\nu}(\zeta) \\ &\quad + \frac{(-1)^p}{2^{2p}p!(1-\nu)_p}[K'_+H_\nu^{(1)}(\zeta) + K'_-H_\nu^{(2)}(\zeta)]. \end{aligned}$$

(ii) *If $\nu = 0$, then we have*

$$S_{-2p-1,0}(\zeta\mathrm{e}^{-m\pi\mathrm{i}}) = S_{-2p-1,0}(\zeta) + \frac{(-1)^p}{2^{2p}(p!)^2}[K''_+H_0^{(1)}(\zeta) + K''_-H_0^{(2)}(\zeta)].$$

(iii) *We define* $\delta_m = 1 + (-1)^{m-1}$ *and, for every polynomial* $P_n(\zeta)$ *of degree* n, *we define* $\hat{P}_n(\zeta)$ *to be the polynomial containing the term of* $P_n(\zeta)$ *with odd powers in* ζ *and* $\bar{P}_n(\zeta) := P_n(\zeta) - \delta_m \hat{P}_n(\zeta)$. *If* $\nu = -n$ *is a positive integer* n, *then we have*

$$\begin{aligned} S_{-n-2p-1,-n}(\zeta e^{-m\pi i}) = (-1)^{mn} S_{-n-2p-1,-n}(\zeta) + \frac{(-1)^{(m+1)n+p}}{2^{2p+n} n!(p!)^2 (1+n)_p} \zeta^{-n} \\ \times \{-\delta_m[\hat{A}_n(\zeta) + \hat{B}_n(\zeta) S_{-1,0}(\zeta) + \zeta \hat{C}_n(\zeta) S'_{-1,0}(\zeta)] \\ + \bar{B}_n(\zeta)[K''_+ H_0^{(1)}(\zeta) + K''_- H_0^{(2)}(\zeta)] \\ - \zeta \bar{C}_n(\zeta)[K''_+ H_1^{(1)}(\zeta) + K''_- H_1^{(2)}(\zeta)]\}, \end{aligned}$$

where $A_n(\zeta)$, $B_n(\zeta)$ *and* $C_n(\zeta)$ *are polynomials in* ζ *of degree at most* n *such that* $A_1(\zeta) = B_1(\zeta) \equiv 0$, $C_1(\zeta) \equiv 1$ *and, when* $n \geqslant 2$, *that they satisfy the following recurrence relations:*

$$\left.\begin{aligned} A_n(\zeta) &= -2(n-1)A_{n-1}(\zeta) + \zeta A'_{n-1}(\zeta) + C_{n-1}(\zeta), \\ B_n(\zeta) &= -2(n-1)B_{n-1}(\zeta) + \zeta B'_{n-1}(\zeta) - \zeta^2 C_{n-1}(\zeta), \\ C_n(\zeta) &= -2(n-1)C_{n-1}(\zeta) + B_{n-1}(\zeta) + \zeta C'_{n-1}(\zeta). \end{aligned}\right\} \tag{3.4}$$

4. Applications of Wright's result

Suppose that a is a non-zero complex number such that $a = Ae^{i\alpha}$, where $-\pi < \alpha \leqslant \pi$ and $A = |a| \neq 0$. Wright [29,30] obtained precise asymptotic locations of the zeros of the equation

$$ze^z = a \tag{4.1}$$

in terms of rapidly convergent series by constructing the Riemann surface of the inverse function of $z + \log z$. This result is of considerable scientific interest, particularly in the theory of, and various applications of, difference–differential equations [5,28]. For further applications of this equation, we refer the interested reader to [7,11,25].

In this section we first describe Wright's result in lemma 4.1. We then apply Wright's result to obtain finer estimates of the real and imaginary parts of the solutions of (4.1) in lemmas 4.3 and 4.4, which we need to construct a certain contour needed in the proof of the main result in proposition 5.3.

Suppose that n is an integer. We let $z(n) = x(n) + iy(n)$ be solutions of equation (4.1), where $x(n)$ and $y(n)$ are real, and are given in the following result.

LEMMA 4.1 (Wright [29,30]). *Let* $a = Ae^{i\alpha}$, *where* $-\pi < \alpha \leqslant \pi$ *and* $A = |a| \neq 0$. *Let* $\operatorname{sgn}(n)$ *be the sign of the non-zero integer* n. *We define*

$$H_n := 2|n|\pi + \operatorname{sgn}(n)\alpha - \tfrac{1}{2}\pi, \qquad \beta_n := \log \frac{A}{H_n}, \tag{4.2}$$

taking $\log A/H_n$ *real. If* $|n|$ *is sufficiently large such that*

$$2H_n|\beta_n| < (H_n - 1)^2, \qquad (\log A)^2 < (H_n - \tfrac{1}{2}\pi)^2 + 2(1 + \log A)\log H_n + 1, \tag{4.3}$$

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then the solutions of equation (4.1) are given by

$$x(n) = (H_n + \eta_n)\tan\eta_n, \qquad y(n) = \operatorname{sgn}(n)(H_n + \eta_n), \tag{4.4}$$

where

$$\eta_n = \sum_{j=0}^{+\infty} (-1)^j Q_{2j+1}(\beta_n) H_n^{-2j-1},$$

and $\{Q_m(t)\}$ *is the sequence of polynomials defined by*

$$Q_1(t) := t, \qquad Q_{m+1}(t) := Q_m(t) + m\int_0^t Q_m(s)\,\mathrm{d}s, \tag{4.5}$$

where m *is a positive integer.*

Remark 4.2. We deduce from the definition (4.2) that $\beta_n < 0$ for all n sufficiently large and $\beta_n \to -\infty$ as $n \to \pm\infty$. We also note that it follows from (4.5) that $Q_m(t)$ is a polynomial of degree at most m. This and the series representation of η_n show that $\eta_n = O(\beta_n/H_n)$, $\eta_n < 0$ and $\eta_n \to 0$ as $n \to \pm\infty$.

As mentioned in § 1, we would like to apply Rouché's theorem to suitable contours. To construct one of these contours, it is necessary to derive accurate bounds for $x(n)$ and $y(n)$ from the following lemma by Wright. We include the argument leading to the inequalities to familiarize our readers for later applications.

Lemma 4.3 (Wright [30, p. 196]). *Suppose that* H_n *and* A *are as defined in (4.2). Then the upper and lower bounds for the real and imaginary parts of the solutions to (4.1) are given, respectively, by*

$$2\log\frac{A}{(2|n|+1)\pi} - 1 < x(n) < \log\frac{A}{2(|n|-1)\pi} + 1 \tag{4.6}$$

and

$$\left.\begin{aligned} (2n-1)\pi + \alpha < y(n) < 2n\pi + \alpha \quad & \text{if } n \text{ is large positively,} \\ 2n\pi + \alpha < y(n) < (2n+1)\pi + \alpha \quad & \text{if } n \text{ is large negatively.} \end{aligned}\right\} \tag{4.7}$$

Proof. It is easy to see that the inequalities (4.7) follow easily from definitions (4.2), (4.4) and the properties of η_n in remark 4.2. For inequality (4.6) representing the real part of $z(n)$, we deduce from the power series of $\tan\eta_n$ and equation (4.4) that

$$x(n) = \eta_n(H_n + \eta_n)\frac{\tan\eta_n}{\eta_n} = \eta_n(H_n + \eta_n)(1 + \tfrac{1}{3}\eta_n^2 + \cdots).$$

This implies that inequalities $\eta_n(H_n + \eta_n)(1 - \eta_n) \leqslant x(n) \leqslant \eta_n(H_n + \eta_n)$ hold when n is sufficiently large. On the other hand, lemma 4.1 and remark 4.2 assert that $\beta_n - \frac{1}{2} < \eta_n H_n < \beta_n + \frac{1}{2}$ when n is sufficiently large. Combining these two inequalities and the fact $0 < 1 - \eta_n < 2$, we deduce

$$x(n) \leqslant \eta_n(H_n + \eta_n) = \eta_n H_n + \eta_n^2 < \beta_n + \tfrac{1}{2} + \eta_n^2 < \beta_n + 1 < 0 \tag{4.8}$$

and

$$x(n) \geqslant \eta_n(H_n + \eta_n)(1 - \eta_n) > 2\eta_n(H_n + \eta_n) > 2\beta_n - 1 + 2\eta_n^2 > 2\beta_n - 1, \tag{4.9}$$

since $\eta_n(H_n + \eta_n) < 0$. Hence, we deduce from inequalities (4.8) and (4.9) the inequalities

$$2\beta_n - 1 < x(n) < \beta_n + 1 < 0. \tag{4.10}$$

Since $-\pi < \alpha \leqslant \pi$, we must have $-\pi < \operatorname{sgn}(n)\alpha \leqslant \pi$ for every non-zero integer n. Then the desired inequalities (4.6) follow from this fact, the inequalities (4.10) and the definition (4.2). This completes the proof of the lemma. □

Lemma 4.4. *Let m be a fixed positive integer such that $\log A - \log m\pi + 1 < -3$. We define $d_r := 2m\pi r^2 - \alpha - \pi$ for every real $r > 0$ and let $n_k = -mk^2$, where k is a sufficiently large positive integer such that n_k satisfies the inequalities (4.3). Then we have*

$$-5\log k < x(n_k) < -2\log k - 2, \qquad -d_{k+1} < y(n_k) < -d_k. \tag{4.11}$$

Proof. Since n_k is large and negative, so the inequalities (4.11) for $y(n_k)$ follow easily from the second set of inequalities in (4.7). We next observe that both inequalities

$$\log(2mk^2 + 1)\pi \leqslant \log mk^2\pi + \log\pi \quad \text{and} \quad \log(2mk^2 - 1)\pi \geqslant \log mk^2\pi - \log\pi$$

hold for k sufficiently large. Hence, it follows from the inequalities (4.6) that

$$\begin{aligned} x(n_k) &> 2\log A - 2\log(2mk^2 + 1)\pi - 1 \\ &> 2\log A - 2\log mk^2\pi - 2\log\pi - 1 \\ &= -4\log k + (2\log A - 2\log m\pi - 2\log\pi - 1) \\ &> -5\log k \end{aligned}$$

and

$$\begin{aligned} x(n_k) &< \log A - \log(2mk^2 - 1)\pi + 1 \\ &< \log A - \log mk^2\pi - \log 2 + \log\pi + 1 \\ &= (\log A - \log m\pi + 1) - 2\log k - \log 2 + \log\pi \\ &< -2\log k - 3 + \log\pi - \log 2 \\ &< -2\log k - 2, \end{aligned}$$

completing the proof of the lemma. □

Remark 4.5. We remark from inequalities (4.11) that the particular set of zeros $z(n_k)$ of equation (4.1) must lie inside the rectangles whose vertices are given by the points $(-5\log k, -d_k)$, $(-2\log k - 2, -d_k)$, $(-5\log k, -d_{k+1})$ and $(-2\log k - 2, -d_{k+1})$ in the complex z-plane.

5. Zeros of an auxiliary function

In the proof of our main results, it will become clear in § 6 that we need to know the locations of zeros of the auxiliary function

$$g(\zeta) = \hat{C}\mathrm{e}^{\mathrm{i}\zeta} + \hat{\sigma}\zeta^{\mu-(1/2)}, \tag{5.1}$$

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where μ, $\hat{C}$, $\hat{\sigma}$ are non-zero complex constants such that $\mu \neq \frac{1}{2}$, and $\zeta^{\mu-(1/2)}$ takes the principal branch.[2] We apply the results from § 4 to investigate the asymptotic locations of zeros for $g(\zeta)$. To do so, we first transform the equation $g(\zeta) = 0$ into the form of (4.1), where

$$z = \frac{\mathrm{i}\zeta}{\frac{1}{2} - \mu}, \qquad a = \frac{\mathrm{i}}{\frac{1}{2} - \mu}(-\hat{\sigma}\hat{C}^{-1})^{1/((1/2)-\mu)}, \quad \text{assuming that } -\pi < \arg a \leqslant \pi. \tag{5.2}$$

Let $\zeta(n) = u(n) + \mathrm{i}v(n)$ and $\frac{1}{2} - \mu = b\mathrm{e}^{\mathrm{i}\phi}$, where n is an integer, $b = |\frac{1}{2} - \mu| > 0$ and $-\pi < \phi \leqslant \pi$. Then it follows from (5.2) and $z(n) = x(n) + \mathrm{i}y(n)$ that, for sufficiently large positive or negative integers n,

$$u(n) = (b\cos\phi)y(n) + (b\sin\phi)x(n), \qquad v(n) = (b\sin\phi)y(n) - (b\cos\phi)x(n). \tag{5.3}$$

In order to find precise asymptotic locations of zeros of the function (5.1), we first consider the particular case that $\phi = \pi$ in (5.3). This forces $\mu > \frac{1}{2}$ and

$$\zeta(n) = u(n) + \mathrm{i}v(n) = b(-y(n) + \mathrm{i}x(n)),$$

where $b = \mu - \frac{1}{2} > 0$. Therefore, we obtain the following lemma from lemma 4.4 and remark 4.5.

Lemma 5.1. *Let $\phi = \pi$ and n_k be defined as in lemma 4.4. Then, for k sufficiently large, we have*

$$0 < bd_k < u(n_k) < bd_{k+1}, \qquad -5b\log k < v(n_k) < -2b\log k - 2b < 0. \tag{5.4}$$

In other words, the zeros $\zeta(n_k)$ must lie inside the rectangles R_k whose vertices are given by the points $(bd_k, -5b\log k)$, $(bd_k, -2b\log k - 2b)$, $(bd_{k+1}, -5b\log k)$ and $(bd_{k+1}, -2b\log k - 2b)$ in the ζ-plane.

Remark 5.2. In view of lemma 5.1, we easily see that, when $\phi = \pi$, *all* such zeros $\zeta(n_k)$ lie in the fourth quadrant of the ζ-plane and the real part of each $\zeta(n_k)$ is increasing much faster than the imaginary part in such a way, so that the argument $\arg\zeta(n_k)$ is always negative and $\arg\zeta(n_k) \to 0$ as $n_k = -mk^2 \to -\infty$ (or as $k \to +\infty$).

Now we are ready to define one of the contours that will be used in the proof of theorem 1.3. For any given function g of the form (5.1), the contour is formed by the curves $\Gamma_1(g)$, $\Gamma_2(g)$ and the line segments $\ell_1(g)$, $\ell_2(g)$, which are defined as follows:

$$\left.\begin{aligned}
\Gamma_1(g) &:= \{b(d_r - 2\mathrm{i}\log r)\colon k \leqslant r \leqslant 2k\},\\
\Gamma_2(g) &:= \{b(d_r - 6i\log r)\colon k \leqslant r \leqslant 2k\},\\
\ell_1(g) &:= \{b(d_k + \mathrm{i}v)\colon -6\log k \leqslant v \leqslant -2\log k\},\\
\ell_2(g) &:= \{b(d_{2k} + \mathrm{i}v)\colon -6\log(2k) \leqslant v \leqslant -2\log(2k)\}.
\end{aligned}\right\} \tag{5.5}$$

We join the line segments $\ell_1(g)$, $\ell_2(g)$ and the curves $\Gamma_1(g)$, $\Gamma_2(g)$ to form the contour $\Omega(g,k)$ for each integer k. We then *glue* the $\Omega(g,k)$ together along each

[2]The remaining case when $\mu = \frac{1}{2}$ will be discussed in lemma 6.2.

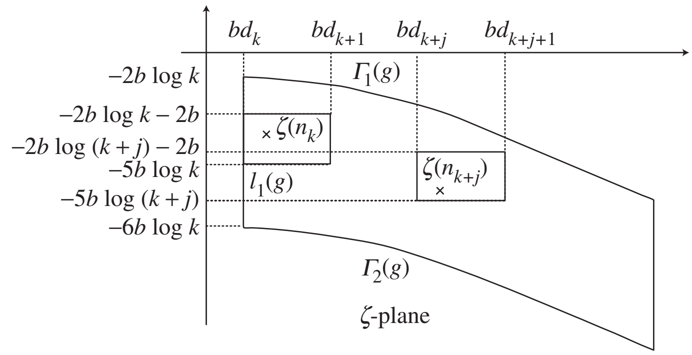

Figure 1. The rectangles R_{k+j} and a part of the contour $\Omega(g,k)$ when $\phi = \pi$.

pair of $\ell_1(g)$, $\ell_2(g)$ and the resulting set is denoted by $\Omega(g) = \bigcup_{k=1}^{+\infty} \Omega(g,k)$. Then we define $e^{i(\pi-\phi)}\Omega(g,k)$ and $e^{i(\pi-\phi)}\Omega(g)$ as follows:

$$\left.\begin{aligned} e^{i(\pi-\phi)}\Omega(g,k) &:= \{e^{i(\pi-\phi)}\zeta : \zeta \in \Omega(g,k)\}, \\ e^{i(\pi-\phi)}\Omega(g) &:= \{e^{i(\pi-\phi)}\zeta : \zeta \in \Omega(g)\}. \end{aligned}\right\} \tag{5.6}$$

Thus, we have the following result.

Proposition 5.3. *Let k be a large positive integer and let $\mu \neq \frac{1}{2}$. Then the function $g(\zeta)$ as defined in (5.1) has at least k distinct zeros lying inside the contour $e^{i(\pi-\phi)}\Omega(g,k)$ and infinitely many zeros lying inside the set $e^{i(\pi-\phi)}\Omega(g)$.*

Proof. It suffices to prove the first statement. We suppose first that $\phi = \pi$ so that the contour and the sets given by (5.6) are $\Omega(g,k)$ and $\Omega(g)$, respectively, and all the zeros $\zeta(n_k)$ lie in the fourth quadrant of the ζ-plane by remark 5.2. We note that, for each $j \in \{0,1,\dots,k-1\}$, the vertices of the rectangle R_{k+j} are given by $(bd_{k+j}, -5b\log(k+j))$, $(bd_{k+j}, -2b\log(k+j)-2b)$, $(bd_{k+j+1}, -5b\log(k+j))$ and $(bd_{k+j+1}, -2b\log(k+j)-2b)$. Since we have

$$-2\log(k+j) - 2 < -2\log r, \qquad -6\log r < -5\log(k+j),$$

where $k+j \leqslant r \leqslant k+j+1$, it means *geometrically* that the upper (respectively, lower) edge of R_{k+j} is *below* (respectively, *above*) the curve $\Gamma_1(g)$ (respectively, $\Gamma_2(g)$) (see figure 1 for an illustration).

Thus, all the k rectangles R_{k+j} for $j \in \{0,1,\dots,k-1\}$ are contained in the contour $\Omega(g,k)$. By lemma 5.1, each rectangle R_{k+j} contains the zero $\zeta(n_{k+j})$ for $j \in \{0,1,\dots,k-1\}$. These zeros are *distinct* because we have $R_{k+j} \cap R_{k+j'} = \emptyset$ whenever $j \neq j'$. Hence, the result follows in this particular case.

Next we suppose that $-\pi < \phi < \pi$. Then it may happen that *not* all zeros $\zeta(n_k)$ lie in the fourth quadrant of the ζ-plane. In this general case, we rotate the ζ-plane through the angle $(\pi - \phi)$ to the ζ'-plane, where $\zeta' := e^{i(\pi-\phi)}\zeta$. Thus, it follows from relations (5.3) that

$$\zeta'(n_k) = e^{i(\pi-\phi)}\zeta(n_k) = b(-y(n_k) + ix(n_k))$$

so that *all* the zeros $\zeta'(n_k)$ of $G(\zeta') = g(e^{i(\phi-\pi)}\zeta')$ lie in the fourth quadrant of the ζ'-plane by lemma 5.1 and remark 5.2. Thus, the argument in the first part applies to this general case with respect to the contour $e^{i(\pi-\phi)}\Omega(g,k)$ and the set $e^{i(\pi-\phi)}\Omega(g)$, thus completing the proof of the proposition. □

6. Proof of theorem 1.3

6.1. Sufficiency part

Suppose that $f(z)$ is subnormal. Then remark 1.5 asserts that we must have $A = B = 0$ and one of the equations in (1.10) holds. Thus, according to (1.11), we have $\lambda(f) < +\infty$. This proves the sufficiency part of the theorem.

6.2. Necessary part

In order to complete the proof of theorem 1.3, that is, in order to find the values of $\mu_j \pm \nu$ under the assumption that $\lambda(f) < +\infty$, we consider the function $f(z)$ in the form (2.3). First, we need the following result.

THEOREM 6.1. *If $C \neq 0$ or $D \neq 0$, then $\lambda(f) = +\infty$.*

Proof of theorem 6.1. We let $y(\zeta)$ in the form (2.2) be the general solution of (1.14). Since lemma A.2 asserts that the Lommel functions $S_{\mu_j,\nu}(\zeta)$, $j = 1, 2, \dots, n$, are linearly independent over $\mathbb{C}$ and that not all σ_j are zero, so the summand in (2.2) is not identically zero.

Without loss of generality, we may assume that $\sigma_n \neq 0$ and the constants μ_j, $j = 1, 2, \dots, n$, in theorem 1.3 satisfy $\mathrm{Re}(\mu_1) < \mathrm{Re}(\mu_2) < \cdots < \mathrm{Re}(\mu_n)$. In order to prove theorem 6.1, we show that the general solution (2.2) has infinitely many zeros *in the principal branch of $H_\nu^{(1)}(\zeta)$, $H_\nu^{(2)}(\zeta)$ and $S_{\mu,\nu}(\zeta)$.*[3] The idea of our proof is to apply asymptotic expansions of the corresponding special functions and Rouché's theorem on suitably chosen contours.

When $-\pi < \arg\zeta < \pi$, we substitute the asymptotic expansions (A 4)–(A 6) into the solution (2.2) to yield[4]

$$\begin{aligned}
\hat{y}(\zeta) &= (\tfrac{1}{2}\pi\zeta)^{1/2} y(\zeta) \\
&= \hat{C}e^{i\zeta}\left[\sum_{k=0}^{p-1}\frac{(\nu,k)}{(-2i\zeta)^k} + O(\zeta^{-p})\right] + \hat{D}e^{-i\zeta}\left[\sum_{k=0}^{p-1}\frac{(\nu,k)}{(2i\zeta)^k} + O(\zeta^{-p})\right] \\
&\quad + \sum_{j=1}^{n}\hat{\sigma}_j\zeta^{\mu_j-(1/2)}\left[\sum_{k=0}^{p-1}\frac{(-1)^k c_{k,j}}{\zeta^{2k}} + O(\zeta^{-2p+1})\right],
\end{aligned}$$

where

$$\begin{aligned}
(\nu,k) &:= \frac{(-1)^k(\frac{1}{2}-\nu)_k(\frac{1}{2}+\nu)_k}{k!}, \\
\hat{C} &:= Ce^{-i((1/2)\nu\pi+(1/4)\pi)}, \\
\hat{D} &:= De^{i((1/2)\nu\pi+(1/4)\pi)}, \\
\hat{\sigma}_j &:= \sigma_j(\tfrac{1}{2}\pi)^{1/2}, \quad \text{where } j = 1, 2, \dots, n.
\end{aligned}$$

[3]That is, $-\pi < \arg\zeta < \pi$.

[4]This refers to the principal branch of the Hankel functions $H_\nu^{(1)}(\zeta)$, $H_\nu^{(2)}(\zeta)$ and the Lommel functions $S_{\mu_j,\nu}(\zeta)$, $j = 1, 2, \dots, n$.

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This gives, when $p = 1$, that

$$\hat{y}(\zeta) = \hat{C}e^{i\zeta}[1 + O(\zeta^{-1})] + \hat{D}e^{-i\zeta}[1 + O(\zeta^{-1})] + \sum_{j=1}^{n} \hat{\sigma}_j \zeta^{\mu_j-(1/2)}[1 + O(\zeta^{-1})]. \quad (6.1)$$

We distinguish two main cases: case 1, $\mu_n \neq \frac{1}{2}$, and case 2, $\mu_n = \frac{1}{2}$, which will then be further split into different subcases.

6.3. Case 1

We suppose that $\mu_n \neq \frac{1}{2}$.

Without loss of generality, we may assume that $C \neq 0$ so that $\hat{C} \neq 0$. We choose the function (5.1) to be

$$g(\zeta) = \hat{C}e^{i\zeta} + \hat{\sigma}_n \zeta^{\mu_n-(1/2)}, \quad (6.2)$$

where $\frac{1}{2} - \mu_n = be^{i\phi}$, $b = |\frac{1}{2} - \mu_n|$ and $-\pi < \phi \leqslant \pi$. Moreover, we assume that the chosen integer m in lemma 4.4 also satisfies the inequality

$$(bm\pi)^{|\operatorname{Re}(\mu_n)-(1/2)|}|\hat{\sigma}_n| > 2|\hat{C}|e^{|\operatorname{Im}(\mu_n)|\pi}. \quad (6.3)$$

There are two subcases in case 1. They are subcase A, $\phi = \pi$, and subcase B, $\phi \neq \pi$.

6.3.1. *Subcase A:* $\phi = \pi$

The definition shows that $\frac{1}{2} - \mu_n = be^{i\pi} = -b$, which implies that μ_n must be a real number such that $\mu_n > \frac{1}{2}$ and $b = \mu_n - \frac{1}{2} > 0$. We obtain from (6.1) and (6.2) that

$$\begin{aligned}|\hat{y}(\zeta) - g(\zeta)| &= \bigg|[\hat{D}e^{-i\zeta} + \hat{C}e^{i\zeta}O(\zeta^{-1}) + \hat{D}e^{-i\zeta}O(\zeta^{-1})] \\ &\qquad + O(\zeta^{\mu_n-(3/2)}) + \sum_{j=1}^{n-1} \hat{\sigma}_j \zeta^{\mu_j-(1/2)}[1 + O(\zeta^{-1})]\bigg| \\ &\leqslant |\hat{D}e^{-i\zeta} + \hat{C}e^{i\zeta}O(\zeta^{-1}) + \hat{D}e^{-i\zeta}O(\zeta^{-1})| + O(|\zeta|^{\kappa}), \end{aligned} \quad (6.4)$$

where κ is defined by

$$\kappa := \max\{\operatorname{Re}(\mu_{n-1}) - \tfrac{1}{2}, \operatorname{Re}(\mu_n) - \tfrac{3}{2}\} < |\mu_n - (1/2)| = b. \quad (6.5)$$

We show that the inequality

$$|\hat{y}(\zeta) - g(\zeta)| < |g(\zeta)| \quad (6.6)$$

holds on the contour $\Omega(g, k)$ for all k sufficiently large. In fact, it is always true that $|\hat{C}e^{i\zeta}O(\zeta^{-1})| < |\hat{C}e^{i\zeta}|$, $|\hat{D}e^{-i\zeta}O(\zeta^{-1})| < |\hat{D}e^{-i\zeta}|$ and

$$|\zeta^{\kappa}| < |\zeta^{(b+\kappa)/2}|, \quad (6.7)$$

so it suffices to compare the values of $|\hat{D}e^{-i\zeta}|$, $|\hat{C}e^{i\zeta}|$ and $|\zeta^{(b+\kappa)/2}|$ along the contour $\Omega(g, k)$. If $\zeta \in \ell_1(g)$, then we have $\zeta = b(d_k - i\gamma \log k)$ (see (5.5)), where $2 \leqslant \gamma \leqslant 6$,

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and if $\zeta \in \Gamma_1(g)$, then we have $\zeta = b(d_r - 2\mathrm{i}\log r)$, where $k \leqslant r \leqslant 2k$. We deduce that

$$|\mathrm{e}^{\pm \mathrm{i}\zeta}| = \begin{cases} k^{\pm b\gamma} & \text{if } \zeta \in \ell_1(g), \\ r^{\pm 2b} & \text{if } \zeta \in \Gamma_1(g), \end{cases} \tag{6.8}$$

and, for k sufficiently large, that

$$\left.\begin{aligned} bm\pi k^2 < |\zeta| < 4bm\pi k^2 \quad \text{if } \zeta \in \ell_1(g), \\ bm\pi r^2 < |\zeta| < 4bm\pi r^2 \quad \text{if } \zeta \in \Gamma_1(g). \end{aligned}\right\} \tag{6.9}$$

On the one hand, for all sufficiently large k, it follows from (6.3) and (6.9) that

$$\left.\begin{aligned} 2|\hat{C}|k^{2b} &< (bm\pi)^b|\hat{\sigma}_n|k^{2b} \\ &< |\hat{\sigma}_n\zeta^b| < (4bm\pi)^b|\hat{\sigma}_n|k^{2b} \quad \text{if } \zeta \in \ell_1(g), \\ 2|\hat{C}|r^{2b} &< (bm\pi)^b|\hat{\sigma}_n|r^{2b} \\ &< |\hat{\sigma}_n\zeta^b| < (4bm\pi)^b|\hat{\sigma}_n|r^{2b} \quad \text{if } \zeta \in \Gamma_1(g). \end{aligned}\right\} \tag{6.10}$$

But the triangle inequality

$$|g(\zeta)| \geqslant ||\hat{C}\mathrm{e}^{\mathrm{i}\zeta}| - |\hat{\sigma}_n\zeta^{\mu_n-(1/2)}|| \tag{6.11}$$

with relations (6.8) and (6.10) imply, for $k \geqslant k_0$ for some positive integer k_0, that

$$\begin{aligned} |g(\zeta)| &\geqslant \begin{cases} ||\hat{C}|k^{b\gamma} - |\hat{\sigma}_n\zeta^b|| & \text{if } \zeta \in \ell_1(g), \\ ||\hat{\sigma}_n\zeta^b| - |\hat{C}|r^{2b}| & \text{if } \zeta \in \Gamma_1(g), \end{cases} \\ &> \begin{cases} \tfrac{1}{2}|\hat{C}|k^{b\gamma} & \text{if } \zeta \in \ell_1(g), \\ \tfrac{1}{2}|\hat{C}|r^{2b} & \text{if } \zeta \in \Gamma_1(g), \end{cases} \end{aligned}$$

where the lower estimate for the case $\zeta \in \ell_1(g)$ is trivial when $\gamma = 2$ and the case when $\gamma > 2$ follows since the factor $(4bm\pi)^b$ from (6.10) is a constant. On the other hand, we obtain from the relations (6.4) and (6.7)–(6.9) that

$$\begin{aligned} &|\hat{y}(\zeta) - g(\zeta)| \\ &\quad\leqslant \begin{cases} |\hat{D}|k^{-b\gamma} + K_1(|\hat{C}|k^{b\gamma} + |\hat{D}|k^{-b\gamma})k^{-2} + K_2|\zeta|^{(b+\kappa)/2} & \text{if } \zeta \in \ell_1(g), \\ |\hat{D}|r^{-2b} + K_3(|\hat{C}|r^{2b} + |\hat{D}|r^{-2b})r^{-2} + K_4|\zeta|^{(b+\kappa)/2} & \text{if } \zeta \in \Gamma_1(g), \end{cases} \qquad (6.12) \\ &\quad\leqslant \begin{cases} |\hat{D}|k^{-b\gamma} + K_5k^{b\gamma-2} + K_6k^{b+\kappa} & \text{if } \zeta \in \ell_1(g), \\ |\hat{D}|r^{-2b} + K_7r^{2b-2} + K_8r^{b+\kappa} & \text{if } \zeta \in \Gamma_1(g), \end{cases} \\ &\quad\leqslant \begin{cases} K_9k^{\kappa_1} & \text{if } \zeta \in \ell_1(g), \\ K_{10}r^{\kappa_2} & \text{if } \zeta \in \Gamma_1(g), \end{cases} \qquad (6.13) \end{aligned}$$

where

$$\kappa_1 := \max\{-b\gamma, b\gamma - 2, b + \kappa\}, \qquad \kappa_2 := \max\{-2b, 2b - 2, b + \kappa\}$$

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and $K_1, K_2, \dots, K_{10}$ are some fixed positive constants depending only on b, m (see lemma 4.4) and k_0. Note that it is easy to check that $b\gamma > \kappa_1$ and $2b > \kappa_2$ hold trivially. We deduce from inequalities (6.12) and (6.13) that inequality (6.6) holds on $\ell_1(g)$ and $\Gamma_1(g)$, then, similarly, it also holds on $\ell_2(g)$ and $\Gamma_2(g)$. Hence, the desired inequality (6.6) holds on the contour $\Omega(g, k)$.

6.3.2. *Subcase B*

If $\phi \neq \pi$, then it may happen, as described in the proof of proposition 5.3, that *not* all zeros $\zeta(n_k)$ lie in the fourth quadrant of the ζ-plane, where the integers n_k are also defined in lemma 4.4. However, one can rotate the ζ-plane through the angle $(\pi - \phi)$ as described in proposition 5.3 (see also its proof) so that all such zeros can only lie in the fourth quadrant of the ζ'-plane.

In this circumstance, we note that the inequalities (6.4) and (6.11) are now replaced by

$$|\hat{Y}(\zeta') - G(\zeta')| \leqslant |\hat{D}\exp\{-\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta'\} + \hat{C}\exp\{\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta'\}O(\zeta'^{-1}) + \hat{D}\exp\{-\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta'\}O(\zeta'^{-1})| + O(|\zeta'|^{\kappa}) \qquad (6.14)$$

and

$$|G(\zeta')| \geqslant ||\hat{C}\exp\{\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta'\}| - |\hat{\sigma}_n\zeta'^{\,\mathrm{Re}(\mu_n)-(1/2)}|\exp\{-\operatorname{Im}(\mu_n)\arg(\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta')\}| \qquad (6.15)$$

respectively, where $\hat{Y}(\zeta') = \hat{y}(\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta')$, $G(\zeta') = g(\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta')$ and the constant κ is given by (6.5). Moreover, the relations (6.8) and the inequalities (6.9) are replaced by

$$|\exp\{\pm\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi-\pi)}\zeta'\}| = \begin{cases} k^{\pm b\gamma} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \\ r^{\pm 2b} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g), \end{cases} \qquad (6.16)$$

and

$$\left.\begin{aligned} bm\pi k^2 < |\zeta'| < 4bm\pi k^2 \quad & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \\ bm\pi r^2 < |\zeta'| < 4bm\pi r^2 \quad & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g) \end{aligned}\right\} \qquad (6.17)$$

respectively, where $b = |\frac{1}{2} - \mu_n| > \mathrm{Re}(\mu_n) - \frac{1}{2}$, $2 \leqslant \gamma \leqslant 6$, $k \leqslant r \leqslant 2k$, where $\ell_1(g)$ and $\Gamma_1(g)$ are defined in (5.5).

Now we further distinguish between

(i) $\mathrm{Re}(\mu_n) > \frac{1}{2}$,

(ii) $\mathrm{Re}(\mu_n) < \frac{1}{2}$.

(i) If $\mathrm{Re}(\mu_n) > \frac{1}{2}$, then $\mathrm{Re}(\mu_n) - \frac{1}{2} > 0$ and it follows from inequalities (6.3) and (6.17) that the inequalities (6.10) are replaced by

$$\begin{aligned} 2|\hat{C}|\mathrm{e}^{|\operatorname{Im}(\mu_n)|\pi}k^{2(\mathrm{Re}(\mu_n)-(1/2))} & \\ < |\hat{\sigma}_n\zeta'^{\,\mathrm{Re}(\mu_n)-(1/2)}| & \\ < (4bm\pi)^{\mathrm{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n|k^{2(\mathrm{Re}(\mu_n)-(1/2))} & \quad \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \end{aligned} \qquad (6.18\,a)$$

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$$
\begin{aligned}
2|\hat{C}|\mathrm{e}^{|\operatorname{Im}(\mu_n)|\pi} r^{2(\operatorname{Re}(\mu_n)-(1/2))}
&< |\hat{\sigma}_n \zeta'^{\operatorname{Re}(\mu_n)-(1/2)}| \\
&< (4bm\pi)^{\operatorname{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n| r^{2(\operatorname{Re}(\mu_n)-(1/2))} \qquad \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g) \qquad (6.18\,b)
\end{aligned}
$$

for sufficiently large k. Thus, inequality (6.15) together with (6.18) yield, for $k \geqslant k_1$ for some sufficiently large positive integer k_1, that

$$
|G(\zeta')| > \begin{cases} \frac{1}{2}|\hat{C}|k^{b\gamma} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \\ \frac{1}{2}|\hat{C}|r^{2b} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g). \end{cases} \tag{6.19}
$$

(ii) If $\operatorname{Re}(\mu_n) < \frac{1}{2}$, then we have $\frac{1}{2} - \operatorname{Re}(\mu_n) > 0$, and the inequalities (6.10) are now replaced by

$$
\begin{aligned}
\frac{(4bm\pi)^{\operatorname{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n|}{k^{2((1/2)-\operatorname{Re}(\mu_n))}} &< |\hat{\sigma}_n \zeta'^{\operatorname{Re}(\mu_n)-(1/2)}| \\
&< \frac{(bm\pi)^{\operatorname{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n|}{k^{2((1/2)-\operatorname{Re}(\mu_n))}} \qquad \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \\
\frac{(4bm\pi)^{\operatorname{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n|}{r^{2((1/2)-\operatorname{Re}(\mu_n))}} &< |\hat{\sigma}_n \zeta'^{\operatorname{Re}(\mu_n)-(1/2)}| \\
&< \frac{(bm\pi)^{\operatorname{Re}(\mu_n)-(1/2)}|\hat{\sigma}_n|}{r^{2((1/2)-\operatorname{Re}(\mu_n))}} \qquad \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g)
\end{aligned}
$$

for k sufficiently large. Therefore, we deduce from these that

$$
\begin{aligned}
\lim_{\substack{k\to+\infty \\ \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g)}} |\hat{\sigma}_n \zeta'^{\operatorname{Re}(\mu_n)-(1/2)}| &= 0, \\
\lim_{\substack{r\to+\infty \\ \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g)}} |\hat{\sigma}_n \zeta'^{\operatorname{Re}(\mu_n)-(1/2)}| &= 0.
\end{aligned}
$$

On the one hand, these limits show that inequality (6.15) implies, for $k \geqslant k_2$ for some sufficiently large positive integer k_2, that the inequalities (6.19) hold in this case. On the other hand, it follows from the relations (6.7), (6.14), (6.16) and (6.17) that the inequalities

$$
|\hat{Y}(\zeta') - G(\zeta')| \leqslant \begin{cases} K_{11}k^{\kappa_1} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\ell_1(g), \\ K_{12}r^{\kappa_2} & \text{if } \zeta' \in \mathrm{e}^{\mathrm{i}(\pi-\phi)}\Gamma_1(g) \end{cases} \tag{6.20}
$$

hold in this subcase B, where K_{11}, K_{12} are some positive constants,

$$
\kappa_1 = \max\{-b\gamma, b\gamma - 2, b + \kappa\} \quad \text{and} \quad \kappa_2 = \max\{-2b, 2b - 2, b + \kappa\}
$$

so that (6.19) and (6.20) imply the inequality

$$
|\hat{Y}(\zeta') - G(\zeta')| < |G(\zeta')|
$$

holds on the contour $\mathrm{e}^{\mathrm{i}(\pi-\phi)}\Omega(g, k)$ for all sufficiently large k. Hence, our desired inequality (6.6) still holds in this general case after we transform the ζ'-plane back to the ζ-plane.

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6.4. Case 2

Next, we suppose that $\mu_n = \frac{1}{2}$.

Unfortunately, the contour $\Omega(g, k)$ and the auxiliary function defined in (5.5) and (6.2), respectively, do not seem to apply in this case. This is because the zeros of (6.2) distribute evenly on a straight line parallel to the real axis, and hence the region enclosed by the contour $\Omega(g, k)$ can only contain *finitely many* such zeros for every positive integer k. We choose the alternative auxiliary function to be

$$\hat{g}(\zeta) = \hat{C}e^{i\zeta} + \hat{D}e^{-i\zeta} + \hat{\sigma}_n. \tag{6.21}$$

Without loss of generality, we continue to assume that $\hat{C} \neq 0$. It remains to construct a suitable contour that contains the zeros of (6.21), which are given by the following lemma.

LEMMA 6.2. *Let k be an integer.*

(i) *If $\hat{D} \neq 0$, then the zeros of (6.21) are given by*

$$\zeta_\pm(k) = 2k\pi + \theta_\pm + i\log|\Delta_\pm|, \tag{6.22}$$

where $\Delta_\pm^{-1}$ are solutions of the quadratic equation $\hat{C}x^2 + \hat{\sigma}_n x + \hat{D} = 0$, which are given by $\Delta_\pm^{-1} := (-\hat{\sigma}_n \pm \sqrt{\hat{\sigma}_n^2 - 4\hat{C}\cdot\hat{D}})/2\hat{C}$ and $\theta_\pm$ are the principal arguments of $\Delta_\pm^{-1}$.

(ii) *If $\hat{D} = 0$, then the zeros of* (6.21) *are given by*

$$\zeta_0(k) = 2k\pi + \theta_0 + i\log|\Delta_0|, \tag{6.23}$$

where $\Delta_0^{-1} := -\hat{\sigma}_n/\hat{C}$ and θ_0 is the principal argument of Δ_0^{-1}.

We omit its proof.

REMARK 6.3. We remark that none of the Δ_+, Δ_- or Δ_0 can be zero. Otherwise, $\hat{D}$ or $\hat{C}$ would be zero, which contradicts the assumption. Moreover, let L_+ and L_- be two horizontal straight lines on which the zeros of equation (6.22) fall, such that L_+ corresponds to the zeros of ζ_+ and L_- corresponds to the zeros of ζ_- in lemma 6.2(i). Similarly, we let L_0 denote the straight line representing the zeros of equation (6.23) in lemma 6.2(ii). Both L_0, L_+ and L_- are parallel to the real axis in the ζ-plane.

The construction of the contour is divided into different cases depending on whether the D vanishes. These are subcase A, $\hat{D} \neq 0$, and subcase B, $\hat{D} = 0$. Subcase A is further divided into

1. $|\Delta_+| \neq |\Delta_-|$,
2. $|\Delta_+| = |\Delta_-|$.

Subsubcase 2 is divided into

(i) $\theta_+ = \theta_-$,

(ii) $\theta_+ \neq \theta_-$.

Now we can start the construction of the contour.

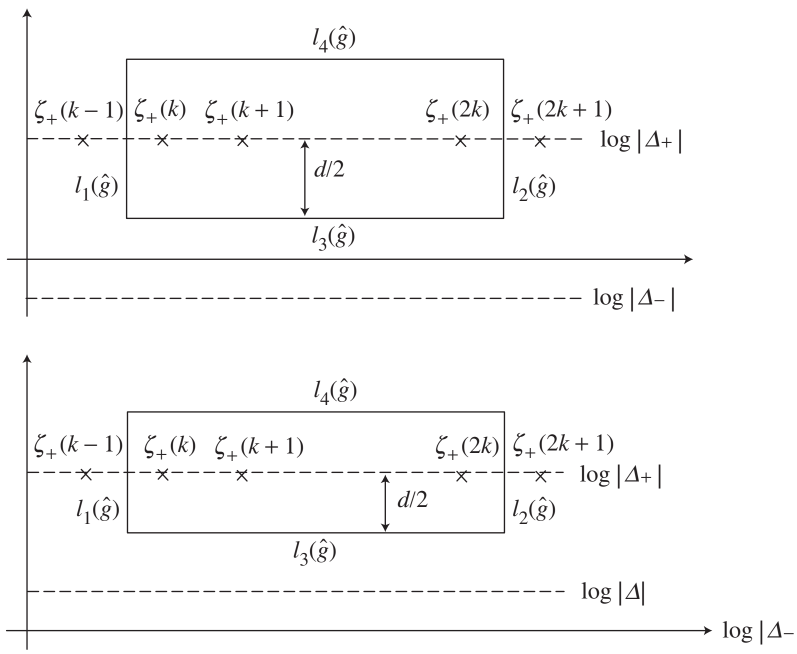

Figure 2. The contour $\Omega(\hat{g}, k)$ when $|\Delta_+| \neq |\Delta_-|$.

6.4.1. *Subcase A*

Suppose that $\hat{D} \neq 0$. By remark 6.3, we define a constant d as follows:

$$d := \begin{cases} |\log|\Delta_+| - \log|\Delta_-|| & \text{if } |\Delta_+| \neq |\Delta_-|, \\ 1 & \text{if } |\Delta_+| = |\Delta_-|. \end{cases} \tag{6.24}$$

It is easy to see from (6.24) that we must have $d > 0$. We distinguish two cases between subsubcase 1, $|\Delta_+| \neq |\Delta_-|$, and subsubcase 2, $|\Delta_+| = |\Delta_-|$.

1. If $|\Delta_+| \neq |\Delta_-|$, then we define, for each integer $k \geqslant k_0$ for some suitably large positive integer k_0, the line segments $\ell_1(\hat{g})$, $\ell_2(\hat{g})$, $\ell_3(\hat{g})$ and $\ell_4(\hat{g})$ as follows:

$$\left.\begin{aligned} \ell_1(\hat{g}) &:= \{(2k-1)\pi + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}dy)\colon\ -1 \leqslant y \leqslant 1\}, \\ \ell_2(\hat{g}) &:= \{(4k+1)\pi + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}dy)\colon\ -1 \leqslant y \leqslant 1\}, \\ \ell_3(\hat{g}) &:= \{x + \theta_+ + \mathrm{i}(\log|\Delta_+| - \tfrac{1}{2}d)\colon (2k-1)\pi \leqslant x \leqslant (4k+1)\pi\}, \\ \ell_4(\hat{g}) &:= \{x + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}d)\colon (2k-1)\pi \leqslant x \leqslant (4k+1)\pi\}. \end{aligned}\right\} \tag{6.25}$$

Then the line segments $\ell_j(\hat{g})$, $j = 1, 2, 3, 4$, are concatenated to form the rectangular contour $\Omega(\hat{g}, k)$. We also form the set $\Omega(\hat{g}) = \bigcup_{k=1}^{+\infty} \Omega(\hat{g}, k)$ (see figure 2).

Instead of inequality (6.6), we shall show that the inequality

$$|\hat{y}(\zeta) - \hat{g}(\zeta)| < |\hat{g}(\zeta)| \tag{6.26}$$

holds on $\Omega(\hat{g}, k)$ for all sufficiently large k.

On one hand, we note from lemma 6.2(i), remark 6.3 and definition (6.25) that, for every integer k, exactly $k+1$ distinct zeros of $\hat{g}(\zeta)$ lie *inside* the $\Omega(\hat{g}, k)$, but

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all ζ_- lie *outside* the $\Omega(\hat{g}, k)$ (see figure 2 for an illustration). Therefore, we must have the fact that $\hat{g}(\zeta)$ *does not* pass through any zero along the $\Omega(\hat{g}, k)$ for every positive integer k. In other words, there exists a positive constant $\Delta_1(k)$, depending only on k, such that the inequality

$$|\hat{g}(\zeta)| > \Delta_1(k) > 0 \tag{6.27}$$

holds on $\Omega(\hat{g}, k)$ for every positive integer k. To obtain the desired inequality (6.26), we must show that the constant $\Delta_1(k)$ can be chosen *independent* of k. To see this, we note that, for each positive integer k, we have

$$\mathrm{e}^{\pm \mathrm{i}\zeta} = \begin{cases} -|\Delta_+|^{\mp 1}\mathrm{e}^{\mp dy/2}\mathrm{e}^{\pm \mathrm{i}\theta_+} & \text{if } \zeta \in \ell_1(\hat{g}) \cup \ell_2(\hat{g}), \\ |\Delta_+|^{\mp 1}\mathrm{e}^{\pm d/2}\mathrm{e}^{\pm \mathrm{i}(x+\theta_+)} & \text{if } \zeta \in \ell_3(\hat{g}), \\ |\Delta_+|^{\mp 1}\mathrm{e}^{\mp d/2}\mathrm{e}^{\pm \mathrm{i}(x+\theta_+)} & \text{if } \zeta \in \ell_4(\hat{g}), \end{cases} \tag{6.28}$$

and so

$$\hat{g}(\zeta) = \begin{cases} -\hat{C}|\Delta_+|^{-1}\mathrm{e}^{-dy/2}\mathrm{e}^{\mathrm{i}\theta_+} - \hat{D}|\Delta_+|\mathrm{e}^{dy/2}\mathrm{e}^{-\mathrm{i}\theta_+} + \hat{\sigma}_n & \text{if } \zeta \in \ell_1(\hat{g}) \cup \ell_2(\hat{g}), \\ \hat{C}|\Delta_+|^{-1}\mathrm{e}^{d/2}\mathrm{e}^{\mathrm{i}(x+\theta_+)} + \hat{D}|\Delta_+|\mathrm{e}^{-d/2}\mathrm{e}^{-\mathrm{i}(x+\theta_+)} + \hat{\sigma}_n & \text{if } \zeta \in \ell_3(\hat{g}), \\ \hat{C}|\Delta_+|^{-1}\mathrm{e}^{-d/2}\mathrm{e}^{\mathrm{i}(x+\theta_+)} + \hat{D}|\Delta_+|\mathrm{e}^{d/2}\mathrm{e}^{-\mathrm{i}(x+\theta_+)} + \hat{\sigma}_n & \text{if } \zeta \in \ell_4(\hat{g}). \end{cases} \tag{6.29}$$

Hence, this implies that $\Delta_1(k)$ can be chosen *independent* of k. We denote this positive number by Δ_1. Thus, we have the inequality

$$|\hat{g}(\zeta)| > \Delta_1 > 0 \tag{6.30}$$

holds on $\Omega(\hat{g}, k)$ for every positive integer k.

On the other hand, it follows from (6.1) and the chosen function (6.21) that

$$\begin{aligned} &|\hat{y}(\zeta) - \hat{g}(\zeta)| \\ &\quad = \left|\hat{C}\mathrm{e}^{\mathrm{i}\zeta}O(\zeta^{-1}) + \hat{D}\mathrm{e}^{-\mathrm{i}\zeta}O(\zeta^{-1}) + O(\zeta^{-1}) + \sum_{j=1}^{n-1}\hat{\sigma}_j\zeta^{\mu_j-(1/2)}[1+O(\zeta^{-1})]\right| \\ &\quad \leqslant \frac{\Delta_2|\mathrm{e}^{\mathrm{i}\zeta}|}{|\zeta|} + \frac{\Delta_3|\mathrm{e}^{-\mathrm{i}\zeta}|}{|\zeta|} + O(|\zeta|^{-1}) + \sum_{j=1}^{n-1} O(|\zeta|^{\mathrm{Re}(\mu_j)-(1/2)}), \end{aligned} \tag{6.31}$$

where Δ_2 and Δ_3 are some fixed positive constants. Since $\mu_n = \frac{1}{2}$, the definition (6.5) implies that κ is negative and $\kappa \geqslant -1$, $\kappa \geqslant \mathrm{Re}(\mu_j) - \frac{1}{2}$ for all $1 \leqslant j \leqslant n-1$. Thus, the relations (6.28) and (6.31) imply that

$$\begin{aligned} |\hat{y}(\zeta) - \hat{g}(\zeta)| &\leqslant \frac{\Delta_2|\mathrm{e}^{\mathrm{i}\zeta}|}{|\zeta|} + \frac{\Delta_3|\mathrm{e}^{-\mathrm{i}\zeta}|}{|\zeta|} + O(|\zeta|^{\kappa}) \\ &< \frac{\Delta_4}{k\pi} + \frac{\Delta_5}{(k\pi)^{|\kappa|}}, \end{aligned} \tag{6.32}$$

holds on the contour $\Omega(\hat{g}, k)$, where Δ_4 and Δ_5 are two fixed positive constants independent of k. Hence, we obtain from (6.30) and (6.32) that the desired inequality (6.26) holds on $\Omega(\hat{g}, k)$ for all sufficiently large k.

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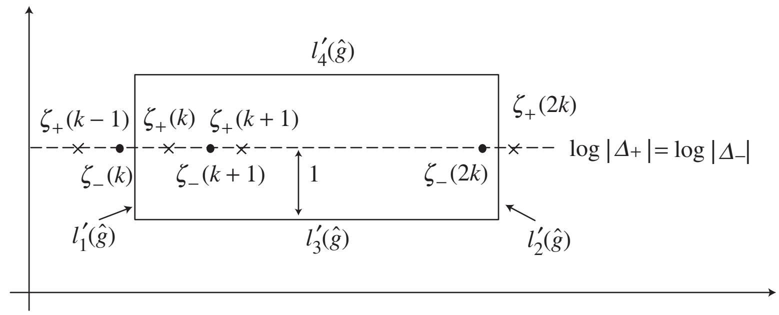

Figure 3. The modified contour $\Omega'(\hat{g}, k)$ when $|\Delta_+| = |\Delta_-|$.

2. If $|\Delta_+| = |\Delta_-|$, then the definition (6.24) gives $d = 1$. In addition, all the $\zeta_+(k)$ and $\zeta_-(k)$ lie on the same straight line $L = L_+ = L_-$ (see remark 6.3) so that, for every integer k, we have $|\zeta_+(k) - \zeta_-(k)| = |\theta_+ - \theta_-|$. Thus, there are two possibilities, (i) $\theta_+ = \theta_-$ and (ii) $\theta_+ \neq \theta_-$.

(i) If $\theta_+ = \theta_-$, then $\zeta_+(k) = \zeta_-(k)$ for every integer k. Hence, the above contour $\Omega(\hat{g}, k)$ and the argument leading to the inequality (6.26) can be applied without any change.

(ii) If $\theta_+ \neq \theta_-$, then it may happen that $\ell_1(\hat{g})$ or $\ell_2(\hat{g})$ passes through the zeros $\zeta_-(k-1)$, $\zeta_-(k)$, $\zeta_-(2k)$ or $\zeta_-(2k+1)$, so we need to modify the contour $\Omega(\hat{g}, k)$ defined in (6.25). In fact, we can replace $(2k-1)\pi$ and $(4k+1)\pi$ by $2k\pi - \frac{1}{2}(\theta_+ - \theta_-)$ and $4k\pi - \frac{1}{2}(\theta_+ - \theta_-)$, respectively, in definitions (6.25). We then denote the modified line segments by $\ell_1'(\hat{g})$, $\ell_2'(\hat{g})$, $\ell_3'(\hat{g})$ and $\ell_4'(\hat{g})$, respectively:

$$\begin{aligned}
\ell_1'(\hat{g}) &:= \{2k\pi - \tfrac{1}{2}(\theta_+ - \theta_-) + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}dy)\colon -1 \leqslant y \leqslant 1\},\\
\ell_2'(\hat{g}) &:= \{4k\pi - \tfrac{1}{2}(\theta_+ - \theta_-) + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}dy)\colon -1 \leqslant y \leqslant 1\},\\
\ell_3'(\hat{g}) &:= \{x + \theta_+ + \mathrm{i}(\log|\Delta_+| - \tfrac{1}{2}d)\colon 2k\pi \\
&\qquad\qquad - \tfrac{1}{2}(\theta_+ - \theta_-) \leqslant x \leqslant 4k\pi - \tfrac{1}{2}(\theta_+ - \theta_-)\},\\
\ell_4'(\hat{g}) &:= \{x + \theta_+ + \mathrm{i}(\log|\Delta_+| + \tfrac{1}{2}d)\colon 2k\pi \\
&\qquad\qquad - \tfrac{1}{2}(\theta_+ - \theta_-) \leqslant x \leqslant 4k\pi - \tfrac{1}{2}(\theta_+ - \theta_-)\}.
\end{aligned}$$

Then the contour and the infinite strip are defined similarly and denoted by $\Omega'(\hat{g}, k)$ and $\Omega'(\hat{g})$, respectively (see figure 3).

Since $d = 1$ in (6.24), the relations (6.28) and (6.29) are replaced by

$$\mathrm{e}^{\pm\mathrm{i}\zeta} = \begin{cases} |\Delta_+|^{\mp 1}\mathrm{e}^{\mp y/2}\mathrm{e}^{\pm\mathrm{i}(\theta_+ + \theta_-/2)/2} & \text{if } \zeta \in \ell_1'(\hat{g}) \cup \ell_2'(\hat{g}),\\ |\Delta_+|^{\mp 1}\mathrm{e}^{\pm 1/2}\mathrm{e}^{\pm\mathrm{i}(x+\theta_+)} & \text{if } \zeta \in \ell_3'(\hat{g}),\\ |\Delta_+|^{\mp 1}\mathrm{e}^{\mp 1/2}\mathrm{e}^{\pm\mathrm{i}(x+\theta_+)} & \text{if } \zeta \in \ell_4'(\hat{g}),\end{cases}$$

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and

$$\hat{g}(\zeta) = \begin{cases} \hat{C}|\Delta_+|^{-1}\mathrm{e}^{-y/2}\mathrm{e}^{\mathrm{i}(\theta_+ + \theta_-)/2} \\ \qquad +\hat{D}|\Delta_+|\mathrm{e}^{y/2}\mathrm{e}^{-\mathrm{i}(\theta_+ + \theta_-)/2} + \hat{\sigma}_n & \text{if } \zeta \in \ell_1'(\hat{g}) \cup \ell_2'(\hat{g}), \\ \hat{C}|\Delta_+|^{-1}\mathrm{e}^{1/2}\mathrm{e}^{\mathrm{i}(x+\theta_+)} \\ \qquad +\hat{D}|\Delta_+|\mathrm{e}^{-1/2}\mathrm{e}^{-\mathrm{i}(x+\theta_+)} + \hat{\sigma}_n & \text{if } \zeta \in \ell_3'(\hat{g}), \\ \hat{C}|\Delta_+|^{-1}\mathrm{e}^{-1/2}\mathrm{e}^{\mathrm{i}(x+\theta_+)} \\ \qquad +\hat{D}|\Delta_+|\mathrm{e}^{1/2}\mathrm{e}^{-\mathrm{i}(x+\theta_+)} + \hat{\sigma}_n & \text{if } \zeta \in \ell_4'(\hat{g}), \end{cases}$$

respectively. Thus, inequalities (6.30)–(6.32) can be similarly deduced with a possibly different set of the positive constants $\Delta_1, \Delta_2, \dots, \Delta_5$.

REMARK 6.4. It is trivial to check that there are exactly $2k+1$ distinct zeros inside the modified contour $\Omega'(\hat{g}, k)$ for every positive integer k.

6.4.2. *Subcase B*

Suppose that $\hat{D} = 0$. Then it is easy to see that this can be regarded as the degenerated case in subcase A(2)(i) with the constant $|\Delta_+|$ and the straight line L_+ replaced by $|\Delta_0|$ and L_0, respectively.

We now continue the proof of theorem 6.1.

So Rouché's theorem implies that the functions $\hat{y}(\zeta)$ and $g(\zeta)$ (respectively, $\hat{g}(\zeta)$) have the same number of zeros inside $\Omega(g, k)$ (respectively, $\Omega(\hat{g}, k)$ or $\Omega'(\hat{g}, k)$). Proposition 5.3 (respectively, lemma 6.2) asserts that $g(\zeta)$ (respectively, $\hat{g}(\zeta)$), and hence $\hat{y}(\zeta)$, has *infinitely many distinct zeros* inside $\Omega(g)$ (respectively, $\Omega(\hat{g})$ or $\Omega'(\hat{g})$). Let $\mathfrak{n}(D, f)$ denote the number of zeros of the function $f(z)$ inside the set D. Then, for any given $0 < \varepsilon < 1$, there exists an infinite sequence $\{\zeta_n\}$ of zeros of $\hat{y}(\zeta)$, and hence of $y(\zeta)$, with $|\zeta_n| = \rho_n$ inside $\Omega(g)$ (respectively, $\Omega(\hat{g})$ or $\Omega'(\hat{g})$) such that

$$\mathfrak{n}(\Omega(g), y(\zeta)) \geqslant \rho_n^{1-\varepsilon}$$
$$(\text{respectively, } \mathfrak{n}(\Omega(\hat{g}), y(\zeta)) \geqslant \rho_n^{1-\varepsilon} \text{ or } \mathfrak{n}(\Omega'(\hat{g}), y(\zeta)) \geqslant \rho_n^{1-\varepsilon})$$

for all sufficiently large n. By the substitution $L\mathrm{e}^{Mz} = \zeta$, where $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $\zeta = \rho\mathrm{e}^{\mathrm{i}\varphi}$. Then, for choosing r_n and θ_n such that $r_n \to +\infty$ as $n \to +\infty$ and $\theta_n + b = 0$ for all positive integers n, where b is the principal argument of M, we must have $\rho_n = |L|\mathrm{e}^{|M|r_n} \to +\infty$ as $n \to +\infty$, and then

$$\begin{aligned} &\frac{\log \mathfrak{n}(\{z\colon |z| \leqslant (1/|M|)\log \rho_n\}, f(z))}{\log r_n} \\ &\qquad \geqslant \frac{\log \mathfrak{n}(\{\zeta\colon (|L|/\rho_n) \leqslant |\zeta| \leqslant |L|\rho_n, \arg\zeta \neq \pi\}, y(\zeta))}{\log r_n} \\ &\qquad \geqslant \frac{\log(\rho_n)^{1-\epsilon}}{\log r_n} \\ &\qquad = \frac{(1-\epsilon)\log \rho_n}{\log\log \rho_n} \\ &\qquad \to +\infty \end{aligned}$$

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as $n \to +\infty$, which implies that $\lambda(f) = +\infty$, thus completing the proof of the theorem. □

We can now continue the proof of theorem 1.3.

Recall that $f(z) = \mathrm{e}^{-Nz}y(L\mathrm{e}^{Mz})$, where $y(\zeta)$ is a solution to equation (1.14). So the requirement $\lambda(f) < +\infty$ is *independent of the branches* of the function $y(\zeta)$. It follows from theorem 6.1 that we must have $C = D = 0$ and, hence, so are $A = B = 0$. Hence, the solution (1.9) is expressed in the form

$$f(z) = \mathrm{e}^{-Nz}\sum_{j=1}^{n}\sigma_j S_{\mu_j,\nu}(L\mathrm{e}^{Mz}). \tag{6.33}$$

To complete the proof of theorem 1.3, we need to prove that when σ_j is non-zero, μ_j and ν must satisfy either

$$\cos(\tfrac{1}{2}(\mu_j+\nu)\pi) = 0 \quad \text{or} \quad 1 + \mathrm{e}^{(-\mu_j+\nu)\pi\mathrm{i}} = 0, \tag{6.34}$$

where $j \in \{1, 2, \dots, n\}$. Following [10, p. 145], we have from remark 1.6 that $S_{\mu_1,\nu}(L\mathrm{e}^{Mz}), S_{\mu_2,\nu}(L\mathrm{e}^{Mz}), \dots, S_{\mu_n,\nu}(L\mathrm{e}^{Mz})$ are entire functions in the z-plane and that each $S_{\mu_j,\nu}(L\mathrm{e}^{Mz})$, $j = 1, 2, \dots, n$, is *independent of the branches* of $S_{\mu_j,\nu}(\zeta)$. We choose for a $j \in \{1, 2, \dots, n\}$ such that $\sigma_j \neq 0$. So we can rewrite the solution (6.33) as

$$f(z) = \sigma_j\mathrm{e}^{-Nz}S_{\mu_j,\nu}(L\mathrm{e}^{Mz}\mathrm{e}^{-m\pi\mathrm{i}}) + \sum_{k=1,\,k\neq j}^{n}\sigma_k\mathrm{e}^{-Nz}S_{\mu_k,\nu}(L\mathrm{e}^{Mz}), \tag{6.35}$$

where the function $S_{\mu_j,\nu}(\zeta)$ belongs to the branch $-(m+1)\pi < \arg\zeta < -(m-1)\pi$ and the other Lommel functions $S_{\mu_1,\nu}(\zeta), \dots, S_{\mu_{j-1},\nu}(\zeta), S_{\mu_{j+1},\nu}(\zeta), \dots, S_{\mu_n,\nu}(\zeta)$ are in the principal branch $-\pi < \arg\zeta < \pi$ and m is an arbitrary but otherwise fixed non-zero integer.

Remark 6.5. We note again that, in the following discussion, we only consider the case where $\mu_j - \nu = -2p_j - 1$. The other case $\mu_j + \nu = -2p_j - 1$ can be dealt with similarly by applying the property that each $S_{\mu_j,\nu}(\zeta)$ is an even function of ν.

Suppose that $\mu_j - \nu = -2p_j - 1$ for some non-negative integer $p_j = p$. If $-\nu \notin \{0, 1, 2, \dots\}$ or $\nu = 0$, then it follows from lemma 3.2(i) and (ii) that the solution (6.35) can be expressed in the form (2.3) with

$$C = \begin{cases} \dfrac{\sigma_j(-1)^pK'_+}{2^{2p}p!(1-\nu)_p} & \text{if } -\nu \notin \{0,1,2,\dots\}, \\[2ex] \dfrac{\sigma_j(-1)^pK''_+}{2^{2p}(p!)^2} & \text{if } \nu = 0, \end{cases}$$

$$D = \begin{cases} \dfrac{\sigma_j(-1)^pK'_-}{2^{2p}p!(1-\nu)_p} & \text{if } -\nu \notin \{0,1,2,\dots\}, \\[2ex] \dfrac{\sigma_j(-1)^pK''_-}{2^{2p}(p!)^2} & \text{if } \nu = 0, \end{cases}$$

where $K'_\pm$ and $K''_\pm$ are the constants defined in (3.1). In order to apply theorem 6.1, we may closely follow the argument used in [10, proposition 4.4(i), (ii)], where if we have $C \neq 0$ or $D \neq 0$ for any integer m, then we will obtain a contradiction to the free choice of the integer m. Hence, $\lambda(f) = +\infty$ and then either C or D must be zero. This implies that (6.34) holds, as required.

If $\nu = -n$ for a positive integer n, then it follows from lemma 3.2(iii) that the solution (6.35) (with m replaced by $2m$) is given by

$$\begin{aligned} f(z) = {} & \frac{(-1)^{n+p}\sigma_j \mathrm{e}^{-Nz}}{2^{2p+n} n!(p!)^2(1+n)_p}(L\mathrm{e}^{Mz})^{-n} \\ & \times \{B_n(L\mathrm{e}^{Mz})[K''_+ H_0^{(1)}(L\mathrm{e}^{Mz}) + K''_- H_0^{(2)}(L\mathrm{e}^{Mz})] \\ & \quad - (L\mathrm{e}^{Mz})C_n(L\mathrm{e}^{Mz})[K''_+ H_1^{(1)}(L\mathrm{e}^{Mz}) + K''_- H_1^{(2)}(L\mathrm{e}^{Mz})]\} \\ & + \sum_{j=1}^{n} \sigma_j \mathrm{e}^{-Nz} S_{\mu_j,\nu}(L\mathrm{e}^{Mz}). \end{aligned}$$

It is obvious that the above expression is *not* in the form (2.3), so theorem 6.1 does not apply in this case. In order to find an alternative approach to show $\lambda(f) = +\infty$, we show that the function $h(\zeta)$ defined by

$$\begin{aligned} h(\zeta) := \zeta^{-n}\{ & B_n(\zeta)[K''_+ H_0^{(1)}(\zeta) + K''_- H_0^{(2)}(\zeta)] \\ & - \zeta C_n(\zeta)[K''_+ H_1^{(1)}(\zeta) + K''_- H_1^{(2)}(\zeta)]\} + \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(\zeta) \end{aligned} \tag{6.36}$$

has *infinitely many* zeros in the principal branch of $H_0^{(1)}(\zeta)$, $H_0^{(2)}(\zeta)$, $H_1^{(1)}(\zeta)$, $H_1^{(2)}(\zeta)$ and $S_{\mu_j,\nu}(\zeta)$. Therefore, we suppose that $-\pi < \arg\zeta < \pi$. Then the asymptotic expansions (A 4)–(A 6), setting $p = 1$, yield

$$\begin{aligned} \hat{h}(\zeta) &:= (\tfrac{1}{2}\pi\zeta)^{1/2} h(\zeta) \\ &= \zeta^{-n}[D_n^+(\zeta)K''_+ \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\zeta} + D_n^-(\zeta)K''_- \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\zeta} \\ &\qquad + D_n^+(\zeta)K''_+ \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\zeta}O(\zeta^{-1}) + D_n^-(\zeta)K''_- \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\zeta}O(\zeta^{-1})] \\ &\quad + \sum_{j=1}^{n} \hat{\sigma}_j \zeta^{\mu_j-(1/2)}[1 + O(\zeta^{-1})], \end{aligned} \tag{6.37}$$

where $D_n^\pm(\zeta) = B_n(\zeta) \pm \mathrm{i}\zeta C_n(\zeta)$. To find the number of zeros of $h(\zeta)$ in $-\pi < \arg\zeta < \pi$, we need the following result.

LEMMA 6.6. *Suppose that n is a positive integer. Then at least one of $D_n^+(\zeta)$ or $D_n^-(\zeta)$ has degree n.*

Proof of lemma 6.6. We let

$$B_n(\zeta) := \sum_{j=0}^{n} \beta_j \zeta^j \quad \text{and} \quad C_n(\zeta) := \sum_{j=0}^{n} \gamma_j \zeta^j,$$

where $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ are complex constants. We prove the lemma by induction on n. When $n = 1$, we have $D_1^\pm(\zeta) = \pm\mathrm{i}\zeta$, so the statement is true. Assume that it is also true when $n = k$ for a positive integer k. Without loss of generality, we may assume that $\deg D_k^+(\zeta) = k$, so that

$$\beta_k + \mathrm{i}\gamma_k \neq 0. \tag{6.38}$$

When $n = k + 1$, it follows from the recurrence relations (3.4) for $B_n(\zeta)$ and $C_n(\zeta)$ that

$$D_{k+1}^{\pm}(\zeta) = -2kD_k^{\pm}(\zeta) + \zeta B_k'(\zeta) \pm \mathrm{i}\zeta^2 C_k'(\zeta) \pm \mathrm{i}\zeta D_k^{\pm}(\zeta).$$

It is easy to check that the coefficients of the ζ^{k+1} in $D_k^{\pm}(\zeta)$ are given by $\pm\mathrm{i}k\gamma_k \pm \mathrm{i}(\beta_k \pm \mathrm{i}\gamma_k)$, respectively. If $\deg D_{k+1}^{\pm}(\zeta) \leqslant k$, then we have $\pm\mathrm{i}k\gamma_k \pm \mathrm{i}(\beta_k \pm \mathrm{i}\gamma_k) = 0$ so that both γ_k and β_k are zero, which certainly contradicts our inductive assumption (6.38). Hence, we must have $\deg D_{k+1}^{+}(\zeta) = k+1$ or $\deg D_{k+1}^{-}(\zeta) = k+1$, completing the proof of the lemma. □

We can now complete the proof of the theorem. We recall that we have assumed $\mu_j - \nu = -2p_j - 1$ for some non-negative integer $p_j = p$ and $\nu = -n$ for a positive integer n, see the paragraphs following remark 6.5. By lemma 6.6, we may suppose that $D_n^{\pm}(\zeta) = C_{\pm}\zeta^n + \cdots$ and $\hat{C}_{\pm} = C_{\pm}\mathrm{e}^{\mp\mathrm{i}\pi/4}$, where $C_+ \neq 0$. Then the expression (6.37) induces

$$\begin{aligned}\hat{h}(\zeta) = \hat{C}_+ K_+'' \mathrm{e}^{\mathrm{i}\zeta}[1 + O(\zeta^{-1})] + \hat{C}_- K_-'' \mathrm{e}^{-\mathrm{i}\zeta}[1 + O(\zeta^{-1})] \\ + \sum_{j=1}^{n} \hat{\sigma}_j \zeta^{\mu_j - (1/2)}[1 + O(\zeta^{-1})],\end{aligned}$$

which is in the form (6.1) with $\hat{C}$ and $\hat{D}$ replaced by $\hat{C}_+K_+''$ and $\hat{C}_-K_-''$, respectively. Therefore, the proof of theorem 6.1 can be applied without change to show that if at least one of $\hat{C}_+K_+'' \neq 0$ or $\hat{C}_-K_-'' \neq 0$, then the function $h(\zeta)$ has infinitely many zeros in $-\pi < \arg\zeta < \pi$ and thus $\lambda(f) = +\infty$, which is a contradiction. Hence, we conclude that $\mu_j - \nu$ cannot be an odd negative integer.

Now we can apply the analytic continuation formula in lemma 3.1 with this fixed integer m to obtain

$$\begin{aligned}S_{\mu_j,\nu}(L\mathrm{e}^{Mz}\mathrm{e}^{-m\pi\mathrm{i}}) = K_+ P_m(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) H_\nu^{(1)}(L\mathrm{e}^{Mz}) \\ + K_+\mathrm{e}^{-\nu\pi\mathrm{i}} P_{m-1}(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) H_\nu^{(2)}(L\mathrm{e}^{Mz}) \\ + (-1)^m \mathrm{e}^{-m\mu_j\pi\mathrm{i}} S_{\mu_j,\nu}(L\mathrm{e}^{Mz}),\end{aligned} \tag{6.39}$$

where $P_m(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}})$ is the polynomial as defined in lemma 3.1. Then the expressions (6.35) and (6.39) give

$$\begin{aligned}f(z) = K_+\sigma_j\mathrm{e}^{-Nz} P_m(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) H_\nu^{(1)}(L\mathrm{e}^{Mz}) \\ + K_+\sigma_j\mathrm{e}^{-Nz}\mathrm{e}^{-\nu\pi\mathrm{i}} P_{m-1}(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) H_\nu^{(2)}(L\mathrm{e}^{Mz}) \\ + (-1)^m\sigma_j\mathrm{e}^{-m\mu_j\pi\mathrm{i}-Nz} S_{\mu_j,\nu}(L\mathrm{e}^{Mz}) + \sum_{k=1,\, k\neq j}^{n} \sigma_k\mathrm{e}^{-Nz} S_{\mu_k,\nu}(L\mathrm{e}^{Mz}).\end{aligned} \tag{6.40}$$

If either of the coefficients of $H_\nu^{(1)}(L\mathrm{e}^{Mz})$ and $H_\nu^{(2)}(L\mathrm{e}^{Mz})$ in (6.40) is non-zero, then theorem 6.1 again implies that $\lambda(f) = +\infty$, which is impossible. Thus, we must have

$$K_+ P_m(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) = 0 = K_+ \mathrm{e}^{-\nu\pi\mathrm{i}} P_{m-1}(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}). \tag{6.41}$$

Now we are ready to derive equations (6.34), we again recall that the value of $\lambda(S_{\mu_j,\nu}(L\mathrm{e}^{Mz}))$ must be independent of branches of the function $S_{\mu_j,\nu}(\zeta)$, which is equivalent to equations (6.41) holding for *each* integer m. It is clear from part (ii) of lemma 3.1 that $P_m(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) \equiv 0$ and $P_{m-1}(\cos\nu\pi, \mathrm{e}^{-\mu_j\pi\mathrm{i}}) \equiv 0$ do not hold simultaneously for any integer m. Thus, $K_+ = 0$ must hold, i.e. when $\sigma_j \neq 0$,

$$\cos(\tfrac{1}{2}(\mu_j + \nu)\pi) = 0 \quad \text{or} \quad 1 + \mathrm{e}^{(-\mu_j+\nu)\pi\mathrm{i}} = 0, \tag{6.34}$$

where $j \in \{1, 2, \dots, n\}$. Hence, we deduce the first and the second conditions in (1.10) from the first and the second equations in (6.34), respectively. A detailed deduction can be found in [10, pp. 154, 155]. This completes the proof of the theorem.

7. Proof of corollary 1.4

If $n = 1$, then the assumption gives $\sigma_1 \neq 0$ so that $F(\zeta) = \sigma_1 S_{\mu_1,\nu}(\zeta) \not\equiv 0$. If $n \geqslant 2$, then it follows from lemma A.2 that $F(\zeta) \not\equiv 0$. Thus, the function $F(\zeta)$ as defined in (1.13) is non-trivial so that we may suppose that the function (1.13) has finitely many zeros in every branch of ζ. Then the entire function

$$f(z) = F(\mathrm{e}^z) = \sum_{j=1}^{n} \sigma_j S_{\mu_j,\nu}(\mathrm{e}^z)$$

is certainly a solution of equation (1.8) with $L = M = 1$ and $\lambda(f) < +\infty$. Hence, theorem 1.3 implies that either $\mu_j + \nu = 2p_j + 1$ or $\mu_j - \nu = 2p_j + 1$ for non-negative integers p_j, where $j = 1, 2, \dots, n$.

Conversely, if either $\mu_j + \nu = 2p_j + 1$ or $\mu_j - \nu = 2p_j + 1$ for non-negative integers p_j, where $j = 1, 2, \dots, n$, then remark A.1 shows that each $S_{\mu_j,\nu}(\zeta)/\zeta^{\mu_j-1}$ is a polynomial in $1/\zeta$ so that $S_{\mu_j,\nu}(\zeta)$ has only finitely many zeros in every branch of ζ, where $j = 1, 2, \dots, n$. Thus, this implies that the function (1.13) has finitely many zeros in every branch of ζ. This completes the proof of corollary 1.4.

8. Non-homogeneous function-theoretic quantization-type results

The explicit representation and the zeros distribution of an entire solution $f(z)$ of either the equation

$$y'' + \mathrm{e}^z y = Ky \tag{8.1}$$

or the equation

$$y'' + (-\tfrac{1}{4}\mathrm{e}^{-2z} + \tfrac{1}{2}\mathrm{e}^{-z})y = Ky \tag{8.2}$$

were studied in [1,3] (see also [23]). Later, these results were strengthened by [8]. In fact, they discovered that the solutions of (8.1) and (8.2) can be solved in terms of Bessel functions and Coulomb wave functions, respectively. Besides, they identified

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Table 1. *Special cases of (8.3).*

	Cases	Corresponding K	Solutions with finite exponent of convergence of zeros
(1)	$\mu = 1$	p^2	$2\sigma e^{z/2} O_{2p}(2e^{z/2})$
(2)	$\mu = 0$	$\frac{1}{4}(2p+1)^2$	$\dfrac{2\sigma}{2p+1} e^{z/2} O_{2p+1}(2e^{z/2})$
(3)	$\mu = -1$	$(p+1)^2$	$\dfrac{\sigma}{4(p+1)} S_{2p+2}(2e^{z/2})$
(4)	$\mu = \nu$	$\frac{1}{16}(2p+1)^2$	$\sigma 2^{p-(1/2)}\sqrt{\pi}p![\boldsymbol{H}_{p+(1/2)}(2e^{z/2}) - Y_{p+(1/2)}(2e^{z/2})]$

that two classes of classical orthogonal polynomials (Bessel and generalized Bessel polynomials, respectively) appeared in the explicit representation of solutions under the *boundary condition* that the exponent of convergence of the zeros of the solution $f(z)$ is finite, i.e.

$$\lambda(f) = \lim_{r\to+\infty} \frac{\log n(r, (1/f))}{\log r} < +\infty.$$

This also results in a complete determination of the eigenvalues and eigenfunctions of the equations. We call such a phenomenon a *function-theoretic quantization result* for the differential equations (8.1) and (8.2).

It is also well known that both equations have important physical applications. For example, equation (8.1) is derived as a reduction of a nonlinear Schrödinger equation in a recent study of the Benjamin–Feir instability phenomenon in deep water [22], while the second equation (8.2) is an exceptional case of a standard classical diatomic model in quantum mechanics introduced in [21][5] and is a basic model in recent $\mathcal{PT}$-symmetric quantum mechanics research [31] (see also [6]).

In [10, theorem 6.1], the following differential equation was considered:

$$f'' + (e^z - K)f = \sigma 2^{\mu-1} e^{(\mu+1)z/2}, \tag{8.3}$$

which is a special case of equation (1.8) when $L = 2$, $M = \frac{1}{2}$, $N = 0$ and $n = 1$ in theorem 1.3, where $K = \frac{1}{4}\nu^2$. They obtained the necessary and sufficient condition on K so that equation (8.3) admits subnormal solutions that are related to classical polynomials and/or functions, i.e. Neumann's polynomials, Gegenbauer's generalization of Neumann's polynomials, Schläfli's polynomials and Struve's functions. This exhibits a kind of function-theoretic quantization phenomenon for non-homogeneous equations.

Now the following result holds trivially by our main theorem 1.3.

Theorem 8.1. *With each choice of parameters as indicated in table 1, we have a necessary and sufficient condition on K that depends on the non-negative integer p so that equation (8.3) admits a solution with finite exponent of convergence of zeros. Furthermore, the forms of such solutions are given explicitly in table 1.*

[5] See [24, pp. 1–4] for a historical background of the Morse potential.

Here $O_{2p}(\zeta)$ and $O_{2p+1}(\zeta)$ are the Neumann polynomials of degrees $2p$ and $2p+1$, respectively; $S_p(\zeta)$ is the Schläfli polynomial and $\boldsymbol{H}_{p+(1/2)}(\zeta)$ is the Struve function [27, §§ 9.1, 9.3, 10.4].

Appendix A. Preliminaries on the Bessel and Lommel functions

A.1. Bessel functions

Let m be an integer. Here we record the following analytic continuation formulae for the Bessel functions [27, 3.62]:

$$J_\nu(\zeta e^{m\pi i}) = e^{m\nu\pi i} J_\nu(\zeta), \tag{A 1}$$

$$Y_\nu(\zeta e^{m\pi i}) = e^{-m\nu\pi i} Y_\nu(\zeta) + 2i\sin(m\nu\pi)\cot(\nu\pi) J_\nu(\zeta). \tag{A 2}$$

We recall that the *Bessel functions of the third kind of order* ν [27, § 3.6] are given by

$$H_\nu^{(1)}(\zeta) = J_\nu(\zeta) + iY_\nu(\zeta), \qquad H_\nu^{(2)}(\zeta) = J_\nu(\zeta) - iY_\nu(\zeta). \tag{A 3}$$

They are also-called the *Hankel functions of order* ν *of the first and second kinds*. The asymptotic expansions of $H_\nu^{(1)}(\zeta)$ and $H_\nu^{(2)}(\zeta)$ are also recorded as follows:

$$(\tfrac{1}{2}\pi\zeta)^{1/2} H_\nu^{(1)}(\zeta) = \exp\{i(\zeta - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\}\biggl[\sum_{k=0}^{p-1} \frac{(\frac{1}{2}-\nu)_k(\frac{1}{2}+\nu)_k}{k!(2i\zeta)^k} + R_p^{(1)}(\zeta)\biggr], \tag{A 4}$$

where $R_p^{(1)}(\zeta) = O(\zeta^{-p})$ in $-\pi < \arg\zeta < 2\pi$;

$$(\tfrac{1}{2}\pi\zeta)^{1/2} H_\nu^{(2)}(\zeta) = \exp\{-i(\zeta - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\}\biggl[\sum_{k=0}^{p-1} \frac{(\frac{1}{2}-\nu)_k(\frac{1}{2}+\nu)_k}{k!(-2i\zeta)^k} + R_p^{(2)}(\zeta)\biggr], \tag{A 5}$$

where $R_p^{(2)}(\zeta) = O(\zeta^{-p})$ in $-2\pi < \arg\zeta < \pi$ (see [27, § 7.2]).

A.2. An asymptotic expansion of $S_{\mu,\nu}(\zeta)$ and linear independence of Lommel's functions

It is known that when $\mu \pm \nu$ are not odd positive integers, then $S_{\mu,\nu}(\zeta)$ has the asymptotic expansion

$$S_{\mu,\nu}(\zeta) = \zeta^{\mu-1}\biggl[\sum_{k=0}^{p-1} \frac{(-1)^k c_k}{\zeta^{2k}}\biggr] + O(\zeta^{\mu-2p}) \tag{A 6}$$

for large $|\zeta|$ and $|\arg\zeta| < \pi$, where p is a positive integer (see also [27, § 10.75]). As a result, we see that the asymptotic expansions (A 4)–(A 6) are valid *simultaneously* in the range $-\pi < \arg\zeta < \pi$.

Remark A.1. It is clear that (A 6) is a series in descending powers of ζ starting from the term $\zeta^{\mu-1}$ and that (A 6) terminates if one of the numbers $\mu \pm \nu$ is an odd positive integer. In particular, if $\mu - \nu = 2p + 1$ for some non-negative integer p, then we have $K_+ = 0$ in the analytic continuation formula (3.2) and thus, in this degenerate case, formula (3.2) becomes

$$S_{2p+1+\nu,\nu}(\zeta e^{-m\pi i}) = e^{-m\nu\pi i} S_{2p+1+\nu,\nu}(\zeta)$$

for every integer m and $|\arg\zeta| < \pi$.

The following lemma concerns the linear independence of the Lommel functions $S_{\mu_j,\nu}(\zeta)$.

Lemma A.2 (Chiang and Yu [10, lemma 3.12]). *Suppose $n \geqslant 2$, and μ_j and ν to be complex numbers such that* $\mathrm{Re}(\mu_j)$ *are all distinct for $j = 1, 2, \dots, n$. Then the Lommel functions $S_{\mu_1,\nu}(\zeta), S_{\mu_2,\nu}(\zeta), \dots, S_{\mu_n,\nu}(\zeta)$ are linearly independent.*

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