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MD SURVEY

FOLIATIONS LEAF THROUGH ECONOMICS

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Foliations provide a general, convenient, geometric way to catalogue information from topics as varied as sufficient statistics, solutions of differential equations, indifference curves for utility functions, distributed computing, and so forth. An introduction to aspects of this area is provided.

Keywords: Foliations, Indifference Sets, Utility Functions, Sufficient Statistics, Apportionment, Message Systems, Excess Demand

1. INTRODUCTION

Although the technical term *foliations* may be unfamiliar to many, this widely used concept is employed almost on a daily basis in economics. Indeed, foliations are used whenever indifference sets for preferences are drawn on a blackboard or the effects of different economic indicators are discussed. They are so common that it is fair to assert that after one learns what to look for, foliations can be found almost everywhere. In part, this is because important special cases of foliations are the level sets of smooth mappings. Thus a convenient geometric way to envision a foliation is with the pages of a folded soft-cover book; each page is a level set, or *leaf*, of the foliation, and the book, the collection of all leafs, is the foliation.

The indifference-sets-of-preferences illustration accurately suggests that one use of foliations is to conveniently catalogue information. Each leaf serves as an equivalence set of information or data that is of equal value relative to a specified objective or goal; for example, this goal could be captured by an agent's preferences or by a specified set of economic indicators. Different leafs, then, describe different equivalence classes of information. This description motivates the following examples:

- The level sets commonly used to express individual preferences. Each indifference set is a leaf; it consists of all alternatives, commodity bundles, etc. that the agent views as being equivalent relative to her preferences. The foliation is the collection of all leafs—of all indifference sets. Changes in leafs correspond to changes in preference levels.

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- All states of an economy corresponding to the same specified index levels define a leaf. Here, relative to the indices, the different states in a leaf provide an equivalent amount of information. The set of all leafs—all possible states of the economy—defines the foliation. Changing from one leaf to another represents changing at least one index level.
- In statistics, all data that determine the same value for a sufficient statistic define a leaf. All possible leafs—all possible data sets—define the foliation. Each leaf consists of all data sets that identify particular parameter values for the sufficient statistic. Consider, for example, the problem of determining the likelihood of "heads" when spinning a penny on edge. Each leaf consists of a particular number of heads obtained by spinning the coin a specified number of times; the nature of runs and the order in which heads occurred are not relevant.
- All posterior distributions obtained by a Bayesian updating of a specified kind of prior defines a leaf. All possible leafs define a foliation.
- The trajectory of an autonomous differential equation as determined by a specified initial condition is a leaf. This defines all states that can be attained from the dynamic and a specified initial condition by going either forward or backward in time. The set of all leafs—all possible solutions—forms the foliation.

The dynamical systems example is particularly important because near the end of the nineteenth century, it already was understood that the study of dynamics could not be advanced significantly just by trying to discover explicit solutions to equations. Moving the field in a different direction, H. Poincaré developed ways to understand dynamical behavior by exploiting the geometry of the associated foliation. This approach is the source of much of modern dynamics [e.g., for applications to economics, see Saari and Simon (1978)] and even chaos [see Medio (1998, 1999)].

Although it is useful to treat foliations as a higher-dimensional dynamical system, I emphasize those other interpretations where the geometry associated with foliations becomes a useful tool for economics. For instance, the geometric traits common to all foliations offer a mathematical framework that allows seemingly unrelated notions to transfer from one topic to another. This is indicated here by showing how natural issues from dynamics introduce new, natural questions about, for example, Bayesian updating. What facilitates this transfer is that foliations start with a common description of how changes among equivalence classes of information can occur. Thus, issues that arise in one context, for example, sufficient statistics, have a foliation analogue in another setting, for example, preferences. As illustrated, the mathematics of foliations provide a technical and conceptual tool to address and predict a variety of topics.

After providing a technical description of foliations (with references), the discussion emphasizes two general themes. An important topic (Section 4) involves the *design* of foliations. To indicate how rich and even traditional this topic is within economics, the design issue is illustrated (Section 5) with Samuelson's weak axiom of revealed preference (WARP), the reason the strong axiom of revealed preferences is needed, Arrow's impossibility theorem and related consequences, L. Hurwicz's information theory, Debreu's proof showing that, beyond Walras's laws, essentially nothing can be stated about the aggregate excess demand function for pure exchange economies, and selective other topics.

A second theme (Sections 3 and 5) concerns the *use* of specified foliations. The geometry of foliation is used to create new tools that motivate, raise, and answer natural issues. Among the illustrations is a brief discussion of strategic behavior.

2. MATHEMATICS OF FOLIATIONS

As Definition 1 (given later) makes clear, foliations partition a specified space or manifold into leafs. What makes foliations more useful than a mere partitioning is their added structure, which requires a smooth transition from one leaf to another. To accomplish this goal, the local structure of leafs is patterned by how p-dimensional coordinate planes are stacked in an R^n coordinate system. This means, for instance, that the local structure of any one-dimensional foliation in a three-dimensional manifold resembles the way that the lines parallel to the x axis are stacked in R^3. To make this notion precise, I review certain concepts about manifolds, which can be thought of as smooth surfaces in a higher-dimensional space.

2.1. Coordinate Systems

To illustrate coordinate systems on a manifold, consider the torus T^2: This is the two-dimensional manifold that resembles the surface of a donut. To suggest how such a construct arises in economics, suppose two agents are planning to locate businesses on the shore of a round lake. Because each potential location for the ith agent can be identified with a point on a circle S^1, the position can be described in terms of an angle θ_i relative to a reference axis. The usual optimization analysis uses the product space $S^1 \times S^1$ where the entries are the couples of angles (θ_1, θ_2). To describe $S^1 \times S^1$, notice that attached to each θ_1 is the circle of points, or possible locations, for agent 2. As indicated in Figure 1a, when agent 1's location changes (i.e., varies θ_1), the circle of agent 2's choices moves around agent 1's circle of choices to define a torus. [To see how T^2 and its foliations arise in a similar fashion in Chichilnisky's (1982) nice extensions of Arrow's theorem, see Saari (1997).]

The torus T^2, illustrated in Figure 1b, is a smooth two-dimensional manifold. One coordinate system uses the θ_1, θ_2 variables; here the names of the coordinates are borrowed from the angular location on each circle. More generally, a local coordinate system on T^2 is defined by "borrowing the coordinates" from R^2. Namely, about each point $p \in T^2$, select an open neighborhood V and an invertible mapping $F : V \to R^2$. Playing the role of the x and y axes in V are, respectively,

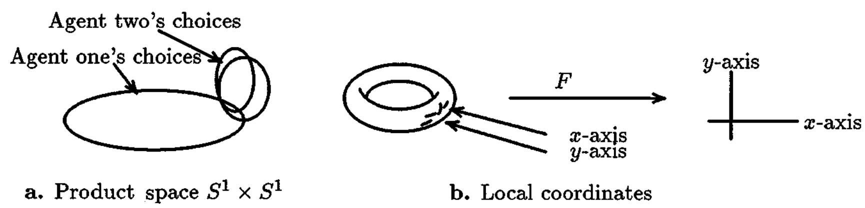

Figure 1. The torus $T^2 = S^1 \times S^1$.

F's inverse image of the x and y axes of R^2. (See Figure 1b.) In this manner, the (F, V) coordinates of a point $p \in V$ on the torus are inherited from the coordinates of $F(p) \in F(V) \subset R^2$. Indeed, by drawing all of the "coordinate" lines in V—that is, all inverse images of the lines parallel to the x and y axes—we obtain a nonlinear version of the usual coordinate mesh of horizontal and vertical lines in R^2. Extending the (F, V) coordinate system to a set V_1, where $V \cap V_1 \neq \emptyset$, means, of course, that the (G, V_1) system must allow the coordinate lines for both (F, V) and (G, V_1) to agree on the intersection $V \cap V_1$. Similarly, local coordinates for an m-dimensional surface are borrowed from R^m via the inverse image of an invertible mapping.

Our discussion of foliations requires the concept of a *smooth-mapping G* from an m-dimensional manifold M to R^n. Here, problems are encountered immediately because, whereas the notion of a continuous-mapping G is well defined as soon as the topology on M is specified, a definition for differentiability on a manifold M is not obvious. This difficulty is resolved in a manner similar to how coordinates are described: The definition of *differentiation* is borrowed from R^m. To do so, "coordinate systems" are used to transfer the issue into a setting where the standard definition of differentiation holds. Namely, although our true interest concerns the mapping $G : M \to R^n$, using the coordinate system $F : V \subset M \to R^m$ allows us to consider the mapping $G \circ F^{-1}$. An advantage of using $G \circ F^{-1} : R^m \to R^n$ is that $G \circ F^{-1}$ has Euclidean spaces for a domain and range where the notion of differentiation is well defined. Thus, $G \circ F^{-1}$ is what we use when discussing the differential structure of G. In other words, an assertion that a function from a manifold to R^n, or to another manifold, is differentiable always is in terms of the adopted coordinate systems. [To side step the obvious problems as to whether these definitions are circular, the idea of a *chart* and an *atlas* are introduced. See, e.g., Spivak (1970), Warner (1970), or Flanders (1989), for more information.]

A particular convenience offered by the coordinate system is that the local properties of a mapping with an m-dimensional manifold M as its domain can be treated as having the more familiar domain of an open set in R^m. So, the assertion that $G : M \to R^n$ is C^r-smooth means that, for each F in a set of admissible coordinate systems, the associated mapping $G \circ F^{-1}$ is differentiable r times and the derivatives are continuous.

2.2. Foliations

Now that the notions of a coordinate system and differentiability are specified, the idea of a foliation can be introduced. A *nonsingular* p*-dimensional foliation* has two parts:

(1) It partitions the space in a manner so that each equivalence set—each leaf—is a p-dimensional manifold.
(2) The partitioning is constructed in a manner to permit a smooth transition between leafs; that is, between the partition classes.

Singular foliations are where some partition sets—that is, some leafs—have lower dimensions. The importance of these foliations already is suggested by dynamical systems where the lower-dimensional leafs correspond to the equilibrium points that are used to determine the overall dynamics of the system. More generally, as described later, it is possible for the geometry of certain surfaces to force *all foliations* of certain dimensions to be singular.

Because a coordinate system allows us to "borrow" from Euclidean space, the notion of a foliation is described first in R^m with coordinates $\boldsymbol{x} = (x_1, \ldots, x_p, \ldots, x_m)$. A natural partitioning of R^m is the set of all p-dimensional subspaces parallel to the p-dimensional coordinate plane defined by the first p coordinates; that is, the p-dimensional planes that are parallel to $x_{p+1} = x_{p+2} = \cdots = x_m = 0$; there are no restrictions on the first p variables. The defined set of stacked p-dimensional planes, then, is completely defined by the level sets

$$x_{p+1} = c_{p+1}, \ldots, x_m = c_m, \text{ for constants } c_{p+1}, \ldots, c_m. \qquad \textbf{(1)}$$

This means, for instance, that a one-dimensional foliation in R^3 is patterned locally after all lines that are parallel to the x axis.

More generally, a p-dimensional foliation on a specified manifold M is defined locally in terms of the coordinate system structure. It is a partitioning of M into p-dimensional submanifolds. The transition property between sets is captured by requiring, about each point, that an appropriate coordinate system can be found {i.e., there is a neighborhood V and an invertible map $F(\boldsymbol{u}) = [F_1(\boldsymbol{u}), \ldots, F_m(\boldsymbol{u})]$} so that the p-dimensional leafs are given by level sets of this coordinate-system structure. Thus, just as with equation (1), *locally* each leaf is given by

$$F(\boldsymbol{u}) = c_{p+1}, \ldots, F_m(\boldsymbol{u}) = c_m$$

for an appropriate open set of choices of the constants $c_{p+1}, \ldots, c_m$. So, locally, a two-dimensional foliation in a three-dimensional manifold resembles the stacking of the planes that are parallel to the x–y coordinate plane of R^3. As an illustration of what does *not* satisfy this definition, the one-dimensional partition illustrated in Figure 2 is *not* a foliation because such a coordinate system cannot be constructed in the region where the horizontal and vertical lines abut.

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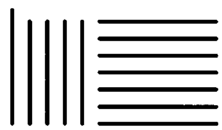

Figure 2. A partition that is not a foliation.

DEFINITION 1. *Let M be a smooth m-dimensional manifold. A C^r nonsingular foliation $\mathcal{F}$ on M with dimension p is a partition $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ of M that satisfies the following property*:

Around every point $x \in M$, there is an open set $V \subset M$ and a C^r-smooth, invertible function

$$F : V \to R^m$$

so that each connected segment $\mathcal{L}_\alpha \cap V$ is identified with a unique set of constants c_j, $j = p+1, \dots, m$ in the following manner. Each component $\mathcal{L}_\alpha \cap V$ is the F inverse image of a p-dimensional plane given by $x_{p+1} = c_{p+1}$, $x_{p+2} = c_{p+2}, \dots,$ $x_m = c_m$. A foliation with dimension p is said to have co-dimension $q = m - p$.

This definition emphasizes that, at least locally, the leafs of a foliation are level sets of a function F. Indeed, when we discuss the design of foliations in Section 4, the goal is to start with a partitioning of a set and then use differential tools to determine whether it is a foliation. For several concerns from economics, an accompanying goal is to find a choice for the corresponding function F.

2.3. Examples

My earlier assertion that the level sets of a set of functions define a foliation is essentially a direct consequence of the implicit function theorem. To illustrate, it follows from the implicit function theorem that if

$$F : R^m \to R^q, \qquad q < m, \tag{2}$$

is a smooth mapping of maximal rank at each point, then its level sets locally define a co-dimension q, or dimension $p = m - q$, surface. To show that these level sets satisfy the definition of a foliation, about some domain point, let $G : R^m \to R^p$ be another mapping so that, locally (i.e., in some open set V), the mapping

$$\hat{F} = (G, F) : V \subset R^m \to R^p \times R^q = R^m$$

has maximal rank. (Because F has maximal rank, there are many such choices of G.) Because this rank condition guarantees that $\hat{F} = (\hat{F}_1, \dots, \hat{F}_m)$ is invertible, it follows that $\hat{F}$ and V define a coordinate system. By construction of $\hat{F}$, the level sets $\hat{F}_{p+1} = c_{p+1}, \dots, \hat{F}_m = c_m$ are the level sets of F, and so, the assertion follows. The role of the level sets of G, then, is to add coordinates that define positions on each leaf.

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Utility functions. To relate Definition 1 to the economic literature, recall the considerable activity directed toward understanding when individual preferences could be represented with a utility function. Restated in the terms given here, this endeavor was to understand the settings where the preferences partition the space of alternatives to define a co-dimension-one foliation. The goal is to discover, or establish the existence of, a utility function U defined over the space of all alternatives, for example, the set $V = R^m_+$. Because this argument defines the mapping

$$U : R^m_+ \to R^1,$$

it becomes a special case of the example given by equation (2). Therefore, the level sets of the utility function define a foliation.

Notice the two different coordinate systems. The first, given by R^m, describes the commodity bundles or the alternatives. The second coordinate system, reflecting how the information is placed into categories as defined by the preference indifference sets, is the level sets of U with an emphasis on the different utility levels. The smoothness of the foliation is determined by the number of times function U is differentiable.

Because a utility function defines a foliation, and because a foliation requires the leafs to line up in a specific manner, the geometric constraints of a foliation make it obvious why certain preferences cannot be described in terms of utility functions. All one needs is to envision the structure of the level sets; if, locally, they do not resemble the stacking of planes in R^n, then a utility function cannot exist. The most obvious illustration is where preferences are determined by a lexicographic ordering: For example, the ranking where (x_1, y_1) is preferred to (x_2, y_2) if and only if $x_1 > x_2$, or if $x_1 = x_2$ and $y_1 > y_2$ cannot be described with a utility function.

General differential equations and extensions. The existence and uniqueness theorems of differential equations ensure that the trajectories of an autonomous system partition phase space. This is because "existence" theorems ensure that a solution passes through each point. Similarly, uniqueness results require that if two curves pass through the same point, then they are the same trajectory. (One might be an extension of the other.) Thus these two conditions guarantee that the trajectories partition phase space.

It remains to ensure that, locally, the trajectories have the appropriate positions relative to one another. This condition is ensured by the "continuity with respect to initial conditions" property of differential equations. Indeed, this property often is expressed exactly in terms of level sets in a manner consistent with Definition 1. [See, e.g., Hartman (1964).]

However, not all differential equations form a *nonsingular* dimension-one foliation. In fact, the more interesting setting is where the solutions create a singular foliation. An example is the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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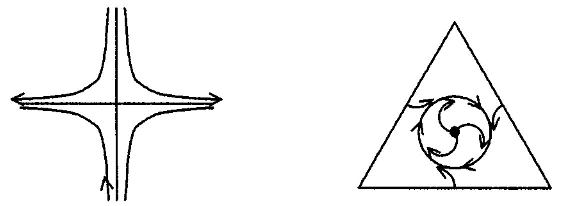


FIGURE 3. Foliations from flows.

with general solution $(x(t), y(t)) = (C_1e^{2t}, C_2e^{-3t})$ for constants C_1, C_2 determined by initial conditions. An important zero-dimensional orbit is the equilibrium point 0. All other trajectories are one-dimensional, as indicated in Figure 3a. Notice how the singular structure of this foliation identifies certain properties of the dynamics. This foliation geometry is emphasized when the dynamics are described in terms of stable and unstable manifolds. [See, e.g., Saari and Simon (1978), Devaney (1989), Katok and Hasselblatt (1995), Robinson (1995).]

Dynamical systems can be defined on a manifold, rather than R^n. For instance, a way to understand the price dynamics of the pure exchange economy with c commodities is to consider the system

$$\boldsymbol{p}' = \xi(\boldsymbol{p}) \tag{3}$$

where $\xi(\boldsymbol{p})$ is the aggregate excess demand function. According to Walras's laws, $\xi(\boldsymbol{p})$ is homogeneous of degree 0, and so, for convenience, the price $\boldsymbol{p}$ can be normalized to have Euclidean length 1. When this is done, equation (3) defines a dynamical system on the portion of the $(c-1)$-dimensional surface of the unit ball in the positive orthant of R^c; that is,

$$S_+^{c-1} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_c) \mid \sum_{j=1}^{c} x_j^2 = 1, \qquad x_j \geq 0 \right\}.$$

All foliations defined by the solutions of equation (3) also are singular. This is because the standard assumption that all commodities are desirable (so the demand will increase with a relatively low price) ensure that equation (3) admits a Walrasian equilibrium. This rest point is a zero-dimensional trajectory.

One way to understand the price dynamics for a pure exchange economy is to examine the geometry of the foliations. For instance, Scarf (1960) proved that the familiar story requiring prices to converge to an equilibrium need not hold. To prove this, Scarf defined utility functions for three agents over three commodities (i.e., he constructed a foliation for each agent's preferences) so that the solution curves of equation (3) behave as indicated in Figure 3b. In particular, the solutions behave such that if the price starts at the equilibrium, it will stay there. However, for any other starting price, the subsequent prices must stay away from the equilibrium. In fact, these other prices tend to the periodic orbit represented by a circle in Figure 3b. (Such a behavior is not a theoretical oddity; in a recent personal conversation, C. Plott stated that such behavior has been observed in experiments.) Notice that the trajectories of a dynamical system form a foliation with the *added property* of a specified direction of movement on each leaf.

2.4. Unit Sphere

The price dynamic discussion emphasizes a portion of the k-dimensional unit sphere S^k. It turns out that if k is an even integer, then all one-dimensional foliations on S^k must be singular. There is an intuitive explanation for the familiar unit sphere S^2 in a three-dimensional space.

A one-dimensional nonsingular foliation always can be converted into a differential equation by using the unit tangent vector at each point on each leaf. (Of course, care must be taken so that vectors point in the same general direction to ensure continuity of the equations.) Therefore, the existence of a nonsingular foliation can be equated with the existence of a continuous tangent vector field on S^k, where each vector has length 1. On the circle S^1, this description is equivalent to starting with a hairy "ring" and then combing it in a continuous manner so that all of the hairs (the unit vectors) are tangent to the ring. Clearly, this is easy to do.

On the sphere S^2, however, the existence of a continuous tangent vector field resembles trying to comb a hairy basketball in a continuous fashion without allowing a "cowlick" (a vector standing up so that it is not tangential). No matter how clever the attempt, it cannot be done because it is impossible to do. However, because this combing is impossible on even-dimensional spheres, it follows that the existence of a one-dimensional nonsingular foliation on these even-dimensional spheres also is impossible. (In turn, this also means that any differential equation defined on an even-dimensional sphere must have an equilibrium.) Slight changes in this argument establish that any p-dimensional foliation on S^k, $k > p$, must be singular when k is even. On the other hand, a one-dimensional nonsingular foliation always exists for S^k, k odd. The answer about the existence of other dimensional foliations on an odd-dimensional sphere is more complicated and surprising.

Flow on a torus. I claimed that it is possible to use the structure of foliations to anticipate new kinds of properties for models from the social sciences. To illustrate this point with concerns from the social sciences, we first need to appreciate the more general geometry allowed by foliations. In particular, although a utility function $U : R^m_+ \to R$ requires a global coordinate representation, the local emphasis of Definition 1 permits the existence of foliations that have a more complex structure.

To develop an example, start with a unit circle where the points are described in terms of angle $2\pi\theta$ defined in a counterclockwise direction from a fixed base point, for example, where the circle meets the positive x axis. Because θ measures multiples of 2π, the integer portion of θ designates the number of times θ has wound around the circle relative to the base point. Thus, for instance, $\theta = 9\frac{1}{2}$

represents nine traversals of the circle, ending at a position diametrically opposed to the reference point.

A particularly simple dynamical system rotates a point by a fixed amount α; this defines the differential equation $\theta' = \alpha$, where the solution is $\theta(t) = \alpha t + \theta(0)$ for initial position $\theta(0)$. Here, nothing surprising happens. So, to create a more interesting description, consider the simultaneous rotation of two angles $\boldsymbol{\theta} = (\theta_1, \theta_2)$ where each θ_j is rotated by a fixed amount according to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ to define

$$\boldsymbol{\theta}' = (\theta_1', \theta_2') = (\alpha_1, \alpha_2) = \boldsymbol{\alpha}. \qquad \textbf{(4)}$$

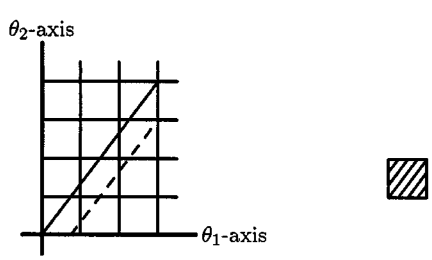
The solution defined by the initial condition $\boldsymbol{\theta}(0) = (\theta_1(0), \theta_2(0))$ is

$$(\theta_1(t), \theta_2(t)) = t\boldsymbol{\alpha} + \boldsymbol{\theta}(0). \qquad \textbf{(5)}$$

For instance, the solution line for the particular initial condition $\boldsymbol{\theta}(0) = 0$ starts at the origin; it is designated by the solid slanted line in Figure 4a.

For purposes of developing intuition, it is useful to think of equation (4) as describing the positions of two agents where each is walking around a circular lake at the constant rate α_j, $j = 1, 2$. Thus, the solution $\boldsymbol{\theta}(t)$ identifies the position of each agent at time t. As described in the earlier lake model, $\boldsymbol{\theta}(t)$ defines a point on the torus T^2. This suggests converting the particular equation (4) solution represented in Figure 4a as a line on torus T^2. To do so, recall that the fractional portion of each $\theta_j(t)$ indicates the jth agent's location on the circle S^1 at time t. Whenever θ_j has an integer value, the solution (or the jth agent's location) is at the base point. Thus, the vertical lines in Figure 4a indicate integer θ_1 values, where the motion (the position of the first agent) has passed around the circle to return to the beginning reference point. Similarly, the horizontal lines correspond to where θ_2 (the position of the second agent) passes through its reference point.

In the particular case of Figure 4a, the solution line first passes through the horizontal line $\theta_2 = 1$. On the two circles, this $(\theta_1(t), 1)$ point has precisely the same location as a $(\theta_1(t), 0)$ point; that is, both $(\theta_1(t), 1)$ and $(\theta_1(t), 0)$ place each agent at the same position along the lake. This suggests cutting the solid line at $\boldsymbol{\theta}(t) = (\theta_1(t), 1)$ and translating it downward [keeping the same slope and $\theta_1(t)$



a. Solution curve **b.** The transposed solution line

FIGURE **4.** Converting solution line to curve on T^2.

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value] so that the new base of the translated line is at $(\theta_1(t), 0)$. Each point on this translated line, which is the parallel dashed line in Figure 4, defines an identical location on each of the two circles as does $\boldsymbol{\theta}(t)$.

Similarly, when $\boldsymbol{\theta}(t)$ meets a vertical integer line, this means that the first component θ_1 passes through its reference point. In Figure 4a, this occurs where the *dashed* line meets the first vertical line $\theta_1 = 1$. At this location, cut the solution line and translate it horizontally from $(1, \theta_2(t))$ to $(0, \theta_2(t))$. By doing this whenever $\boldsymbol{\theta}(t)$ passes through a horizontal or vertical line, the solution of Figure 4a is represented by the collection of lines in the square of Figure 4b.

This construction explicitly uses the identification of horizontal and of vertical lines in Figure 4a. So, by identifying (i.e., gluing together) the two vertical edges of the square in Figure 4b, a cylinder is formed. To understand this construction, notice that each vertical edge corresponds to where the first agent is at the base point. So, as it should, the gluing reattaches the solution lines that were cut and then translated when passing through vertical edges. Next, identify the former two horizontal edges (now circles at the top and bottom of the cylinder) to form the torus T^2. The manner in which the lines in Figure 4b are constructed means that they now are reconnected to form a smooth, continuous line on T^2. This line represents the actual location of the solution on each circle; that is, each point on this line on the torus identifies a position of each agent on the lakeshore.

The Figure 4a solution on T^2 corresponds to the initial condition $\boldsymbol{\theta}(0) = 0$. However, each initial solution defines a corresponding line on the torus, and so, each T^2 point is on at least one such line. To see that the collection of all solution lines defines a foliation for T^2, we appeal, again, to the uniqueness properties of solutions of autonomous differential equations, which require that if any two solution lines agree at any point, they agree everywhere. Thus, the solutions partition T^2. That the solutions form a foliation follows, again, from the usual theorems about the smoothness of solutions with respect to initial conditions; as already asserted, these theorems define this dependency in a manner exactly as specified by Definition 1 for a foliation.

Figure 4b makes it clear that the choice of $\boldsymbol{\alpha}$ determines how often each leaf passes around T^2 before it meets—if it ever does. To explain, start with the special case in which $\alpha_1 = \alpha_2$. With the example of two agents, this assumption means they walk at the same speed, and so, they always are separated by the same distance. If they start together, they remain together forever. Here, the Figure 4a line would be $y = x$ passing through the two diametric vertices of the relevant squares. Corresponding to the fact that the agents simultaneously return to their starting positions each time around the lake is the fact that the line on T^2 connects each time it circles the torus. In a special case, this is captured by the fact that the transposed line in Figure 4b would be a single diagonal line. In contrast, if $2\alpha_1 = \alpha_2$, then the corresponding Figure 4a line is $y = 2x$ starting at $(0, 0)$ and passing through $(1, 2)$, where this structure is repeated forever in blocks of two; the associated Figure 4b would have two line segments. [The first passes through the top edge at $(\frac{1}{2}, 1)$.]

More generally, it follows that the only way the curve will reconnect is if the ratio of the two components of α is a rational number $\alpha_1/\alpha_2 = a/b$. Assume that a/b is in reduced form, where $b > a$. (If $a > b$, then consider $\alpha_2/\alpha_1 = b/a$.) The same argument shows that the two agents always return to their initial positions after the second agent has gone around the lake b times, and the first agent has gone around a times. Because the corresponding version of Figure 4b has b lines, it follows that when b has a sufficiently large value, then a portion of each line comes close to another portion of the same line.

The extreme case is if α_1/α_2 is irrational. *Here the curve never reconnects, but it passes infinitely often, arbitrarily close to any other specified point on the curve.* In other words, the closure of the curve is the full torus. To explain this fact in terms of the two walking agents, choose any location along the lake for each agent. If α_1/α_2 is irrational, then, eventually, there is a time when simultaneously each agent is arbitrarily close to their specified location. This is similar to two children swinging on swings with incommensurable rates; for any position specified for each child, eventually there is a time when each is arbitrarily close to the specified location.

Notice what this structure means about the coordinate representation required by Definition 1. Because each leaf $\mathcal{L}$ can intersect the specified open neighborhood V many times—even an infinite number of times should α_1/α_2 be irrational — it is possible for F to identify different portions of each leaf with many (even an infinite number of) different p-dimensional coordinate planes. This property also shows why, in general, we cannot expect to define the foliation with just one function F.

Incidentally, this "irrational flow on a torus," where α_1/α_2 is an irrational number, determines one of the simpler examples of chaotic behavior. [See, e.g., Devaney (1989).] The standard "sensitivity with respect to initial conditions" requirement is captured partially by the geometry of each leaf that passes arbitrarily close to other portions of the same leaf.

3. USING FOLIATIONS

The geometric structure of foliations provides insight into several concerns coming from the social sciences. A sample is provided.

3.1. Congressional Apportionments

This example of a foliation defined by a flow on a torus may appear to be sufficiently pathological to ensure that it never will occur in the social sciences. However, this is not the case. Instead, as I now indicate with a rounding-off example, this flow on a torus and the associated properties of the foliation provide an important tool for understanding, and even predicting, a reasonably wide class of problems from the social sciences.

The problem of rounding off a fractional allocation to an integer one arises in statistics, integer programming, optimization settings, and anywhere that only integer answers make sense. A politically important example, which I am willing to predict will result in at least one U.S. Supreme Court case during the next decade, involves the allocation of U.S. congressional seats to states on the basis of the decennial census. With 50 states, the population figures define a 50-dimensional vector $\boldsymbol{p}$, where the jth component, p_j, is the fraction representing the population of the jth state relative to the total U.S. population. Let $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_{50}(t))$ be the vector denoting the number of seats assigned to each state with house size t. Reality requires the total number of representatives t to be an integer.

This problem can be identified with the equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{p}, \tag{6}$$

where, perhaps, the initial condition $\boldsymbol{x}(50) = (1, 1, \ldots, 1)$ is imposed to satisfy the U.S. Constitutional requirement that each state is entitled to at least one congressional seat. However, the solution

$$\boldsymbol{x}(t) = (t - 50)\boldsymbol{p} + \boldsymbol{x}(50) \tag{7}$$

creates difficulties because, in general, integer values of t (remember, each t value identifies the total numbers of representatives in the house) designate *fractional* apportionments $x_j(t)$ for the states. Because a "fraction of a representative" cannot be sent to Congress, the issue becomes one of rounding off; it is to replace the vector $\boldsymbol{x}(t)$ of fractional values with an apportionment vector of integer components.

Hamilton's method. There are many ways to handle this integer approximation concern. For instance, because the sum of the components of $\boldsymbol{p}$ is unity, at house size $t = h$ the vector $\boldsymbol{x}(h) = h\boldsymbol{p}$ is on the simplex

$$S_h^{50} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_{50}) \in R^{50} \,\middle|\, x_j \geq 0, \sum_{j=1}^{50} x_j = h \right\}.$$

By adopting a choice of a distance, for example, the l_1 metric (which is the sum of the absolute value of the components), it is natural to declare the actual apportionment of $h\boldsymbol{p}$ to be the nearest S_h^{50} point with integer coefficients. Such an approach, now known as the *Hamilton method*, was proposed by Alexander Hamilton during the formative years of the United States.

To illustrate with $h = 25$ seats and three states, where $25\boldsymbol{p} = (11.30, 10.34, 3.36) \in S_{25}^3$ is the exact fractional apportionment, notice that a simple way to determine the closest integer point is first to assign each state the integer value of each component of $h\boldsymbol{p}$; this is computed for the example in the second to last column of Table 1. Any remaining seats are assigned to states according to the size of the fractional portions, where "larger is better." In the example, one extra seat

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Table 1. Hamilton method for apportionment of congressional seats

State	Population	Exact Rep.	Min.	Hamilton
A	4,520	11.30	11	11
B	4,136	10.34	10	10
C	1,344	3.36	3	4
Total	10,000	25	24	25

remains and, because C has the largest fractional portion of 0.36, state C receives the extra seat.

As this example demonstrates, the rounding-off procedure uses only the relative sizes of the fractional parts; the integer part plays no role. However, placing emphasis on the fractional portions converts the problem to a flow on a torus. This is because as soon as a new integer value for any state is reached, the fractional part becomes zero. Consequently, following the arguments in the section "Flow on a torus," it follows immediately that the relevant fractional portions of $\boldsymbol{x}(t)$ evolve according to a flow on a *50-dimensional torus* for the United States, or as a flow on a three-dimensional torus for Table 1. The role of the different initial conditions is to identify different leafs for the foliation.

By equating rounding off with a flow on a torus, we immediately discover the source for many of the apportionment paradoxes: the more accurate the population figures, the closer we can expect leafs on this torus to approach other portions of the *same leaf*. To see why this is so, consider the simpler setting of two states with the population figures 4,000 and 16,000. The associated $\boldsymbol{p} = (\frac{1}{5}, \frac{4}{5})$ requires the fractional part of an apportionment for both states to be zero for any house size that is a multiple of 5; that is, for every $t = 5n$, $n = 0, 1, 2, \ldots$. In turn, because the line on the torus quickly returns to the starting reference point for both states, it follows from the argument used with Figure 4 that different portions of a leaf always remain reasonably far apart on the torus. However, by adding just one person to the first state creates $\boldsymbol{p}' = (4{,}001/20{,}001,\ 16{,}000/20{,}001)$, where the fractional parts of each state are simultaneously zero if and only if t is an integer multiple of 20,001. In turn, this means that the portions of the same leaf (defining the fractional parts) must come very close to one another to pack the torus in a very crowded manner.

As described next, this incommensurability property of the foliation that allows the fractional parts of each state simultaneously to be near any specified value is what creates the likelihood of all sorts of apportionment paradoxes, paradoxes sufficiently severe to have caused Supreme Court cases [e.g., see Saari (1995b, Chapters 5.4, 5.5)] and most surely will do so again during the first decade of the new millennium. The geometric source of these problems is that whenever different portions of an apportionment curve are packed close enough together, they can enter *any* reasonably sized open set on the torus.

To carry the argument to the next step, recall that the continuity of a mapping is defined in terms of open sets. Thus, if any paradoxical behavior can be identified with continuity, then we must expect *any* leaf from a foliation corresponding to "more accurate or large population figures" to enter these open regions. In other words, on the basis of the structure of foliations, we must expect sharper census figures (independent of the choice of initial conditions) to generate unexpected difficulties, that is, apportionment paradoxes. I illustrate this observation with an historically important problem, a problem in which ad hoc solutions resulted in the current size of 435 members for the U.S. House of Representatives. The main point, however, is that this foliation structure ensures that much more can go wrong; indeed, it predicts that *any paradoxical behavior where a local description uses continuity can occur, and most surely already has occurred, somewhere.*

Alabama paradox. The problem described here concerns the change in allocation with House size. Geometrically, this change is easy to understand. With the equation (7) representation, to go from House size $t = h$ to $t = h + 1$, just add $\boldsymbol{p}$ to the $h\boldsymbol{p}$ term. In Table 1, for instance,

$$
\begin{aligned}
26\boldsymbol{p} = 25\boldsymbol{p} + \boldsymbol{p} &= (11.30, 10.34, 3.36) + (0.4520, 0.4136, 0.1344)\\
&= (11.752, 10.7536, 3.4944).
\end{aligned}
$$

This observation means that, to understand what can go wrong with apportionments, our analysis can emphasize just the geometry of an increment $\boldsymbol{p}$. To illustrate with just one kind of problem, on a simplex S_h^n, place the butt of $\boldsymbol{p}$ at a midpoint of fractional values; that is, place it at a fractional point that is equidistant from all neighboring S_h^n integer points. For instance, with three states and $h = 25$, one choice is to place the butt of $\boldsymbol{p}$ at $(11\frac{1}{3}, 10\frac{1}{3}, 3\frac{1}{3})$. The tip of vector $\boldsymbol{p}$, then, is in the S_{h+1}^n region, which favors states with the largest populations. (That is, the tip of $\boldsymbol{p}$ is closer to an integer coordinate that adds an extra seat to a state with the larger population.) Continuity considerations allow the base of $\boldsymbol{p}$ to be slightly moved off of the midpoint in *any desired direction* while keeping the tip in the same S_{h+1}^n region. In particular, the base of $\boldsymbol{p}$ can be moved into an S_h^n region where a state with a low population is granted the extra seat, but increasing the House size by 1 forces this seat *to be taken away and assigned to another state*.

Can we expect this to occur? We surely can because continuity ensures the existence of a small, open region of opportunities for this paradox to occur. One such region has the property that if $\boldsymbol{x}(h)$ is in this small S_h^n region, then a state with a small population is granted an extra seat. Changing the size of the House to $\boldsymbol{x}(h+1)$ has the effect of seizing this seat away from the small-population state to award it to a state with a larger population. This scenario, however, depends upon whether the $\boldsymbol{x}(t)$ solution—for a leaf defined by the solution—ever can enter this region. We now know that when population terms involve enough decimal values, this must occur. To illustrate with the example in Table 1, the House size of $h = 26$ gives the number of representatives shown in Table 2.

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Table 2. Alabama paradox in apportionment of congressional seats

State	Population	Exact Rep.	Min.	Hamilton
A	4,520	11.752	11	12
B	4,136	10.7536	10	11
C	1,344	3.4944	3	3
Total	10,000	26	24	26

Table 2 illustrates that by increasing the size of the House from 25 to 26, state *C loses* a representative. Such a phenomenon, called the *Alabama paradox*, has occurred often in the apportionment of the seats to the various states in the United States. [See Huntington (1928) for the clever, original arguments in this area, Balinski and Young (1975) for the paper that revived interest in this topic, and Saari (1995b Chapters 5.4 and 5.5) for an explanation of why this and other difficulties must occur with apportionment concerns, list methods, and almost any multiple-variable rounding-off problem.]

By combining the continuity property with the structure of foliations once a more accurate census is taken leads to the following conclusion.

THEOREM 1 (Saari 1978, 1995b). *Let there be at least three states, and suppose for each House size h, the same continuous method based only on the fractional parts of $h\mathbf{p}$ is used to find the "closest" integer apportionment for $h\mathbf{p}$. Furthermore, assume for each state, that there are choices of $\mathbf{p}$ so that that state receives an extra representative. For almost all choices of $\mathbf{p}$, there exists an h so that some state has one less representative at House size $(h+1)$ than at House size h.*

To complete the story, based on the 1910 Census, an appropriate number of congressional seats to avoid this "chaotic dynamics" phenomenon was 433. Reserving a seat for each of the two territories, we reach the current number of representatives of 435. (For various political and historical reasons, this number was not adjusted in 1920 or in 1930.) In other words, the current size of the U.S. House of Representatives was determined as an ad hoc way to avoid difficulties imposed by a natural foliation structure. Moreover, by invoking the structure of foliations, it now is not difficult to establish that any "reform" procedure also has serious faults. No wonder the mathematics of the social sciences is so intriguing!

3.2. Strategic Behavior

A particularly striking result discovered about a quarter century ago is the theorem of Gibbard (1973) and Satterthwaite (1975), asserting that a certain class of voting procedures always admits settings where the outcome can be manipulated. Namely, a voter can obtain a more favorable outcome by not voting according to her true preferences. This result raises several questions, many of which have not been answered. For instance,

- Is there a straightforward method to determine whether a given procedure can be manipulated?
- How can we identify all settings where a procedure can or cannot be manipulated?
- For such a setting, which agent(s) can manipulate the outcome? What is each agent's set of possible, successful strategies?

Answers for these questions follow from the structure of foliations. [See Saari (1995b) for the special case of voting procedures.] To explain, first notice that voting and allocation problems involving a agents can be represented as a mapping

$$P : R^{n_1} \times \cdots \times R^{n_a} \to R^A, \qquad \textbf{(8)}$$

where R^{n_j} is the n_j-dimensional space of parameters characterizing the jth agent, $j = 1, \ldots, a$; R^A represents the space of allocations or outcomes; and the *performance function* P identifies the agents' characteristics with the desired allocation. So, P defines a foliation over the space of agent characteristics where each leaf is the set of combined characteristics that cause a particular P outcome. To illustrate with an election with A candidates, each component of $\boldsymbol{y} \in R^A$ is the election tally for a particular candidate. If P is the voting procedure, then each leaf identifies all possible combinations of voter preferences that cause the specified election tally $\boldsymbol{y} \in R^A$.

An important part of the definition of a foliation concerns the relative positioning of leafs. In differential equations, this aspect is captured by the continuity with respect to initial conditions; that is, a small change in initial conditions results in a solution that remains near the original one, at least for awhile. In the allocation framework of equation (8), a change in leafs corresponds to how a change in individual characteristics changes the P outcome. The foliation structure captures the effects of this change; the *reason* for the change of individual characteristics is a separate issue. It may represent a strategic attempt, voters choosing to abstain, or even mistakes in marking a ballot. This suggests (and it is the case) that all of these seemingly disparate topics admit a similar analysis.

A standard approach to analyze these concerns emphasizes combinatorics by considering different combinations of voters preferences. The approach advocated here emphasizes the structure of the leafs of P: Namely, because the goal is to determine how the outcome of P can change, a natural first step is to study how to move from one leaf to another. The structure of changes in leafs (i.e., in the allocation or outcome represented by P) identifies how changes in individual characteristics can change the outcome. To find the changes in the $P = (P_1, \ldots, P_A)$ outcome at point $\boldsymbol{x}$, just compute

$$DP = (\nabla P_1(\boldsymbol{x}), \ldots, \nabla P_A(\boldsymbol{x})). \qquad \textbf{(9)}$$

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This equation (9) structure of DP completely characterizes all possible changes—strategic or otherwise–that can occur at the current position $\boldsymbol{x}$.

Rather than offering an abstract presentation of this approach, it probably is of more value to illustrate it with a example. The example comes from the attempt in recent years to characterize all ways to manipulate the plurality vote. (This is the widely used procedure in which each voter votes for his top-ranked candidate and the ranking of each candidate depends on the number of points the candidate receives.) I now outline with three candidates how to completely analyze this topic not only for the plurality vote, but for all positional procedures.

DEFINITION 2. *A three-candidate positional procedure is given by* $\boldsymbol{w}_s = (1, s, 0)$, *where in tallying ballots*, 1, s, *and* 0 *points are given*, *respectively*, *to a voter's top-*, *middle-*, *and bottom-ranked candidates. The candidates are ranked according to the number of points received.*

To illustrate this definition, the plurality vote corresponds to $\boldsymbol{w}_0 = (1, 0, 0)$, the Borda count, which normally assigns 2, 1, 0 points to a voter's first-, second-, and third-ranked candidate, is identified with $\boldsymbol{w}_{1/2} = (2/2, 1/2, 0/2)$, and the antiplurality vote, where a voter votes for two candidates, is associated with $\boldsymbol{w}_1 = (1, 1, 0)$.

In computing voting outcomes, it suffices to use the fraction of all voters who have a particular ranking rather than the actual integer values. The advantage of this approach is that it reduces the product space of equation (8) to the simplex

$$Si(6) = \left\{ \boldsymbol{p} \in R^6 \;\middle|\; \sum_{j=1}^{6} p_j = 1, \qquad p_j \geq 0 \right\},$$

where p_j is the fraction of all voters with the j ranking type as given by Table 3.

To analyze the strategic use of $\boldsymbol{w}_s$, it suffices to study the strategies involving candidates A and B with sincere outcome $A \succ B$. This involves comparing the A and B outcomes by subtracting B's $\boldsymbol{w}_s$ tally of $sp_1 + sp_4 + p_5 + p_6$ from A's tally of $p_1 + p_2 + sp_3 + sp_6$ to obtain

$$T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s) = (1 - s)p_1 + p_2 + sp_3 - sp_4 - p_5 - (1 - s)p_6.$$

(In B's tally, for instance, a type-1 voter has B second ranked, and so, he gives her s points. Because p_1 is the fraction of all voters with a type-1 preference, the type-1 voters give B $p_1 s$ points.)

TABLE 3. Voter ranking types

Type	Preferences	Type	Preferences
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

Each leaf of the foliation defined by $T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)$ identifies all profiles $\boldsymbol{p}$ causing the specified difference in A, B tallies. Because the $A-B$ relative election ranking changes when this difference is zero, an analysis of strategic behavior must emphasize changes near the leaf $T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s) = 0$ as given by the gradient

$$\nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s) = (s, 1, s, -s, -1, -(1-s)). \qquad \textbf{(10)}$$

By construction, the gradient points in the direction of profiles assisting A.

To understand how to use equation (10) to identify all possible strategic behavior, notice that with n voters, if a type-j voter votes as though type k, this causes a

$$\boldsymbol{d}_{j\to k} = \frac{1}{n}(\boldsymbol{E}_k - \boldsymbol{E}_j)$$

change in the profile, where $\boldsymbol{E}_j \in R^6$ is the unit vector with 1 in the jth component. The effects of any such change are given by the directional derivative

$$\frac{\partial T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)}{\partial \boldsymbol{d}_{j\to k}} = \nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s) \cdot \boldsymbol{d}_{j\to k}. \qquad \textbf{(11)}$$

Because we assume that the sincere outcome is $A \succ B$, a change in the election outcome can occur only if the adopted strategy $\boldsymbol{d}_{j\to k}$ makes the sign of equation (11) negative.

By using equation (11), the strategic analysis now reduces to elementary computations involving the dot product. For instance, voters of types 1, 2, and 3 prefer A to B, and so, they have no interest in changing the outcome. To find the options of a type-5 voter, notice that the product of the fifth components of $\nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)$ and $\boldsymbol{d}_{5\to k}$ is unity. Therefore, a strategic type-5 voter needs to vote like a type-k voter, where the kth component of $\nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)$ is less than -1. Because no such term exists, *a type-5 voter has no available strategic behavior.*

Now consider the fate of a type-4 voter. Again, because the product of the fourth components of $\nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)$ and $\boldsymbol{d}_{4\to k}$ is s, this voter needs to vote like a type-k voter, where the kth component of $\nabla T_{A-B}(\boldsymbol{p}; \boldsymbol{w}_s)$ is less than $-s$. So, this voter can vote as a type-5 voter where $\boldsymbol{d}_{4\to 5}$ creates an equation (11) value of $s-1$; this has the desired negative value if $s < 1$. Alternatively, the type-4 voter can vote as though type 6 where $\boldsymbol{d}_{4\to 6}$ creates an equation (11) value of $s-(1-s) = 2s-1$, which is negative for $\boldsymbol{w}_s$ for $s < \frac{1}{2}$.

THEOREM 2. *In a three-candidate $\mathbf{w}_s$ election with sincere election outcome $A \succ B$, only type-*4 *and* -6 *voters have strategic options to try to change the election outcome. A type-*4 *voter can vote as though type-*5 *for $\mathbf{w}_s$ procedures where $0 \le s < 1$ and as though type* 6 *for procedures where $0 \le s < \frac{1}{2}$. The strategic options available to a type-*6 *voter are to vote as though type* 5 *for $\mathbf{w}_s$ where $0 < s \le 1$, and as though type* 4 *for $\frac{1}{2} < s \le 1$. For a strategic action to be successful, the election tally must be sufficiently close to an $A \sim B$ tie.*

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In the same manner using $a \geq 2$ agents in a pure exchange economy, where each has a utility function of the form $U = x^{\alpha} y^{\beta}$, a mapping can be constructed that identifies the price equilibria with the coefficients. Thus, each leaf of this mapping identifies all combinations of the coefficients that create a specified price equilibrium. Therefore, by using gradients to study possible changes in this foliation, we can quickly identify all possible strategic actions.

3.3. Bayesian Updating

To see how the structure of foliations raises new questions, recall the essentials of Bayesian updating. Observations are taken from a distribution with a probability distribution function $f(x \mid \theta)$, where parameter $\theta \in \Omega$ has an unknown value. The goal is to use the p.d.f. $f(x \mid \theta)$ and observations to determine the most likely value for $\theta \in \Omega$.

To do so, start with a *prior distribution* $\xi(\theta)$; this is a probability distribution over Ω for the likely value of θ. Once observations $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ are obtained, the prior distribution is updated to obtain a refined, *posterior distribution* for the value of θ according to the relationship

$$\xi(\theta \mid \boldsymbol{x}) \propto f(\boldsymbol{x} \mid \theta)\xi(\theta). \tag{12}$$

It is clear from equation (12) that the form of the posterior distribution depends upon the form of the p.d.f. $f(x \mid \theta)$ and the prior distribution $\xi(\theta)$. So, certain prior distributions conveniently simplify the analysis when the samples come from certain distributions. For instance, if $f(x \mid \theta)$ is a Bernoulli distribution, then the analysis of the posterior distribution becomes elementary by using a beta distribution for a prior. This is because equation (12) becomes

$$\xi(\theta \mid \boldsymbol{x}) \propto f(\boldsymbol{x} \mid \theta)\xi(\theta) \propto \left[\theta^{S(\boldsymbol{x})}(1-\theta)^{1-S(\boldsymbol{x})}\right]\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

where $S(\boldsymbol{x})$ is the number of "successes" in $\boldsymbol{x}$. Because this representation requires the posterior also to be a beta distribution, the computation of the posterior reduces to adding the appropriate exponents. In other words, we can think of the *conjugate family of prior distributions* for samples coming from a given p.d.f. $f(x \mid \theta)$ as a family that is closed under sampling.

However, are these conjugacy classes mere conveniences, or do they have a deeper meaning? This is a natural question to raise when Bayesian updating is viewed from the perspective of foliations. To explain, it is not difficult to show that with certain distributions, the Bayesian updating creates a foliation for the space of continuous priors: Namely, just as an initial condition defines a particular leaf for a differential equation, a prior identifies a leaf for the class of resulting posterior distributions. Now, a natural question from dynamics concerns *stability.* To illustrate, in the Scarf's example of Figure 3b, the equilibrium point is unstable because the motion on all nearby leafs moves away. On the other hand, the periodic orbit depicted by the circle is *stable*; all nearby orbits tend toward this periodic orbit.

To define *stability*, then, we just need a foliation augmented with a sense of direction. This we have with the Bayesian updating foliation; the prior defines the leaf and the updating defines the motion along the leaf. Thus, stability becomes a natural issue. In particular, replacing the circular periodic orbit from Scarf's example with a conjugacy class, a natural question is *to determine which conjugacy classes are stable in the sense that* (*in some metric*) *the posterior distributions from other leafs tend to the conjugacy class with updating*, and which conjugacy classes are unstable. It is clear that the conjugacy classes that enjoy stability have value and importance beyond convenience; in a real sense these conjugacy classes are the "natural" choice. Conversely, an unstable conjugacy class offers convenience at an expense; much like a pendulum standing upright, one should doubt whether these classes capture any lasting, actual behavior.

4. DESIGNING FOLIATIONS

Because foliations partition information, we should expect that several issues and tools from the social sciences revolve about the design of foliations. The basic problem is to determine the kinds of structures that define foliations. Geometry suggests two natural "differential" ways to accomplish this goal. A way to think about them is to recall how level sets of a utility function can be described in terms of derivatives; this is either by specifying at each point the tangent-plane for the level set, or the gradient. Both approaches are used to determine foliations.

The tangent-plane approach can be viewed as describing a higher-dimensional differential equation. Indeed, a common way to determine solutions of a differential equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \qquad \textbf{(13)}$$

is to plot at each point in R^2 the tangent line defined by the right-hand side of equation (13). To find the solutions, a collection of curves is sought so that

- passing through each point is a unique curve, and
- at each point, the tangent line to the curve is the specified line.

Each solution curve is a one-dimensional leaf; the set of all solution curves is the foliation.

More generally, as a way to define a p-dimensional foliation, at each point in a given space, specify a p-dimensional plane. This collection of planes is called a *p-dimensional distribution.* The goal is to find a collection of p-dimensional submanifolds so that

- passing through each point is a unique submanifold, and
- at each point, the tangent plane of the p-dimensional surface agrees with the specified p-dimensional plane.

Each submanifold is a p-dimensional leaf; the set of all leafs is the foliation.

For a differential equation to admit a solution, it suffices to have a sufficiently smooth right-hand side for equation (13). Unfortunately, this simple condition does not suffice for higher-dimensional settings of p-dimensional foliations; here, more stringent requirements are needed to handle the many possible ways to orient a plane. (As illustrated later, these extra requirements correspond to the added conditions of the Strong Axiom of Revealed Preferences.) Namely, to define a foliation, these tangent planes need to satisfy certain structural rules of how they are stacked. For instance, it is clear that a Figure 2 type of configuration is not permitted. After all, after smoothly positioning "lines" in a space, there is not much flexibility for something to go wrong. However, accompanying the positioning of planes are the stacking conditions needed to ensure that the different directions agree. When a distribution satisfies these technical *integrability conditions* [see, e.g., Spivak (1970) and Warner (1970)], it is called an *integrable distribution.*

Rather than distributions, the emphasis of this paper involves the dual approach where, rather than the tangent vectors, the set of vectors orthogonal to level sets is specified. An advantage of using this approach is that it involves generalizations of concepts, such as the gradient, that are familiar to most readers. A related advantage of this dual approach comes from a standard goal in the design of foliations, which is to find a function that defines the level sets. Because the gradient of such a function is orthogonal to the level sets, the gradient must be in this set of normal vectors. Therefore, this dual approach brings us at least one step closer to the objective of finding a functional representation of a foliation.

4.1. Normal Vectors

A way to design utility functions is to specify, at each point in R^m, the line defined by the desired gradient direction. For instance, in a pure exchange economy, this line is defined by the price vector. The goal is to find a collection of $(m-1)$-dimensional surfaces (the level sets) so that

- passing through each point is a unique $(m-1)$-dimensional surface, and
- at each point the line defined by the gradient of the surface is the specified line.

More generally, the normal vector approach assigns an $m - p = q$ dimensional vector space at each point in an m-dimensional manifold M. (More accurately, this vector space is in the tangent space of M.) So, at each specified point of M, this vector space identifies all vectors orthogonal to the leaf of the foliation. Once these orthogonal spaces are specified, the goal is to find a collection of p-dimensional submanifolds of M so that

- passing through each point of M is a unique submanifold, and
- at each point of M, the submanifold has a uniquely defined tangent plane; each vector in this tangent plane is orthogonal to each vector in the specified q-dimensional plane.

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Each submanifold is a p-dimensional leaf; the set of all leafs is the foliation.

Just as a distribution must satisfy appropriate structural conditions to permit the existence of an associated foliation, dual conditions exist for this setting. These constraints impose appropriate restrictions on the choice of normal vector spaces to ensure the existence of a solution foliation.

The motivation for these conditions uses the fact that, at least locally, the leafs are level sets of a function. Therefore, start by examining the level sets of

$$F : R^n \to R. \tag{14}$$

At any point $\boldsymbol{x}$, a normal for the particular level set is given by

$$\nabla F(\boldsymbol{x}) = \left(\frac{\partial F(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial F(\boldsymbol{x})}{\partial x_n}\right). \tag{15}$$

In the co-dimension-one foliation setting characterized by equation (14), this gradient defines the one-dimensional space of orthogonal vectors. Because it also is well known, with sufficient smoothness, the mixed partials satisfy

$$\frac{\partial}{\partial x_i}\left[\frac{\partial F(\boldsymbol{x})}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[\frac{\partial F(\boldsymbol{x})}{\partial x_i}\right] \quad \text{for all } i, j. \tag{16}$$

Our concern is to go in the opposite direction; it is to start with a vector-valued function

$$\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), \ldots, v_n(\boldsymbol{x})) \tag{17}$$

and then determine whether $\boldsymbol{v}(\boldsymbol{x})$ can be expressed as a gradient of some function. As established in standard calculus courses, this is true, at least locally, if the $\boldsymbol{v}(\boldsymbol{x})$ components in equation (17) behave in the manner indicated by equation (16). That is, a sufficient condition for a function $F(\boldsymbol{x})$ to exist so that, in some neighborhood, $\nabla F(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})$ is if the integrability conditions

$$\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} = \frac{\partial v_i(\boldsymbol{x})}{\partial x_j} \tag{18}$$

hold for all i, j about each point $\boldsymbol{x}$. More generally, if a vector field $\boldsymbol{v}(\boldsymbol{x})$ satisfies condition (18) at each point, then $\boldsymbol{v}(\boldsymbol{x})$ defines a foliation, and $\boldsymbol{v}(\boldsymbol{x})$ serves as a normal vector for the foliation. (Recall that a foliation is defined, but a function that defines the foliation everywhere might not exist.)

It follows from the definition of a foliation that if the designated normal bundles define a foliation, then it always is possible to find vectors $\boldsymbol{v}(\boldsymbol{x})$ that satisfy equation (18). However, most choices of $\boldsymbol{v}(\boldsymbol{x})$ do *not* satisfy equation (18) even if they do define a foliation. To explain this assertion, start with the foliation defined by $F(x, y) = x^3y^2$ and the gradient $\nabla F(x, y) = (3x^2y^2, 2yx^3)$. Even after factoring out the common x^2y from each component of ∇F, the remaining smooth vector function $\boldsymbol{v}(x, y) = (3y, 2x)$ defines a normal vector for each leaf of F's foliation.

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On the other hand, because

$$\frac{\partial v_1(\boldsymbol{x})}{\partial y} = \frac{\partial 3y}{\partial y} = 3 \neq 2 = \frac{\partial 2x}{\partial x} = \frac{\partial v_2(\boldsymbol{x})}{\partial x},$$

$\boldsymbol{v}(\boldsymbol{x})$ fails to satisfy equation (18).

This kind of problem occurs when finding foliations associated with constrained optimization problems; as such, this failure of equation (18) is a natural concern for economics. For instance, when optimizing an individual's utility in a pure exchange economy, the demand $\boldsymbol{x}$ at price $\boldsymbol{p}$ satisfies

$$\lambda \boldsymbol{p} = \nabla U(\boldsymbol{x}).$$

Therefore, when using the demand to find the preferences, the unknown $\nabla U(\boldsymbol{x})$ can be replaced by the direction $\boldsymbol{p}(\boldsymbol{x}) = (\|\nabla U(\boldsymbol{x})\|)^{-1}\nabla U(\boldsymbol{x})$. As true with the illustration, although $\boldsymbol{p}(\boldsymbol{x})$ may define the desired foliation, we must expect $\boldsymbol{p}(\boldsymbol{x})$ to fail equation (18).

To correct this integrability problem, we could seek an appropriate function $f(\boldsymbol{x})$ so that $f(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})$ *does* satisfy equation (18). In designing a utility function, this requires finding a function $f(\boldsymbol{x}) = \|\nabla U(\boldsymbol{x})\|$ so that $f(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}) = \nabla U(\boldsymbol{x})$. However, seeking such an integrating factor tends to result in complicated partial differential equations that are difficult to solve. Therefore, there is a crucial need to find a computationally simpler approach. This is introduced next.

4.2. Ideals and Wedge Products

To handle the integrability concerns, concepts such as differential forms, wedge products, and ideals are introduced.

Forms and wedge products. Recall from integral calculus that the differential dx_j can be viewed as a measure of an element of length—it captures the sense of an incremental change in the x_j direction. So, vector field $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), \ldots, v_n(\boldsymbol{x}))$ can be identified with a *one-form* by defining

$$v = \sum_{i=1}^{n} v_i(\boldsymbol{x})\, dx_i.$$

In this manner, a gradient $\nabla F(\boldsymbol{x})$ is identified with the one-form

$$dF = \frac{\partial F(\boldsymbol{x})}{\partial x_1} dx_1 + \cdots + \frac{\partial F(\boldsymbol{x})}{\partial x_n}\, dx_n. \qquad \textbf{(19)}$$

Higher-dimensional integrals from calculus standardly use terms such as $dx\ dy\ dz$ to measure rectangular increments of area, volume, etc. [See, e.g., Flanders (1989).] This differential term captures the sense of a volume given by the orthogonal x, y, z directions. To handle the more subtle aspects when the

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natural area or volume measure may not be rectangular, or when it may change structure with the base point, the *wedge product* $\wedge$ is used. This product can be thought of as combining two incremental elements of length in different directions by building in the appropriate trigonometric factors. One wedge-product rule is an orientation whereby each of the *two-forms*

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{20}$$

defines a two-dimensional element of area, but with different orientations. (The orientation is important for, e.g., fluid flow problems where we need to know which way the fluid is passing through an element of area of a surface.)

A convenient consequence of including this orientation is that it requires $dx_j \wedge dx_j$ to agree with $-dx_j \wedge dx_j$—the negative of itself. However, the expression $dx_j \wedge dx_j = -dx_j \wedge dx_j$ can be true only if

$$dx_j \wedge dx_j = 0.$$

This conclusion reflects the obvious fact that a two-dimensional area cannot come from incremental changes in the same direction—area is not "length times length"; it is "length times width."

The wedge product also satisfies the usual distributive rules. Consequently, by use of the wedge product, two-forms can be constructed from any 2 one-forms, for example, $\omega_1 = \sum_{j=1}^{3} a_j(\boldsymbol{x})\, dx_j$ and $\omega_2 = \sum_{j=1}^{3} b_j(\boldsymbol{x})\, dx_j$, where

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (a_1 dx_1 + a_2 dx_2 + a_3 dx_3) \wedge (b_1 dx_1 + b_2 dx_2 + b_3 dx_3)\\ &= a_1 dx_1 \wedge (b_1 dx_1 + b_2 dx_2 + b_3 dx_3) + a_2 dx_2\\ &\quad \wedge (b_1 dx_1 + b_2 dx_2 + b_3 dx_3) + a_3 dx_3 \wedge (b_1 dx_1 + b_2 dx_2 + b_3 dx_3),\end{aligned}$$

or, after collecting terms and using the orientation rules (20)

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (a_2 b_3 - b_2 a_3) dx_2 \wedge dx_3\\ &\quad + (a_3 b_1 - a_1 b_3) dx_3 \wedge dx_1 + (a_1 b_2 - b_1 a_2) dx_1 \wedge dx_2.\end{aligned} \tag{21}$$

To appreciate the type of area measured by this two-form $\omega_1 \wedge \omega_2$, identify ω_1, ω_2, respectively, with the vectors $\boldsymbol{V}_1 = (a_1, a_2, a_3)$, $\boldsymbol{V}_2 = (b_1, b_2, b_3)$. At each $\boldsymbol{x}$ the vectors $\boldsymbol{V}_1, \boldsymbol{V}_2$ define a parallelogram with area given by the magnitude of the vector

$$\boldsymbol{V}_1 \times \boldsymbol{V}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \tag{22}$$

where $\boldsymbol{e}_i$ represents the unit vector with unity in the ith component. The identified combination of a_j, b_i components captures the necessary trigonometric terms required to compute this area. It is no accident that the coefficients in equations (21) and (22) agree. [To achieve agreement, $dx_3 \wedge dx_1$ is used in equation (21) instead

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of $dx_1 \wedge dx_3$. To have agreement with the right-hand rule of the cross product, the $dx_3 \wedge dx_1$ term represents the $\boldsymbol{e}_2$ direction.] Thus, this two-form captures an area element defined by the parallelogram, but it is an area element that can change with $\boldsymbol{x}$.

In general, the two-dimensional area element

$$\omega_1 \wedge \omega_2 = \left(\sum a_i dx_i\right) \wedge \left(\sum b_j dx_j\right) = \sum_{i<j} (a_i b_j - b_i a_j) dx_i \wedge dx_j \quad \textbf{(23)}$$

has the same meaning. In part, this can be seen by the fact that it has a determinant interpretation for each coefficient. To see this similarity, for $i < j$ let $A_{i,j}$ be the 2×2 determinant using the ith and jth columns from

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}.$$

Then,

$$\omega_1 \wedge \omega_2 = \sum_{i<j} A_{i,j} dx_i \wedge dx_j.$$

In the same fashion, the k-form $\omega_1 \wedge \omega_2 \cdots \wedge \omega_k$ formed from the indicated k one-forms computes the k-dimensional volume element defined by the k vectors that are naturally associated with the k forms ω_i, $i = 1, \ldots, k$. Because the k-dimensional volume is nonzero if and only if the k vectors are linearly independent, an important consequence of this interpretation is that the k one-forms $\{\omega_j\}_{j=1}^k$ are *linearly independent* at $\boldsymbol{x}$ if and only if

$$\omega_1(\boldsymbol{x}) \wedge \omega_2(\boldsymbol{x}) \wedge \cdots \wedge \omega_k(\boldsymbol{x}) \neq 0. \quad \textbf{(24)}$$

Ideals. Starting with specified forms $\{\omega_j\}_{j=1}^k$ and using algebraic combinations such as $f_1(\boldsymbol{x})\omega_1 + f_2(\boldsymbol{x})\omega_2$, where smooth functions are used as coefficients and wedge products create a mathematical object that is more general than a vector space. This construct, known as an *ideal* plays a central role in our considerations.

DEFINITION 3. *The Ideal generated by the one-forms $\omega_1, \ldots, \omega_k$, denoted by $I = \langle \omega_1, \ldots, \omega_k \rangle$, is the collection of all possible forms that can be obtained with wedge products of forms with $\omega_1, \ldots, \omega_k$ and any linear combination where the coefficients are smooth functions. If*

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \neq 0,$$

then the ideal is said to have dimension k.

To illustrate with R^4 with variables (x, y, z, u), the ideal $\langle dx, dy \rangle$ consists of all one-forms $f(x, y, z, u)\, dx + g(x, y, z, u)\, dy$, all forms of the type $dx \wedge dz$, $dx \wedge du, dy \wedge dz$, $dy \wedge du, dx \wedge dy$, $dx \wedge dz \wedge du$, etc. along with all linear combinations with smooth functions as coefficients. The definition of dimension in Definition 3 comes, of course, from equation (24). To obtain a sense of how ideals assist in understanding foliations, start with the fact that, according to the implicit function theorem, the two functions

$$F_1(x, y, z, u) = x^2u^2z^3, \qquad F_2(x, y, z, u) = u^4y^2x^3 \tag{25}$$

define a two-dimensional foliation in R^4: Each leaf is the intersection of a level set from each function. At each point of each leaf of the foliation, the space of normal vectors is spanned by

$$\boldsymbol{v}_1 = \nabla F_1 \big/ (2xuz^2), \qquad \boldsymbol{v}_2 = \nabla F_2 \big/ (u^3yx^2). \tag{26}$$

However, starting with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ or, even worse, starting with some functional combinations

$$\boldsymbol{w}_1(x, y, zu) = f_1(x, y, z, u)\boldsymbol{v}_1 + f_2(x, y, z, u)\boldsymbol{v}_2,$$

$$\boldsymbol{w}_2(x, y, z, u) = g_1(x, y, z, u)\boldsymbol{v}_1 + g_2(x, y, z, u)\boldsymbol{v}_2,$$

there is no way (yet) to determine whether they define a foliation. Most surely neither vector (in either case) satisfies the equation (16) integrability condition. Moreover, we have yet even to discuss the kinds of integrability conditions needed when the normal directions have a higher dimension.

To resolve this difficulty, let ω_j be the one-form associated with $\boldsymbol{v}_j$, $j = 1, 2$. By definition of the ideal $I = \langle \omega_1, \omega_2 \rangle$, any functional multiple of ω_1 is in I. In particular, the appropriate (but, in general, unknown) multiple $2xuz^2\omega_1 = dF_1$ is in I. Because the same argument shows that $dF_1, dF_2 \in I$, this suggests that the sought-after integrability conditions might be obtained in terms of the more general structure of the ideal. This is the case.

A far more complicated setting is where ω_1, ω_2 are defined by the above $\boldsymbol{w}_1, \boldsymbol{w}_2$. It is highly unlikely that the unknown forms dF_1, dF_2 can be discovered with any amount of algebraic manipulation. However, this is not necessary. Just the fact that the ideal includes all possible algebraic combinations with all possible smooth functions still ensures that $dF_1, dF_2 \in I$. Again, as outlined next, integrability conditions exist for this setting that are defined in terms of the *structure* of the ideal.

Exterior derivative. As the simpler equation (16) suggests, we must suspect that integrability conditions are expressed in terms of the higher derivative conditions. This is because, although the first-order derivatives describe the tangent plane, the second-order conditions describe how these planes move. As such, it is clear that some combination of these derivatives is needed to capture the required stacking condition of a foliation.

In general, integrability conditions involve the *exterior derivative*, which is defined as follows [see, e.g., Spivak (1970), Warner (1970), and Flanders (1989)]:

Given a form $\omega = \sum_{j=1}^{n} a_j(\boldsymbol{x})\, dx_j$, define

$$d\omega = \sum_{j=1}^{n} [da_j(\boldsymbol{x})] \wedge dx_j \tag{27}$$

where, from equation (19),

$$da_j(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial a_j(\boldsymbol{x})}{\partial x_i}\, dx_i.$$

Applying this exterior derivative to $\omega = dF$ leads to

$$\begin{aligned} d\omega = d^2F &= d\left[\frac{\partial F(\boldsymbol{x})}{\partial x_1}\right] \wedge dx_1 + \cdots + d\left[\frac{\partial F(\boldsymbol{x})}{\partial x_n}\right] \wedge dx_n \\ &= \sum_{i<j} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \frac{\partial^2 F}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j. \end{aligned} \tag{28}$$

Because the mixed partials must agree, we have $d^2F \equiv 0$. In other words, the orientation rule of the wedge product, along with the fact that mixed partials are equal, forces all of the terms to cancel. This leads to the valuable fact that $d^2 \equiv 0$.

Important for our considerations is the fact that the converse of the $d^2F \equiv 0$ assertion also is true; *if*

$$d\omega \equiv 0, \tag{29}$$

then, at least locally, $\omega = dF$ *for some function* F. This is just the integrability condition (16) expressed in terms of differentials.

To indicate how this ideal structure provides a simpler way to find the more general integrability conditions, start with the weaker integrable situation $\omega = h(\boldsymbol{x})\, dF(\boldsymbol{x})$ where both $h(\boldsymbol{x}) \neq 0$ and F are unknown functions. The new problem, then, is to identify when a given ω can be expressed as

$$\omega = h(\boldsymbol{x})\, dF(\boldsymbol{x}). \tag{30}$$

For intuition, start with the desired setting of equation (30) and differentiate. According to the product rule,

$$d\omega = dh \wedge dF + hd^2F = dh \wedge dF.$$

Unfortunately, $d\omega$ still involves the unknown functions h and F. To find an expression not dependent upon h and F, use the assumption $\omega = h(\boldsymbol{x})\, dF$ to obtain $dF = h^{-1}\omega$, or that

$$d\omega = (h^{-1} dF) \wedge \omega. \tag{31}$$

So, if $d\omega$ has the equation (31) form, we must expect the existence of unknown functions h and F that satisfy equation (30). Unfortunately, equation (31) also depends upon the unknown functions, and so, to eliminate them from the expression,

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represent $(h^{-1}(\boldsymbol{x})\, dF(\boldsymbol{x}))$ as an unknown one-form β. Thus, with the resulting new representation of equation (31), we must expect that if $d\omega$ can be represented as

$$d\omega = \beta \wedge \omega \tag{32}$$

for some one-form β, then there exist appropriate functions allowing equation (30) to hold.

However, determining whether equation (32) holds can still involve considerable, nontrivial algebraic computations. The wedge product, however, helps to avoid these difficulties. For intuition, notice that $\beta \wedge \omega$ represents an area element that includes the direction associated with ω. Therefore, the three-dimensional volume determined by $\beta \wedge \omega$ and ω must be zero. This observation is supported by the computation

$$\omega \wedge d\omega = \omega \wedge (h^{-1} dh \wedge \omega) = -h^{-1} dh \wedge (\omega \wedge \omega) \equiv 0$$

(Recall that $\omega \wedge \omega \equiv 0$; this is because $\omega \wedge \omega$ corresponds to an area element of a degenerate parallelogram.)

The important fact is that the converse holds; *if* $\omega = \sum_{j=1}^{n} a_j(\boldsymbol{x})\, dx_j$ *satisfies*

$$\omega \wedge d\omega \equiv 0, \tag{33}$$

then, at least locally, equation (30) *holds. Namely, if equation* (33) *holds, there exist functions* $h(\boldsymbol{x})$, $F(\boldsymbol{x})$, *so that, at least locally, one has the integrable setting* $h(\boldsymbol{x})\omega = dF$.

To illustrate, to determine whether $\omega = 2yz\, dx + 3xz\, dy + 4xy\, dz$ defines a two-dimensional foliation, it suffices to determine whether $\omega \wedge d\omega \equiv 0$. Because $d\omega = z\, dx \wedge dy + 2y\, dx \wedge dz + x\, dy \wedge dz$, a direct computation proves that this is true.

Note that equation (33) is a compact representation that includes as a special case a standard integrability condition used in economics. To see this for the special case of $n = 3$ and a vector field $\boldsymbol{p}(\boldsymbol{x}) = (p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), p_3(\boldsymbol{x}))$, the associated one-form is $\omega = \sum_{j=1}^{3} p_j\, dx_j$. The integrability conditions from equation (33) become the standard

$$p_3\left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2}\right) + p_2\left(\frac{\partial p_1}{\partial x_3} - \frac{\partial p_3}{\partial x_1}\right) + p_1\left(\frac{\partial p_3}{\partial x_2} - \frac{\partial p_2}{\partial x_3}\right) = 0$$

requirements found in, for example, Samuelson (1950), Varian (1978), etc.

Differential ideals. It remains to express conditions of equation (33) and its generalization to any dimensions in terms of the structure of ideals. This is accomplished by adding a generalized form of condition (33) to the earlier structure of an ideal.

DEFINITION 4. *A k-dimensional ideal $I = \langle \omega_1, \ldots, \omega_k \rangle$ is called a differential ideal if the k-form independence condition*

$$r = [\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k] \neq 0 \tag{34}$$

is satisfied along with the following integrability condition:

$$d\omega_j \wedge r = d\omega_j \wedge [\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k] \equiv 0 \quad \textit{for} \quad j = 1, \ldots, k. \tag{35}$$

The importance of Definition 4 is that it completely characterizes (smooth) foliations.

THEOREM 3 (Frobenius). *Associated with a smooth p-dimensional foliation in a (smooth) m-dimensional manifold M is a $k = (m - p)$ dimensional differential ideal I. Each one-form in I can be identified with a vector field that, at each point, is orthogonal to a leaf in the foliation.*

Conversely, a smooth k-dimensional differential ideal I defines a $p = m - k$ dimensional foliation. Each one-form in I can be identified with a vector field that, at each point, is orthogonal to a leaf in the foliation.

This important conclusion is a differential-form version of the Frobenius theorem. See Spivak (1970) or Warner (1970) for a proof.

Example. To illustrate the use of Theorem 3, suppose the goal is to determine whether the two vector fields

$$\boldsymbol{w}_1 = (5yzu, 2xzu, 3xyu, 6xyz), \qquad \boldsymbol{w}_2 = (3yu, 2ux, 0, 4xy)$$

define the normal bundle for each point of a two-dimensional foliation. Because neither vector satisfies condition (16) use Theorem 3 by representing these vector fields as the differentials

$$\omega_1 = 5yzu\, dx + 2xzu\, dy + 3xyu\, dz + 6xyz\, du,$$

$$\omega_2 = 3yu\, dx + 2ux\, dy + 4xy\, du.$$

To determine whether $I = \langle \omega_1, \omega_2 \rangle$ is a differential ideal, notice that because

$$r = \omega_1 \wedge \omega_2 = 4xyzu^2\, dx \wedge dy - 9xy^2u^2\, dx \wedge dz + 2xy^2zu\, dx \wedge du$$
$$- 6x^2yu^2\, dy \wedge dz - 4x^2yzu\, dy \wedge du + 12x^2y^2u\, dz \wedge du$$

is nonzero, the independence condition is satisfied. Because

$$d\omega_1 = -3zu\, dx \wedge dy - 2yu\, dx \wedge dz + yz\, dx \wedge du + xu\, dy \wedge dz$$
$$+ 4xz\, dy \wedge du + 3xy\, dz \wedge du,$$

a computation proves that

$$d\omega_1 \wedge r = [(-3)(12) - (-2)(-4) + (1)(-6) + (1)(2) - (4)(-9)$$
$$+ (3)(4)]x^2y^2zu^d x \wedge dy \wedge dz \wedge du \equiv 0.$$

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Similarly, because $d\omega_2 = -u\,dx \wedge dy + y\,dx \wedge du + 2x\,dy \wedge du$, a computation proves that $d\omega_2 \wedge r \equiv 0$.

Thus, because $\langle \omega_1, \omega_2 \rangle$ is a differential ideal, it follows from Theorem 3 that the two vector fields $\boldsymbol{w}_1, \boldsymbol{w}_2$ define a two-dimensional foliation. (The space $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ span at each point contains all possible normal vectors.) Indeed, $\boldsymbol{w}_1 = 2y\boldsymbol{v}_1 + z\boldsymbol{v}_2$, $\boldsymbol{w}_2 = \boldsymbol{v}_2$ from equation (26), so the foliation is defined by the intersections of the level sets of the two functions in equation (25).

5. APPLICATIONS

Certain applications of Theorem 3 follow immediately just from dimensional considerations. A particularly important setting involves the design of one-dimensional foliations in R^n—a setting closely related to dynamical systems. This is because specifying a vector at each point is equivalent to specifying an $(n-1)$-dimensional space of normal vectors. When the basis vectors are expressed in terms of the one-forms and the ideal $I = \langle \omega_1, \ldots, \omega_{n-1} \rangle$, the dimensional condition requires $r = \omega_1 \wedge \cdots \wedge \omega_{n-1}$ to be a nonzero $(n-1)$-form. It remains to determine whether I is a differential ideal.

Proving that I is a differential ideal is immediate simply because each $d\omega_j$ is a two-form. Consequently, $d\omega_j \wedge r$ is an $(n+1)$-form. Because it is impossible to have a nondegenerate $(n+1)$-dimensional element of measure in an n-dimensional space, we have that

$$d\omega_j \wedge r \equiv 0 \quad \text{for all} \quad j = 1, \ldots, n-1.$$

In other words, the conditions of Theorem 3 are satisfied because of dimensional considerations. Therefore, it follows that if the original vector field is smooth, then it defines a one-dimensional foliation.

COROLLARY 1. *For an n-dimensional manifold M, suppose there is a set of $(n-1)$ smooth vector fields $\{\boldsymbol{v}_j(\boldsymbol{x})\}_{j=1}^{n-1}$ (where each $\boldsymbol{v}_i(\boldsymbol{x})$ is in the tangent space of M) so that at each $\boldsymbol{x} \in M$ they span an $(n-1)$-dimensional space. There exists a one-dimensional foliation of M that is orthogonal to each $\boldsymbol{v}_j(\boldsymbol{x})$ at each $\boldsymbol{x} \in M$.*

Geometrically, Corollary 1 means that the careful stacking problem required for a foliation does not occur for one-dimensional foliations. However, it is easy to use Theorem 3 to create examples where the same assertion fails even for two-dimensional foliations or for co-dimension-one foliations. As an immediate example, consider the vector field

$$\boldsymbol{v}(\boldsymbol{x}) = (y, z, x), \tag{36}$$

which has a cyclic flavor. The corresponding one-form is $\omega = x\,dz + z\,dy + x\,dz$. Even though $\boldsymbol{v}(\boldsymbol{x})$ and ω are smooth, by using $d\omega = -dx \wedge dy - dy \wedge dz +$

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$dx \wedge dz$, it follows that

$$d\omega \wedge \omega = (x + y + z)\, dx \wedge dy \wedge dz \neq 0,$$

which proves that $I = \langle \omega \rangle$ is *not* a differential ideal. Consequently, the associated tangent planes defined by the normal vector $\boldsymbol{v}(\boldsymbol{x})$ do not line up in a manner to permit a two-dimensional foliation. Instead, the twist provided by the cyclic nature of $\boldsymbol{v}(\boldsymbol{x})$ prohibits the planes from allowing solution manifolds.

5.1. WARP and SARP

Theorem 3 and Corollary 1 shed light on the widely discussed differences between the weak and strong axioms of revealed preferences. Recall how Samuelson (1938) argued that decisions made by individuals capture their preferences. More precisely, if at price $\boldsymbol{p}_1$, bundle $\boldsymbol{x}_1$ was selected over bundle $\boldsymbol{x}_2$, which was affordable because $(\boldsymbol{x}_2, \boldsymbol{p}_1) \leq (\boldsymbol{x}_1, \boldsymbol{p}_1)$, then this individual has revealed the preferences $\boldsymbol{x}_1 \succ \boldsymbol{x}_2$. Consequently, a way to express the WARP is that if $\boldsymbol{p}_2$ should be the price associated with bundle $\boldsymbol{x}_2$, then $\boldsymbol{x}_1$ is not selected only because it is too expensive; that is,

$$\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \text{ and } (\boldsymbol{x}_2, \boldsymbol{p}_1) \leq (\boldsymbol{x}_1, \boldsymbol{p}_1) \Rightarrow (\boldsymbol{x}_2, \boldsymbol{p}_2) < (\boldsymbol{x}_1, \boldsymbol{p}_2). \tag{37}$$

In the special setting of two commodities, Samuelson (1950) showed how this structure determines at each point (each commodity bundle) $\boldsymbol{x}$ a tangent line. He then appealed to the theory of differential equations to establish the existence of utility functions. To redescribe this formulation in terms applicable to Theorem 3, notice that the tangent line through $\boldsymbol{x}$ (the budget line) can be used to determine the normal line at each $\boldsymbol{x}$; it is the line defined by the associated price $\boldsymbol{p}(\boldsymbol{x})$. Just from dimensional considerations, it now follows from Corollary 1 that in a two-commodity setting, a smooth $\boldsymbol{p}(\boldsymbol{x})$ defines a co-dimension-one foliation–the leafs of the foliation are the desired level sets of the utility function.

Can this same approach be used for higher dimensions? Samuelson's arguments show that, for any number of commodities, for each $\boldsymbol{x}$ an associated price $\boldsymbol{p}(\boldsymbol{x})$ is defined which, in turn, defines the budget plane. However, does $\boldsymbol{p}(\boldsymbol{x})$ or WARP suffice to ensure the existence of the foliation? Houthakker (1950) claimed that it does not "for although [Samuelson's condition] can be derived from utility considerations it does not entail integrability, which is an essential property of utility functions." Houthakker replaced Samuelson's WARP with a Strong Axiom of Revealed Preferences (SARP) requiring

> *If for every finite t and T $(t = 1, 2, \ldots, T)$ the inequality $(\boldsymbol{p}_{t-1}, \boldsymbol{x}_t) \leq (\boldsymbol{p}_{t-1}, \boldsymbol{x}_{t-1})$ holds, and if there are numbers i and j such that $0 \leq i < j \leq T$ and $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, then $(\boldsymbol{p}_T, \boldsymbol{x}_T) \leq (\boldsymbol{p}_T, \boldsymbol{x}_0)$.*

Houthakker then showed how replacing WARP with the stronger WARP establishes the existence of utility functions.

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Although SARP does establish the existence of utility functions, doubt remains about Houthakker's assertion that WARP is a necessary condition, but not sufficient to satisfy the needed integrability conditions [e.g., Arrow (1959)]. Gale (1960) resolved this question by constructing an example of a demand function that satisfies WARP but is not integrable.

Without resorting to details, I now use Theorem 3 to offer intuition why WARP works for two commodities but not for more. As described above by use of Corollary 1, dimensional considerations combined with the smoothness of $\boldsymbol{p}(\boldsymbol{x})$ immediately ensure that the integrability conditions are satisfied for two commodities. However, for three or more commodities, the stacking conditions require more of $\boldsymbol{p}(\boldsymbol{x})$ as specified by the second derivative conditions.

The important point is that WARP is a *binary condition*; it compares two situations. Thus, as in the definition of a derivative

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x})}{\partial x_j} = \lim_{h \to 0} \frac{\boldsymbol{v}(\boldsymbol{x} + h\boldsymbol{e}_j) - \boldsymbol{v}(\boldsymbol{x})}{h}, \qquad \textbf{(38)}$$

where $\boldsymbol{e}_j$ is the vector with unity in the jth position, *binary comparisons* can be used to approximate tangent vector positions. Indeed, this observation and equation (38) are consistent with the spirit of Samuelson's argument.

However, separated binary comparisons cannot be used to approximate second-derivative conditions. This is because, to obtain such approximations, more terms are needed. To see what they are, by using h and $-h$ with

$$\boldsymbol{v}(\boldsymbol{x} + h\boldsymbol{e}_j) - \boldsymbol{v}(\boldsymbol{x}) = h\frac{\partial \boldsymbol{v}(\boldsymbol{x})}{\partial x_j} + \frac{h^2}{2}\frac{\partial^2 \boldsymbol{v}(\boldsymbol{x})}{\partial x_j^2} + O(h^3)$$

and adding the results leads to

$$\frac{\partial^2 \boldsymbol{v}(\boldsymbol{x})}{\partial x_j^2} = \lim_{h \to 0} \frac{\boldsymbol{v}(\boldsymbol{x} + h\boldsymbol{e}_j) - 2\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x} - h\boldsymbol{e}_j)}{h^2/2}.$$

(Similar expressions can be derived for the other partial derivatives.) The extra comparisons needed to approximate the higher-order derivative terms are ensured by SARP; indeed, this observation also is in the spirit of Houthakker's argument (but which used integrability conditions different from Theorem 3). So, although WARP provides information about $\boldsymbol{p}(\boldsymbol{x})$, not enough terms are involved to deliver the required information about the change in $\boldsymbol{p}(\boldsymbol{x})$; that is, about $d\boldsymbol{p}(\boldsymbol{x})$. The added information required to use results such as Theorem 3 requires the added terms of SARP. I return to these differences between WARP and SARP in the brief "foliation" outline of Arrow's impossibility theorem (Section 5.3).

5.2. Statistics, Debreu, and Continuous Foliations

At this point, I should correct any thought that the differential approaches over smooth surfaces are the only way to create useful foliations. One important example to the contrary comes from sufficient statistics, where the goal is to find an estimator $\hat{\theta}$ for parameter $\theta \in \Omega$, which applies simultaneously for a given class $\mathcal{P} = \{P_\theta\}$ of distributions of a given random variable X.

By viewing this concern from the perspective of foliations, an approach becomes clear. Because the object is to determine the θ value, we treat the distributions as functions that implicitly define θ as a function of the observations of the random variable. However, once we recognize that a function can be defined, we know that we should search for a foliation. This converts the goal to finding a foliation, or a mapping $T : R^m \to \Omega$, where $T(\boldsymbol{x}) = \theta$ is based on observations $\boldsymbol{x} = (x_1, \ldots, x_m) \in R^m$ of the random variables. By construction, each leaf of the resulting foliation is identified with a specific θ value. Thus, the characterization of the leaf identifies the sufficient statistic. For instance, to find the probability of heads by spinning a coin 200 times, each leaf has all possible ways that the same number of heads occurs.

It remains to characterize these leafs, or sufficient statistics. According to Theorem 3, we should seek an answer in terms of a differential expression. Unfortunately, many of the data spaces do not admit the clean differentials of Theorem 3. However, by using generalized versions of derivatives and the structure of foliations, we should anticipate that the widely used *factorization criterion* is expressed with derivative conditions related to those of Theorem 3. [See Savage (1954) and Lehmann (1986) for the precise statements.]

Another important example illustrating the importance of nondifferentiable foliations comes from Sonnenschein's (1972, 1973) concern about whether an aggregate excess demand must satisfy properties other than the usual Walras laws. The importance of this question can be seen from the usual "Invisible Hand Story," if this story is true, then all aggregate excess demand functions must satisfy appropriate conditions, or, to prevent the price mechanism from experiencing highly chaotic behavior, the aggregate excess demand function must satisfy appropriate properties.

Mantel (1974) improved upon Sonnenschein's results, and then Debreu (1974) found a particularly sharp result for this Sonnenschein–Mantel–Debreu conclusion, which asserts that, essentially, with at least as many agents as commodities, the aggregate excess demand function can be anything; it need not satisfy any other conditions. Thus, all of the negative scenarios that I suggested can occur.

THEOREM 4 [SMD; Debreu (1974)]. *Assume that there are $c \geq 2$ commodities and assume that the prices are normalized to have Euclidean length* 1. *Let $f(\boldsymbol{p})$ be a continuous, tangential vector field on S_+^{c-1}. For any $\epsilon > 0$, an example can be created of an initial endowment and convex preferences for each of $a \geq c$ agents so that for any price $\boldsymbol{p}$, where each $p_j \geq \epsilon$, the aggregate excess demand function $\xi(\boldsymbol{p})$ equals $f(\boldsymbol{p})$.*

The "continuity" assumption of Theorem 4 makes the result more difficult to prove; namely, rather than designing utility functions that generate a *smooth* foliation, Debreu tackled the more challenging task of designing utility functions

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that determine *continuous* foliations. Because smoothness is explicitly not assumed, the tools of Theorem 3 are not applicable.

It is worth outlining the approach that Debreu created to construct the needed foliation. To understand the first step, start by considering three agents, where the initial endowment for each is all of one of the three commodities. Thus, at any price, each agent's demand is some combination of the other two commodities. Therefore, by adjusting the individual demands, it is easy to show that the sum can be made to agree with any desired aggregate vector. In the reverse direction, by treating a vector as a sum of individual demands coming from agents of this type, it becomes possible to take a given demand function and construct individual excess demands with a simpler structure.

With technical arguments of this type, individual demands can be constructed from a given demand function, which have certain desired properties such as WARP. The important fact is that each point $\boldsymbol{x}$ on the individual excess demand function is identified with a unique $\boldsymbol{p}(\boldsymbol{x})$. So, the first step of designing a utility function is to consider the surface of a sphere passing through $\boldsymbol{x}$, where the center of the sphere is sufficiently far in the direction $\boldsymbol{p}(\boldsymbol{x})$. The radius is chosen to ensure that the surface does not hit the demand function in any other place. Then, more complicated arguments involving smaller spheres are used to ensure that the level sets do not cross.

So, the Sonnenschein–Mantel–Debreu result implies that anything can occur. For instance, rather than converging to an equilibrium, the usual "supply and demand" story captured by the equation

$$\boldsymbol{p}' = \xi(\boldsymbol{p})$$

can be far wilder than Scarf's example; it can exhibit any desired chaotic behavior (which is allowed by the dimension $c-1$ of the unit price sphere). On the other hand, if we know the aggregate excess demand function for the economy of c commodities, then what does this tell us about the associated economy of $(c-1)$ commodities? (Assume that there is a commodity that each agent must hold for, possibly, individual consumption; or, going in the opposite direction, assume that we are comparing the "before" and "after" story when a new commodity is offered on the market.) For instance, if the original economy is well behaved in the sense that $\xi_c(\boldsymbol{p}_c)$ (where the subscript identifies the number of commodities) satisfies the highly restrictive "gross substitutes" condition, can the associated but restricted economy become highly random in the sense that the solution to

$$\boldsymbol{p}'_{c-1} = \xi_{c-1}(\boldsymbol{p}_{c-1})$$

is highly chaotic? It can.

THEOREM 5 (Saari 1992). *For $c \geq 3$ commodities, let $S_1, S_2, \ldots, S_{2^c-(c+1)}$ denote all possible subsets of two or more commodities. For set S_j, let $f_{S_j}(\boldsymbol{p}_{S_j})$ be a continuous, tangential vector field on the unit price sphere for commodities in S_j; $j = 1, \ldots, 2^c - (c+1)$. For any $\epsilon > 0$, an example can be created of an*

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initial endowment and convex preferences for each of a $\geq c$ agents so that for each subset of commodities S_j and any price $\boldsymbol{p}_{S_j}$, where each $p_j \geq \epsilon$, the aggregate excess demand function $\xi_{S_j}(\boldsymbol{p}_{S_j})$ equals $f_{S_j}(\boldsymbol{p}_{S_j})$.

In other words, once there are at least as many agents as commodities, there need not be any relationship whatsoever among the aggregate excess demand functions of the different subsets of commodities! Although the proof of this theorem is technically difficult, by appealing to the structure and theory of foliations, Theorem 5 becomes a natural, even expected, extension of Theorem 4. To explain, notice that Debreu's construction specifies the position of only one point for each leaf (each indifference set); Consequently, away from the specified point, the rest of the leaf is free to be molded to satisfy a large class of other conditions. So, by using this freedom to use a Debreu-type construction to design the other portion of the leafs in a manner that ensures that the aggregate excess demand for each subset of commodities is the desired one, we have to expect a result like Theorem 5. All that remains is to develop a global, connecting argument that glues the different portions together into one foliation structure.

5.3. Discrete Foliations and Arrow's Theorem

The social sciences also uses foliations that are more general than the continuous ones described earlier. For instance, a major social choice goal is to discover a welfare function that satisfies appropriate properties. Whenever the desired function exists, it defines a foliation. Each leaf is one of the level sets of the social welfare mapping. If the domain of voters' preferences is a discrete space, then a *discrete foliation* results.

So, a first step in this social choice goal of finding a social welfare function is to determine whether an appropriate foliation exists. In a setting of smooth foliations, tools such as the Frobenius result can be used. However, although Theorem 3 is not applicable when the underlying space is discrete, the structure of foliations can provide valued assistance. To illustrate, I outline a foliation approach to analyze Arrow's impossibility theorem (1963). [Details are in Saari (1995b).]

THEOREM 6 [Special case of Arrow (1963)]. *Suppose there two voters, each of whom has a complete, strict, transitive ranking of the three alternatives $\{A, B, C\}$. Suppose they use a decision rule F that satisfies the following:*

(i) *The societal ranking of the three alternatives always is transitive.*
(ii) *For each pair, if the two voters rank the pair in the same way, then that is the societal ranking of the pair.*
(iii) *In fact, the societal ranking of each pair is strictly determined by each voter's relative ranking of the pair.*

The only procedure satisfying this condition is where the outcome is strictly determined by one of the voters.

In designing a proof, notice that the third condition allows the desired mapping F to be described in terms of three mappings $\{F_{A,B}(\boldsymbol{p}), F_{B,C}(\boldsymbol{p}), F_{A,C}(\boldsymbol{p})\}$, where

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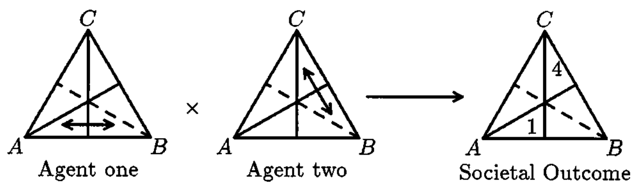

FIGURE 5. Foliation for Arrow's theorem.

each map defines the societal ranking for the indicated pair of alternatives. This means that each leaf of F is defined by the intersection of the appropriate leafs of each of the three binary outcome functions. To represent these rankings, the three equilateral triangles of Figure 5 are used. The ranking identified with a point in the triangle is determined by its distance to each of the labeled vertices, where "closer is better." Thus, each of the six smaller triangles within each large triangle corresponds to a particular strict, transitive ranking of the alternatives; for example, the small triangle on the right with a 1 represents $A \succ B \succ C$, whereas the ranking with a 4 represents $C \succ B \succ A$.

The two possible societal rankings for $\{A, B\}$ define two leafs in the foliation determined by $F_{A,B}$. For instance, the second condition requires the leaf associated with the societal outcome $A \succ B$ to contain the setting where each agent has this same ranking. For either agent, this $A \succ B$ region is the large right triangle where one leg is the vertical bisector.

The leaf structure is influenced by assuming that not one agent makes all decisions. It means that for each agent, a situation exists in which the agent determines the societal outcome for a particular pair. Without loss of generality, assume that when agent 2 prefers $B \succ A$, then agent 1's preferences determine the $\{A, B\}$ outcome. Similarly, assume that when agent 1 prefers $B \succ C$, then agent 2's preferences determine the $\{B, C\}$ outcomes.

This assumption specifies properties of leafs for two pairs of alternatives. Taking their intersections, we find, as indicated by the arrows in Figure 5, that both conditions can be satisfied simultaneously. For instance, if the first agent moves between the two indicated ranking regions, she keeps the $B \succ C$ ranking, but changes her $\{A, B\}$ ranking to change the $\{A, B\}$ societal outcome. Similarly, by the second agent moving between the two indicated rankings, he satisfies the specified conditions while changing the $\{B, C\}$ societal outcome. Two facts about this foliation are important:

(1) Each agent can change rankings independent of what the other agent does.
(2) The moves for each agent stay in the same $\{A, C\}$ ranking region, and so, by the third condition, the societal $\{A, C\}$ outcome remains fixed for any combination of these changes.

To complete the proof, choose preference (on the arrows) where the first agent forces an $A \succ B$ societal outcome at the same time the second agent forces a $B \succ C$ conclusion. According to Figure 5, the intersection of these two regions requires the societal ranking to be $A \succ B \succ C$, as indicated by 1. Similarly, there are other preferences where each voter forces the opposite conclusion; this forces the societal outcome to be in region 4, or to have a $C \succ B \succ A$ conclusion. However, even though each voter kept the $\{A, C\}$ ranking fixed, the outcome changed. This contradiction means that some voter is a dictator in the societal decisions.

As a parting comment, note that the third condition prevents sufficient information from being used to ensure a transitive societal outcome. The reason, as true with WARP, is that binary information is not sufficient to capture the desired structure. In fact, for almost the same conceptual reasons that WARP needed to be replaced by SARP, once the binary assumption in Arrow's theorem is replaced by adding some information about other alternatives, positive conclusions occur.

5.4. Who Says What to Whom?

As a final, somewhat more complicated, example of how Theorem 3 can be used to identify the kinds of information needed to achieve a desired goal, I use the message systems introduced by L. Hurwicz [e.g., see Hurwicz (1960, 1986)] and advanced by Mount and Reiter (1974) and many others. As an oversimplification, consider the solutions concepts based on the parameters of the agents; for example, the set of Walrasian equilibria is determined by individual preferences and initial endowments. An idealized setting describes the outcomes as a mapping

$$P : R^{k_1} \times \cdots R^{k_a} \to R^A \qquad \textbf{(39)}$$

where R^{k_j} is the k_j-dimensional space of parameters characterizing the jth agent, $j = 1, \ldots, a$; R^A is the space of allocationn; and the *performance function* P identifies the agents' characteristics with the desired allocation (as given by the solution concept).

The question is to understand how this outcome can be realized. Clearly, the agents need to convey information about their individual characteristics, but what information, and to whom? Moreover, the type and kind of information needed from each agent changes with the performance function P. So, relative to the choice of P, we need to characterize "who says what to whom" in a manner that allows the P outcome to be achieved.

At this stage, the kinds of information as well as the manner in which the information is to be conveyed remain unknown. To provide structure that suggests how to tackle this issue, let $\boldsymbol{m}_j$ be the yet-to-be-determined message that the j agent communicates about his characteristics $\boldsymbol{x}_j \in R^{k_j}$. This message belongs to a yet-to-be-determined message space M. Although $\boldsymbol{m}_j$ must depend upon the agent's characteristics $\boldsymbol{x}_j \in R^{k_j}$, it also may depend upon what the other agents say. So, assume that $\boldsymbol{m}_j$ is implicitly defined from the yet-to-be-determined equation

$$G_j(\boldsymbol{x}_j; \boldsymbol{m}_1, \ldots, \boldsymbol{m}_j, \ldots, \boldsymbol{m}_a) = 0, \qquad j = 1, \ldots, a. \qquad \textbf{(40)}$$

In other words, what the jth agent communicates about his characteristics, $\boldsymbol{m}_j$ is implicitly determined by (the unknown) G_j. The message space M is the

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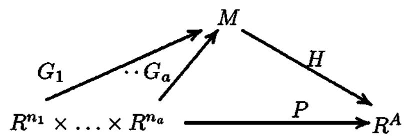

FIGURE 6. Mount–Reiter message diagram.

set of all possible $(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_j, \ldots, \boldsymbol{m}_a)$. Notice how this structure can generate a dynamic; this is a natural consequence of the fact that what one agent communicates can change another agent's message. So, concentrate on the *equilibrium messages* $(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_j, \ldots, \boldsymbol{m}_a) \in M$, which simultaneously satisfies all equations $G_j,\ j = 1, \ldots, a,$ in system (40).

If the equilibrium message is to achieve its purpose, it must determine the "correct" allocation; Thus, for this communication structure to achieve the stated goal, there must exist a mapping $H : M \to R^A$ so that $H(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_j, \ldots, \boldsymbol{m}_a) = P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a)$. In terms of a diagram, we seek mappings $\{G_j\}$, an appropriate message space M, and a mapping H so that the natural diagram [of a type probably first used by Mount and Reiter (1974)] in Figure 6 commutes.

Even for a relatively simple choice of a performance function P, it is not clear how to design the associated message system $[(G_1, G_2, \ldots, G_a), M, H]$. However, guidance comes from Figure 6:

(1) The level sets of P define a foliation for the product space $R^{n_1} \times R^{n_2} \times \cdots \times R^{n_a}$. In particular, each leaf identifies all appropriate combinations of agents' characteristics that give rise to the specified allocation.
(2) However H may be defined, it generates a foliation on the (yet to be determined) message space M. Each leaf identifies all possible equilibrium messages that give rise to the specified allocation.
(3) According to equation (40), each G_j defines a foliation in R^{n_j}. Each leaf is the set of the jth agent's characteristics that give rise to the message $\boldsymbol{m}_j$. So, each leaf from this foliation identifies the kind of information needed from this agent to realize the performance function P; it identifies all of the agent's characteristics that provide the same outcome.
(4) Condition 3 provides an interpretation for the message $\boldsymbol{m}_j$; it is nothing more than a label identifying the particular leaf of characteristics of the jth agent. This, however, introduces a technical problem. If a message just identifies a leaf of the jth agent's characteristics, and if [as allowed by equation (40)] each message can depend upon other agents' messages, then, rather than residing in R^{n_j}, the jth agent's foliation is in the product space $R^{n_1} \times R^{n_2} \times \cdots \times R^{n_a}$. In turn, conditions must be imposed to ensure that each agent's message depends upon the agent's characteristics and not that of the other agents. Hurwicz (1960) calls this *privacy preserving.*
(5) The agents' characteristics define both the P outcome and the messages that each agent conveys, and so, that each leaf defined by each G_j is in the appropriate leaf of P. This connection among the foliations captures the assertion that Figure 6 commutes.

Because this structure can be described with foliations, this suggests that information about the unknown message network can be discovered by using the

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differential approach for the design of foliations. This approach was started by Hurwicz et al. (1978) [also see Saari (1995a) for some of the history] with the distribution approach; that is, our earlier arguments used the tangent-plane approach. What I outline next is a later approach using differential ideals and Theorem 3 that was developed in Saari (1984, 1995a). Because my goal here is to illustrate the use of foliations, I only outline how to incorporate the main aspects of the model.

A foliation needs to be determined for each of the a agents, and so, let $I_j,\ j = 1, \dots, a$, denote the ideal that determines the jth agent's foliation. The design problem is to determine the entries in each I_j. A first entry reflects that the purpose of the message system is to realize the performance function; thus condition 5 requires

$$dP = (dP_1, \dots, dP_A) \in I_j, \qquad j = 1, \dots, a. \tag{41}$$

This condition forces each agent's message to be related to the specified goal of P.

Next, we need to ensure the privacy-preserving condition that the jth agent's messages are independent of the other agents' characteristics. To be independent, the characteristics of the other agents must be orthogonal to each leaf of the jth agent's foliation. Thus, if x_k^i is a component of another agent's characteristics, then $dx_k^i \in I_j$. So, if $[d\mathbf{x}]_j$ denotes the set of differentials of all coordinate functions *except* those of the jth agent, then

$$I_j \supset \langle [d\mathbf{x}]_j \rangle. \tag{42}$$

Because any linear combination of entries in an ideal are also in the ideal, conditions 42 and 43 allow for a reduction. To explain by using $dP = (dP_1, \dots, dP_A) \in I_j$, recall that each coordinate function, for example, dP_1, is a sum involving the differentials of coordinates for the jth agent's characteristics and a sum involving differentials of all other coordinates:

$$dP_1 = \sum_k \frac{\partial P_1}{\partial x_j^k}\, dx_j^k + \sum_i \sum_s \frac{\partial P_1}{\partial x_i^s}\, dx_i^s.$$

Because the second summation is a linear combination of privacy-preserving terms $[d\mathbf{x}]_j$ already entered into I_j, this summation can be eliminated. Thus, only the sum

$$\sum \frac{\partial P_1}{\partial x_j^k}\, dx_j^k$$

has any relevance for I_j. Denote this sum as $d_j P_1$, and let $d_j P = (d_j P_1, \dots, d_j P_A)$. We then have

$$I_j \supset \langle d_j P;\ [d\mathbf{x}]_j \rangle. \tag{43}$$

So far, the design only describes conditions for each agent, but it imposes no conditions ensuring an *effective interaction*; for example, a condition is needed to ensure that the jth agent's message helps and coordinates with the other agents' messages in realizing P. Because this coordination is intended to capture the sense of how agent's messages interact in condition (40), it requires the foliations from the different agents to define another foliation. To understand the associated condition, notice that each agent's foliation corresponds to the level sets of G_j, so the foliation for all a agents must correspond to a foliation defined by the set $[G_1, G_2, \ldots, G_a]$. This places emphasis on the ideal

$$I = \cap_{k=1}^{n} I_k. \tag{44}$$

Based on the entries determined for each I_j, so far we have that $I \supset \langle d_1 P, \ldots, d_n P \rangle$.

Finally, equation (40) requires the message systems to be based on the G_j functions, and so, the appropriate *integrability conditions* must be imposed to ensure that the ideals define foliations. This means that $\{I_j\}_{j=1}^{n}$ and I need to be differential ideals. It is easy to show that if I is a differential ideal, then so are the ideals I_j. Therefore, the mechanism design problem hinges on whether I is a differential ideal. This is not a mere technical detail; as I indicate next, it is the crux of the design problem. After all, the integrability of I determines whether or not it is possible to coordinate the agents' messages to realize P.

It is overly optimistic to believe that just the preceding entries suffice. The more common experience is that I is *not* a differential ideal. When this happens, it means that the information needed from the agents is so interconnected that, rather than just a single message from each agent, several are needed; for example, $\boldsymbol{m}_j$ is a vector rather than a scalar. From a technical approach, this obstacle requires adding more one-forms to appropriate I_j ideals. This is as far as the analysis is carried out here. However, although it is not immediate how to find the new entries, notice that we already have discovered information about the complexity of the message system and that the search for these extra ω forms is assisted by conditions already imposed on the ideals, such as requiring them to be differential ideals. [A more detailed discussion of this point and how to discover the appropriate ω forms is in Saari (1995a), where ideas from Gardner (1968) are used.] When these ω forms are found, then the associated consequences are specified in the following basic conclusion. [A related result described in terms of distributions is in Hurwicz et al. (1978).]

THEOREM 7 [Privacy-Preserving Characterization Theorem, Saari (1984)]. *Let*

$$P : \prod_{j=1}^{a} R^{k_j} \to R^A$$

be a smooth performance function. The following are necessary and sufficient conditions that a privacy-preserving message system with a message space of dimension

$$\dim(M) = \sum_{j=1}^{n} n_j$$

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exists that realizes P in a neighborhood of $\mathbf{x} \in \prod R^{k_j}$.

(i) *For each j, there is a differential ideal $I_j = \langle d_j P, \omega_{j,1}, \ldots, \omega_{j,s_j}; [d\mathbf{x}]_j \rangle$, which is of dimension $n_j + \sum_{i \neq j} k_i$. The $\omega_{j,i}$ are smooth one-forms.*

(ii) *The set $I = \cap_{j=1}^{n} I_j$ is a differential ideal of dimension $\sum_{j=1}^{n} n_j$.*

All sorts of extensions are possible. For instance, it now is possible to characterize what happens if some information is shared among certain agents, or if certain agents must pass their information to other specified agents, or if we wish to model a dialogue in which some agents respond only after they receive further information. [See, e.g., Saari (1984, 1988, 1990, 1995a).]

6. SUMMARY

We started with the assertion that foliations are almost everywhere. They are, and this adds to their importance. This is because, by understanding when an issue is a foliation, we also uncover new tools and ways to analyze the concerns.

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