Ergod. Th. & Dynam. Sys., (2023), **43**, 1785–1828
doi:10.1017/etds.2022.24

Critical values for the β-transformation with a hole at 0

PIETER ALLAART† and DERONG KONG‡

† *Mathematics Department, University of North Texas, 1155 Union Cir #311430, Denton, TX 76203-5017, USA*
(e-mail: allaart@unt.edu)

‡ *College of Mathematics and Statistics, Chongqing University, Chongqing 401331, PR China*
(e-mail: derongkong@126.com)

(*Received* 18 *November* 2021 *and accepted in revised form* 20 *February* 2022)

Abstract. Given $\beta \in (1, 2]$, let T_β be the β-transformation on the unit circle $[0, 1)$ such that $T_\beta(x) = \beta x (\text{mod } 1)$. For each $t \in [0, 1)$, let $K_\beta(t)$ be the survivor set consisting of all $x \in [0, 1)$ whose orbit $\{T_\beta^n(x) : n \geq 0\}$ never hits the open interval $(0, t)$. Kalle *et al* [*Ergod. Th. & Dynam. Sys.* **40**(9) (2020) 2482–2514] proved that the Hausdorff dimension function $t \mapsto \dim_H K_\beta(t)$ is a non-increasing Devil's staircase. So there exists a critical value $\tau(\beta)$ such that $\dim_H K_\beta(t) > 0$ if and only if $t < \tau(\beta)$. In this paper, we determine the critical value $\tau(\beta)$ for all $\beta \in (1, 2]$, answering a question of Kalle *et al* (2020). For example, we find that for the *Komornik–Loreti constant* $\beta \approx 1.78723$, we have $\tau(\beta) = (2 - \beta)/(\beta - 1)$. Furthermore, we show that (i) the function $\tau : \beta \mapsto \tau(\beta)$ is left continuous on $(1, 2]$ with right-hand limits everywhere, but has countably infinitely many discontinuities; (ii) τ has no downward jumps, with $\tau(1+) = 0$ and $\tau(2) = 1/2$; and (iii) there exists an open set $O \subset (1, 2]$, whose complement $(1, 2] \setminus O$ has zero Hausdorff dimension, such that τ is real-analytic, convex, and strictly decreasing on each connected component of O. Consequently, the dimension $\dim_H K_\beta(t)$ is not jointly continuous in β and t. Our strategy to find the critical value $\tau(\beta)$ depends on certain substitutions of Farey words and a renormalization scheme from dynamical systems.

Key words: β-transformation, survivor set, Farey word, Lyndon word, substitution operator
2020 Mathematics Subject Classification: 37B10, 28A78 (Primary); 68R15, 26A30, 37E05 (Secondary)

Contents

1	Introduction	1786
2	Farey words and Farey intervals	1793
	2.1 Farey words	1793
	2.2 Quasi-greedy expansions, Farey intervals, and Lyndon intervals	1795
	2.3 Greedy expansions and the symbolic survivor set	1797
3	Substitution of Lyndon words	1798
	3.1 An equivalent definition of the substitution	1798
	3.2 Properties of the substitution	1800
4	Critical values in a basic interval	1804
5	Geometrical structure of the basic intervals and exceptional sets	1809
	5.1 Tree structure of the Lyndon intervals and relative exceptional sets	1809
	5.2 The infinitely Farey set	1815
6	Critical values in the exceptional sets	1817
7	Càdlàg property of the critical value function	1823
Acknowledgements		1827
References		1827

1. *Introduction*

The mathematical study of dynamical systems with holes, called *open dynamical systems*, was first proposed by Pianigiani and Yorke [27] in 1979. In recent years, open dynamical systems have received considerable attention from both theoretical and applied perspectives (cf. [13–15]). In the general setting, one considers a discrete dynamical system (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map having positive topological entropy. Let $H \subset X$ be an open connected set, called the *hole*. It is interesting to study the set of points $x \in X$ whose orbit $\{T^n(x) : n \geq 0\}$ never hits the hole H. In other words, we are interested in the *survivor set*

$$K(H) = \{x \in X : T^n(x) \notin H \text{ for all } n \geq 0\} = X \setminus \bigcup_{n=0}^{\infty} T^{-n}(H).$$

It is known that the size of $K(H)$ depends not only on the size but also on the position of the hole H (cf. [7]). In [29, 30], Urbański considered C^2-expanding, orientation-preserving circle maps with a hole of the form $(0, t)$. In particular, he proved that for the doubling map T_2 on the circle $\mathbb{R}/\mathbb{Z} \sim [0, 1)$, that is, $T_2 : [0, 1) \to [0, 1); x \mapsto 2x (\text{mod } 1)$, the Hausdorff dimension of the survivor set $K_2(t) := \{x \in [0, 1) : T_2^n(x) \notin (0, t) \text{ for all } n \geq 0\}$ depends continuously on the parameter $t \in [0, 1)$. Furthermore, he showed that the dimension function $\eta_2 : t \mapsto \dim_H K_2(t)$ is a devil's staircase, and studied its bifurcation set. Carminati and Tiozzo [9] showed that the function η_2 has an interesting analytic property: the local Hölder exponent of η_2 at any bifurcation point t is equal to $\eta_2(t)$. For the doubling map T_2 with an arbitrary hole $(a, b) \subset [0, 1)$, Glendinning and Sidorov [18] studied (i) when the survivor set $K_2(a, b) = \{x \in [0, 1) : T_2^n(x) \notin (a, b) \text{ for all } n \geq 0\}$ is non-empty; (ii) when $K_2(a, b)$ is infinite; and (iii) when $K_2(a, b)$ has positive Hausdorff dimension. They proved that when the size of the hole (a, b) is strictly smaller than 0.175092, the

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

survivor set $K_2(a, b)$ has positive Hausdorff dimension. The work of Glendinning and Sidorov was partially extended by Clark [10] to the β-dynamical system $([0, 1), T_\beta)$ with a hole (a, b), where $\beta \in (1, 2]$ and $T_\beta(x) := \beta x \pmod 1$.

Motivated by the above works, Kalle *et al* [20] considered the survivor set in the β-dynamical system $([0, 1), T_\beta)$ with a hole at zero. More precisely, for $t \in [0, 1)$, they determined the Hausdorff dimension of the survivor set

$$K_\beta(t) = \{x \in [0, 1) : T_\beta^n(x) \notin (0, t) \text{ for all } n \geq 0\},$$

and showed that the dimension function $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ is a non-increasing Devil's staircase. So there exists a critical value $\tau(\beta) \in [0, 1)$ such that $\dim_H K_\beta(t) > 0$ if and only if $t < \tau(\beta)$. Kalle *et al* [20] gave general lower and upper bounds for $\tau(\beta)$. In particular, they showed that $\tau(\beta) \leq 1 - 1/\beta$ for all $\beta \in (1, 2]$, and the equality $\tau(\beta) = 1 - 1/\beta$ holds for infinitely many $\beta \in (1, 2]$. They left open the interesting question to determine $\tau(\beta)$ for all $\beta \in (1, 2]$. In this paper, we give a complete description of the critical value

$$\tau(\beta) = \sup\{t : \dim_H K_\beta(t) > 0\} = \inf\{t : \dim_H K_\beta(t) = 0\} \tag{1.1}$$

for each $\beta \in (1, 2]$. Qualitatively, our main result is the following.

THEOREM 1.

(i) *The function $\tau : \beta \mapsto \tau(\beta)$ is left continuous on $(1, 2]$ with right-hand limits everywhere (càdlàg), and, as a result, has only countably many discontinuities.*

(ii) *τ has no downward jumps.*

(iii) *There is an open set $O \subset (1, 2]$, whose complement $(1, 2] \setminus O$ has zero Hausdorff dimension, such that τ is real-analytic, convex, and strictly decreasing on each connected component of O.*

Quantitatively, the main results are Theorem 2 and Propositions 6.2 and 6.3. Together with Proposition 1.12, they specify the value of $\tau(\beta)$ for all $\beta \in (1, 2]$. In Proposition 1.9 below, we give an explicit description of the discontinuities of the map τ, which shows that the dimension $\dim_H K_\beta(t)$ is not jointly continuous in β and t. The closures of the connected components of the set O in Theorem 1(iii) form a pairwise disjoint collection $\{I_\alpha\}$ of closed intervals which we call *basic intervals* (see Definition 1.5). In the remainder of this introduction, we describe these basic intervals by using certain substitutions on Farey words. We then give a formula for $\tau(\beta)$ on each basic interval (see Theorem 2) and decompose the complement $(1, 2] \setminus \bigcup_\alpha I_\alpha$ into countably many disjoint subsets (see Theorem 3), which are of two essentially different types. We then calculate $\tau(\beta)$ on each subset.

To describe the critical value $\tau(\beta)$, we first introduce the Farey words, also called standard words (see [24, Ch. 2.2]). Following a recent paper of Carminati, Isola, and Tiozzo [8], we define recursively a sequence of ordered sets $(F_n)_{n=0}^\infty$. Let $F_0 = (0, 1)$, and for $n \geq 0$, the ordered set $F_{n+1} = (v_1, \ldots, v_{2^{n+1}+1})$ is obtained from $F_n = (w_1, \ldots, w_{2^n+1})$ by inserting for each $1 \leq j \leq 2^n$ the new word $w_j w_{j+1}$ between the two neighboring

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

words w_j and w_{j+1}. So,

$$F_1 = (0, 01, 1), \quad F_2 = (0, 001, 01, 011, 1),$$
$$F_3 = (0, 0001, 001, 00101, 01, 01011, 011, 0111, 1), \tag{1.2}$$

and so on (see §2 for more details on Farey words). Set $\Omega_F^* := \bigcup_{n=1}^{\infty} F_n \setminus F_0$. Then each word in Ω_F^* is called a non-degenerate *Farey word*. Note that any word in Ω_F^* has length at least two, and begins with digit 0 and ends with digit 1. We will use the Farey words as basic bricks to construct infinitely many pairwise disjoint closed intervals so that we can explicitly determine $\tau(\beta)$ for β in each of these intervals. Furthermore, we will show that these closed intervals cover $(1, 2]$ up to a set of zero Hausdorff dimension.

The construction of these basic intervals depends on certain substitutions of Farey words. For this reason, we need to introduce a larger class of words, called Lyndon words; see [**20**, Lemma 3.2].

Definition 1.1. A word $\mathbf{s} = s_1 \ldots s_m \in \{0, 1\}^*$ is *Lyndon* if

$$s_{i+1} \ldots s_m \succ s_1 \ldots s_{m-i} \quad \text{for all } 0 < i < m.$$

Here and throughout the paper, we use lexicographical order $\succ$ between sequences and words; see §2. The words 0 and 1 are (vacuously) Lyndon. Let Ω_L^* denote the set of all Lyndon words of length at least two. Then by Definition 1.1, each $\mathbf{s} \in \Omega_L^*$ has a prefix 0 and a suffix 1. It is well known that each Farey word is Lyndon (cf. [**8**, Proposition 2.8]). Thus $\Omega_F^* \subset \Omega_L^*$.

Now we define a substitution operator $\bullet$ in Ω_L^*. This requires the following notation. By a *word* we mean a finite string of zeros and ones. For any two words, $\mathbf{u} = u_1 \ldots u_m$, $\mathbf{v} = v_1 \ldots v_n$, we denote by $\mathbf{uv} = u_1 \ldots u_m v_1 \ldots v_n$ their concatenation. Furthermore, we write $\mathbf{u}^\infty$ for the periodic sequence with periodic block $\mathbf{u}$. For a word $\mathbf{w} = w_1 \ldots w_n \in \{0, 1\}^n$, we denote $\mathbf{w}^- := w_1 \ldots w_{n-1}0$ if $w_n = 1$, and $\mathbf{w}^+ := w_1 \ldots w_{n-1}1$ if $w_n = 0$. Furthermore, we denote by $\mathbb{L}(\mathbf{w})$ the lexicographically largest cyclic permutation of $\mathbf{w}$. Now for two words $\mathbf{s} = s_1 \ldots s_m \in \Omega_L^*$ and $\mathbf{r} = r_1 \ldots r_\ell \in \{0, 1\}^\ell$, we define

$$\mathbf{s} \bullet \mathbf{r} := c_1 \ldots c_{\ell m}, \tag{1.3}$$

where

$$c_1 \ldots c_m = \begin{cases} \mathbf{s}^- & \text{if } r_1 = 0, \\ \mathbb{L}(\mathbf{s})^+ & \text{if } r_1 = 1, \end{cases}$$

and for $1 \le j < \ell$,

$$c_{jm+1} \ldots c_{(j+1)m} = \begin{cases} \mathbb{L}(\mathbf{s}) & \text{if } r_j r_{j+1} = 00, \\ \mathbb{L}(\mathbf{s})^+ & \text{if } r_j r_{j+1} = 01, \\ \mathbf{s}^- & \text{if } r_j r_{j+1} = 10, \\ \mathbf{s} & \text{if } r_j r_{j+1} = 11. \end{cases}$$

For an equivalent definition of the substitution operator $\bullet$, see §3.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Example 1.2. Let $\mathbf{r} = 01$, $\mathbf{s} = 001$, and $\mathbf{t} = 011$ be three words in Ω_F^*. Then $\mathbb{L}(\mathbf{r}) = 10$ and $\mathbb{L}(\mathbf{s}) = 100$. So, by equation (1.3), it follows that

$$\mathbf{r} \bullet \mathbf{s} = \mathbf{r} \bullet 001 = \mathbf{r}^- \mathbb{L}(\mathbf{r}) \mathbb{L}(\mathbf{r})^+ = 001011 \in \Omega_L^*,$$
$$\mathbf{s} \bullet \mathbf{t} = \mathbf{s} \bullet 011 = \mathbf{s}^- \mathbb{L}(\mathbf{s})^+ \mathbf{s} = 000\ 101\ 001 \in \Omega_L^*.$$

Then $\mathbb{L}(\mathbf{r} \bullet \mathbf{s}) = 110010$, and thus

$$(\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t} = (\mathbf{r} \bullet \mathbf{s}) \bullet 011 = (\mathbf{r} \bullet \mathbf{s})^- \mathbb{L}(\mathbf{r} \bullet \mathbf{s})^+ (\mathbf{r} \bullet \mathbf{s}) = 001010\ 110011\ 001011,$$

and

$$\begin{aligned} \mathbf{r} \bullet (\mathbf{s} \bullet \mathbf{t}) &= \mathbf{r} \bullet 000101001 \\ &= \mathbf{r}^- \mathbb{L}(\mathbf{r}) \mathbb{L}(\mathbf{r}) \mathbb{L}(\mathbf{r})^+ \mathbf{r}^- \mathbb{L}(\mathbf{r})^+ \mathbf{r}^- \mathbb{L}(\mathbf{r}) \mathbb{L}(\mathbf{r})^+ = 00\ 10\ 10\ 11\ 00\ 11\ 00\ 10\ 11. \end{aligned}$$

Hence, $(\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t} = \mathbf{r} \bullet (\mathbf{s} \bullet \mathbf{t})$, suggesting that the operator $\bullet$ is associative. However, observe that $\mathbf{r} \bullet \mathbf{s} = 00\ 10\ 11 \neq 000\ 101 = \mathbf{s} \bullet \mathbf{r}$. So $\bullet$ is not commutative.

From Example 1.2, we see that Ω_F^* is not closed under the substitution operator $\bullet$, since $\mathbf{r} \bullet \mathbf{s} = 001011 \notin \Omega_F^*$. Hence we need the larger collection Ω_L^*. It turns out that Ω_L^* is a non-Abelian semi-group under the substitution operator $\bullet$.

PROPOSITION 1.3. $(\Omega_L^*, \bullet)$ *forms a non-Abelian semi-group.*

Remark 1.4. The substitution operator $\bullet$ defined in equation (1.3) is similar to that introduced by Allaart [1], who used it to study the entropy plateaus in unique q-expansions.

Let

$$\Lambda := \{\mathbf{S} = \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k : \mathbf{s}_i \in \Omega_F^* \text{ for any } 1 \le i \le k; k \in \mathbb{N}\} \tag{1.4}$$

be the set of all substitutions of Farey words from Ω_F^*. Then by Proposition 1.3 it follows that $\Omega_F^* \subset \Lambda \subset \Omega_L^*$. Moreover, both inclusions are strict. For instance, $001011 = 01 \bullet 001 \in \Lambda \backslash \Omega_F^*$ by Example 1.2 and Proposition 2.4 below, and $0010111 \in \Omega_L^* \backslash \Lambda$.

Given $\beta \in (1, 2]$, for a sequence $(c_i) \in \{0, 1\}^{\mathbb{N}}$, we write

$$((c_i))_\beta := \sum_{i=1}^{\infty} \frac{c_i}{\beta^i}.$$

Now we define the basic intervals.

Definition 1.5. A closed interval $I = [\beta_\ell, \beta_*] \subset (1, 2]$ is called a *basic interval* if there exists a word $\mathbf{S} \in \Lambda$ such that

$$(\mathbb{L}(\mathbf{S})^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (\mathbb{L}(\mathbf{S})^+ \mathbf{S}^- \mathbb{L}(\mathbf{S})^\infty)_{\beta_*} = 1.$$

The interval $I = I^{\mathbf{S}}$ is also called a basic interval generated by $\mathbf{S}$.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

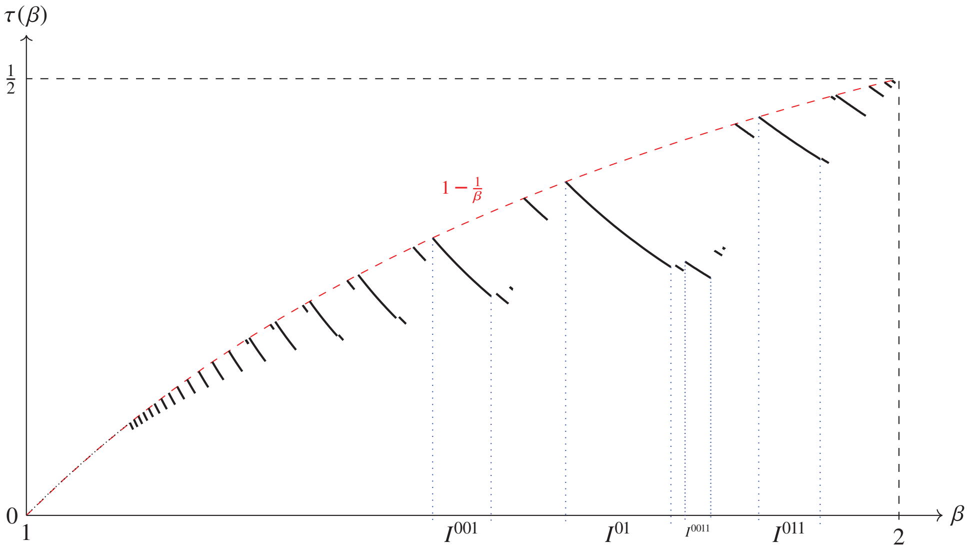

FIGURE 1. Graph of the critical value function $\tau(\beta)$ for $\beta \in (1, 2]$. We see that $\tau(\beta) \leq 1 - 1/\beta$ for all $\beta \in (1, 2]$, and the function τ is strictly decreasing in each basic interval $I^{\mathbf{S}}$. For example, the basic interval generated by the Farey word 01 is given by $I^{01} = [\beta_\ell, \beta_*] \approx [1.61803, 1.73867]$ with $((10)^\infty)_{\beta_\ell} = (1100(10)^\infty)_{\beta_*} = 1$. Furthermore, for any $\beta \in I^{01}$, we have $\tau(\beta) = (00(10)^\infty)_\beta = 1/\beta(\beta^2 - 1)$; see Example 1.7 for more details.

The subscripts for the endpoints β_ℓ and β_* of a basic interval will be clarified when we define the Lyndon intervals (see Definition 1.8 below). Our second main result gives a formula for $\tau(\beta)$ when β lies in a basic interval $I^{\mathbf{S}}$.

THEOREM 2.

(i) *The basic intervals $I^{\mathbf{S}}, \mathbf{S} \in \Lambda$ are pairwise disjoint.*

(ii) *If $I^{\mathbf{S}}$ is a basic interval generated by $\mathbf{S} \in \Lambda$, then*

$$\tau(\beta) = (\mathbf{S}^- \mathbb{L}(\mathbf{S})^\infty)_\beta \quad \text{for every } \beta \in I^{\mathbf{S}}. \tag{1.5}$$

(iii) *The function τ is strictly decreasing on $I^{\mathbf{S}}$, and is real-analytic and strictly convex in the interior of $I^{\mathbf{S}}$.*

Remark 1.6. Note that (iii) follows immediately from (ii). For the special case when $\mathbf{S} \in \Omega_F^*$, the formula (1.5) was stated without proof by Kalle *et al* [20].

Example 1.7.

(i) Let $\mathbf{s} = 01 \in \Omega_F^*$. Then by Definition 1.5, the basic interval $I^{01} = [\beta_\ell, \beta_*]$ satisfies

$$(\mathbb{L}(01)^\infty)_{\beta_\ell} = ((10)^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (\mathbb{L}(01)^+(01)^- \mathbb{L}(01)^\infty)_{\beta_*} = (1100(10)^\infty)_{\beta_*} = 1.$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

By numerical calculation, we get $I^{01} \approx [1.61803, 1.73867]$ (see Figure 1). In fact, $\beta_\ell = (1+\sqrt{5})/2$. Theorem 2 yields that

$$\tau(\beta) = (00(10)^\infty)_\beta = \frac{1}{\beta(\beta^2-1)} \quad \text{for all } \beta \in I^{01}.$$

(ii) Let $\mathbf{s}_1 = \mathbf{s}_2 = 01 \in \Omega_F^*$. Then $\mathbf{S} = \mathbf{s}_1 \bullet \mathbf{s}_2 = 01 \bullet 01 = 0011$. By Definition 1.5, the basic interval $I^{\mathbf{s}_1 \bullet \mathbf{s}_2} = I^{0011} = [\beta_\ell, \beta_*]$ is given implicitly by

$$(\mathbb{L}(0011)^\infty)_{\beta_\ell} = ((1100)^\infty)_{\beta_\ell} = 1,$$
$$(\mathbb{L}(0011)^+(0011)^-\mathbb{L}(0011)^\infty)_{\beta_*} = (11010010(1100)^\infty)_{\beta_*} = 1.$$

Numerical calculation gives $I^{0011} \approx [1.75488, 1.78431]$ (see Figure 1), and Theorem 2 implies

$$\tau(\beta) = (\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_\beta = (0010(1100)^\infty)_\beta = \frac{1}{\beta^3} + \frac{1+\beta}{\beta^2(\beta^4-1)} \quad \text{for all } \beta \in I^{0011}.$$

Next, we introduce the Lyndon intervals.

Definition 1.8. For each Lyndon word $\mathbf{S} \in \Omega_L^*$, the interval $J^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_r^{\mathbf{S}}] \subset (1, 2]$ is called a *Lyndon interval* generated by $\mathbf{S}$ if

$$(\mathbb{L}(\mathbf{S})^\infty)_{\beta_\ell^{\mathbf{S}}} = 1 \quad \text{and} \quad (\mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty)_{\beta_r^{\mathbf{S}}} = 1.$$

If in particular $\mathbf{S} \in \Omega_F^*$, we call $J^{\mathbf{S}}$ a *Farey interval*.

We remark that the Farey intervals defined in [20, Definition 4.5] are half-open intervals, which is slightly different from our definition. It turns out that the discontinuity points of τ are precisely the right endpoints of the Lyndon intervals $J^{\mathbf{S}}$ with $\mathbf{S} \in \Lambda$.

PROPOSITION 1.9. *The function* τ *is continuous on* $(1, 2]\backslash\{\beta_r^{\mathbf{S}} : \mathbf{S} \in \Lambda\}$. *However, for each* $\mathbf{S} \in \Lambda$, *we have*

$$\lim_{\beta \searrow \beta_r^{\mathbf{S}}} \tau(\beta) = (\mathbf{S}^\infty)_{\beta_r^{\mathbf{S}}} > (\mathbf{S}0^\infty)_{\beta_r^{\mathbf{S}}} = \tau(\beta_r^{\mathbf{S}}). \tag{1.6}$$

Remark 1.10. Proposition 1.9 implies that although the dimension $\dim_H K_\beta(t)$ is continuous in t for fixed β, it is not jointly continuous in β and t. In particular, when $t = \tau(\beta_r^{\mathbf{S}})$ for $\mathbf{S} \in \Lambda$, the function $\beta \mapsto \dim_H K_\beta(t)$ has a jump at $\beta_r^{\mathbf{S}}$.

It was shown in [20, §4] that the Farey intervals $J^{\mathbf{s}}$, $\mathbf{s} \in \Omega_F^*$ are pairwise disjoint, and the *exceptional set*

$$E := (1, 2] \setminus \bigcup_{\mathbf{s} \in \Omega_F^*} J^{\mathbf{s}}$$

has zero Hausdorff dimension. We strengthen this result slightly and show in Proposition 5.6(i) that E is uncountable and has zero packing dimension.

From Definitions 1.5 and 1.8, it follows that $I^{\mathbf{S}} \subset J^{\mathbf{S}}$ for any $\mathbf{S} \in \Lambda$, and the two intervals $I^{\mathbf{S}}$ and $J^{\mathbf{S}}$ have the same left endpoint (see Proposition 5.1.) In Proposition 5.6(ii),

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

we show that for any $\mathbf{S} \in \Lambda$, the Lyndon intervals $J^{\mathbf{S}\bullet\mathbf{r}}$, $\mathbf{r} \in \Omega_F^*$ are pairwise disjoint subsets of $J^{\mathbf{S}} \setminus I^{\mathbf{S}}$, and the *relative exceptional set*

$$E^{\mathbf{S}} := (J^{\mathbf{S}} \setminus I^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}$$

is also uncountable and has zero box-counting dimension. In Proposition 5.1, we show that the Lyndon intervals $J^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$ have a tree structure. This gives rise to the set

$$E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{S}\in\Lambda(k)} J^{\mathbf{S}}, \tag{1.7}$$

where $\Lambda(k) := \{\mathbf{S} = \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k : \mathbf{s}_i \in \Omega_F^* \text{ for all } 1 \le i \le k\}$. We call E_∞ the *infinitely Farey set*, because its elements arise from substitutions of an infinite sequence of Farey words. It follows at once that E_∞ is uncountable; we show in Proposition 5.8 that it has zero Hausdorff dimension.

Combining the above results, we obtain our last main theorem.

THEOREM 3. *The interval* $(1, 2]$ *can be partitioned as*

$$(1, 2] = E \cup E_\infty \cup \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}} \cup \bigcup_{\mathbf{S}\in\Lambda} I^{\mathbf{S}},$$

and the basic intervals $\{I^{\mathbf{S}} : \mathbf{S} \in \Lambda\}$ *cover* $(1, 2]$ *up to a set of zero Hausdorff dimension.*

Remark 1.11. It is worth mentioning that the Lyndon intervals $J^{\mathbf{S}}$ and the relative exceptional sets $E^{\mathbf{S}}$ constructed in our paper have similar geometrical structure as the relative entropy plateaus and relative bifurcation sets studied in [**2**], where they were used to describe the local dimension of the set of univoque bases.

The following result was established in the proof of [**20**, Theorem D].

PROPOSITION 1.12. *For any* $\beta \in (1, 2]$, *we have* $\tau(\beta) \le 1 - 1/\beta$. *Furthermore,*

$$\tau(\beta) = 1 - \frac{1}{\beta} \quad \textit{for any } \beta \in E.$$

Thus, in view of Theorem 3, it remains to determine $\tau(\beta)$ for $\beta \in E^{\mathbf{S}}$ with $\mathbf{S} \in \Lambda$ and for $\beta \in E_\infty$. In Proposition 6.2, we compute $\tau(\beta)$ for $\beta \in E^{\mathbf{S}}$ by relating the relative exceptional set $E^{\mathbf{S}}$ to the exceptional set E via a renormalization map $\Psi_{\mathbf{S}}$. Proposition 6.3 gives an expression for $\tau(\beta)$ when $\beta \in E_\infty$. As an illustration of the latter, in Proposition 6.4, we construct in each Farey interval $J^{\mathbf{s}}$ a transcendental base $\beta_\infty^{\mathbf{s}} \in E_\infty$ and give an explicit formula for $\tau(\beta_\infty^{\mathbf{s}})$. Here we point out an interesting connection with unique β-expansions: Let $\beta \approx 1.78723$ be the *Komornik–Loreti constant* (cf. [**21**]); that is, β is the smallest base in which 1 has a unique expansion. Then it follows from Proposition 6.4 that $\beta = \beta_\infty^{01} \in E_\infty$, and $\tau(\beta) = (2 - \beta)/(\beta - 1) \approx 0.270274$.

The rest of the paper is organized as follows. In §2, we recall some properties of Farey words and Farey intervals, as well as greedy and quasi-greedy β-expansions. In §3, we give an equivalent definition of the substitution operator $\bullet$, and prove Proposition 1.3. The

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

proof of Theorem 2 is given in §4. At the heart of the argument is Proposition 4.1, which clarifies the role of the special Lyndon words $\mathbf{S} \in \Lambda$ and is used in several settings to derive the upper bound for $\tau(\beta)$. The relative exceptional sets $E^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$ and the infinitely Farey set E_∞ are studied in detail in §5, where we show that all of these sets have zero Hausdorff dimension, proving Theorem 3. In §6, we determine the critical value $\tau(\beta)$ for β in the relative exceptional sets $E^{\mathbf{S}}$ and the infinitely Farey set E_∞. Finally, in §7, we show that the function $\beta \mapsto \tau(\beta)$ is càdlàg, and prove Proposition 1.9 and Theorem 1.

2. *Farey words and Farey intervals*

In this section, we recall some properties of Farey words, which are vital in determining the critical value $\tau(\beta)$. We also recall from [20] the Farey intervals, and review basic properties of greedy and quasi-greedy β-expansions.

First we introduce some terminology from symbolic dynamics (cf. [23]). Let $\{0, 1\}^{\mathbb{N}}$ be the set of all infinite sequences of zeros and ones. Denote by σ the left shift map. Then $(\{0, 1\}^{\mathbb{N}}, \sigma)$ is a full shift. By a *word* we mean a finite string of zeros and ones. Let $\{0, 1\}^*$ be the set of all words over the alphabet $\{0, 1\}$ together with the empty word ϵ. For a word $\mathbf{c} \in \{0, 1\}^*$, we denote its length by $|\mathbf{c}|$, and for a digit $a \in \{0, 1\}$, we denote by $|\mathbf{c}|_a$ the number of occurrences of a in the word $\mathbf{c}$. For two words $\mathbf{c} = c_1 \ldots c_m$ and $\mathbf{d} = d_1 \ldots d_n$ in $\{0, 1\}^*$, we write $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$ for their concatenation. For $n \in \mathbb{N}$, we denote by $\mathbf{c}^n$ the n-fold concatenation of $\mathbf{c}$ with itself, and by $\mathbf{c}^\infty$ the periodic sequence with period block $\mathbf{c}$.

Throughout the paper, we will use the lexicographical order '$\prec, \preccurlyeq, \succ$' or '$\succcurlyeq$' between sequences and words. For example, for two sequences $(c_i), (d_i) \in \{0, 1\}^{\mathbb{N}}$, we say $(c_i) \prec (d_i)$ if $c_1 < d_1$, or there exists $n \in \mathbb{N}$ such that $c_1 \ldots c_n = d_1 \ldots d_n$ and $c_{n+1} < d_{n+1}$. For two words $\mathbf{c}, \mathbf{d}$, we say $\mathbf{c} \prec \mathbf{d}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$. We also recall from §1 that if $\mathbf{c} = c_1 \ldots c_m$ with $c_m = 0$, then $\mathbf{c}^+ = c_1 \ldots c_{m-1}1$; and if $\mathbf{c} = c_1 \ldots c_m$ with $c_m = 1$, then $\mathbf{c}^- = c_1 \ldots c_{m-1}0$. Finally, for a word $\mathbf{c} = c_1 c_2 \ldots c_n$, we denote its *reflection* by $\overline{\mathbf{c}} := (1 - c_1)(1 - c_2) \ldots (1 - c_n)$.

2.1. *Farey words.*

Farey words have attracted much attention in the literature due to their intimate connection with rational rotations on the circle (see [24, Ch. 2]) and their one-to-one correspondence with the rational numbers in $[0, 1]$ (see equation (2.1) below). In the following, we adopt the definition from a recent paper of Carminati, Isola, and Tiozzo [8].

First we recursively define a sequence of ordered sets $F_n, n = 0, 1, 2, \ldots$. Let $F_0 = (0, 1)$; and for $n \geq 0$, the ordered set $F_{n+1} = (v_1, \ldots, v_{2^{n+1}+1})$ is obtained from $F_n = (w_1, \ldots, w_{2^n+1})$ by

$$\begin{cases} v_{2i-1} = w_i & \text{for } 1 \leq i \leq 2^n + 1, \\ v_{2i} = w_i w_{i+1} & \text{for } 1 \leq i \leq 2^n. \end{cases}$$

In other words, F_{n+1} is obtained from F_n by inserting for each $1 \leq j \leq 2^n$ the new word $w_j w_{j+1}$ between the two neighboring words w_j and w_{j+1}. See equation (1.2) for examples. Note that for each $n \geq 0$, the ordered set F_n consists of $2^n + 1$ words which are listed from the left to the right in lexicographically increasing order. We call $w \in \{0, 1\}^*$ a *Farey word* if $w \in F_n$ for some $n \geq 0$, and we denote by $\Omega_F := \bigcup_{n=1}^{\infty} F_n$ the set of all Farey words. As shown in [8, Proposition 2.3], the set Ω_F can be bijectively mapped to $\mathbb{Q} \cap [0, 1]$ via the map

$$\xi : \Omega_F \to \mathbb{Q} \cap [0, 1]; \quad \mathbf{s} \mapsto \frac{|\mathbf{s}|_1}{|\mathbf{s}|}. \tag{2.1}$$

So, $\xi(\mathbf{s})$ is the frequency of the digit 1 in $\mathbf{s}$.

For each $n \geq 1$, set

$$F_n^* := F_n \setminus \{0, 1\},$$

and

$$F_n^0 := \{w \in F_n^* : |w|_0 > |w|_1\}, \quad F_n^1 := \{w \in F_n^* : |w|_1 > |w|_0\}.$$

For example, $F_1^* = (01)$, $F_2^* = (001, 01, 011)$, and $F_2^0 = (001)$, $F_2^1 = (011)$. The following decomposition can be deduced from [8, Proposition 2.3].

LEMMA 2.1. *For any $n \geq 2$, we have $F_n^* = F_n^0 \cup F_1^* \cup F_n^1$.*

The ordered sets F_n^*, $n \geq 1$ can also be obtained via substitutions. We define the two substitution operators by

$$U_0 : \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 01, \end{cases} \quad \text{and} \quad U_1 : \begin{cases} 0 \mapsto 01, \\ 1 \mapsto 1. \end{cases} \tag{2.2}$$

Then U_0 and U_1 naturally induce a map on $\{0, 1\}^*$ or $\{0, 1\}^{\mathbb{N}}$. For example,

$$U_0 : \{0, 1\}^* \to \{0, 1\}^*; \quad c_1 \ldots c_n \mapsto U_0(c_1) \ldots U_0(c_n).$$

The following result was proven in [8, Proposition 2.9].

LEMMA 2.2. *For each $a \in \{0, 1\}$, the map $U_a : F_n^* \to F_{n+1}^a$ is bijective.*

By Lemmas 2.1 and 2.2, it follows that the ordered sets F_n^* can be obtained by the substitution operators U_0 and U_1 on the set $F_1^* = (01)$. We will clarify this in the next proposition. Let Ω_F^* be the set of all non-degenerate Farey words, that is,

$$\Omega_F^* = \bigcup_{n=1}^{\infty} F_n^*.$$

For a word $\mathbf{c} = c_1 \ldots c_m \in \{0, 1\}^*$, let $\mathbb{S}(\mathbf{c})$ and $\mathbb{L}(\mathbf{c})$ be the lexicographically smallest and largest cyclic permutations of $\mathbf{c}$, respectively. In other words, $\mathbb{S}(\mathbf{c})$ is the lexicographically smallest word among

$$c_1 c_2 \ldots c_m, \quad c_2 \ldots c_m c_1, \quad c_3 \ldots c_m c_1 c_2, \quad \ldots, \quad c_m c_1 \ldots c_{m-1};$$

and $\mathbb{L}(\mathbf{c})$ is the lexicographically largest word in the above list. The following properties of Farey words are well known (see, e.g., [8, Proposition 2.5]).

LEMMA 2.3. *Let $\mathbf{s} = s_1 \ldots s_m \in \Omega_F^*$. Then the following hold.*

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

(i) $\mathbb{S}(\mathbf{s}) = \mathbf{s}$ *and* $\mathbb{L}(\mathbf{s}) = s_m s_{m-1} \ldots s_1$.

(ii) $\mathbf{s}^-$ *is a palindrome; that is,* $s_1 \ldots s_{m-1}(s_m - 1) = (s_m - 1)s_{m-1}s_{m-2} \ldots s_1$.

(iii) *The word* $\mathbf{s}$ *has a conjugate* $\tilde{\mathbf{s}} \in \Omega_F^*$, *given by*

$$\tilde{\mathbf{s}} := \overline{\mathbb{L}(\mathbf{s})} = 0\, \overline{s_2 \ldots s_{m-1}}\, 1. \tag{2.3}$$

The last equality in equation (2.3) follows from statements (i) and (ii). In terms of the correspondence equation (2.1), if $\xi(\mathbf{s}) = r \in \mathbb{Q} \cap [0, 1]$, then $\xi(\tilde{\mathbf{s}}) = 1 - r$. Note also that the conjugate of $\tilde{\mathbf{s}}$ is simply $\mathbf{s}$ itself.

The following explicit description of Ω_F^* will be useful in §4 to prove the upper bound for $\tau(\beta)$.

PROPOSITION 2.4. Ω_F^* *consists of all words in one of the following forms:*

(i) 01^p *or* 0^p1 *for some* $p \in \mathbb{N}$;

(ii) $01^p01^{p+t_1} \ldots 01^{p+t_N}01^{p+1}$ *for some* $p \in \mathbb{N}$ *and Farey word* $0t_1 \ldots t_N1 \in \Omega_F^*$;

(iii) $0^{p+1}10^{p+t_1}1 \ldots 0^{p+t_N}10^p1$ *for some* $p \in \mathbb{N}$ *and Farey word* $0t_1 \ldots t_N1 \in \Omega_F^*$.

Proof. Note that $01 = U_1(0) = U_0(1) \in F_1^* \subset \Omega_F^*$. Furthermore, for $p \in \mathbb{N}$ and $0t_1 \ldots t_N1 \in \Omega_F^*$, we have

$$01^p = U_1(01^{p-1}) = U_1^{p-1}(U_0(1)),$$

$$0^p1 = U_0(0^{p-1}1) = U_0^{p-1}(U_1(0)),$$

$$01^p01^{p+t_1} \ldots 01^{p+t_N}01^{p+1} = U_1^p(U_0(0t_1 \ldots t_N1)),$$

$$0^{p+1}10^{p+t_1}1 \ldots 0^{p+t_N}10^p1 = U_0^p(U_1(0\,\overline{t_1 \ldots t_N}\,1)).$$

By Lemma 2.3(iii), if $0t_1 \ldots t_N1 \in \Omega_F^*$, then $0\,\overline{t_1 \ldots t_N}\,1 \in \Omega_F^*$ as well. Hence by Lemma 2.2, all the above words lie in Ω_F^*.

To prove the converse, it suffices to show that each word in Ω_F^* is of the form $U_0^p(U_1(\mathbf{t}))$ or $U_1^p(U_0(\mathbf{t}))$ for some $p \geq 0$ and Farey word $\mathbf{t} \in \Omega_F$. This is clearly true for $01 = U_0^0(U_1(0))$, where U_0^0 denotes the identity map. Let $n \geq 1$ and suppose the statement is true for all Farey words in F_n^*. Take $\mathbf{s} \in F_{n+1}^*$ with $\mathbf{s} \neq 01$. By Lemmas 2.1 and 2.2, $\mathbf{s} = U_0(\mathbf{t})$ or $\mathbf{s} = U_1(\mathbf{t})$ for some Farey word $\mathbf{t} \in F_n^*$. We assume the former, as the argument for the second case is similar. By the induction hypothesis, either $\mathbf{t} = U_0^p(U_1(\mathbf{u}))$ for some $\mathbf{u} \in \Omega_F$ and $p \geq 0$, in which case $\mathbf{s} = U_0^{p+1}(U_1(\mathbf{u}))$; or $\mathbf{t} = U_1^p(U_0(\mathbf{u}))$ for some $\mathbf{u} \in \Omega_F$ and $p \geq 1$, in which case $\mathbf{s} = U_0(U_1(\mathbf{v}))$, where $\mathbf{v} = U_1^{p-1}(U_0(\mathbf{u})) \in \Omega_F$. In both cases, $\mathbf{s}$ is of the required form. □

Observe that the two types of words in Proposition 2.4(i) are each others conjugates, and the conjugate of a Farey word of type (ii) is a Farey word of type (iii), and vice versa. For more properties of Farey words, we refer to the book of Lothaire [**24**] and the references therein.

2.2. *Quasi-greedy expansions, Farey intervals, and Lyndon intervals.* Given $\beta \in (1, 2]$, let $\delta(\beta) = \delta_1(\beta)\delta_2(\beta) \ldots \in \{0, 1\}^{\mathbb{N}}$ be the *quasi-greedy* β-expansion of 1 (cf. [**11**]),

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

that is, $\delta(\beta)$ is the lexicographically largest sequence not ending with 0^∞ such that $(\delta_i(\beta))_\beta = 1$. The following property of $\delta(\beta)$ is well known (cf. [6]).

LEMMA 2.5.

(i) *The map $\beta \mapsto \delta(\beta)$ is an increasing bijection from $\beta \in (1, 2]$ to the set of sequences $(a_i) \in \{0, 1\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying*

$$\sigma^n((a_i)) \preccurlyeq (a_i) \quad \text{for all } n \geq 0.$$

(ii) *The map $\beta \mapsto \delta(\beta)$ is left continuous everywhere on $(1, 2]$ with respect to the order topology, and it is right continuous at $\beta_0 \in (1, 2)$ if and only if $\delta(\beta_0)$ is not periodic. Furthermore, if $\delta(\beta_0) = (a_1 \ldots a_m)^\infty$ with minimal period m, then $\delta(\beta) \searrow a_1 \ldots a_m^+ 0^\infty$ as $\beta \searrow \beta_0$.*

Recall from Definition 1.1 that for a word $\mathbf{s} = s_1 \ldots s_m \in \Omega_L^*$, we have $s_{i+1} \ldots s_m \succ s_1 \ldots s_{m-i}$ for all $1 \leq i < m$. The following basic fact can be found in [5, Theorem 1.5.3].

LEMMA 2.6. *Let $\mathbf{c} = c_1 \ldots c_m \in \{0, 1\}^*$, and suppose two cyclic permutations of $\mathbf{c}$ are equal (that is, $c_{i+1} \ldots c_m c_1 \ldots c_i = c_{j+1} \ldots c_m c_1 \ldots c_j$, where $i \neq j$). Then $\mathbf{c}$ is periodic; in other words, $\mathbf{c} = \mathbf{b}^k$ for some word $\mathbf{b}$ and $k \geq 2$.*

In fact, the length of $\mathbf{b}$ in Lemma 2.6 can be taken to equal $\gcd(|i - j|, m)$.

LEMMA 2.7. *Let $\mathbf{s} \in \Omega_L^*$ and $\mathbf{a} = \mathbb{L}(\mathbf{s}) = a_1 \ldots a_m$. Then*

$$a_{i+1} \ldots a_m \prec a_1 \ldots a_{m-i} \quad \text{for all } 1 \leq i < m. \tag{2.4}$$

Furthermore,

$$\sigma^n(\mathbf{a}^+\mathbf{s}^-\mathbf{a}^\infty) \preccurlyeq \mathbf{a}^+\mathbf{s}^-\mathbf{a}^\infty \quad \text{for all } n \geq 0. \tag{2.5}$$

Proof. First we prove equation (2.4). Since $\mathbf{s}$ is Lyndon, it is not periodic. Hence $\mathbf{a} = \mathbb{L}(\mathbf{s})$ is not periodic, because any cyclic permutation of a periodic word is periodic. Since $\mathbf{a} = \mathbb{L}(\mathbf{s})$, we have

$$a_{i+1} \ldots a_m \preccurlyeq a_1 \ldots a_{m-i} \quad \text{for all } 1 \leq i < m.$$

Suppose equality holds for some i. Then

$$a_{i+1} \ldots a_m a_1 \ldots a_i = a_1 \ldots a_{m-i} a_1 \ldots a_i \succcurlyeq a_1 \ldots a_{m-i} a_{m-i+1} \ldots a_m = \mathbf{a},$$

so $a_{i+1} \ldots a_m a_1 \ldots a_i = \mathbb{L}(\mathbf{s}) = \mathbf{a}$ by definition of $\mathbb{L}(\mathbf{s})$. By Lemma 2.6, this cannot happen, since $\mathbf{a}$ is not periodic.

Next we prove equation (2.5). Since $\mathbf{s} = s_1 \ldots s_m$ is a Lyndon word, any word of length $k \in \{1, \ldots, m-1\}$ occurring in $\mathbf{a} = \mathbb{L}(\mathbf{s})$ is lexicographically larger than or equal to $s_1 \ldots s_k$. By equation (2.4), it follows that

$$a_{k+1} \ldots a_m^+ s_1 \ldots s_k \preccurlyeq a_1 \ldots a_{m-k} a_{m-k+1} \ldots a_m \prec a_1 \ldots a_m^+ \tag{2.6}$$

for all $0 < k < m$. Hence, by equations (2.6) and (2.4), we conclude that $\sigma^n(\mathbf{a}^+\mathbf{s}^-\mathbf{a}^\infty) \prec \mathbf{a}^+\mathbf{s}^-\mathbf{a}^\infty$ for all $n \geq 1$. This completes the proof. □

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

LEMMA 2.8. *Let $\beta \in (1,2)$. Then $\delta(\beta)$ is periodic if and only if $\delta(\beta) = \mathbb{L}(\mathbf{s})^\infty$ for some Lyndon word* **s** *of length at least two.*

Proof. Suppose $\delta(\beta) = (a_1 \ldots a_m)^\infty$ with minimal period block $\mathbf{a} = a_1 \ldots a_m$. Then $m \geq 2$ since $\beta < 2$. Take $\mathbf{s} := \mathbb{S}(\mathbf{a})$. Then $\mathbf{a} = \mathbb{L}(\mathbf{s})$, and

$$s_{i+1} \ldots s_m \succcurlyeq s_1 \ldots s_{m-i} \quad \text{for all } 1 \leq i < m. \tag{2.7}$$

If equality holds for some i, then we deduce just as in the proof of Lemma 2.7 that **s** is periodic. However, then **a** is also periodic, contradicting that m is the minimal period of $\delta(\beta)$. Hence, strict inequality holds in equation (2.7), and **s** is Lyndon. The converse is trivial. □

Recall the Farey intervals and Lyndon intervals from Definition 1.8. The following properties of Lyndon intervals and Farey intervals were established in the proof of [**20**, Theorem C].

LEMMA 2.9.

(i) *The Farey intervals $J^{\mathbf{s}}, \mathbf{s} \in \Omega_F^*$ are pairwise disjoint, and their union is dense in* $(1, 2]$.

(ii) *Any two Lyndon intervals are either disjoint or one is contained in the other.*

(iii) *For any Lyndon interval $J^{\mathbf{S}}, \mathbf{S} \in \Omega_L^*$, there exists a unique Farey interval $J^{\mathbf{r}}$ such that $J^{\mathbf{S}} \subset J^{\mathbf{r}}$.*

Note that (iii) follows immediately from (i) and (ii).

2.3. *Greedy expansions and the symbolic survivor set.* Given $\beta \in (1, 2]$ and $t \in [0, 1)$, we call the sequence $(d_i) \in \{0, 1\}^{\mathbb{N}}$ a *β-expansion* of t if $((d_i))_\beta = t$. Note that a point $t \in [0, 1)$ may have multiple β-expansions. We denote by $b(t, \beta) = (b_i(t, \beta)) \in \{0, 1\}^{\mathbb{N}}$ the *greedy* β-expansion of t, which is the lexicographically largest expansion of t in base β. Since $T_\beta(t) = \beta t \pmod 1$, it follows that $b(T_\beta^n(t), \beta) = \sigma^n(b(t, \beta)) = b_{n+1}(t, \beta) b_{n+2}(t, \beta) \ldots$. The following result was established by Parry [**26**] and de Vries and Komornik [**12**, Lemma 2.5 and Proposition 2.6].

LEMMA 2.10. *Let $\beta \in (1, 2]$. The map $t \mapsto b(t, \beta)$ is an increasing bijection from* $[0, 1)$ *to*

$$\{(d_i) \in \{0, 1\}^{\mathbb{N}} : \sigma^n((d_i)) \prec \delta(\beta) \text{ for all } n \geq 0\}.$$

Furthermore:

(i) *the map $t \mapsto b(t, \beta)$ is right-continuous everywhere in* $[0, 1)$ *with respect to the order topology in* $\{0, 1\}^{\mathbb{N}}$*;*

(ii) *if $b(t_0, \beta)$ does not end with 0^∞, then the map $t \mapsto b(t, \beta)$ is continuous at t_0;*

(iii) *if $b(t_0, \beta) = b_1 \ldots b_m 0^\infty$ with $b_m = 1$, then $b(t, \beta) \nearrow b_1 \ldots b_m^- \delta(\beta)$ as $t \nearrow t_0$.*

Recall that the survivor set $K_\beta(t)$ consists of all $x \in [0, 1)$ whose orbit $\{T_\beta^n(x) : n \geq 0\}$ avoids the hole $(0, t)$. To describe the dimension of $K_\beta(t)$, we introduce the topological

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

entropy of a symbolic set. For a subset $X \subset \{0, 1\}^{\mathbb{N}}$, its *topological entropy* $h_{\text{top}}(X)$ is defined by

$$h_{\text{top}}(X) := \liminf_{n\to\infty} \frac{\log \#B_n(X)}{n},$$

where $\#B_n(X)$ denotes the number of all length n words occurring in sequences of X. The following result for the Hausdorff dimension of $K_\beta(t)$ can be essentially deduced from Raith [28] (see also [20]).

LEMMA 2.11. *Given $\beta \in (1, 2]$ and $t \in [0, 1)$, the Hausdorff dimension of $K_\beta(t)$ is given by*

$$\dim_H K_\beta(t) = \frac{h_{\text{top}}(\mathbf{K}_\beta(t))}{\log \beta},$$

where

$$\mathbf{K}_\beta(t) = \{(d_i) \in \{0, 1\}^{\mathbb{N}} : b(t, \beta) \preccurlyeq \sigma^n((d_i)) \prec \delta(\beta) \text{ for all } n \geq 0\}.$$

To determine the critical value $\tau(\beta)$ for β inside any Farey interval $J^{\mathbf{s}}$, we first need to develop some properties of the substitution operator $\bullet$ from equation (1.3). We do this in the next section.

3. *Substitution of Lyndon words*

In this section, we give an equivalent definition of the substitution operator in Ω_L^* introduced in equation (1.3), and prove that Ω_L^* forms a semi-group under this substitution operator. This will play a crucial role in the rest of the paper.

3.1. *An equivalent definition of the substitution.*

Given a Lyndon word $\mathbf{s} \in \Omega_L^*$ with $\mathbf{a} = \mathbb{L}(\mathbf{s})$, we construct a directed graph $G = (V, E)$ as in Figure 2. The directed graph G has two starting vertices 'Start-0' and 'Start-1'. The directed edges in the graph G take labels from $\{0, 1\}$, and the vertices in G take labels from $\{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. Denote by $\mathcal{L}^E$ the edge labeling and by $\mathcal{L}^V$ the vertex labeling. Then for each directed edge $e \in E(G)$, we have $\mathcal{L}^E(e) \in \{0, 1\}$, and for each vertex $v \in V(G)$, we have $\mathcal{L}^V(v) \in \{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. The labeling maps $\mathcal{L}^E$ and $\mathcal{L}^V$ naturally induce the maps on the infinite edge paths and infinite vertex paths in G, respectively. For example, for an infinite edge path $e_1e_2\ldots$, we have

$$\mathcal{L}^E(e_1e_2\ldots) = \mathcal{L}^E(e_1)\mathcal{L}^E(e_2)\ldots \quad \in \{0, 1\}^{\mathbb{N}}.$$

Here we call $e_1e_2\ldots$ an *infinite edge path* in G if the initial vertex of e_1 is one of the starting vertices and for any $i \geq 1$, the *terminal vertex* $t(e_i)$ equals the *initial vertex* $i(e_{i+1})$. Similarly, for an infinite vertex path $v = v_1v_2\ldots$, we have

$$\mathcal{L}^V(v_1v_2\ldots) = \mathcal{L}^V(v_1)\mathcal{L}^V(v_2)\ldots \quad \in \{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}^{\mathbb{N}},$$

where we call $v_1v_2\ldots$ an *infinite vertex path* in G if v_1 is one of the starting vertices and for any $i \geq 1$, there exists a directed edge $e \in E(G)$ such that $i(e) = v_i$ and $t(e) = v_{i+1}$.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

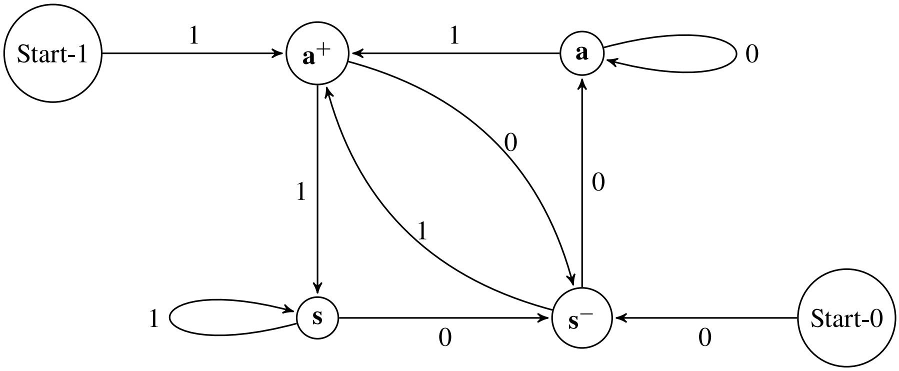

FIGURE 2. The directed graph $G = (V, E)$ with the edge labels from $\{0, 1\}$ and vertex labels from $\{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$, where $\mathbf{s} \in \Omega_L^*$ and $\mathbf{a} = \mathbb{L}(\mathbf{s})$.

Let $X_E = X_E(G)$ be the *edge shift* consisting of all labelings of infinite edge paths in G, that is,

$$X_E := \{\mathcal{L}^E(e_1e_2\ldots) : e_1e_2\ldots \text{ is an infinite edge path in } G\}.$$

One can verify easily that $X_E = \{0, 1\}^{\mathbb{N}}$. Also, let $X_V = X_V(G)$ be the *vertex shift* which consists of all labelings of infinite vertex paths in G, that is,

$$X_V := \{\mathcal{L}^V(v_1v_2\ldots) : v_1v_2\ldots \text{ is an infinite vertex path in } G\}.$$

Then any sequence in X_V is an infinite concatenation of words from $\{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. Observe that the edge shift X_E is *right-resolving*, which means that out-going edges from the same vertex have different labels (cf. [23]). Moreover, different vertices have different labels. So for each $(d_i) \in X_E$, there is a unique infinite edge path $e_1e_2\ldots$ in G such that $d_1d_2\ldots = \mathcal{L}^E(e_1e_2\ldots)$.

Definition 3.1. The substitution map $\Phi_{\mathbf{s}}$ from X_E to X_V is defined by

$$\Phi_{\mathbf{s}} : X_E \to X_V; \quad \mathcal{L}^E(e_1e_2\ldots) \mapsto \mathcal{L}^V(t(e_1)t(e_2)\ldots),$$

where $t(e_i)$ denotes the terminal vertex of the directed edge e_i.

We can extend the substitution map $\Phi_{\mathbf{s}}$ to a map from $B_*(X_E)$ to $B_*(X_V)$ by

$$\Phi_{\mathbf{s}} : B_*(X_E) \to B_*(X_V); \quad \mathcal{L}^E(e_1\ldots e_n) \mapsto \mathcal{L}^V(t(e_1)\ldots t(e_n)), \tag{3.1}$$

where $B_*(X_E)$ consists of all labelings of finite edge paths in G and $B_*(X_V)$ consists of all labelings of finite vertex paths in G. So, by equations (1.3) and (3.1), it follows that for any two words $\mathbf{s} \in \Omega_L^*$ and $\mathbf{r} \in \{0, 1\}^*$, we have

$$\mathbf{s} \bullet \mathbf{r} = \Phi_{\mathbf{s}}(\mathbf{r}). \tag{3.2}$$

Example 3.2. Let $\mathbf{s} = 01$ and $\mathbf{r} = 001011$. Then $\mathbf{s} \in \Omega_F^*$ and $\mathbf{r} \in \Omega_L^* \setminus \Omega_F^*$. Furthermore, $\mathbf{s}^- = 00$, $\mathbf{a} = \mathbb{L}(\mathbf{s}) = 10$, $\mathbf{a}^+ = 11$. So by the definition of $\Phi_{\mathbf{s}}$, it follows that

$$\Phi_{\mathbf{s}}(\mathbf{r}) = \Phi_{\mathbf{s}}(001011) = \mathbf{s}^-\mathbf{a}\mathbf{a}^+\mathbf{s}^-\mathbf{a}^+\mathbf{s} = 001011001101,$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

$$\Phi_{\mathbf{s}}(\mathbf{r}^-) = \Phi_{\mathbf{s}}(001010) = \mathbf{s}^-\mathbf{a}\mathbf{a}^+\mathbf{s}^-\mathbf{a}^+\mathbf{s}^- = 001011001100.$$

Observe that $\Phi_{\mathbf{s}}(\mathbf{r}^-) = \Phi_{\mathbf{s}}(\mathbf{r})^-$. By Definition 1.1, one can check that $\Phi_{\mathbf{s}}(\mathbf{r}) \in \Omega_L^*$. Furthermore,

$$\Phi_{\mathbf{s}}(\mathbb{L}(\mathbf{r})) = \Phi_{\mathbf{s}}(110010) = \mathbf{a}^+\mathbf{s}\mathbf{s}^-\mathbf{a}\mathbf{a}^+\mathbf{s}^- = 110100101100 = \mathbb{L}(\Phi_{\mathbf{s}}(\mathbf{r})),$$

and

$$\Phi_{\mathbf{s}}(\mathbf{r}^\infty) = \Phi_{\mathbf{s}}((001011)^\infty) = (\mathbf{s}^-\mathbf{a}\mathbf{a}^+\mathbf{s}^-\mathbf{a}^+\mathbf{s})^\infty = (001011001101)^\infty = \Phi_{\mathbf{s}}(\mathbf{r})^\infty.$$

3.2. *Properties of the substitution.* Motivated by Examples 1.2 and 3.2, we study the properties of the substitution $\Phi_{\mathbf{s}}$. We will show that Ω_L^* forms a semi-group under the substitution operator defined in Definition 3.1. First we prove the monotonicity of $\Phi_{\mathbf{s}}$.

LEMMA 3.3. *Let* $\mathbf{s} \in \Omega_L^*$. *Then the map* $\Phi_{\mathbf{s}}$ *is strictly increasing in* $X_E = \{0, 1\}^{\mathbb{N}}$.

Proof. Let (d_i) and (d_i') be two sequences in X_E, and let (e_i), (e_i') be their corresponding edge paths; thus, $(d_i) = \mathcal{L}^E((e_i))$ and $(d_i') = \mathcal{L}^E((e_i'))$. Suppose $(d_i) \prec (d_i')$. Then there exists $k \in \mathbb{N}$ such that $d_1 \ldots d_{k-1} = d_1' \ldots d_{k-1}'$ and $d_k < d_k'$. If $k = 1$, then $d_1 = 0$ and $d_1' = 1$. So, $\mathcal{L}^V(t(e_1)) = \mathbf{s}^-$ and $\mathcal{L}^V(t(e_1')) = \mathbf{a}^+$. By Definition 3.1, it follows that $\Phi_{\mathbf{s}}((d_i)) \prec \Phi_{\mathbf{s}}((d_i'))$.

If $k > 1$, then $e_1 \ldots e_{k-1} = e_1' \ldots e_{k-1}'$, which implies that the initial vertices of e_k and e_k' coincide. Since $d_k < d_k'$, by the definition of $\mathcal{L}^V$, it follows that (see Figure 2)

$$\mathcal{L}^V(t(e_k)) \prec \mathcal{L}^V(t(e_k')).$$

By Definition 3.1, we also have $\Phi_{\mathbf{s}}((d_i)) \prec \Phi_{\mathbf{s}}((d_i'))$. This completes the proof. □

LEMMA 3.4. *Let* $\mathbf{s} \in \Omega_L^*$. *Then for any word* $\mathbf{d} = d_1 \ldots d_k \in B_*(X_E)$ *with* $k \geq 2$, *we have*

$$\begin{cases} \Phi_{\mathbf{s}}(\mathbf{d}^-) = \Phi_{\mathbf{s}}(\mathbf{d})^- & \text{if } d_k = 1, \\ \Phi_{\mathbf{s}}(\mathbf{d}^+) = \Phi_{\mathbf{s}}(\mathbf{d})^+ & \text{if } d_k = 0. \end{cases}$$

Proof. Since $\mathbf{d} = d_1 \ldots d_k \in B_*(X_E)$, there exists a unique finite edge path $e_1 \ldots e_k$ such that $\mathcal{L}^E(e_1 \ldots e_k) = \mathbf{d}$. If $d_k = 1$, then $\mathbf{d}^- = d_1 \ldots d_{k-1}0$ can be represented by a unique finite edge path $e_1' \ldots e_k'$ with $e_1' \ldots e_{k-1}' = e_1 \ldots e_{k-1}$. By the definition of $\mathcal{L}^V$, it follows that $\mathcal{L}^V(t(e_k')) = \mathcal{L}^V(t(e_k))^-$. Therefore, by Definition 3.1, it follows that

$$\begin{aligned} \Phi_{\mathbf{s}}(\mathbf{d}^-) = \Phi_{\mathbf{s}}(\mathcal{L}^E(e_1' \ldots e_k')) &= \Phi_{\mathbf{s}}(\mathcal{L}^E(e_1 \ldots e_{k-1}e_k')) \\ &= \mathcal{L}^V(t(e_1) \ldots t(e_{k-1})t(e_k')) \\ &= \mathcal{L}^V(t(e_1) \ldots t(e_k))^- = \Phi_{\mathbf{s}}(\mathbf{d})^-. \end{aligned}$$

This proves the first equality of the lemma. The second equality follows analogously. □

Recall the operator $\bullet$ from equation (3.2). In the following, we prove Proposition 1.3 by showing that Ω_L^* is closed under $\bullet$ and that $\bullet$ is associative. The proof will be split into a sequence of lemmas. First we prove that Ω_L^* is closed under $\bullet$.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

LEMMA 3.5. *For any* $\mathbf{s}, \mathbf{r} \in \Omega_L^*$, *we have* $\mathbf{s} \bullet \mathbf{r} \in \Omega_L^*$.

Proof. Let $\mathbf{s} = s_1 \ldots s_m \in \Omega_L^*$ and $\mathbf{a} = \mathbb{L}(\mathbf{s})$. Then there exists $j \in \{1, \ldots, m-1\}$ such that

$$\mathbf{a} = s_{j+1} \ldots s_m s_1 \ldots s_j. \tag{3.3}$$

Let $\mathbf{r} = r_1 \ldots r_\ell \in \Omega_L^*$. Then we can write $\mathbf{s} \bullet \mathbf{r} = \Phi_{\mathbf{s}}(\mathbf{r}) = b_1 \ldots b_{m\ell}$. Furthermore, there exists a finite edge path $e_1 \ldots e_\ell$ representing $\mathbf{r}$ such that

$$\Phi_{\mathbf{s}}(\mathbf{r}) = \mathcal{L}^V(t(e_1) \ldots t(e_\ell)) =: \mathbf{b}_1 \ldots \mathbf{b}_\ell,$$

where each block $\mathbf{b}_i \in \{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. Note that $\mathbf{b}_1 = \mathbf{s}^-$ since the block $\mathbf{r}$ begins with $r_1 = 0$. By Definition 1.1, it suffices to prove

$$b_{i+1} \ldots b_{m\ell} \succ b_1 \ldots b_{m\ell - i} \quad \text{for any } 0 < i < m\ell. \tag{3.4}$$

We split the proof of equation (3.4) into two cases.

Case I. $i = km$ for some $k \in \{1, 2, \ldots, \ell - 1\}$. Then $b_{i+1} \ldots b_{m\ell} = \mathbf{b}_{k+1} \ldots \mathbf{b}_\ell$. Since $\mathbf{r}$ is a Lyndon word, we have $r_{k+1} \ldots r_\ell \succ r_1 \ldots r_{\ell-k}$. So, equation (3.4) follows directly by Lemma 3.3.

Case II. $i = km + p$ for some $k \in \{0, 1, \ldots, \ell - 1\}$ and $p \in \{1, \ldots, m-1\}$. Then $b_{i+1} \ldots b_{m\ell} = b_{i+1} \ldots b_{i+m-p} \mathbf{b}_{k+2} \ldots \mathbf{b}_\ell$. In the following, we prove equation (3.4) by considering the four possible choices of $\mathbf{b}_{k+1} \in \{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. If $\mathbf{b}_{k+1} = \mathbf{s}$, then by using that $\mathbf{s} \in \Omega_L^*$, we conclude that

$$b_{i+1} \ldots b_{i+m-p} = s_{p+1} \ldots s_m \succ s_1 \ldots s_{m-p} = b_1 \ldots b_{m-p},$$

proving equation (3.4). Similarly, if $\mathbf{b}_{k+1} = \mathbf{a}^+$, then by equation (3.3), one can also prove that $b_{i+1} \ldots b_{i+m-p} \succ b_1 \ldots b_{m-p}$. Now we assume $\mathbf{b}_{k+1} = \mathbf{s}^-$. Then by using $\mathbf{s} \in \Omega_L^*$, it follows that

$$b_{i+1} \ldots b_{i+m-p} = s_{p+1} \ldots s_m^- \succcurlyeq s_1 \ldots s_{m-p} = b_1 \ldots b_{m-p}. \tag{3.5}$$

Observe that the word $\mathbf{s}^-$ can only be followed by $\mathbf{a}$ or $\mathbf{a}^+$ in G (see Figure 2). So $\mathbf{b}_{k+2} \in \{\mathbf{a}, \mathbf{a}^+\}$. Since $\mathbf{a} = \mathbb{L}(\mathbf{s})$, we obtain that

$$b_{i+m-p+1} \ldots b_{i+m} = a_1 \ldots a_p \succcurlyeq s_{m-p+1} \ldots s_m \succ s_{m-p+1} \ldots s_m^- = b_{m-p+1} \ldots b_m. \tag{3.6}$$

Thus, by equations (3.5) and (3.6), we conclude that $b_{i+1} \ldots b_{i+m} \succ b_1 \ldots b_m$, proving equation (3.4). Finally, suppose $\mathbf{b}_{k+1} = \mathbf{a}$. Note that the word $\mathbf{a}$ can only be followed by $\mathbf{a}$ or $\mathbf{a}^+$ in G. Then by equation (3.3) and using $\mathbf{s} \in \Omega_L^*$, we have

$$b_{i+1} \ldots b_{i+m} \succcurlyeq s_1 \ldots s_m \succ b_1 \ldots b_m.$$

This completes the proof. □

Say a finite or infinite sequence of words $\mathbf{b}_1, \ldots, \mathbf{b}_n$ or $\mathbf{b}_1, \mathbf{b}_2, \ldots$ is *connectible* if for each i, the last digit of $\mathbf{b}_i$ differs from the first digit of $\mathbf{b}_{i+1}$. Thus, for instance, the sequence 1101, 00111 is connectible whereas the sequence 11010, 0111 is not.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

LEMMA 3.6.

(i) *Let* $\mathbf{b}_1, \mathbf{b}_2, \ldots$ *be a (finite or infinite) connectible sequence of words. Then for any* $\mathbf{s} \in \Omega_L^*$,

$$\Phi_{\mathbf{s}}(\mathbf{b}_1\mathbf{b}_2\ldots) = \Phi_{\mathbf{s}}(\mathbf{b}_1)\Phi_{\mathbf{s}}(\mathbf{b}_2)\ldots.$$

(ii) *Let* $\mathbf{s}, \mathbf{r} \in \Omega_L^*$. *Then* $\Phi_{\mathbf{s}}(\mathbf{r}^\infty) = \Phi_{\mathbf{s}}(\mathbf{r})^\infty$ *and* $\Phi_{\mathbf{s}}(\mathbb{L}(\mathbf{r})^\infty) = \Phi_{\mathbf{s}}(\mathbb{L}(\mathbf{r}))^\infty$.

Proof. To prove (i), it suffices to show that if $\mathbf{b}_1, \mathbf{b}_2$ is a connectible sequence, then $\Phi_{\mathbf{s}}(\mathbf{b}_1\mathbf{b}_2) = \Phi_{\mathbf{s}}(\mathbf{b}_1)\Phi_{\mathbf{s}}(\mathbf{b}_2)$; the statement then extends to arbitrary connectible sequences by induction.

Without loss of generality, by the symmetry of the edge-labels in Figure 2, we may assume that $\mathbf{b}_1$ ends in the digit 0 and $\mathbf{b}_2$ begins with the digit 1. However, note that in the directed graph in Figure 2, if we travel along an edge labeled 0 followed by an edge labeled 1, we always end up at the vertex labeled $\mathbf{a}^+$, which is also the first vertex visited after traveling along an edge labeled 1 from the 'Start-1' vertex. Thus, $\Phi_{\mathbf{s}}(\mathbf{b}_1\mathbf{b}_2) = \Phi_{\mathbf{s}}(\mathbf{b}_1)\Phi_{\mathbf{s}}(\mathbf{b}_2)$.

Statement (ii) follows from (i) since $\mathbf{r}$ begins with digit 0 and ends with digit 1, so $\mathbf{r}$ is connectible to itself; and similarly, $\mathbb{L}(\mathbf{r})$ begins with digit 1 and ends with digit 0, so $\mathbb{L}(\mathbf{r})$ is connectible to itself. □

To prove that $\bullet$ is associative, we need the following result, which says that the two operators $\bullet$ and $\mathbb{L}$ commute.

LEMMA 3.7. *For any* $\mathbf{s}, \mathbf{r} \in \Omega_L^*$, *we have* $\mathbb{L}(\mathbf{s} \bullet \mathbf{r}) = \mathbf{s} \bullet \mathbb{L}(\mathbf{r})$.

Proof. The proof is similar to that of Lemma 3.5. Let $\mathbf{r} = r_1 \ldots r_\ell \in \Omega_L^*$. First we show that $\mathbf{s} \bullet \mathbb{L}(\mathbf{r})$ is a cyclic permutation of $\mathbf{s} \bullet \mathbf{r}$. Note that $\mathbb{L}(\mathbf{r}) = r_{j+1} \ldots r_\ell r_1 \ldots r_j$ for some $1 < j < \ell$. Then $r_j = 0$ and $r_{j+1} = 1$, so Lemma 3.6(i) implies that

$$\mathbf{s} \bullet \mathbf{r} = \Phi_{\mathbf{s}}(r_1 \ldots r_\ell) = \Phi_{\mathbf{s}}(r_1 \ldots r_j)\Phi_{\mathbf{s}}(r_{j+1} \ldots r_\ell). \tag{3.7}$$

However, since $\mathbf{r} \in \Omega_L^*$, we have $r_\ell = 1$ and $r_1 = 0$, so by Lemma 3.6(i), we obtain that

$$\mathbf{s} \bullet \mathbb{L}(\mathbf{r}) = \Phi_{\mathbf{s}}(r_{j+1} \ldots r_\ell r_1 \ldots r_j) = \Phi_{\mathbf{s}}(r_{j+1} \ldots r_\ell)\Phi_{\mathbf{s}}(r_1 \ldots r_j).$$

This, together with equation (3.7), proves that $\mathbf{s} \bullet \mathbb{L}(\mathbf{r})$ is indeed a cyclic permutation of $\mathbf{s} \bullet \mathbf{r}$. It remains to prove that $\mathbf{s} \bullet \mathbb{L}(\mathbf{r})$ is the lexicographically largest cyclic permutation of itself.

Write $\mathbf{s} = s_1 \ldots s_m \in \Omega_L^*$ with $\mathbf{a} = \mathbb{L}(\mathbf{s}) = a_1 \ldots a_m$, and write $\mathbb{L}(\mathbf{r}) = c_1 \ldots c_\ell$. Then by Lemma 2.7, it follows that

$$\begin{aligned} a_{i+1} \ldots a_m &\prec a_1 \ldots a_{m-i} \quad \text{for all } 0 < i < m; \\ c_{i+1} \ldots c_\ell &\prec c_1 \ldots c_{\ell-i} \quad \text{for all } 0 < i < \ell. \end{aligned} \tag{3.8}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Write $\mathbf{s} \bullet \mathbb{L}(\mathbf{r}) = \mathbf{b}_1 \ldots \mathbf{b}_\ell = b_1 \ldots b_{m\ell}$, where each $\mathbf{b}_i \in \{\mathbf{s}^-, \mathbf{s}, \mathbf{a}, \mathbf{a}^+\}$. Then it suffices to prove that

$$b_{i+1} \ldots b_{m\ell} \prec b_1 \ldots b_{m\ell-i} \quad \text{for all } 0 < i < m\ell. \tag{3.9}$$

Since $\mathbb{L}(\mathbf{r})$ has a prefix $c_1 = 1$, we see that $b_1 \ldots b_m = \mathbf{b}_1 = \mathbf{a}^+$. So, by using equation (3.8) and the same argument as in the proof of Lemma 3.5, we can prove equation (3.9). □

The next lemma will be used in the proof of Lemma 5.3 and Proposition 6.2.

LEMMA 3.8. *Let* $\mathbf{s} \in \Omega_L^*$, *and take two sequences* $(c_i), (d_i) \in \{0, 1\}^{\mathbb{N}}$.

(i) *If* $d_1 = 1$, *then*

$$\sigma^n((c_i)) \prec (d_i) \quad \textit{for all } n \geq 0 \quad \Longrightarrow \quad \sigma^n(\Phi_{\mathbf{s}}((c_i))) \prec \Phi_{\mathbf{s}}((d_i)) \quad \textit{for all } n \geq 0.$$

(ii) *If* $d_1 = 0$, *then*

$$\sigma^n((c_i)) \succ (d_i) \quad \textit{for all } n \geq 0 \quad \Longrightarrow \quad \sigma^n(\Phi_{\mathbf{s}}((c_i))) \succ \Phi_{\mathbf{s}}((d_i)) \quad \textit{for all } n \geq 0.$$

Proof. (i) Suppose $d_1 = 1$ and $\sigma^n((c_i)) \prec (d_i)$ for all $n \geq 0$. Then $\Phi_{\mathbf{s}}((d_i))$ begins with $\mathbb{L}(\mathbf{s})^+$. If $n \equiv 0 (\text{mod } |\mathbf{s}|)$, then by Lemma 3.3, it follows that $\sigma^n(\Phi_{\mathbf{s}}((c_i))) \prec \Phi_{\mathbf{s}}((d_i))$. If $n \not\equiv 0 (\text{mod } |\mathbf{s}|)$, then by using $\Phi_{\mathbf{s}}(d_1) = \mathbb{L}(\mathbf{s})^+$ and the same argument as in the proof of Lemma 3.7, one can verify that $\sigma^n(\Phi_{\mathbf{s}}((c_i))) \prec \Phi_{\mathbf{s}}((d_i))$. The proof of (ii) is similar. □

Finally, we show that $\bullet$ is associative.

LEMMA 3.9. *For any three words* $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \Omega_L^*$, *we have* $(\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t} = \mathbf{r} \bullet (\mathbf{s} \bullet \mathbf{t})$.

Proof. Let $\mathbf{r} = r_1 \ldots r_m$, $\mathbf{s} = s_1 \ldots s_n$, and $\mathbf{t} = t_1 \ldots t_\ell$. Then we can write$(\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t}$ as

$$(\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t} = \mathbf{B}_1 \mathbf{B}_2 \ldots \mathbf{B}_\ell, \tag{3.10}$$

where each $\mathbf{B}_i \in \{(\mathbf{r} \bullet \mathbf{s})^-, \mathbf{r} \bullet \mathbf{s}, \mathbb{L}(\mathbf{r} \bullet \mathbf{s}), \mathbb{L}(\mathbf{r} \bullet \mathbf{s})^+\}$. Since $t_1 = 0$, we have $\mathbf{B}_1 = (\mathbf{r} \bullet \mathbf{s})^-$. Furthermore, by the definition of $\Phi_{\mathbf{r}\bullet\mathbf{s}}$ it follows that for $1 < i \leq \ell$,

$$\mathbf{B}_i = \begin{cases} \mathbb{L}(\mathbf{r} \bullet \mathbf{s}) & \text{if } t_{i-1}t_i = 00, \\ \mathbb{L}(\mathbf{r} \bullet \mathbf{s})^+ & \text{if } t_{i-1}t_i = 01, \\ (\mathbf{r} \bullet \mathbf{s})^- & \text{if } t_{i-1}t_i = 10, \\ \mathbf{r} \bullet \mathbf{s} & \text{if } t_{i-1}t_i = 11. \end{cases} \tag{3.11}$$

Similarly, we can write

$$\mathbf{s} \bullet \mathbf{t} = \mathbf{b}_1 \mathbf{b}_2 \ldots \mathbf{b}_\ell,$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

where each $\mathbf{b}_i \in \{\mathbf{s}^-, \mathbf{s}, \mathbb{L}(\mathbf{s}), \mathbb{L}(\mathbf{s})^+\}$, and it follows from the definition of $\Phi_\mathbf{s}$ that

$$\mathbf{b}_i = \begin{cases} \mathbb{L}(\mathbf{s}) & \text{if } t_{i-1}t_i = 00, \\ \mathbb{L}(\mathbf{s})^+ & \text{if } t_{i-1}t_i = 01, \\ \mathbf{s}^- & \text{if } t_{i-1}t_i = 10, \\ \mathbf{s} & \text{if } t_{i-1}t_i = 11, \end{cases} \tag{3.12}$$

for $1 < i \le \ell$. Comparing equations (3.11) and (3.12) and using Lemmas 3.4 and 3.7, it follows that

$$\Phi_\mathbf{r}(\mathbf{b}_i) = \mathbf{B}_i \quad \text{for all } i \ge 1.$$

Moreover, the sequence $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_\ell$ is connectible because $\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_\ell = \Phi_\mathbf{s}(\mathbf{t})$ arises from a walk along the directed graph in Figure 2. Hence, Lemma 3.6 and equation (3.10) yield

$$\mathbf{r} \bullet (\mathbf{s} \bullet \mathbf{t}) = \Phi_\mathbf{r}(\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_\ell) = \Phi_\mathbf{r}(\mathbf{b}_1)\Phi_\mathbf{r}(\mathbf{b}_2) \ldots \Phi_\mathbf{r}(\mathbf{b}_\ell) = \mathbf{B}_1\mathbf{B}_2 \ldots \mathbf{B}_\ell = (\mathbf{r} \bullet \mathbf{s}) \bullet \mathbf{t},$$

as desired. □

Proof of Proposition 1.3. The proposition follows by Lemmas 3.5 and 3.9 and Example 1.2, which shows that $\bullet$ is not commutative. □

4. *Critical values in a basic interval*

In this section, we will prove Theorem 2. Recall from equation (1.4) that Λ consists of all words $\mathbf{S}$ of the form

$$\mathbf{S} = \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k, \quad k \in \mathbb{N},$$

where each $\mathbf{s}_i \in \Omega_F^*$. By Proposition 1.3, it follows that $\Lambda \subset \Omega_L^*$, and each $\mathbf{S} \in \Lambda$ can be uniquely represented in the above form. Take $\mathbf{S} \in \Lambda$. As in Definition 1.5, we let $I^\mathbf{S} := [\beta_\ell^\mathbf{S}, \beta_*^\mathbf{S}]$ be the basic interval generated by $\mathbf{S}$. Then by Lemmas 2.5 and 2.7, it follows that

$$\delta(\beta_\ell^\mathbf{S}) = \mathbb{L}(\mathbf{S})^\infty \quad \text{and} \quad \delta(\beta_*^\mathbf{S}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty. \tag{4.1}$$

To prove Theorem 2, we first prove the following proposition, which provides one of the key tools in this paper and will be used again in §6.

PROPOSITION 4.1. *For any $\mathbf{S} \in \Lambda$, the set*

$$\Gamma(\mathbf{S}) := \{(x_i) : \mathbf{S}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbb{L}(\mathbf{S})^\infty \text{ for all } n \ge 0\} \tag{4.2}$$

is countable.

We point out that the specific form of $\mathbf{S}$ is essential in this proposition: it is not enough to merely assume that $\mathbf{S}$ is a Lyndon word. For instance, take $\mathbf{S} = 0010111 \in \Omega_L^*$. Then $\mathbb{L}(\mathbf{S}) = 1110010$, and it is easy to see that $\Gamma(\mathbf{S}) \supset \{10, 110\}^\mathbb{N}$.

For $\mathbf{S} = \mathbf{s} \in \Omega_F^*$, Proposition 4.1 follows from the following stronger result, proved in [20, Proposition 4.4].

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

LEMMA 4.2. *For any* $\mathbf{s} = s_1 \ldots s_m \in \Omega_F^*$, *the set*

$$\Gamma(\mathbf{s}) := \{(x_i) : \mathbf{s}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbb{L}(\mathbf{s})^\infty \text{ for all } n \geq 0\}$$

consists of exactly m different elements.

To reduce the technicalities in the proof of Proposition 4.1, we extend the definition from Lemma 2.3(iii) and define the *conjugate* of any word $\mathbf{S} \in \Lambda$ by

$$\varphi(\mathbf{S}) := \overline{\mathbb{L}(\mathbf{S})}.$$

LEMMA 4.3. *The function* $\varphi : \Lambda \to \{0, 1\}^*$; $\mathbf{S} \mapsto \varphi(\mathbf{S})$ *is a semigroup automorphism on* $(\Lambda, \bullet)$. *That is,* φ *maps* Λ *bijectively onto itself, and*

$$\varphi(\mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k) = \varphi(\mathbf{s}_1) \bullet \cdots \bullet \varphi(\mathbf{s}_k) \quad \text{for all } \mathbf{s}_1, \ldots, \mathbf{s}_k \in \Omega_F^*. \tag{4.3}$$

Furthermore, φ *is its own inverse:*

$$\varphi(\varphi(\mathbf{S})) = \mathbf{S} \quad \text{for all } \mathbf{S} \in \Lambda. \tag{4.4}$$

Proof. We prove equations (4.3) and (4.4) simultaneously by induction on the degree k of $\mathbf{S} = \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k$. For $k = 1$, equation (4.3) is trivial and equation (4.4) follows from Lemma 2.3(iii). Now suppose equations (4.3) and (4.4) both hold for any word $\mathbf{S} = \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k$ of degree k, and consider $\mathbf{S} \bullet \mathbf{r}$ with $\mathbf{r} \in \Omega_F^*$. Set $\widetilde{\mathbf{S}} := \varphi(\mathbf{S})$. We claim first that for any word $\mathbf{t} \in \{0, 1\}^*$,

$$\Phi_{\mathbf{S}}(\overline{\mathbf{t}}) = \overline{\Phi_{\widetilde{\mathbf{S}}}(\mathbf{t})}. \tag{4.5}$$

The expression on the right is well defined since, by equation (4.3), $\widetilde{\mathbf{S}} = \varphi(\mathbf{s}_1) \bullet \cdots \bullet \varphi(\mathbf{s}_k) \in \Omega_L^*$.

Write $\mathbf{A} := \mathbb{L}(\mathbf{S})$ and $\widetilde{\mathbf{A}} := \mathbb{L}(\widetilde{\mathbf{S}})$. By equation (4.4), $\varphi(\widetilde{\mathbf{S}}) = \mathbf{S}$, so we have

$$\overline{\mathbf{A}} = \widetilde{\mathbf{S}} \quad \text{and} \quad \widetilde{\mathbf{A}} = \overline{\mathbf{S}}. \tag{4.6}$$

Now note the rotational skew-symmetry in the edge labels of the directed graph in Figure 2. The edge path corresponding to the word $\overline{\mathbf{t}}$ is just the 180° rotation about the center of the figure of the edge path corresponding to $\mathbf{t}$. However, replacing the vertex labels $\mathbf{S}, \mathbf{S}^-, \mathbf{A}$, and $\mathbf{A}^+$ by $\widetilde{\mathbf{S}} = \overline{\mathbf{A}}$, $\widetilde{\mathbf{S}}^- = \overline{\mathbf{A}^+}$, $\widetilde{\mathbf{A}} = \overline{\mathbf{S}}$, and $\widetilde{\mathbf{A}}^+ = \overline{\mathbf{S}^-}$, respectively, and rotating the whole graph by 180°, we get the original graph back except that all the vertex labels and edge labels are reflected. This implies equation (4.5).

Now we can apply equation (4.5) to $\mathbf{t} = \overline{\mathbb{L}(\mathbf{r})}$ and obtain:

$$\varphi(\mathbf{S}) \bullet \varphi(\mathbf{r}) = \widetilde{\mathbf{S}} \bullet \varphi(\mathbf{r}) = \Phi_{\widetilde{\mathbf{S}}}(\overline{\mathbb{L}(\mathbf{r})}) = \overline{\Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r}))} = \overline{\mathbf{S} \bullet \mathbb{L}(\mathbf{r})} = \overline{\mathbb{L}(\mathbf{S} \bullet \mathbf{r})} = \varphi(\mathbf{S} \bullet \mathbf{r}). \tag{4.7}$$

Since $\mathbf{r} \in \Omega_F^*$ was arbitrary, the induction hypothesis of equations (4.3) and (4.7) give

$$\varphi(\mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k \bullet \mathbf{s}_{k+1}) = \varphi(\mathbf{s}_1) \bullet \cdots \bullet \varphi(\mathbf{s}_k) \bullet \varphi(\mathbf{s}_{k+1}) \quad \text{for all } \mathbf{s}_1, \ldots, \mathbf{s}_{k+1} \in \Omega_F^*. \tag{4.8}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Thus, equation (4.3) holds for $k+1$ in place of k. Next, by Lemma 2.3(iii), $\varphi(\mathbf{s}_i) \in \Omega_F^*$ and $\varphi(\varphi(\mathbf{s}_i)) = \mathbf{s}_i$ for each i, so applying equation (4.8) with $\varphi(\mathbf{s}_i)$ in place of $\mathbf{s}_i$ for each i, we conclude that $\varphi(\varphi(\mathbf{S}')) = \mathbf{S}'$ for every $\mathbf{S}' \in \Lambda$ of degree $k+1$ also.

Thus, we have proved equations (4.3) and (4.4) by induction. Now the remaining statements of the lemma follow immediately: by equation (4.3), Lemma 2.3(iii), and Proposition 1.3, it follows that $\varphi(\mathbf{S}) \in \Lambda$ for every $\mathbf{S} \in \Lambda$, whereas equation (4.4) implies that $\varphi : \Lambda \to \Lambda$ is bijective. Therefore, φ is an automorphism of $(\Lambda, \bullet)$. □

Define

$$\overline{\Gamma(\mathbf{S})} := \{\overline{(x_i)} : (x_i) \in \Gamma(\mathbf{S})\}, \quad \mathbf{S} \in \Lambda.$$

It is clear that $\overline{\Gamma(\mathbf{S})}$ has the same cardinality as $\Gamma(\mathbf{S})$. Observe also by equation (4.6) that

$$\begin{aligned}\overline{\Gamma(\mathbf{S})} &= \{(y_i) : \mathbf{S}^\infty \preccurlyeq \sigma^n(\overline{(y_i)}) \preccurlyeq \mathbb{L}(\mathbf{S})^\infty \ \text{ for all } n \geq 0\} \\ &= \{(y_i) : \overline{\mathbf{S}}^\infty \succcurlyeq \sigma^n((y_i)) \succcurlyeq \overline{\mathbb{L}(\mathbf{S})}^\infty \ \text{ for all } n \geq 0\} \\ &= \{(y_i) : \mathbb{L}(\varphi(\mathbf{S}))^\infty \succcurlyeq \sigma^n((y_i)) \succcurlyeq \varphi(\mathbf{S})^\infty \ \text{ for all } n \geq 0\} \\ &= \Gamma(\varphi(\mathbf{S})). \end{aligned} \tag{4.9}$$

Proof of Proposition 4.1. For $\mathbf{S} = \mathbf{s} \in \Omega_F^*$, the proposition follows from Lemma 4.2. So it suffices to prove that if $\Gamma(\mathbf{S})$ is countable for an $\mathbf{S} \in \Lambda$, then $\Gamma(\mathbf{S} \bullet \mathbf{r})$ is also countable for any $\mathbf{r} \in \Omega_F^*$.

Fix $\mathbf{S} \in \Lambda$ with $\Gamma(\mathbf{S})$ countable; fix $\mathbf{r} \in \Omega_F^*$, and note that $\mathbf{r}$ begins with 0 and $\mathbb{L}(\mathbf{r})$ begins with 1. Therefore, $\mathbf{S} \bullet \mathbf{r}$ begins with $\mathbf{S}^-$ and $\mathbb{L}(\mathbf{S} \bullet \mathbf{r}) = \mathbf{S} \bullet \mathbb{L}(\mathbf{r})$ begins with $\mathbb{L}(\mathbf{S})^+$. So

$$(\mathbf{S} \bullet \mathbf{r})^\infty \prec \mathbf{S}^\infty \quad \text{and} \quad \mathbb{L}(\mathbf{S} \bullet \mathbf{r})^\infty \succ \mathbb{L}(\mathbf{S})^\infty. \tag{4.10}$$

By equations (4.2) and (4.10), it follows that

$$\Gamma(\mathbf{S}) \subseteq \{(x_i) : (\mathbf{S} \bullet \mathbf{r})^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbb{L}(\mathbf{S} \bullet \mathbf{r})^\infty \text{ for all } n \geq 0\} = \Gamma(\mathbf{S} \bullet \mathbf{r}).$$

Since $\Gamma(\mathbf{S})$ is countable, it suffices to prove that the difference set $\Gamma(\mathbf{S} \bullet \mathbf{r}) \setminus \Gamma(\mathbf{S})$ is countable. By Proposition 2.4, the word $\mathbf{r}$ must be of one of the following four types:

(I) $\mathbf{r} = 01^p$ for some $p \in \mathbb{N}$;
(II) $\mathbf{r} = 0^p 1$ for some $p \in \mathbb{N}$;
(III) $\mathbf{r} = 01^p 01^{p+t_1} \ldots 01^{p+t_N} 01^{p+1}$ for some $p \in \mathbb{N}$ and $0t_1 \ldots t_N 1 \in \Omega_F^*$;
(IV) $\mathbf{r} = 0^{p+1} 10^{p+t_1} 1 \ldots 0^{p+t_N} 10^p 1$ for some $p \in \mathbb{N}$ and $0t_1 \ldots t_N 1 \in \Omega_F^*$;

Since the words in (II) and (IV) are the conjugates of the words in (I) and (III), respectively, it suffices by Lemma 4.3 and the relationship of equation (4.9) to consider cases (I) and (III). Let $\mathbf{A} := \mathbb{L}(\mathbf{S})$.

Case I. $\mathbf{r} = 01^p$ for some $p \in \mathbb{N}$. Note that $\mathbf{S} \bullet \mathbf{r} = \Phi_{\mathbf{S}}(01^p) = \mathbf{S}^- \mathbf{A}^+ \mathbf{S}^{p-1}$ and $\mathbb{L}(\mathbf{S} \bullet \mathbf{r}) = \mathbf{S} \bullet \mathbb{L}(\mathbf{r}) = \Phi_{\mathbf{S}}(1^p 0) = \mathbf{A}^+ \mathbf{S}^{p-1} \mathbf{S}^-$. Then $\Gamma(\mathbf{S} \bullet \mathbf{r})$ consists of all sequences $(x_i) \in \{0,1\}^{\mathbb{N}}$ satisfying

$$(\mathbf{S}^- \mathbf{A}^+ \mathbf{S}^{p-1})^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (\mathbf{A}^+ \mathbf{S}^{p-1} \mathbf{S}^-)^\infty \quad \text{for all } n \geq 0. \tag{4.11}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Take a sequence $(x_i) \in \Gamma(\mathbf{S} \bullet \mathbf{r}) \setminus \Gamma(\mathbf{S})$. Then by equations (4.2) and (4.11), it follows that $x_{k+1} \ldots x_{k+m} = \mathbf{S}^-$ or $\mathbf{A}^+$ for some $k \ge 0$. If $x_{k+1} \ldots x_{k+m} = \mathbf{S}^-$, then by taking $n = k$ in equation (4.11), we obtain

$$x_{k+m+1}x_{k+m+2} \ldots \succcurlyeq (\mathbf{A}^+\mathbf{S}^{p-1}\mathbf{S}^-)^\infty.$$

However, by taking $n = k + m$ in equation (4.11), we see that the above inequality is indeed an equality. So, $x_{k+1}x_{k+2} \ldots = (\mathbf{S}^-\mathbf{A}^+\mathbf{S}^{p-1})^\infty$.

If $x_{k+1} \ldots x_{k+m} = \mathbf{A}^+$, then by taking $n = k$ in equation (4.11), we have

$$x_{k+m+1}x_{k+m+2} \ldots \preccurlyeq (\mathbf{S}^{p-1}\mathbf{S}^-\mathbf{A}^+)^\infty. \tag{4.12}$$

Note by equation (4.11) that $x_{i+1} \ldots x_{i+m} \succcurlyeq \mathbf{S}^-$ for all $i \ge 0$. So by equation (4.12), there must exist a $j \in \{k + m, k + 2m, \ldots, k + pm\}$ such that $x_{j+1} \ldots x_{j+m} = \mathbf{S}^-$. Then by the same argument as above, we conclude that $x_{j+1}x_{j+2} \ldots = (\mathbf{S}^-\mathbf{A}^+\mathbf{S}^{p-1})^\infty$. So, $\Gamma(\mathbf{S} \bullet \mathbf{r}) \setminus \Gamma(\mathbf{S})$ is at most countable.

Case III. $\mathbf{r} = 01^p01^{p+t_1} \ldots 01^{p+t_N}01^{p+1}$, where $p \in \mathbb{N}$ and $\hat{\mathbf{r}} := 0t_1 \ldots t_N1 \in \Omega_F^*$. Consider the substitution

$$\eta_p := U_1^p \circ U_0 : 0 \mapsto 01^p; \quad 1 \mapsto 01^{p+1}.$$

Then $\mathbf{r} = \eta_p(\hat{\mathbf{r}})$, as shown in the proof of Proposition 2.4. Note by Lemma 2.3 that $\mathbb{L}(\mathbf{r}) = 1^{p+1}01^{p+t_1}01^{p+t_2} \ldots 01^{p+t_N}01^p0$ and $\mathbb{L}(\hat{\mathbf{r}}) = 1t_1 \ldots t_N0$. Then

$$\mathbb{L}(\mathbf{r})^\infty = \sigma((01^{p+1}01^{p+t_1} \ldots 01^{p+t_N}01^p)^\infty) = \sigma(\eta_p((1t_1 \ldots t_N0)^\infty)) = \sigma(\eta_p(\mathbb{L}(\hat{\mathbf{r}})^\infty)). \tag{4.13}$$

Claim. If $(x_i) \in \Gamma(\mathbf{S} \bullet \mathbf{r})$ begins with $x_1 \ldots x_m = \mathbf{S}^-$, then there exists a unique sequence $(z_i) \in \{0, 1\}^{\mathbb{N}}$ such that $(x_i) = \Phi_{\mathbf{S}}(\eta_p(z_1z_2 \ldots))$.

Note that $\mathbf{r}$ begins with 01^p0 and $\mathbb{L}(\mathbf{r})$ begins with $1^{p+1}0$. Then $\mathbf{S} \bullet \mathbf{r}$ begins with $\Phi_{\mathbf{S}}(01^p0) = \mathbf{S}^-\mathbf{A}^+\mathbf{S}^{p-1}\mathbf{S}^-$ and $\mathbb{L}(\mathbf{S} \bullet \mathbf{r}) = \mathbf{S} \bullet \mathbb{L}(\mathbf{r})$ begins with $\Phi_{\mathbf{S}}(1^{p+1}0) = \mathbf{A}^+\mathbf{S}^p\mathbf{S}^-$. Let $(x_i) \in \Gamma(\mathbf{S} \bullet \mathbf{r})$ with $x_1 \ldots x_m = \mathbf{S}^-$. Then

$$\mathbf{S}^-\mathbf{A}^+\mathbf{S}^{p-1}\mathbf{S}^- \preccurlyeq x_{n+1} \ldots x_{n+m(p+2)} \preccurlyeq \mathbf{A}^+\mathbf{S}^p\mathbf{S}^- \quad \text{for all } n \ge 0. \tag{4.14}$$

By taking $n = 0$ in equation (4.14), it follows that

$$x_{m+1} \ldots x_{m(p+2)} \succcurlyeq \mathbf{A}^+\mathbf{S}^{p-1}\mathbf{S}^-. \tag{4.15}$$

However, by taking $n = m$ in equation (4.14), we have

$$x_{m+1} \ldots x_{m(p+3)} \preccurlyeq \mathbf{A}^+\mathbf{S}^p\mathbf{S}^-. \tag{4.16}$$

By equations (4.14)–(4.16), it follows that

$$\text{either} \quad x_{m+1} \ldots x_{m(p+2)} = \mathbf{A}^+\mathbf{S}^{p-1}\mathbf{S}^- \quad \text{or} \quad x_{m+1} \ldots x_{m(p+3)} = \mathbf{A}^+\mathbf{S}^p\mathbf{S}^-.$$

Observe that in both cases, we obtain a block ending with $\mathbf{S}^-$. Then we can repeat the above argument indefinitely, and conclude that

$$(x_i) \in \{\mathbf{S}^-\mathbf{A}^+\mathbf{S}^{p-1}, \mathbf{S}^-\mathbf{A}^+\mathbf{S}^p\}^{\mathbb{N}} = \{\Phi_{\mathbf{S}}(\eta_p(0)), \Phi_{\mathbf{S}}(\eta_p(1))\}^{\mathbb{N}}. \tag{4.17}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Since $\eta_p(0) = 01^p$ and $\eta_p(1) = 01^{p+1}$, it follows from equation (4.17) and Lemma 3.6 that

$$(x_i) = \Phi_{\mathbf{S}}(\eta_p(z_1))\Phi_{\mathbf{S}}(\eta_p(z_2)) \ldots = \Phi_{\mathbf{S}}(\eta_p(z_1)\eta_p(z_2) \ldots) = \Phi_{\mathbf{S}}(\eta_p(z_1 z_2 \ldots)) \tag{4.18}$$

for some sequence $(z_i) \in \{0, 1\}^{\mathbb{N}}$. The uniqueness of (z_i) follows by the definition of the substitutions η_p and $\Phi_{\mathbf{S}}$. This proves the claim.

Now take a sequence $(x_i) \in \Gamma(\mathbf{S} \bullet \mathbf{r}) \setminus \Gamma(\mathbf{S})$. Then by equations (4.2) and (4.14), we can find an $n_0 \geq 0$ such that $x_{n_0+1} \ldots x_{n_0+m} = \mathbf{S}^-$ or $\mathbf{A}^+$. If $x_{n_0+1} \ldots x_{n_0+m} = \mathbf{A}^+$, then by equation (4.14), there must exist $n_1 > n_0$ such that $x_{n_1+1} \ldots x_{n_1+m} = \mathbf{S}^-$. So, without loss of generality, we may assume $x_{n_0+1} \ldots x_{n_0+m} = \mathbf{S}^-$. Then by the claim there is a unique sequence $(z_i) \in \{0, 1\}^{\mathbb{N}}$ that $x_{n_0+1}x_{n_0+2} \ldots = \Phi_{\mathbf{S}}(\eta_p(z_1 z_2 \ldots)) \in \Gamma(\mathbf{S} \bullet \mathbf{r})$. By the definition of $\Gamma(\mathbf{S} \bullet \mathbf{r})$, it follows that

$$(\mathbf{S} \bullet \mathbf{r})^\infty \preccurlyeq \sigma^n(\Phi_{\mathbf{S}}(\eta_p(z_1 z_2 \ldots))) \preccurlyeq \mathbb{L}(\mathbf{S} \bullet \mathbf{r})^\infty \quad \text{for all } n \geq 0. \tag{4.19}$$

Note by $\mathbf{r} = \eta_p(\hat{\mathbf{r}})$ and Lemma 3.6(ii) that $(\mathbf{S} \bullet \mathbf{r})^\infty = \Phi_{\mathbf{S}}(\mathbf{r}^\infty) = \Phi_{\mathbf{S}}(\eta_p(\hat{\mathbf{r}}^\infty))$. Similarly, by Lemmas 3.6(ii), 3.7, and equation (4.13), it follows that $\mathbb{L}(\mathbf{S} \bullet \mathbf{r})^\infty = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^\infty) = \Phi_{\mathbf{S}}(\sigma(\eta_p(\mathbb{L}(\hat{\mathbf{r}})^\infty)))$. Thus, equation (4.19) can be rewritten as

$$\Phi_{\mathbf{S}}(\eta_p(\hat{\mathbf{r}}^\infty)) \preccurlyeq \sigma^n(\Phi_{\mathbf{S}}(\eta_p(z_1 z_2 \ldots))) \preccurlyeq \Phi_{\mathbf{S}}(\sigma(\eta_p(\mathbb{L}(\hat{\mathbf{r}})^\infty))) \quad \text{for all } n \geq 0.$$

By Lemma 3.3, this implies that

$$\eta_p(\hat{\mathbf{r}}^\infty) \preccurlyeq \sigma^n(\eta_p(z_1 z_2 \ldots)) \preccurlyeq \sigma(\eta_p(\mathbb{L}(\hat{\mathbf{r}})^\infty)) \quad \text{for all } n \geq 0. \tag{4.20}$$

Note $(c_i) \prec (d_i)$ is equivalent to $\eta_p((c_i)) \prec \eta_p((d_i))$. So, by equation (4.20) and the definition of η_p, it follows that

$$\hat{\mathbf{r}}^\infty \preccurlyeq \sigma^n(z_1 z_2 \ldots) \preccurlyeq \mathbb{L}(\hat{\mathbf{r}})^\infty \quad \text{for all } n \geq 0,$$

and hence $(z_i) \in \Gamma(\hat{\mathbf{r}})$. Since $\hat{\mathbf{r}} \in \Omega_F^*$, we know that $\Gamma(\hat{\mathbf{r}})$ is finite by Lemma 4.2. Hence there are only countably many choices for the sequence (z_i), and thus by the claim, there are only countably many choices for the tail sequence of (x_i). Therefore, $\Gamma(\mathbf{S} \bullet \mathbf{r}) \setminus \Gamma(\mathbf{S})$ is at most countable. □

Recall from Lemma 2.11 the symbolic survivor set $\mathbf{K}_\beta(t)$. To prove Theorem 2, we also recall the following result from [20, Lemma 3.7].

LEMMA 4.4. *Let $\beta \in (1, 2]$ and $t \in [0, 1)$. If $\sigma^m(\delta(\beta)) \preccurlyeq b(t, \beta)$, then*

$$\mathbf{K}_\beta(t) = \{(d_i) : b(t, \beta) \preccurlyeq \sigma^n((d_i)) \preccurlyeq (\delta_1(\beta) \ldots \delta_m(\beta)^-)^\infty \text{ for all } n \geq 0\}.$$

Proof of Theorem 2. That the basic intervals $I^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$ are pairwise disjoint will be shown in Proposition 5.1 below. In what follows, we fix a basic interval $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$. Take $\beta \in I^{\mathbf{S}}$, and let $t^* = (\Phi_{\mathbf{S}}(0^\infty))_\beta = (\mathbf{S}^-\mathbf{A}^\infty)_\beta$, where $\mathbf{A} = \mathbb{L}(\mathbf{S}) = A_1 \ldots A_m$. Then by equation (4.1) and Lemma 2.5, it follows that

$$\mathbf{A}^\infty = \delta(\beta_\ell^{\mathbf{S}}) \preccurlyeq \delta(\beta) \preccurlyeq \delta(\beta_*^{\mathbf{S}}) = \mathbf{A}^+\mathbf{S}^-\mathbf{A}^\infty. \tag{4.21}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Since $\mathbf{S} = \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k$ with each $\mathbf{s}_i \in \Omega_F^*$, by Proposition 1.3, we have $\mathbf{S} \in \Omega_L^*$. If $\beta \in (\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$, then by Lemma 2.7, we have

$$\sigma^n(\mathbf{S}^-\mathbf{A}^\infty) \preccurlyeq \mathbf{A}^\infty \prec \delta(\beta) \quad \text{for all } n \geq 0;$$

and by Lemma 2.10, it follows that $\mathbf{S}^-\mathbf{A}^\infty$ is the greedy β-expansion of t^*, that is, $b(t^*, \beta) = \mathbf{S}^-\mathbf{A}^\infty$. If $\beta = \beta_\ell^{\mathbf{S}}$, then $\sigma^n(\mathbf{S}^-\mathbf{A}^\infty) \preccurlyeq \mathbf{A}^\infty = \delta(\beta)$ for all $n \geq 0$; and in this case, one can verify that the greedy β-expansion of t^* is given by $b(t^*, \beta) = \mathbf{S}0^\infty$.

First we prove $\tau(\beta) \geq t^*$. Note that $\mathbf{A} = A_1 \ldots A_m$. Let $t_N := ((\mathbf{S}^-\mathbf{A}^N A_1 \ldots A_j)^\infty)_\beta$, where the index $j \in \{1, \ldots, m\}$ satisfies $\mathbf{S} = \mathbb{S}(\mathbf{A}) = A_{j+1} \ldots A_m A_1 \ldots A_j$. Note by Lemma 2.7 that $A_{i+1} \ldots A_m \prec A_1 \ldots A_{m-i}$ for any $0 < i < m$. Then by equation (4.21), one can verify that

$$\begin{aligned}\sigma^n((\mathbf{S}^-\mathbf{A}^N A_1 \ldots A_j)^\infty) &= \sigma^n(\mathbf{S}^-\mathbf{A}^{N+1}(A_1 \ldots A_j^-\mathbf{A}^{N+1})^\infty)\\ &\prec \mathbf{A}^\infty \preccurlyeq \delta(\beta) \quad \text{for all } n \geq 0.\end{aligned}$$

So, $b(t_N, \beta) = (\mathbf{S}^-\mathbf{A}^N A_1 \ldots A_j)^\infty$. This implies that any sequence (x_i) constructed by arbitrarily concatenating blocks of the form

$$\mathbf{S}^-\mathbf{A}^k A_1 \ldots A_j, \quad k > N$$

satisfies $(\mathbf{S}^-\mathbf{A}^N A_1 \ldots A_j)^\infty \preccurlyeq \sigma^n((x_i)) \prec \delta(\beta)$ for all $n \geq 0$. So,

$$\{\mathbf{S}^-\mathbf{A}^{N+1}A_1 \ldots A_j, \mathbf{S}^-\mathbf{A}^{N+2}A_1 \ldots A_j\}^{\mathbb{N}} \subset \mathbf{K}_\beta(t_N).$$

By Lemma 2.11, this implies that $\dim_H K_\beta(t_N) > 0$ for all $N \geq 1$. Thus, $\tau(\beta) \geq t_N$ for all $N \geq 1$. Note that $t_N \nearrow t^*$ as $N \to \infty$. We then conclude that $\tau(\beta) \geq t^*$.

Next we prove $\tau(\beta) \leq t^*$. By equation (4.21) and Lemma 4.4, it follows that

$$\begin{aligned}\mathbf{K}_\beta(t^*) &\subset \{(x_i) : \mathbf{S}^-\mathbf{A}^\infty \preccurlyeq \sigma^n((x_i)) \prec \mathbf{A}^+\mathbf{S}^-\mathbf{A}^\infty \text{ for all } n \geq 0\}\\ &= \{(x_i) : \mathbf{S}^-\mathbf{A}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbf{A}^\infty \text{ for all } n \geq 0\} =: \Gamma.\end{aligned} \tag{4.22}$$

Note by Proposition 4.1 that $\Gamma(\mathbf{S}) = \{(x_i) : \mathbf{S}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbf{A}^\infty \text{ for all } n \geq 0\}$ is a countable subset of Γ. Furthermore, any sequence in the difference set $\Gamma \setminus \Gamma(\mathbf{S})$ must end with $\mathbf{S}^-\mathbf{A}^\infty$. As a result, Γ is also countable. By equation (4.22), this implies that $\dim_H K_\beta(t^*) = 0$, and thus $\tau(\beta) \leq t^*$. This completes the proof. □

5. *Geometrical structure of the basic intervals and exceptional sets*

In this section, we will prove Theorem 3. The proof will be split into two subsections. In §5.1, we demonstrate the tree structure of the Lyndon intervals $J^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$, and the relative exceptional sets $E^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$, from which it follows that the basic intervals $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$, $\mathbf{S} \in \Lambda$ are pairwise disjoint. We show that each relative exceptional set $E^{\mathbf{S}}$ has zero box-counting dimension, and the exceptional set E has zero packing dimension. In §5.2, we prove that the infinitely Farey set E_∞ has zero Hausdorff dimension.

5.1. *Tree structure of the Lyndon intervals and relative exceptional sets.* Given $\mathbf{S} \in \Lambda$, recall from Definitions 1.5 and 1.8 the basic interval $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$ and the Lyndon interval

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

$J^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_r^{\mathbf{S}}]$ generated by $\mathbf{S}$, respectively. Then by Lemmas 2.5 and 2.7, it follows that

$$\delta(\beta_\ell^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^\infty, \quad \delta(\beta_*^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty \quad \text{and} \quad \delta(\beta_r^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty. \tag{5.1}$$

First we show that the Lyndon intervals $J^{\mathbf{S}}, \mathbf{S} \in \Lambda$ have a tree structure.

PROPOSITION 5.1. *Let $\mathbf{S} \in \Lambda$. Then $I^{\mathbf{S}} \subset J^{\mathbf{S}}$. Furthermore,*

(i) *for any $\mathbf{r} \in \Omega_F^*$, we have $J^{\mathbf{S}\bullet\mathbf{r}} \subset J^{\mathbf{S}} \setminus I^{\mathbf{S}}$;*

(ii) *for any two different words $\mathbf{r}, \mathbf{r}' \in \Omega_F^*$, we have $J^{\mathbf{S}\bullet\mathbf{r}} \cap J^{\mathbf{S}\bullet\mathbf{r}'} = \emptyset$.*

Proof. Let $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$ and $J^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_r^{\mathbf{S}}]$. Then by equation (5.1), it follows that

$$\delta(\beta_*^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty \prec \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty = \delta(\beta_r^{\mathbf{S}}),$$

which implies $\beta_*^{\mathbf{S}} < \beta_r^{\mathbf{S}}$ by Lemma 2.5. So $I^{\mathbf{S}} \subset J^{\mathbf{S}}$.

For (i), let $\mathbf{r} \in \Omega_F^*$. Then $\mathbf{r}$ begins with digit 0 and ends with digit 1. By Lemmas 3.6(ii) and 3.7, this implies that

$$\begin{aligned}\delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}}) &= \mathbb{L}(\mathbf{S}\bullet\mathbf{r})^\infty = (\mathbf{S}\bullet\mathbb{L}(\mathbf{r}))^\infty = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^\infty) \\ &\succ \Phi_{\mathbf{S}}(10^\infty) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty = \delta(\beta_*^{\mathbf{S}}).\end{aligned}$$

So, $\beta_\ell^{\mathbf{S}\bullet\mathbf{r}} > \beta_*^{\mathbf{S}}$. Furthermore, by Lemmas 3.4, 3.6(ii), and 3.7, it follows that

$$\begin{aligned}\delta(\beta_r^{\mathbf{S}\bullet\mathbf{r}}) &= \mathbb{L}(\mathbf{S}\bullet\mathbf{r})^+(\mathbf{S}\bullet\mathbf{r})^\infty \\ &= \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^+)\Phi_{\mathbf{S}}(\mathbf{r}^\infty) = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty) \\ &\prec \Phi_{\mathbf{S}}(1^\infty) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty = \delta(\beta_r^{\mathbf{S}}).\end{aligned}$$

This proves $\beta_r^{\mathbf{S}\bullet\mathbf{r}} < \beta_r^{\mathbf{S}}$. Hence, $J^{\mathbf{S}\bullet\mathbf{r}} = [\beta_\ell^{\mathbf{S}\bullet\mathbf{r}}, \beta_r^{\mathbf{S}\bullet\mathbf{r}}] \subset (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}] = J^{\mathbf{S}} \setminus I^{\mathbf{S}}$.

Next we prove (ii). Let $\mathbf{r}, \mathbf{r}'$ be two different Farey words in Ω_F^*. By Lemma 2.9(i), it follows that $J^{\mathbf{r}} \cap J^{\mathbf{r}'} = \emptyset$. Write $J^{\mathbf{r}} = [\beta_\ell^{\mathbf{r}}, \beta_r^{\mathbf{r}}]$ and $J^{\mathbf{r}'} = [\beta_\ell^{\mathbf{r}'}, \beta_r^{\mathbf{r}'}]$. Since $J^{\mathbf{r}}$ and $J^{\mathbf{r}'}$ are disjoint, we may assume $\beta_r^{\mathbf{r}} < \beta_\ell^{\mathbf{r}'}$. By equation (5.1) and Lemma 2.5, it follows that

$$\mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty = \delta(\beta_r^{\mathbf{r}}) \prec \delta(\beta_\ell^{\mathbf{r}'}) = \mathbb{L}(\mathbf{r}')^\infty. \tag{5.2}$$

Then by equations (5.1), (5.2), and Lemma 3.3, we obtain that

$$\begin{aligned}\delta(\beta_r^{\mathbf{S}\bullet\mathbf{r}}) &= \mathbb{L}(S\bullet\mathbf{r})^+(S\bullet\mathbf{r})^\infty = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty) \\ &\prec \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r}')^\infty) = \mathbb{L}(\mathbf{S}\bullet\mathbf{r}')^\infty = \delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}'}).\end{aligned}$$

It follows that $\beta_r^{\mathbf{S}\bullet\mathbf{r}} < \beta_\ell^{\mathbf{S}\bullet\mathbf{r}'}$, and hence $J^{\mathbf{S}\bullet\mathbf{r}} \cap J^{\mathbf{S}\bullet\mathbf{r}'} = \emptyset$. □

Remark 5.2. Proposition 5.1 implies that the Lyndon intervals $J^{\mathbf{S}}, \mathbf{S} \in \Lambda$ have a tree structure. More precisely, we say $J^{\mathbf{R}}$ is an *offspring* of $J^{\mathbf{S}}$ if there exists a word $\mathbf{T} \in \Lambda$ such that $\mathbf{R} = \mathbf{S}\bullet\mathbf{T}$. Then any offspring of $J^{\mathbf{S}}$ is a subset of $J^{\mathbf{S}}$. Furthermore, if $J^{\mathbf{S}'}$ is not an offspring of $J^{\mathbf{S}}$ and $J^{\mathbf{S}}$ is not an offspring of $J^{\mathbf{S}'}$, then Proposition 5.1 implies that $J^{\mathbf{S}'} \cap J^{\mathbf{S}} = \emptyset$. Consequently, the basic intervals $I^{\mathbf{S}}, \mathbf{S} \in \Lambda$ are pairwise disjoint.

Recall from §1 the exceptional set $E = (1,2] \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{r}}$ and the relative exceptional sets $E^{\mathbf{S}} = (J^{\mathbf{S}} \setminus I^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}$ with $\mathbf{S} \in \Lambda$. Next we will show that E is bijectively mapped to $E^{\mathbf{S}}$ via the map

$$\Psi_{\mathbf{S}} : (1,2] \to J^{\mathbf{S}} \setminus I^{\mathbf{S}} = (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}]; \quad \beta \mapsto \delta^{-1} \circ \Phi_{\mathbf{S}} \circ \delta(\beta), \tag{5.3}$$

where $\delta(\beta)$ is the quasi-greedy β-expansion of 1.

We mention that $\Psi_{\mathbf{S}}$ is not surjective, which somewhat complicates the proof of Proposition 5.4 below. For example, let $\mathbf{S} = 011$. Then $\delta(\beta_*^{\mathbf{S}}) = 111010(110)^\infty$ and $\delta(\beta_r^{\mathbf{S}}) = 111(011)^\infty$. Take $\beta \in (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}]$ such that $\delta(\beta) = 1110110^\infty$. One can verify that $\beta \notin \Psi_{\mathbf{S}}((1,2])$, since $\delta(\beta)$ cannot be written as a concatenation of words from $\{\mathbf{S}, \mathbf{S}^-, \mathbb{L}(\mathbf{S}), \mathbb{L}(\mathbf{S})^+\}$.

LEMMA 5.3. *For any $\mathbf{S} \in \Lambda$, the map $\Psi_{\mathbf{S}}$ is well defined and strictly increasing.*

Proof. Let $\mathbf{S} \in \Lambda$ with $\mathbf{A} = \mathbb{L}(\mathbf{S})$. First we show that the map $\Psi_{\mathbf{S}} : (1,2] \to J^{\mathbf{S}} \setminus I^{\mathbf{S}} = (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}]$ is well defined. Note that

$$\delta(\beta_*^{\mathbf{S}}) = \mathbf{A}^+\mathbf{S}^-\mathbf{A}^\infty = \Phi_{\mathbf{S}}(10^\infty) \quad \text{and} \quad \delta(\beta_r^{\mathbf{S}}) = \mathbf{A}^+\mathbf{S}^\infty = \Phi_{\mathbf{S}}(1^\infty). \tag{5.4}$$

Take $\beta \in (1,2]$. Then $10^\infty \prec \delta(\beta) \preccurlyeq 1^\infty$. By Lemma 3.3 and equation (5.4), it follows that

$$\delta(\beta_*^{\mathbf{S}}) \prec \Phi_{\mathbf{S}}(\delta(\beta)) \preccurlyeq \delta(\beta_r^{\mathbf{S}}).$$

Thus, by Lemma 2.5, it suffices to prove that

$$\sigma^n(\Phi_{\mathbf{S}}(\delta(\beta))) \preccurlyeq \Phi_{\mathbf{S}}(\delta(\beta)) \quad \text{for all } n \geq 0. \tag{5.5}$$

Note by Lemma 2.5 that $\sigma^n(\delta(\beta)) \preccurlyeq \delta(\beta)$ for all $n \geq 0$, and $\delta(\beta)$ begins with digit 1. Thus, equation (5.5) follows by Lemma 3.8(i), and we conclude that the map $\Psi_{\mathbf{S}}$ is well defined.

The monotonicity of $\Psi_{\mathbf{S}} = \delta^{-1} \circ \Phi_{\mathbf{S}} \circ \delta$ follows since both maps δ and $\Phi_{\mathbf{S}}$ are strictly increasing by Lemmas 2.5 and 3.3, respectively. This completes the proof. □

PROPOSITION 5.4. *For any $\mathbf{S} \in \Lambda$, we have $\Psi_{\mathbf{S}}(E) = E^{\mathbf{S}}$.*

Proof. We first prove that

$$\Psi_{\mathbf{S}}(\beta_\ell^{\mathbf{r}}) = \beta_\ell^{\mathbf{S}\bullet\mathbf{r}} \quad \text{and} \quad \Psi_{\mathbf{S}}(\beta_r^{\mathbf{r}}) = \beta_r^{\mathbf{S}\bullet\mathbf{r}} \quad \text{for all } \mathbf{r} \in \Omega_L^*. \tag{5.6}$$

Observe that $\delta(\beta_\ell^{\mathbf{r}}) = \mathbb{L}(\mathbf{r})^\infty$. Then by Lemmas 3.6(ii) and 3.7, it follows that

$$\Phi_{\mathbf{S}}(\delta(\beta_\ell^{\mathbf{r}})) = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^\infty) = (\mathbf{S} \bullet \mathbb{L}(\mathbf{r}))^\infty = \mathbb{L}(\mathbf{S} \bullet \mathbf{r})^\infty = \delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}}),$$

so $\Psi_{\mathbf{S}}(\beta_\ell^{\mathbf{r}}) = \beta_\ell^{\mathbf{S}\bullet\mathbf{r}}$. Similarly, since $\delta(\beta_r^{\mathbf{r}}) = \mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty$, Lemmas 3.4, 3.6(ii), and 3.7 imply that

$$\begin{aligned}\Phi_{\mathbf{S}}(\delta(\beta_r^{\mathbf{r}})) &= \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty) = \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r})^+)\Phi_{\mathbf{S}}(\mathbf{r}^\infty) \\ &= (\mathbf{S} \bullet \mathbb{L}(\mathbf{r}))^+\Phi_{\mathbf{S}}(\mathbf{r})^\infty = \mathbb{L}(\mathbf{S} \bullet \mathbf{r})^+(\mathbf{S} \bullet \mathbf{r})^\infty = \delta(\beta_r^{\mathbf{S}\bullet\mathbf{r}}).\end{aligned}$$

We conclude that $\Psi_{\mathbf{S}}(\beta_r^{\mathbf{r}}) = \beta_r^{\mathbf{S}\bullet\mathbf{r}}$. This proves equation (5.6).

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Note by Lemma 2.5(ii) that the map $\beta \mapsto \delta(\beta)$ is left continuous in $(1, 2]$, and is right continuous at a point β_0 if and only if $\delta(\beta_0)$ is *not* periodic. Hence by Lemma 2.8, it follows that the map $\beta \mapsto \delta(\beta)$ is continuous at β_0 if and only if $\delta(\beta_0)$ is not of the form $\mathbb{L}(\mathbf{r})^\infty$ for a Lyndon word $\mathbf{r}$. Since $\Phi_\mathbf{S}$ is clearly continuous with respect to the order topology and the map δ^{-1} is continuous, it follows that $\Psi_\mathbf{S}$ is continuous at β_0 if and only if $\delta(\beta_0)$ is not of the form $\mathbb{L}(\mathbf{r})^\infty$ for a Lyndon word $\mathbf{r}$. Moreover, $\Psi_\mathbf{S}$ is left continuous everywhere.

Note that if $\delta(\beta_0) = \mathbb{L}(\mathbf{r})^\infty$, then by Lemma 2.5(ii), it follows that as β decreases to β_0, the sequence $\delta(\beta)$ converges to $\mathbb{L}(\mathbf{r})^+0^\infty$ with respect to the order topology, that is, $\lim_{\beta \searrow \beta_0} \delta(\beta) = \mathbb{L}(\mathbf{r})^+0^\infty$. Since $\Psi_\mathbf{S}$ is increasing, it follows that

$$\Psi_\mathbf{S}((1, 2]) = (J^\mathbf{S} \backslash I^\mathbf{S}) \backslash \bigcup_{\mathbf{r} \in \Omega_L^*} (p^\mathbf{r}, q^\mathbf{r}],$$

where $\delta(p^\mathbf{r}) = \Phi_\mathbf{S}(\mathbb{L}(\mathbf{r})^\infty)$ and $\delta(q^\mathbf{r}) = \Phi_\mathbf{S}(\mathbb{L}(\mathbf{r})^+0^\infty)$. Note that $(p^\mathbf{r}, q^\mathbf{r}] \subset J^{\mathbf{S} \bullet \mathbf{r}}$. By Lemma 2.9(iii), there is a (unique) Farey word $\hat{\mathbf{r}}$ such that $J^\mathbf{r} \subset J^{\hat{r}}$. Applying equation (5.6) to both $\mathbf{r}$ and $\hat{\mathbf{r}}$, and using Lemma 5.3, we conclude that $J^{\mathbf{S} \bullet \mathbf{r}} \subset J^{\mathbf{S} \bullet \hat{\mathbf{r}}}$. Hence, $(p^\mathbf{r}, q^\mathbf{r}] \subset J^{\mathbf{S} \bullet \hat{\mathbf{r}}}$. Therefore, if $\beta \in E^\mathbf{S} = (J^\mathbf{S} \backslash I^\mathbf{S}) \backslash \bigcup_{\mathbf{r} \in \Omega_F^*} J^{\mathbf{S} \bullet \mathbf{r}}$, then β lies in the range of $\Psi_\mathbf{S}$. This implies that

$$E^\mathbf{S} \subset \Psi_\mathbf{S}((1, 2]). \tag{5.7}$$

Now assume first that $\beta \in \Psi_\mathbf{S}(E)$. Then $\beta = \Psi_\mathbf{S}(\hat{\beta})$, where $\hat{\beta} \notin J^\mathbf{r}$ for any $\mathbf{r} \in \Omega_F^*$. Hence, $\beta \notin J^{\mathbf{S} \bullet \mathbf{r}}$ for any $\mathbf{r} \in \Omega_F^*$ by equation (5.6) and since $\Psi_\mathbf{S}$ is increasing. Therefore, $\beta \in E^\mathbf{S}$.

Conversely, suppose $\beta \in E^\mathbf{S}$. By equation (5.7), $\beta = \Psi_\mathbf{S}(\hat{\beta})$ for some $\hat{\beta} \in (1, 2]$. If $\hat{\beta} \in J^\mathbf{r}$ for some $\mathbf{r} \in \Omega_F^*$, then $\beta \in \Psi_\mathbf{S}(J^\mathbf{r}) \subset J^{\mathbf{S} \bullet \mathbf{r}}$ by equation (5.6), contradicting that $\beta \in E^\mathbf{S}$. Hence, $\hat{\beta} \in E$ and then $\beta \in \Psi_\mathbf{S}(E)$. This completes the proof. □

Kalle *et al* proved in [20, Theorem C] that the Farey intervals $J^\mathbf{r}, \mathbf{r} \in \Omega_F^*$ cover the whole interval $(1, 2]$ up to a set of zero Hausdorff dimension. Here we strengthen this result and show that the exceptional set E is uncountable and has zero packing dimension. Furthermore, we show that each relative exceptional set $E^\mathbf{S}$ is uncountable and has zero box-counting dimension. The proof uses the following simple lemma.

LEMMA 5.5. *Let $J^\mathbf{S} = [\beta_\ell^\mathbf{S}, \beta_r^\mathbf{S}] =: [p, q]$ be any Lyndon interval. Then the length of $J^\mathbf{S}$ satisfies*

$$|J^\mathbf{S}| \le \frac{q}{q-1} q^{-|\mathbf{S}|}.$$

Proof. Since $\delta(p) = \mathbb{L}(\mathbf{S})^\infty$ and $\delta(q) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty$, we have

$$(\mathbb{L}(\mathbf{S})^+0^\infty)_p = 1 = (\mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty)_q =: ((c_i))_q.$$

It follows that

$$|J^\mathbf{S}| = q - p = \sum_{i=1}^{|\mathbf{S}|} \frac{c_i}{q^{i-1}} + \sum_{i=|\mathbf{S}|+1}^{\infty} \frac{c_i}{q^{i-1}} - \sum_{i=1}^{|\mathbf{S}|} \frac{c_i}{p^{i-1}} \le \sum_{i=|\mathbf{S}|+1}^{\infty} \frac{1}{q^{i-1}} = \frac{q}{q-1} q^{-|\mathbf{S}|},$$

as required. □

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

PROPOSITION 5.6.

(i) *The exceptional set*

$$E = (1, 2] \setminus \bigcup_{\mathbf{r} \in \Omega_F^*} J^{\mathbf{r}}$$

is uncountable and has zero packing dimension.

(ii) *For any* $\mathbf{S} \in \Lambda$, *the relative exceptional set*

$$E^{\mathbf{S}} = (J^{\mathbf{S}} \setminus I^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r} \in \Omega_F^*} J^{\mathbf{S} \bullet \mathbf{r}}$$

is uncountable and has zero box-counting dimension.

Proof. (i) First we prove $\dim_P E = 0$. Let $\rho_N \in (1, 2]$ such that $\delta(\rho_N) = (10^{N-1})^\infty$. Then by Lemma 2.5, it follows that $\rho_N \searrow 1$ as $N \to \infty$. Thus $E = \bigcup_{N=1}^{\infty}(E \cap [\rho_N, 2])$. By the countable stability of packing dimension (cf. [16]), it suffices to prove that

$$\dim_B(E \cap [\rho_N, 2]) = 0 \quad \text{for all } N \in \mathbb{N}. \tag{5.8}$$

Let $N \in \mathbb{N}$. Take a Farey interval $J^{\mathbf{s}} := [p, q] \subset [\rho_N, 2]$ with $\mathbf{s} = s_1 \ldots s_m \in \Omega_F^*$ such that

$$m > N + 2 - 3 \log_2(\rho_N - 1). \tag{5.9}$$

Write $\mathbb{L}(\mathbf{s}) = a_1 \ldots a_m$. Then $\delta(p) = (a_1 \ldots a_m)^\infty$. Since $p \geq \rho_N$, by Lemma 2.5, we have $(a_1 \ldots a_m)^\infty = \delta(p) \succcurlyeq \delta(\rho_N) = (10^{N-1})^\infty$, which implies that $a_1 \ldots a_{N+1} \succcurlyeq 10^{N-1}1$. Then by Proposition 2.4, we conclude that

$$s_1 \ldots s_{N+1} \succcurlyeq 0^N 1. \tag{5.10}$$

Note that

$$(\mathbb{L}(\mathbf{s})^+ 0^\infty)_p = 1 = (\mathbb{L}(\mathbf{s})^+ \mathbf{s}^\infty)_q =: ((c_i))_q.$$

So, by equation (5.10), it follows that

$$\sum_{i=1}^{m} \frac{c_i}{p^i} = 1 = \sum_{i=1}^{m} \frac{c_i}{q^i} + \sum_{i=m+1}^{\infty} \frac{c_i}{q^i} > \sum_{i=1}^{m} \frac{c_i}{q^i} + \frac{1}{q^{m+N+1}},$$

which implies

$$\frac{1}{q^{m+N+1}} < \sum_{i=1}^{\infty} \left(\frac{1}{p^i} - \frac{1}{q^i} \right) = \frac{q - p}{(p-1)(q-1)}.$$

Whence,

$$|J^{\mathbf{s}}| = q - p > \frac{(p-1)(q-1)}{q^{N+1}} q^{-m} \geq \frac{(\rho_N - 1)^2}{2^{N+1}} q^{-m}. \tag{5.11}$$

However, by Lemma 5.5, it follows that

$$|J^{\mathbf{s}}| \leq \frac{q}{q-1} q^{-m} \leq \frac{2}{\rho_N - 1} q^{-m} \leq \frac{2}{\rho_N - 1} \rho_N^{-m}. \tag{5.12}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Now we list all of the Farey intervals in $[\rho_N, 2]$ in a decreasing order according to their length, say $J^{\mathbf{s}_1}, J^{\mathbf{s}_2}, \ldots$. In other words, $|J^{\mathbf{s}_i}| \geq |J^{\mathbf{s}_j}|$ for any $i < j$. For a Farey interval $J^{\mathbf{s}}$, if $J^{\mathbf{s}} = J^{\mathbf{s}_k}$, we then define its *order index* as $o(J^{\mathbf{s}}) = k$.

Set $C_N := 2\log 2/\log \rho_N$. Let $J^{\mathbf{s}'}$ be a Farey interval with $|\mathbf{s}'| > C_N m$. Then by equations (5.9), (5.11), and (5.12), it follows that

$$|J^{\mathbf{s}'}| \leq \frac{2}{\rho_N - 1}\rho_N^{-C_N m} = \frac{2}{\rho_N - 1}2^{-2m} < \frac{(\rho_N - 1)^2}{2^{N+1}}2^{-m} \leq |J^{\mathbf{s}}|.$$

This implies that

$$o(J^{\mathbf{s}}) \leq \sum_{k=1}^{\lfloor C_N m \rfloor} \#\{\mathbf{s}' \in \Omega_F^* : |\mathbf{s}'| = k\} \leq \sum_{k=1}^{\lfloor C_N m \rfloor} (k-1) < C_N^2 m^2, \tag{5.13}$$

where the second inequality follows by equation (2.1) since the number of non-degenerate Farey words of length k is at most $k-1$ (see [8, Proposition 2.3]). Together with equation (5.12), equation (5.13) implies that

$$\liminf_{i\to\infty} \frac{-\log |J^{\mathbf{s}_i}|}{o(J^{\mathbf{s}_i})} = +\infty.$$

Note that $[\rho_N, 2] \cap E = [\rho_N, 2] \setminus \bigcup_{\mathbf{s}\in\Omega_F^*} J^{\mathbf{s}}$. So, by [17, Proposition 3.6], we conclude equation (5.8). This proves $\dim_P E = 0$.

Next we prove that E is uncountable. For $\mathbf{s} \in \Omega_F^*$, let $\hat{J}^{\mathbf{s}} = (\beta_\ell^{\mathbf{s}}, \beta_r^{\mathbf{s}})$ be the interior of the Farey interval $J^{\mathbf{s}} = [\beta_\ell^{\mathbf{s}}, \beta_r^{\mathbf{s}}]$. By Lemma 2.9(i), it follows that the compact set

$$\hat{E} := [1, 2] \setminus \bigcup_{\mathbf{s}\in\Omega_F^*} \hat{J}^{\mathbf{s}}$$

is non-empty and has no isolated points. Hence, $\hat{E}$ is a perfect set and is therefore uncountable. Since $\hat{E}\setminus E$ is countable, it follows that E is uncountable as well.

(ii) In a similar way, we prove $\dim_B E^{\mathbf{S}} = 0$. Note that $E^{\mathbf{S}} = (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}] \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{S\bullet\mathbf{r}}$. Fix a Farey word $\mathbf{r} = r_1 \ldots r_m$. Then the Lyndon interval $J^{\mathbf{S}\bullet\mathbf{r}} = [\beta_\ell^{\mathbf{S}\bullet\mathbf{r}}, \beta_r^{\mathbf{S}\bullet\mathbf{r}}] =: [p_{\mathbf{r}}, q_{\mathbf{r}}]$ satisfies

$$(\mathbb{L}(\mathbf{S}\bullet\mathbf{r})^+ 0^\infty)_{p_{\mathbf{r}}} = 1 = (\mathbb{L}(\mathbf{S}\bullet\mathbf{r})^+\ (\mathbf{S}\bullet\mathbf{r})^\infty)_{q_{\mathbf{r}}} =: ((d_i))_{q_{\mathbf{r}}}.$$

So,

$$\sum_{i=1}^{m|\mathbf{S}|} \frac{d_i}{p_{\mathbf{r}}^i} = 1 = \sum_{i=1}^{m|\mathbf{S}|} \frac{d_i}{q_{\mathbf{r}}^i} + \sum_{i=m|\mathbf{S}|+1}^{\infty} \frac{d_i}{q_{\mathbf{r}}^i} > \sum_{i=1}^{m|\mathbf{S}|} \frac{d_i}{q_{\mathbf{r}}^i} + \frac{1}{q_{\mathbf{r}}^{(m+1)|\mathbf{S}|+1}},$$

where the inequality follows by observing that $\mathbf{S} \in \Omega_L^*$ and thus $\mathbf{S}\bullet\mathbf{r}^\infty \succcurlyeq 0^{|\mathbf{S}|}10^\infty$. Therefore,

$$\frac{1}{q_{\mathbf{r}}^{(m+1)|\mathbf{S}|+1}} \leq \sum_{i=1}^{\infty}\left(\frac{1}{p_{\mathbf{r}}^i} - \frac{1}{q_{\mathbf{r}}^i}\right) = \frac{q_{\mathbf{r}} - p_{\mathbf{r}}}{(p_{\mathbf{r}} - 1)(q_{\mathbf{r}} - 1)},$$

which implies

$$|J^{\mathbf{S}\bullet\mathbf{r}}| = q_{\mathbf{r}} - p_{\mathbf{r}} \geq \frac{(p_{\mathbf{r}}-1)(q_{\mathbf{r}}-1)}{q_{\mathbf{r}}^{|\mathbf{S}|+1}} q_{\mathbf{r}}^{-|\mathbf{S}\bullet\mathbf{r}|} > \frac{(\beta_*^{\mathbf{S}}-1)^2}{2^{|\mathbf{S}|+1}} q_{\mathbf{r}}^{-|\mathbf{S}\bullet\mathbf{r}|}.$$

However, by Lemma 5.5, it follows that

$$|J^{\mathbf{S}\bullet\mathbf{r}}| \leq \frac{q_{\mathbf{r}}}{q_{\mathbf{r}}-1} q_{\mathbf{r}}^{-m|\mathbf{S}|} \leq \frac{2}{\beta_*^{\mathbf{S}}-1} q_{\mathbf{r}}^{-|\mathbf{S}\bullet\mathbf{r}|}.$$

Now let $(J^{\mathbf{S}\bullet\mathbf{r}_i})$ be an enumeration of the intervals $J^{\mathbf{S}\bullet\mathbf{r}}$, $\mathbf{r} \in \Omega_F^*$, arranged in order by decreasing length. Then by a similar argument as in (i) above, we obtain

$$\liminf_{i\to\infty} \frac{-\log |J^{\mathbf{S}\bullet\mathbf{r}_i}|}{\log o(J^{\mathbf{S}\bullet\mathbf{r}_i})} = +\infty.$$

Thus, $\dim_B E^{\mathbf{S}} = 0$.

Finally, since we showed in (i) that E is uncountable, we conclude by Lemma 5.3 and Proposition 5.4 that $E^{\mathbf{S}} = \Psi_{\mathbf{S}}(E)$ is also uncountable. This completes the proof. □

5.2. *The infinitely Farey set.* Recall from equation (1.7) that

$$E_\infty = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{S}\in\Lambda(n)} J^{\mathbf{S}},$$

where

$$\Lambda(n) = \{\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_n : \mathbf{s}_i \in \Omega_F^* \text{ for all } 1 \leq i \leq n\}.$$

In particular, $\Lambda(1) = \Omega_F^*$ and $\Lambda = \bigcup_{n=1}^{\infty} \Lambda(n)$. Note that $(1, 2] = E \cup \bigcup_{\mathbf{s}\in\Omega_F^*} J^{\mathbf{s}}$. Furthermore, for each word $\mathbf{S} \in \Lambda$, we have

$$J^{\mathbf{S}} \setminus I^{\mathbf{S}} = E^{\mathbf{S}} \cup \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}.$$

By iteration of the above equation, we obtain the following partition of the interval (1, 2].

LEMMA 5.7. *The interval* (1, 2] *can be partitioned as*

$$(1, 2] = E \cup E_\infty \cup \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}} \cup \bigcup_{\mathbf{S}\in\Lambda} I^{\mathbf{S}}.$$

To complete the proof of Theorem 3, we still need the following dimension result for E_∞.

PROPOSITION 5.8. *We have* $\dim_H E_\infty = 0$.

Proof. Note by equation (1.7) that

$$E_\infty = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{S}\in\Lambda(n)} J^{\mathbf{S}} \subset \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{S}\in\Lambda:|\mathbf{S}|\geq n} J^{\mathbf{S}}. \tag{5.14}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

This suggests covering E_∞ by the intervals $J^{\mathbf{S}}$ for $\mathbf{S} \in \Lambda$ with $|\mathbf{S}| \geq n$ for a sufficiently large n. To this end, we first estimate the diameter of $J^{\mathbf{S}}$. Take $\mathbf{S} \in \Lambda$ with $|\mathbf{S}| = m$, and write $J^{\mathbf{S}} = [p, q]$. Then $\delta(p) = \mathbb{L}(\mathbf{S})^\infty$ and $\delta(q) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty$, and it follows from Lemma 5.5 that

$$|J^{\mathbf{S}}| \leq \frac{q}{q-1} \cdot q^{-m}. \tag{5.15}$$

Let (β_n) be an arbitrary sequence in $(1, 2)$ decreasing to 1. We will use equation (5.15) to show that $\dim_H(E_\infty \cap (\beta_n, 2]) = 0$ for all $n \in \mathbb{N}$, so the result will follow from the countable stability of Hausdorff dimension (cf. [16]). Fix $n \in \mathbb{N}$. Observe that if $J^{\mathbf{S}} = [p, q]$ intersects $(\beta_n, 2]$, then $q > \beta_n$ and so by equation (5.15),

$$|J^{\mathbf{S}}| \leq \frac{2}{\beta_n - 1}\beta_n^{-m} =: C_n\beta_n^{-m}. \tag{5.16}$$

Next, we count how many words $\mathbf{S} \in \Lambda$ there are with $|\mathbf{S}| = m$. Call this number N_m. Observe that if $\mathbf{S} = \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k$ and $|\mathbf{s}_i| = l_i$ for $i = 1, \ldots, k$, then $|\mathbf{S}| = l_1 l_2 \ldots l_k$. Note by [8, Proposition 2.3] that $\#\{\mathbf{r} \in \Omega_F^* : |\mathbf{r}| = l\} \leq l - 1$ for any $l \geq 2$. Thus, for any given tuple $(l_1, \ldots, l_k)$, the number of possible choices for the words $\mathbf{s}_1, \ldots, \mathbf{s}_k$ is at most $l_1 l_2 \ldots l_k = |\mathbf{S}| = m$. It remains to estimate how many *ordered factorizations* of m there are, that is, to estimate the number

$$f_m := \#\{(l_1, \ldots, l_k) : k \in \mathbb{N},\ l_i \in \mathbb{N}_{\geq 2} \text{ for all } i \text{ and } l_1 l_2 \ldots l_k = m\}.$$

By considering the possible values of l_1, it is easy to see that f_m satisfies the recursion

$$f_m = \sum_{d|m, d>1} f_{m/d},$$

where we set $f_1 := 1$. (See [19].) We claim that $f_m \leq m^2$. This is trivial for $m = 1$, so let $m \geq 2$ and assume $f_n \leq n^2$ for all $n < m$; then

$$f_m = \sum_{d|m, d>1} f_{m/d} \leq \sum_{d|m, d>1} \left(\frac{m}{d}\right)^2 \leq m^2 \sum_{d=2}^{\infty} \frac{1}{d^2} = m^2\left(\frac{\pi^2}{6} - 1\right) < m^2.$$

This proves the claim, and we thus conclude that $N_m \leq m^3$. Now, given $\varepsilon > 0$ and $\delta > 0$, choose N large enough so that $C_n(\beta_n)^{-N} < \delta$. Using equations (5.14) and (5.16), we obtain

$$\mathcal{H}_\delta^\varepsilon(E_{\infty \cap (\beta_n, 2]}) \leq \sum_{\mathbf{S} \in \Lambda : |\mathbf{S}| \geq N,\ J^{\mathbf{S}} \cap (\beta_n, 2] \neq \emptyset} |J^{\mathbf{S}}|^\varepsilon \leq \sum_{m=N}^{\infty} m^3 C_n^\varepsilon \beta_n^{-m\varepsilon} \to 0$$

as $N \to \infty$. This shows that $\dim_H\ (E_\infty \cap (\beta_n, 2]) = 0$, as desired. □

Proof of Theorem 3. The theorem follows by Proposition 5.6, Lemma 5.7, and Proposition 5.8. □

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

6. *Critical values in the exceptional sets*

By Proposition 1.12, Theorem 2, and Theorem 3, it suffices to determine the critical value $\tau(\beta)$ for

$$\beta \in \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}} \cup E_\infty.$$

First we compute $\tau(\beta)$ for $\beta \in \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}}$. Recall from Lemma 5.3 and Proposition 5.4 that for each $\mathbf{S} \in \Lambda$, the map $\Psi_{\mathbf{S}}$ bijectively maps the exceptional set $E = (1, 2] \setminus \bigcup_{\mathbf{s}\in\Omega_F^*} J^{\mathbf{s}}$ to the relative exceptional set $E^{\mathbf{S}} = (J^{\mathbf{S}} \setminus I^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}$.

LEMMA 6.1. *Let $\hat\beta \in E\setminus\{2\}$ with $\delta(\hat\beta) = \delta_1\delta_2\ldots$. Also let $\mathbf{S} \in \Lambda$, and set $\beta := \Psi_{\mathbf{S}}(\hat\beta)$. Then*

(i) $b(\tau(\hat\beta), \hat\beta) = 0\delta_2\delta_3\ldots$; *and*
(ii) *the map $\hat t \mapsto (\Phi_{\mathbf{S}}(b(\hat t, \hat\beta)))_\beta$ is continuous at $\tau(\hat\beta)$.*

Proof. First we prove (i). Note by Proposition 1.12 that $\tau(\hat\beta) = 1 - 1/\hat\beta = (0\delta_2\delta_3\ldots)_{\hat\beta}$. So by Lemma 2.10, it suffices to verify that

$$\sigma^n(0\delta_2\delta_3\ldots) \prec \delta_1\delta_2\ldots \quad \text{for all } n \ge 0. \tag{6.1}$$

By Lemma 2.5, it is immediate that $\sigma^n(0\delta_2\delta_3\ldots) \preccurlyeq (\delta_i)$ for all $n \ge 0$. If equality holds for some n, then $\delta(\hat\beta)= (\delta_i)$ is periodic with period $m \ge 2$ (since $\hat\beta \ne 2$), so by Lemma 2.8, $\delta(\hat\beta) = \mathbb{L}(\mathbf{r})^\infty$ for some Lyndon word $\mathbf{r}$. This implies $\hat\beta = \beta_\ell^{\mathbf{r}} \in J^{\mathbf{r}}$. However, then by Lemma 2.9, $\hat\beta \in J^{\mathbf{s}}$ for some Farey word $\mathbf{s}$, and so $\hat\beta \notin E$, a contradiction. This proves equation (6.1), and then yields statement (i).

For (ii), note by Lemma 2.10(ii) that the map $\hat t \mapsto b(\hat t, \hat\beta)$ is continuous at all points $\hat t$ for which $b(\hat t, \hat\beta)$ does not end with 0^∞. Furthermore, the map $\Phi_{\mathbf{S}}$ is continuous with respect to the order topology. So, by statement (i), it follows that the map $\hat t \mapsto (\Phi_{\mathbf{S}}(b(\hat t, \hat\beta)))_\beta$ is continuous at $\tau(\hat\beta)$, completing the proof. □

PROPOSITION 6.2. *Let $\mathbf{S} \in \Lambda$. Then for any $\beta \in E^{\mathbf{S}}$, we have*

$$\tau(\beta) = (\Phi_{\mathbf{S}}(0\delta_2\delta_3\ldots))_\beta,$$

where $1\delta_2\delta_3\ldots$ is the quasi-greedy expansion of 1 in base $\hat\beta := \Psi_{\mathbf{S}}^{-1}(\beta)$.

Proof. Let $\beta \in E^{\mathbf{S}}$. Then by Lemma 5.3 and Proposition 5.4, there exists a unique $\hat\beta \in E$ such that $\hat\beta = \Psi_{\mathbf{S}}^{-1}(\beta) \in E$, in other words, $\delta(\beta) = \Phi_{\mathbf{S}}(\delta(\hat\beta))$. Write $\delta(\hat\beta) = \delta_1\delta_2\ldots$ and set $t^* := (\Phi_{\mathbf{S}}(0\delta_2\delta_3\ldots))_\beta$. We will show that $\tau(\beta) = t^*$, by proving that $h_{top}(\mathbf{K}_\beta(t)) > 0$ for $t < t^*$, and $\mathbf{K}_\beta(t)$ is countable for $t > t^*$. We consider separately the two cases: (i) $\hat\beta < 2$ and (ii) $\hat\beta = 2$.

Case I. $\hat\beta < 2$. First, for notational convenience, we define the map

$$\Theta_{\mathbf{S},\hat\beta} : (0, 1) \to (0, 1); \quad \hat t \mapsto (\Phi_{\mathbf{S}}(b(\hat t, \hat\beta)))_\beta.$$

Since by Lemma 6.1 the map $\Theta_{\mathbf{S},\hat\beta}$ is continuous at $\tau(\hat\beta)$ and $t^* = \Theta_{\mathbf{S},\hat\beta}(\tau(\hat\beta)) = (\Phi_{\mathbf{S}}(0\delta_2\delta_3\ldots))_\beta$, it is, by the monotonicity of the set-valued map $t \mapsto \mathbf{K}_\beta(t)$, sufficient

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

to prove the following two things:

$$\hat{t} < \tau(\hat{\beta}) \quad \Longrightarrow \quad h_{\text{top}}(\mathbf{K}_\beta(\Theta_{\mathbf{S},\hat{\beta}}(\hat{t}))) > 0, \tag{6.2}$$

and

$$\hat{t} > \tau(\hat{\beta}) \quad \Longrightarrow \quad \mathbf{K}_\beta(\Theta_{\mathbf{S},\hat{\beta}}(\hat{t})) \text{ is countable.} \tag{6.3}$$

First, take $\hat{t} < \tau(\hat{\beta})$ and set $t := \Theta_{\mathbf{S},\hat{\beta}}(\hat{t}) = (\Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta})))_\beta$. Since $\sigma^n(b(\hat{t}, \hat{\beta})) \prec \delta(\hat{\beta})$ for all $n \geq 0$, Lemma 3.8 implies that

$$\sigma^n(\Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta}))) \prec \Phi_{\mathbf{S}}(\delta(\hat{\beta})) = \delta(\beta) \quad \text{for all } n \geq 0.$$

Hence, $b(t, \beta) = \Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta}))$. Now

$$\begin{aligned}
\Phi_{\mathbf{S}}(\mathbf{K}_{\hat{\beta}}(\hat{t})) &= \{\Phi_{\mathbf{S}}((x_i)) : b(\hat{t}, \hat{\beta}) \preccurlyeq \sigma^n((x_i)) \prec \delta(\hat{\beta}) \text{ for all } n \geq 0\} \\
&\subset \{\Phi_{\mathbf{S}}((x_i)) : \Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta})) \preccurlyeq \sigma^n(\Phi_{\mathbf{S}}((x_i))) \prec \Phi_{\mathbf{S}}(\delta(\hat{\beta})) \text{ for all } n \geq 0\} \\
&= \{\Phi_{\mathbf{S}}((x_i)) : b(t, \beta) \preccurlyeq \sigma^n(\Phi_{\mathbf{S}}((x_i))) \prec \delta(\beta) \text{ for all } n \geq 0\} \\
&\subset \{(y_i) : b(t, \beta) \preccurlyeq \sigma^n((y_i)) \prec \delta(\beta) \text{ for all } n \geq 0\} = \mathbf{K}_\beta(t),
\end{aligned}$$

where the first inclusion again follows by Lemma 3.8. We deduce that

$$h_{\text{top}}(\mathbf{K}_\beta(t)) \geq h_{\text{top}}(\Phi_{\mathbf{S}}(\mathbf{K}_{\hat{\beta}}(\hat{t}))) = |\mathbf{S}|^{-1} h_{\text{top}}(\mathbf{K}_{\hat{\beta}}(\hat{t})) > 0,$$

where the last inequality follows since $\hat{t} < \tau(\hat{\beta})$. This gives equation (6.2).

Next, let $\hat{t} > \tau(\hat{\beta})$ and set $t := \Theta_{\mathbf{S},\hat{\beta}}(\hat{t})$. Then, by the same argument as above, we have $b(t, \beta) = \Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta}))$. Since $\hat{\beta} \in E$, there exists a sequence of Farey intervals $J^{\mathbf{r}_k} = [\beta_\ell^{\mathbf{r}_k}, \beta_r^{\mathbf{r}_k}]$ with $\mathbf{r}_k \in \Omega_F^*$ such that $\hat{q}_k := \beta_\ell^{\mathbf{r}_k} \searrow \hat{\beta}$ as $k \to \infty$.

We claim that $b(\hat{t}, \hat{\beta}) \succ (\mathbf{r}_k)^\infty$ for all sufficiently large k. This can be seen as follows. As explained in the proof of Lemma 6.1, $\delta(\hat{\beta})$ is not periodic, and therefore by Lemma 2.5(ii), the map $\beta' \mapsto \delta(\beta')$ is continuous at $\hat{\beta}$ (where we use β' to denote a generic base). This implies $\delta(\hat{q}_k) \searrow \delta(\hat{\beta}) = 1\delta_2\delta_3 \ldots$ as $k \to \infty$. However, $\delta(\hat{q}_k) = \delta(\beta_\ell^{\mathbf{r}_k}) = \mathbb{L}(\mathbf{r}_k)^\infty$, and by Lemma 2.3, $\mathbb{L}(\mathbf{r}_k)$ is the word obtained from $\mathbf{r}_k$ by flipping the first and last digits. Thus, $(\mathbf{r}_k)^\infty$ converges to $0\delta_2\delta_3 \ldots$ in the order topology. Since $\hat{t} > \tau(\hat{\beta})$ implies $b(\hat{t}, \hat{\beta}) \succ b(\tau(\hat{\beta}), \hat{\beta}) = 0\delta_2\delta_3 \ldots$, the claim follows.

We can now deduce that for all sufficiently large k,

$$\begin{aligned}
\mathbf{K}_\beta(t) &= \{(y_i) : b(t, \beta) \preccurlyeq \sigma^n((y_i)) \prec \delta(\beta) \text{ for all } n \geq 0\} \\
&= \{(y_i) : \Phi_{\mathbf{S}}(b(\hat{t}, \hat{\beta})) \preccurlyeq \sigma^n((y_i)) \prec \Phi_{\mathbf{S}}(\delta(\hat{\beta})) \text{ for all } n \geq 0\} \\
&\subset \{(y_i) : \Phi_{\mathbf{S}}((\mathbf{r}_k)^\infty) \preccurlyeq \sigma^n((y_i)) \prec \Phi_{\mathbf{S}}(\mathbb{L}(\mathbf{r}_k)^\infty) \text{ for all } n \geq 0\} \\
&= \{(y_i) : (\mathbf{S} \bullet \mathbf{r}_k)^\infty \preccurlyeq \sigma^n((y_i)) \prec \mathbb{L}(\mathbf{S} \bullet \mathbf{r}_k)^\infty \text{ for all } n \geq 0\},
\end{aligned}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

where the inclusion follows using the claim and $\delta(\hat{\beta}) \prec \delta(\hat{q}_k) = \mathbb{L}(\mathbf{r}_k)^\infty$. Hence, $\mathbf{K}_\beta(t)$ is countable by Proposition 4.1. This establishes equation (6.3).

Case II. $\hat{\beta} = 2$. In this case, $\delta(\hat{\beta}) = 1^\infty$, so $\delta(\beta) = \Phi_{\mathbf{S}}(\delta(\hat{\beta})) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty$ and $t^* = (\Phi_{\mathbf{S}}(01^\infty))_\beta = (\mathbf{S}^-\mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty)_\beta = (\mathbf{S}0^\infty)_\beta$. Recall that $\beta = \beta_r^{\mathbf{S}}$ is the right endpoint of the Lyndon interval $J^{\mathbf{S}}$.

If $t < t^*$, then $b(t,\beta) \prec b(t^*,\beta) = \mathbf{S}0^\infty$, so by Lemma 2.10(iii), there exists $k \in \mathbb{N}$ such that $b(t,\beta) \preccurlyeq \mathbf{S}^-\mathbb{L}(\mathbf{S})^+\mathbf{S}^k 0^\infty$. It follows that

$$\begin{aligned}\mathbf{K}_\beta(t) &= \{(x_i) : b(t,\beta) \preccurlyeq \sigma^n((x_i)) \prec \delta(\beta) \text{ for all } n \geq 0\}\\ &\supset \{\mathbb{L}(\mathbf{S})^+\mathbf{S}^k\mathbf{S}^-, \mathbb{L}(\mathbf{S})^+\mathbf{S}^{k+1}\mathbf{S}^-\}^{\mathbb{N}},\end{aligned}$$

and hence $h_{top}(\mathbf{K}_\beta(t)) > 0$.

Now suppose $t > t^*$. Then $b(t,\beta) \succ b(t^*,\beta) = \mathbf{S}0^\infty$, so by Lemma 4.4,

$$\begin{aligned}\mathbf{K}_\beta(t) &= \{(x_i) : b(t,\beta) \preccurlyeq \sigma^n((x_i)) \prec \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty \text{ for all } n \geq 0\}\\ &\subset \{(x_i) : \mathbf{S}0^\infty \preccurlyeq \sigma^n((x_i)) \prec \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty \text{ for all } n \geq 0\}\\ &= \{(x_i) : \mathbf{S}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty \text{ for all } n \geq 0\}\\ &= \{(x_i) : \mathbf{S}^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq \mathbb{L}(\mathbf{S})^\infty \text{ for all } n \geq 0\},\end{aligned}$$

where the second equality follows by using $\mathbf{S} \in \Omega_L^*$, so $\sigma^n((x_i)) \succcurlyeq \mathbf{S}0^\infty$ for all $n \geq 0$ if and only if $\sigma^n((x_i)) \succcurlyeq \mathbf{S}^\infty$ for all $n \geq 0$. Therefore, $\mathbf{K}_\beta(t)$ is countable by Proposition 4.1. This completes the proof. □

Next we will determine the critical value $\tau(\beta)$ for $\beta \in E_\infty$. Recall from equation (1.7) that

$$E_\infty = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{S} \in \Lambda(n)} J^{\mathbf{S}},$$

where for each $n \in \mathbb{N}$, the Lyndon intervals $J^{\mathbf{S}}$, $\mathbf{S} \in \Lambda(n)$ are pairwise disjoint. Thus, for any $\beta \in E_\infty$, there exists a unique sequence of words $(\mathbf{s}_k)$ with each $\mathbf{s}_k \in \Omega_F^*$ such that

$$\{\beta\} = \bigcap_{n=1}^{\infty} J^{\mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_n}.$$

We call $(\mathbf{s}_k)$ the *coding* of β.

Proposition 6.3. *For any $\beta \in E_\infty$ with its coding $(\mathbf{s}_k)$, we have*

$$\tau(\beta) = \lim_{n\to\infty} (\mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_n 0^\infty)_\beta.$$

Proof. Take $\beta \in E_\infty$. For $k \geq 1$, let $\mathbf{S}_k := \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k$ and write $t_k := (\mathbf{S}_k 0^\infty)_\beta$. Note that $\beta \in J^{\mathbf{S}_k} = [\beta_\ell^{\mathbf{S}_k}, \beta_r^{\mathbf{S}_k}]$ for all $k \geq 1$. Hence,

$$\delta(\beta) \succ \delta(\beta_\ell^{\mathbf{S}_k}) = \mathbb{L}(\mathbf{S}_k)^\infty, \tag{6.4}$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

which implies that $b(t_k,\beta)=\mathbf{S}_k0^\infty$ for all $k\ge 1$. Observe that $\mathbf{S}_{k+1}=\mathbf{S}_k\bullet\mathbf{s}_{k+1}$ begins with $\mathbf{S}_k^-$. Therefore,

$$t_{k+1}=(\mathbf{S}_{k+1}0^\infty)_\beta<(\mathbf{S}_k0^\infty)_\beta=t_k,$$

so the sequence (t_k) is decreasing. Since $t_k\ge 0$ for all $k\ge 1$, the limit $t^*:=\lim_{k\to\infty}t_k$ exists. We will now show that $\tau(\beta)=t^*$.

First we prove $\tau(\beta)\le t^*$. Since t_k decreases to t^* as $k\to\infty$, it suffices to prove that $\tau(\beta)\le t_k$ for all $k\ge 1$. Let $q_k:=\beta_r^{\mathbf{S}_k}$ for all $k\ge 1$. Then $q_k>\beta$ since $\beta\in J^{\mathbf{S}_k}$, and $q_k\searrow\beta$ as $k\to\infty$. Set $t_k':=(\mathbf{S}_k0^\infty)_{q_k}$. Since $q_k>\beta$, one can verify that $b(t_k',q_k)=\mathbf{S}_k0^\infty=b(t_k,\beta)$. So,

$$\begin{aligned}\mathbf{K}_\beta(t_k)&=\{(x_i):b(t_k,\beta)\preccurlyeq\sigma^n((x_i))\prec\delta(\beta)\text{ for all }n\ge 0\}\\&\subset\{(x_i):\mathbf{S}_k0^\infty\preccurlyeq\sigma^n((x_i))\prec\delta(q_k)\text{ for all }n\ge 0\}=\mathbf{K}_{q_k}(t_k').\end{aligned}\tag{6.5}$$

Note by Proposition 6.2 and Case II of its proof that $\tau(q_k)=(\mathbf{S}_k0^\infty)_{q_k}=t_k'$. This implies that $\dim_H K_{q_k}(t_k')=0$, and thus by equation (6.5), we have $\dim_H K_\beta(t_k)=0$. Hence, $\tau(\beta)\le t_k$ for any $k\ge 1$. Letting $k\to\infty$, we obtain that $\tau(\beta)\le t^*$.

Next we prove $\tau(\beta)\ge t^*$. Note that $\beta=\Psi_{\mathbf{S}_k}(\beta_k)$, where $\beta_k\in E_\infty$ has coding $(\mathbf{s}_{k+1},\mathbf{s}_{k+2},\ldots)$. Let $a(\hat{t},\beta_k)$ denote the quasi-greedy expansion of $\hat{t}$ in base β_k (cf. [12, Lemma 2.3]). Observe that the map $\hat{t}\mapsto a(\hat{t},\beta_k)$ is strictly increasing and left continuous everywhere in $(0,1)$, and thus the map $\hat{t}\mapsto(\Phi_{\mathbf{S}_k}(a(\hat{t},\beta_k)))_\beta$ is also left continuous in $(0,1)$. So, by the same argument as in the proof of equation (6.2), it follows that

$$\tau(\beta)\ge(\Phi_{\mathbf{S}_k}(a(\tau(\beta_k),\beta_k)))_\beta\ge(\Phi_{\mathbf{S}_k}(0^\infty))_\beta>(\mathbf{S}_k^-0^\infty)_\beta$$

for every $k\in\mathbb{N}$, and letting $k\to\infty$ gives $\tau(\beta)\ge t^*$. □

To illustrate Proposition 6.3, we construct in each Farey interval $J^{\mathbf{s}}$ a transcendental base $\beta\in E_\infty$ and give an explicit formula for the critical value $\tau(\beta)$. Recall from [4] that the classical *Thue–Morse sequence* $(\theta_i)_{i=0}^\infty=01101001\ldots$ is defined recursively as follows. Let $\theta_0=0$; and if $\theta_0\ldots\theta_{2^n-1}$ is defined for some $n\ge 0$, then

$$\theta_{2^n}\ldots\theta_{2^{n+1}-1}=\overline{\theta_0\ldots\theta_{2^n-1}}.\tag{6.6}$$

By the definition of (θ_i), it follows that

$$\theta_{2k+1}=1-\theta_k,\quad\theta_{2k}=\theta_k\quad\text{for any }k\ge 0.\tag{6.7}$$

Komornik and Loreti [21] showed that

$$\theta_{i+1}\theta_{i+2}\ldots\prec\theta_1\theta_2\ldots\quad\text{for all }i\ge 1.\tag{6.8}$$

PROPOSITION 6.4. *Given $\mathbf{s}=s_1\ldots s_m=0\mathbf{c}1\in\Omega_F^*$, let $\beta:=\beta_\infty^{\mathbf{s}}\in(1,2]$ such that*

$$(\theta_1\mathbf{c}\theta_2\,\theta_3\mathbf{c}\theta_4\,\ldots\,\theta_{2k+1}\mathbf{c}\theta_{2k+2}\ldots)_\beta=1.$$

Then $\beta\in E_\infty\cap J^{\mathbf{s}}$ is transcendental, and

$$\tau(\beta)=\frac{2\sum_{j=2}^m s_j\beta^{m-j}+\beta^{m-1}-\beta^m}{\beta^m-1}.$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

We point out that in the above proposition, $\mathbf{c}$ may be the empty word. To prove the transcendence of β, we recall the following result due to Mahler [25].

LEMMA 6.5. (Mahler, 1976) *If z is an algebraic number in the open unit disc, then the number $Z := \sum_{i=1}^{\infty} \theta_i z^i$ is transcendental.*

Proof of Proposition 6.4. Let $\mathbf{s} = s_1 \ldots s_m = 0\mathbf{c}1 \in \Omega_F^*$. First we prove that

$$\delta(\beta) = \theta_1 \mathbf{c} \theta_2 \; \theta_3 \mathbf{c} \theta_4 \; \ldots \; \theta_{2k+1} \mathbf{c} \theta_{2k+2} \ldots =: (\delta_i). \tag{6.9}$$

By Lemma 2.5, it suffices to prove that $\sigma^n((\delta_i)) \preccurlyeq (\delta_i)$ for all $n \geq 1$. Note by Lemma 2.3 that $\delta_1 \ldots \delta_m = 1\mathbf{c}1 = \mathbb{L}(\mathbf{s})^+ =: a_1 \ldots a_m^+$. Take $n \in \mathbb{N}$, and write $n = mk + j$ with $k \in \mathbb{N} \cup \{0\}$ and $j \in \{1, 2, \ldots, m\}$. We will prove $\sigma^n((\delta_i)) \prec (\delta_i)$ in the following three cases.

Case I. $j \in \{1, 2, \ldots, m-2\}$. Note by equation (6.7) that $\theta_{2k} = 1 - \theta_{2k+1}$. This implies that $\sigma^n((\delta_i))$ begins with either $a_{j+1} \ldots a_m$ or $a_{j+1} \ldots a_m^+ s_1 \ldots s_{m-1}$. By Lemma 2.7, it follows that $\sigma^n((\delta_i)) \prec (\delta_i)$.

Case II. $j = m - 1$. Then $\sigma^n((\delta_i))$ begins with $\theta_{2k}\theta_{2k+1}\mathbf{c}$ for some $k \in \mathbb{N}$. If $\theta_{2k} = 0$, then it is clear that $\sigma^n((\delta_i)) \prec (\delta_i)$ since $\delta_1 = 1$. Otherwise, equation (6.7) implies that $\theta_{2k}\theta_{2k+1}\mathbf{c} = 10\mathbf{c} = 1s_1 \ldots s_{m-1}$. Hence,

$$\sigma^n((\delta_i)) = 1s_1 \ldots s_{m-1} \; \delta_{n+m+1}\delta_{n+m+2} \ldots \prec 1a_2 \ldots a_m^+ \delta_{m+1}\delta_{m+2} = (\delta_i),$$

where the strict inequality follows since $s_1 \ldots s_{m-1} \preccurlyeq a_2 \ldots a_m$.

Case III. $j = m$. Then

$$\sigma^n((\delta_i)) = \theta_{2k+1}\mathbf{c}\theta_{2k+2} \; \theta_{2k+3}\mathbf{c}\theta_{2k+4} \ldots \prec \theta_1\mathbf{c}\theta_2 \; \theta_3\mathbf{c}\theta_4 \ldots = (\delta_i),$$

where the strict inequality is a consequence of equation (6.8).

Therefore, by Cases I–III, we establish equation (6.9). Next we show that $\beta \in E_\infty$. For $k \in \mathbb{N}$, let $\mathbf{S}_k := \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k$ with $\mathbf{s}_1 = \mathbf{s}$ and $\mathbf{s}_i = 01$ for all $2 \leq i \leq k$. Then $\mathbf{S}_k \in \Lambda$ for all $k \in \mathbb{N}$. So it suffices to show that $\beta \in J^{\mathbf{S}_k}$ for all $k \geq 1$. First we claim that

$$\mathbf{S}_k = \overline{\theta_1}\mathbf{c}\overline{\theta_2} \; \overline{\theta_3}\mathbf{c}\overline{\theta_4} \ldots \overline{\theta_{2^k-1}}\mathbf{c}\overline{\theta_{2^k}}^+, \quad \mathbb{L}(\mathbf{S}_k) = \theta_1\mathbf{c}\theta_2 \; \theta_3\mathbf{c}\theta_4 \; \ldots \theta_{2^k-1}\mathbf{c}\theta_{2^k}^- \tag{6.10}$$

for all $k \geq 1$.

Since $\mathbf{S}_1 = \mathbf{s} = 0\mathbf{c}1 = \overline{\theta_1}\mathbf{c}\overline{\theta_2}^+$ and $\mathbb{L}(\mathbf{S}_1) = 1\mathbf{c}0 = \theta_1\mathbf{c}\theta_2^-$, equation (6.10) holds for $k = 1$. Now suppose equation (6.10) holds for a given $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathbf{S}_{k+1} = \mathbf{S}_k \bullet (01) = \mathbf{S}_k^- \mathbb{L}(\mathbf{S}_k)^+ &= \overline{\theta_1}\mathbf{c}\overline{\theta_2} \ldots \overline{\theta_{2^k-1}}\mathbf{c}\overline{\theta_{2^k}} \; \theta_1\mathbf{c}\theta_2 \; \ldots \theta_{2^k-1}\mathbf{c}\theta_{2^k} \\ &= \overline{\theta_1}\mathbf{c}\overline{\theta_2} \ldots \overline{\theta_{2^{k+1}-1}}\mathbf{c}\overline{\theta_{2^{k+1}}}^+,\end{aligned}$$

where the last equality follows since, by the definition of (θ_i) in equation (6.6), $\theta_{2^k+1} \ldots \theta_{2^{k+1}} = \overline{\theta_1 \ldots \theta_{2^k}}^+$. Similarly, by the induction hypothesis and Lemma 3.7,

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

we obtain

$$\begin{aligned}\mathbb{L}(\mathbf{S}_{k+1}) = \mathbb{L}(\mathbf{S}_k \bullet (01)) = \mathbf{S}_k \bullet \mathbb{L}(01) = \mathbf{S}_k \bullet (10) &= \mathbb{L}(\mathbf{S}_k)^+\mathbf{S}_k^- \\ &= \theta_1\mathbf{c}\theta_2 \ \ldots \theta_{2^k-1}\mathbf{c}\theta_{2^k}\ \overline{\theta_1}\mathbf{c}\overline{\theta_2} \ldots \overline{\theta_{2^k-1}}\mathbf{c}\overline{\theta_{2^k}} \\ &= \theta_1\mathbf{c}\theta_2 \ \ldots \theta_{2^{k+1}-1}\mathbf{c}\theta_{2^{k+1}}^-.\end{aligned}$$

Hence, by induction, equation (6.10) holds for all $k \geq 1$.

Next, recall that the Lyndon interval $J^{\mathbf{S}_k} = [\beta_\ell^{\mathbf{S}_k}, \beta_r^{\mathbf{S}_k}]$ satisfies

$$\delta(\beta_\ell^{\mathbf{S}_k}) = \mathbb{L}(\mathbf{S}_k)^\infty \quad \text{and} \quad \delta(\beta_r^{\mathbf{S}_k}) = \mathbb{L}(\mathbf{S}_k)^+\mathbf{S}_k^\infty. \tag{6.11}$$

By equations (6.9) and (6.10), it follows that $\delta(\beta)$ begins with $\mathbb{L}(\mathbf{S}_k)^+$, so by equation (6.11), $\delta(\beta) \succ \mathbb{L}(\mathbf{S}_k)^\infty = \delta(\beta_\ell^{\mathbf{S}_k})$. Thus, $\beta > \beta_\ell^{\mathbf{S}_k}$ for all $k \geq 1$. However, by equations (6.9), (6.10), and Lemmas 3.7 and 3.9, we see that $\delta(\beta)$ also begins with

$$\begin{aligned}\mathbb{L}(\mathbf{S}_{k+2})^+ &= \mathbb{L}(\mathbf{S}_k \bullet (01 \bullet 01))^+ = \mathbb{L}(\mathbf{S}_k \bullet (0011))^+ \\ &= (\mathbf{S}_k \bullet \mathbb{L}(0011))^+ = (\mathbf{S}_k \bullet (1100))^+ = \mathbb{L}(\mathbf{S}_k)^+\mathbf{S}_k\mathbf{S}_k^-\mathbb{L}(\mathbf{S}_k)^+,\end{aligned}$$

which is strictly smaller than a prefix of $\delta(\beta_r^{\mathbf{S}_k})^+ = \mathbb{L}(\mathbf{S}_k)^+\mathbf{S}_k^\infty$. This implies that $\beta < \beta_r^{\mathbf{S}_k}$ for all $k \geq 1$. Hence, $\beta \in J^{\mathbf{S}_k}$ for all $k \geq 1$, and thus $\beta \in E_\infty \cap J^{\mathbf{s}}$. Furthermore, by equations (6.9), (6.10), and Proposition 6.3, it follows that

$$\begin{aligned}\tau(\beta) &= \lim_{k\to\infty}(\mathbf{S}_k 0^\infty)_\beta = (\overline{\theta_1}\mathbf{c}\overline{\theta_2}\ \overline{\theta_3}\mathbf{c}\overline{\theta_4}\ldots)_\beta \\ &= \sum_{k=0}^{\infty}\bigg(\frac{1}{\beta^{mk+1}} + 2\sum_{j=2}^{m-1}\frac{s_j}{\beta^{mk+j}} + \frac{1}{\beta^{mk+m}}\bigg) - (\theta_1\mathbf{c}\theta_2\theta_3\mathbf{c}\theta_4\ldots)_\beta \\ &= \frac{\beta^{m-1} + 2\sum_{j=2}^{m-1}s_j\beta^{m-j} + 1}{\beta^m - 1} - 1 \\ &= \frac{2\sum_{j=2}^{m}s_j\beta^{m-j} + \beta^{m-1} - \beta^m}{\beta^m - 1},\end{aligned}$$

where we recall that $\mathbf{c} = s_2 \ldots s_{m-1}$, and the last equality uses that $s_m = 1$.

Finally, the transcendence of β follows by using equation (6.9), Lemma 6.5, and a similar argument as in the proof of [22, Proposition 5.2]. □

Remark 6.6.

(i) When $\mathbf{s} = 01$, the base $\beta_\infty^{01} \approx 1.78723$ given in Proposition 6.4 is the *Komornik–Loreti constant* (cf. [21]), whose transcendence was first proved by Allouche and Cosnard [3]. In this case, we obtain $\tau(\beta_\infty^{01}) = (2 - \beta_\infty^{01})/(\beta_\infty^{01} - 1) \approx 0.270274$.

(ii) When $\mathbf{s} = 001$, the base $\beta_\infty^{001} \approx 1.55356$ is a critical value for the fat Sierpinski gaskets studied by Li and the second author in [22]. In this case, we have $\tau(\beta_\infty^{001}) \approx 0.241471$.

TABLE 1. The triples $(\mathbf{s}, \beta_\infty^{\mathbf{s}}, \tau(\beta_\infty^{\mathbf{s}}))$ with $\mathbf{s} \in F_3^* \subset \Omega_F^*$.

$\mathbf{s} =$	0001	001	00101	01	01011	011	0111
$\beta_\infty^{\mathbf{s}} \approx$	1.43577	1.55356	1.59998	1.78723	1.83502	1.91988	1.96452
— $\tau(\beta_\infty^{\mathbf{s}}) \approx$	0.218562	0.241471	0.336114	0.270274	0.432175	0.40305	0.455933

By Proposition 6.4 and numerical calculation, we give in Table 1 the triples $(\mathbf{s}, \beta_\infty^{\mathbf{s}}, \tau(\beta_\infty^{\mathbf{s}}))$ for all $\mathbf{s} \in F_3^* \subset \Omega_F^*$. Based on Proposition 6.4, we conjecture that each base $\beta \in E_\infty$ is transcendental.

7. *Càdlàg property of the critical value function*

In this section, we prove Proposition 1.9 and Theorem 1. Recall by Lemma 5.7 that the interval (1, 2] can be partitioned as

$$(1, 2] = E \cup \bigcup_{\mathbf{s}\in\Omega_F^*} J^{\mathbf{s}} = E \cup E_\infty \cup \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}} \cup \bigcup_{\mathbf{S}\in\Lambda} I^{\mathbf{S}}. \tag{7.1}$$

Here we emphasize that the exceptional set E, the relative exceptional sets $E^{\mathbf{S}}$ and the infinitely Farey set E_∞ featuring in equation (7.1) all have Lebesgue measure zero in view of Propositions 5.6 and 5.8. Hence the basic intervals $I^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$, and then certainly the Lyndon intervals $J^{\mathbf{S}}$, $\mathbf{S} \in \Lambda$, are dense in (1, 2]. This allows for approximation of points in E, $E^{\mathbf{S}}$, and E_∞ by left and/or right endpoints of such Lyndon intervals. We also recall from Theorem 2 that for any basic interval $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$ with $\delta(\beta_\ell^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^\infty$ and $\delta(\beta_*^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty$, the critical value is given by

$$\tau(\beta) = (\Phi_{\mathbf{S}}(0^\infty))_\beta = (\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_\beta \quad \text{for any } \beta \in I^{\mathbf{S}}. \tag{7.2}$$

Moreover, by Proposition 6.2, it follows that for each $\beta \in E^{\mathbf{S}}$, we have

$$\tau(\beta) = (\Phi_{\mathbf{S}}(0\delta_2\delta_3 \ldots))_\beta, \tag{7.3}$$

where $1\delta_2\delta_3 \ldots = \delta(\hat{\beta})$ with $\hat{\beta} = \Psi_{\mathbf{S}}{}^{-1}(\beta) \in E$. In particular, when $\beta \in E$, we have $\tau(\beta) = 1 - 1/\beta$ (see Proposition 1.12). When $\beta \in E_\infty$, it follows by Proposition 6.3 that

$$\tau(\beta) = \lim_{n\to\infty} (\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_n 0^\infty)_\beta, \tag{7.4}$$

where $(\mathbf{s}_k)$ is the unique coding of β (that is, $\beta \in J^{\mathbf{s}_1\bullet\cdots\bullet\mathbf{s}_k}$ for all $k \in \mathbb{N}$).

From equation (7.2), it is clear that the critical value function τ is continuous inside each basic interval $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$. So, in view of equation (7.1), we still need to consider the continuity of τ for $\beta \in E \cup E_\infty \cup \bigcup_{\mathbf{S}\in\Lambda}(E^{\mathbf{S}} \setminus \{\beta_r^{\mathbf{S}}\})$, the left continuity of τ at $\beta = \beta_\ell^{\mathbf{S}}$ and $\beta = \beta_r^{\mathbf{S}}$, and the right continuity of τ at $\beta = \beta_*^{\mathbf{S}}$. We need the following lemma.

LEMMA 7.1. *If $\beta \in (1, 2)$ and $\delta(\beta)$ is periodic, then $\beta \in \bigcup_{\mathbf{S}\in\Lambda} I^{\mathbf{S}}$.*

Proof. Assume $\delta(\beta)$ is periodic. In view of equation (7.1), it suffices to prove

$$\beta \notin E \cup E_\infty \cup \bigcup_{\mathbf{S}\in\Lambda} E^{\mathbf{S}}.$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

First we prove $\beta \notin E$. By Lemma 2.8, $\delta(\beta) = \mathbb{L}(\mathbf{S}')^\infty$ for some Lyndon word $\mathbf{S}'$. This means β is the left endpoint of a Lyndon interval, so by Lemma 2.9, $\beta \in J^{\mathbf{s}}$ for some Farey word $\mathbf{s}$. Hence, $\beta \notin E$.

Next, suppose $\beta \in E^{\mathbf{S}}$ for some $\mathbf{S} \in \Lambda$. Clearly, $\beta \neq \beta_r^{\mathbf{S}}$ since $\delta(\beta_r^{\mathbf{S}}) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty$ is not periodic. Thus $\beta \in (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}) \setminus \bigcup_{r \in \Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}$. So, by Proposition 5.6, there is a sequence $(\mathbf{r}_k)$ of Farey words such that $\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k} \searrow \beta$. Write $\delta(\beta) = (a_1 \ldots a_n)^\infty$ with minimal period n. Then by Lemma 2.5, it follows that $\delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k}) \searrow a_1 \ldots a_n^+ 0^\infty$, so for all sufficiently large k, $\delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k})$ contains a block of more than $2m$ zeros, where $m := |\mathbf{S}|$. However, this is impossible, since $\delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k}) = \mathbb{L}(\mathbf{S} \bullet \mathbf{r}_k)^\infty$ is a concatenation of blocks from $\mathbf{S}$, $\mathbf{S}^-$, $\mathbb{L}(\mathbf{S})$, and $\mathbb{L}(\mathbf{S})^+$. These blocks all have length m, and only $\mathbf{S}^-$ could possibly consist of all zeros, while $\mathbf{S}^-$ can only be followed by $\mathbb{L}(\mathbf{S})$ or $\mathbb{L}(\mathbf{S})^+$. Thus, $\delta(\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k})$ cannot contain a block of $2m$ zeros. This contradiction shows that $\beta \notin E^{\mathbf{S}}$.

Finally, suppose $\beta \in E_\infty$. Then there is a sequence $(\mathbf{S}_k)$ of words in Λ such that β lies in the interior of $J^{\mathbf{S}_k}$ for each k. Note that $\beta = \beta_\ell^{\mathbf{S}'}$ is the left endpoint of $J^{\mathbf{S}'}$. Thus, $J^{\mathbf{S}'} \cap J^{\mathbf{S}_k} \neq \emptyset$, and therefore by Lemma 2.9, it must be the case that $J^{\mathbf{S}'} \subset J^{\mathbf{S}_k}$ for all k. However this is impossible, since $|J^{\mathbf{S}_k}| \to 0$. Hence, $\beta \notin E_\infty$. □

Proof of Proposition 1.9. First fix $\beta_0 \in (1, 2] \setminus \{\beta_r^{\mathbf{S}} : \mathbf{S} \in \Lambda\}$. It is sufficient to prove that

(*) for each $N \in \mathbb{N}$, there exists $r > 0$ such that if $\beta \in (1, 2]$ satisfies $|\beta - \beta_0| < r$, then there is a word $s_1 \ldots s_N$ such that $\tau(\beta)$ has a β-expansion beginning with $s_1 \ldots s_N$, and $\tau(\beta_0)$ has a β_0-expansion beginning with $s_1 \ldots s_N$.

For, if $\tau(\beta) = (s_1 \ldots s_N c_1 c_2 \ldots)_\beta$ and $\tau(\beta_0) = (s_1 \ldots s_N d_1 d_2 \ldots)_{\beta_0}$, then

$$\begin{aligned} |\tau(\beta) - \tau(\beta_0)| &\le |(s_1 \ldots s_N c_1 c_2 \ldots)_\beta - (s_1 \ldots s_N c_1 c_2 \ldots)_{\beta_0}| \\ &\quad + |(s_1 \ldots s_N c_1 c_2 \ldots)_{\beta_0} - (s_1 \ldots s_N d_1 d_2 \ldots)_{\beta_0}| \\ &\le \sum_{i=1}^\infty \left| \frac{1}{\beta^i} - \frac{1}{\beta_0^i} \right| + \sum_{i=N+1}^\infty \frac{1}{\beta_0^i} \\ &= \frac{|\beta - \beta_0|}{(\beta - 1)(\beta_0 - 1)} + \frac{1}{(\beta_0 - 1)\beta_0^N} \\ &< \frac{r}{(\beta - 1)(\beta_0 - 1)} + \frac{1}{(\beta_0 - 1)\beta_0^N}, \end{aligned}$$

and this can be made as small as desired by choosing N sufficiently large and r sufficiently small. In view of equation (7.1), we prove $(*)$ by considering several cases.

Case I. $\beta_0 \in (\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}})$ for some basic interval $I^{\mathbf{S}} = [\beta_\ell^{\mathbf{S}}, \beta_*^{\mathbf{S}}]$ with $\mathbf{S} \in \Lambda$. It is clear from Theorem 2 that $(*)$ holds in this case.

Case II. $\beta_0 \in E$. Then by Proposition 5.6, there exists a sequence of Farey intervals $J^{\mathbf{s}_k} = [\beta_\ell^{\mathbf{s}_k}, \beta_r^{\mathbf{s}_k}]$ such that $\beta_\ell^{\mathbf{s}_k} \to \beta_0$ as $k \to \infty$. Furthermore, $|J^{\mathbf{s}_k}| \to 0$ as $k \to \infty$. This implies that the length $|\mathbf{s}_k|$ of the Farey word $\mathbf{s}_k$ goes to infinity as $k \to \infty$. Let $N \in \mathbb{N}$ be given. We can choose $r > 0$ small enough so that if a Farey interval $J^{\mathbf{s}}$ intersects $(\beta_0 - r, \beta_0 + r)$,

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

then $|\mathbf{s}| > N$ and

$$\delta_1(\beta)\ldots\delta_N(\beta) = \delta_1(\beta_0)\ldots\delta_N(\beta_0). \tag{7.5}$$

We can guarantee equation (7.5) because for $\beta_0 \in E \setminus \{2\}$, the expansion $\delta(\beta_0)$ is not periodic by Lemma 7.1, so by Lemma 2.5, the map $\beta \mapsto \delta(\beta)$ is continuous at β_0. Furthermore, for $\beta_0 = 2$, the map $\beta \mapsto \delta(\beta)$ is left continuous at β_0.

Let $\beta \in (\beta_0 - r, \beta_0 + r)$. By equation (7.1), we have either $\beta \in E$ or $\beta \in J^{\mathbf{s}}$ for some $\mathbf{s} \in \Omega_F^*$. If $\beta \in E$, then by Proposition 1.12, it follows that $\tau(\beta) = 1 - 1/\beta = (0\delta_2(\beta)\delta_3(\beta)\ldots)_\beta$ and

$$\tau(\beta_0) = 1 - \frac{1}{\beta_0} = (0\delta_2(\beta_0)\delta_3(\beta_0)\ldots)_{\beta_0}, \tag{7.6}$$

so $(*)$ holds by equation (7.5).

Next we assume $\beta \in J^{\mathbf{s}}$ with $\mathbf{s} = s_1\ldots s_m \in \Omega_F^*$. By our choice of r, it follows that $m = |\mathbf{s}| > N$, and equation (7.5) holds. Since $\beta \in J^{\mathbf{s}} = [\beta_\ell^{\mathbf{s}}, \beta_r^{\mathbf{s}}]$, we have $\mathbb{L}(\mathbf{s})^\infty \preccurlyeq \delta(\beta) \preccurlyeq \mathbb{L}(\mathbf{s})^+\mathbf{s}^\infty$. Write $\mathbb{L}(\mathbf{s}) = a_1\ldots a_m$; then by equation (7.5), it follows that

$$\delta_1(\beta_0)\ldots\delta_N(\beta_0) = \delta_1(\beta)\ldots\delta_N(\beta) = a_1\ldots a_N. \tag{7.7}$$

Observe by equations (7.1)–(7.4) and Lemma 2.3 that $\tau(\beta)$ has a β expansion beginning with $\mathbf{s}^- = 0a_2\ldots a_m$. Hence, by equation (7.7), $\tau(\beta)$ has a β-expansion with prefix

$$s_1\ldots s_N = 0a_2\ldots a_N = 0\delta_2(\beta_0)\ldots\delta_N(\beta_0). \tag{7.8}$$

This, together with equation (7.6), gives $(*)$.

Case III. $\beta_0 \in E^{\mathbf{S}}\setminus\{\beta_r^{\mathbf{S}}\}$ for some $\mathbf{S} \in \Lambda$. The proof is similar to that of Case II, but there are some extra details involving the substitution operator. By Proposition 5.6, it follows that

$$\beta_0 \in (\beta_*^{\mathbf{S}}, \beta_r^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r}\in\Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}, \tag{7.9}$$

and there exists a sequence $(\mathbf{r}_k)$ of Farey words such that $\beta_\ell^{\mathbf{S}\bullet\mathbf{r}_k} \to \beta_0$ as $k \to \infty$. This implies that $|\mathbf{r}_k| \to \infty$ as $k \to \infty$.

Let $N \in \mathbb{N}$ be given; without loss of generality, we may assume that $N = N'|\mathbf{S}|$ for some integer N'. By Lemma 7.1, we can choose $r > 0$ sufficiently small so that if a Lyndon interval $J^{\mathbf{S}\bullet\mathbf{r}}$ intersects $(\beta_0 - r, \beta_0 + r)$, then $|\mathbf{r}| > N'$ and

$$\delta_1(\beta)\ldots\delta_{N'|\mathbf{S}|}(\beta) = \delta_1(\beta_0)\ldots\delta_{N'|\mathbf{S}|}(\beta_0) = \mathbf{S}\bullet(\delta_1(\hat\beta_0)\ldots\delta_{N'}(\hat\beta_0)) \tag{7.10}$$

for any $\beta \in (\beta_0 - r, \beta_0 + r)$, where $\hat\beta_0 = \Psi_{\mathbf{S}}^{-1}(\beta_0) \in E$.

Now take $\beta \in (\beta_0 - r, \beta_0 + r)$. By equation (7.9), it follows that either $\beta \in E^{\mathbf{S}}$ or $\beta \in J^{\mathbf{S}\bullet\mathbf{r}}$ for some $\mathbf{r} \in \Omega_F^*$. If $\beta \in E^{\mathbf{S}}$, we let $\hat\beta = \Psi_{\mathbf{S}}^{-1}(\beta)$. Then equation (7.10) yields

$$\mathbf{S}\bullet(\delta_1(\hat\beta)\ldots\delta_{N'}(\hat\beta)) = \delta_1(\beta)\ldots\delta_{N'|\mathbf{S}|}(\beta) = \mathbf{S}\bullet(\delta_1(\hat\beta_0)\ldots\delta_{N'}(\hat\beta_0)),$$

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

which implies $\delta_1(\hat{\beta})\ldots\delta_{N'}(\hat{\beta})=\delta_1(\hat{\beta}_0)\ldots\delta_{N'}(\hat{\beta}_0)$. So, by equation (7.3), it follows that $\tau(\beta)$ has a β-expansion with a prefix

$$\mathbf{S}\bullet(0\delta_2(\hat{\beta})\ldots\delta_{N'}((\hat{\beta}))=\mathbf{S}\bullet(0\delta_2(\hat{\beta}_0)\ldots\delta_{N'}(\hat{\beta}_0)),$$

which coincides with a prefix of a β_0-expansion of $\tau(\beta_0)$. This gives $(*)$.

If $\beta\in J^{\mathbf{S}\bullet\mathbf{r}}$ for some $\mathbf{r}=r_1\ldots r_m\in\Omega_F^*$, then by our assumption on r, we have $m>N'$. Furthermore,

$$(\mathbf{S}\bullet\mathbb{L}(\mathbf{r}))^\infty=\mathbb{L}(\mathbf{S}\bullet\mathbf{r})^\infty\preccurlyeq\delta(\beta)\preccurlyeq\mathbb{L}(\mathbf{S}\bullet\mathbf{r})^+(\mathbf{S}\bullet\mathbf{r})^\infty=(\mathbf{S}\bullet\mathbb{L}(\mathbf{r}))^+(\mathbf{S}\bullet\mathbf{r})^\infty.$$

Therefore, writing $\mathbb{L}(\mathbf{r})=b_1\ldots b_m$, it follows from equation (7.10) that

$$\mathbf{S}\bullet(b_1\ldots b_{N'})=\delta_1(\beta)\ldots\delta_{N'|\mathbf{S}|}(\beta)=\mathbf{S}\bullet(\delta_1(\hat{\beta}_0)\ldots\delta_{N'}(\hat{\beta}_0)). \tag{7.11}$$

This shows that $b_1\ldots b_{N'}=\delta_1(\hat{\beta}_0)\ldots\delta_{N'}(\hat{\beta}_0)$. Now observe by equations (7.1)–(7.4) and Lemma 2.3 that $\tau(\beta)$ has a β-expansion beginning with $(\mathbf{S}\bullet\mathbf{r})^-=\mathbf{S}\bullet(0b_2\ldots b_m)$. So, by equations (7.3) and (7.11), it follows that $\tau(\beta)$ has a β-expansion with a prefix

$$\mathbf{S}\bullet(0b_2\ldots b_{N'})=\mathbf{S}\bullet(0\delta_2(\hat{\beta}_0)\ldots\delta_{N'}(\hat{\beta}_0))$$

of length N, which is also a prefix of a β_0-expansion of $\tau(\beta_0)$. This again gives $(*)$.

Case IV. $\beta_0=\beta_\ell^{\mathbf{S}}$ for some $\mathbf{S}\in\Lambda$. Here the right continuity of τ at β_0 follows from Theorem 2. The left continuity can be seen as follows. If $\mathbf{S}\in\Omega_F^*$, then $\tau(\beta_0)$ has a β_0-expansion $0\delta_2(\beta_\ell^{\mathbf{S}})\delta_3(\beta_\ell^{\mathbf{S}})\ldots$, and the left continuity follows by the argument in Case II, using the left continuity of the map $\beta\mapsto\delta(\beta)$ at β_0. Otherwise, $\mathbf{S}=\mathbf{S}'\bullet\mathbf{r}$ for some $\mathbf{S}'\in\Lambda$ and $\mathbf{r}\in\Omega_F^*$, and $\tau(\beta_0)$ has a β_0-expansion $\Phi_{\mathbf{S}'}(0\delta_2(\beta_\ell^{\mathbf{r}})\delta_3(\beta_\ell^{\mathbf{r}})\ldots)$. In this case the left continuity follows from the argument in Case III.

Case V. $\beta_0=\beta_*^{\mathbf{S}}$ for some $\mathbf{S}\in\Lambda$. Here the left continuity at β_0 follows from Theorem 2. The right continuity can be seen as follows. First note that

$$\delta(\beta_*^{\mathbf{S}})=\mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty\quad\text{and}\quad\tau(\beta_*^{\mathbf{S}})=(\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_{\beta_*^{\mathbf{S}}}.$$

Observe also that $\beta_*^{\mathbf{S}}=\lim_{\hat{\beta}\searrow 1}\Psi_{\mathbf{S}}(\hat{\beta})$, so by Lemma 2.5, $\Phi_{\mathbf{S}}(\delta(\hat{\beta}))\searrow\delta(\beta_*^{\mathbf{S}})$ as $\hat{\beta}\searrow 1$.

Now let $N\in\mathbb{N}$ be given. As in Case III, we may assume that $N=N'|\mathbf{S}|$ for some integer N'. We choose $r>0$ small enough so that if $\mathbf{r}\in\Omega_F^*$ and $J^{\mathbf{S}\bullet\mathbf{r}}$ intersects (β_0,β_0+r), then $|\mathbf{r}|>N'$. Now take $\beta\in(\beta_0,\beta_0+r)$. If $\beta\in E^{\mathbf{S}}$, then $\beta=\Psi_{\mathbf{S}}(\hat{\beta})$ for some $\hat{\beta}\in E$, and $\delta(\beta)=\Phi_{\mathbf{S}}(\delta(\hat{\beta}))$. Note that if $\beta\searrow\beta_0$, then $\hat{\beta}\searrow 1$ and so $\delta(\hat{\beta})\searrow 10^\infty$. Thus, we may also assume r is small enough so that $\delta(\hat{\beta})$ begins with $10^{N'-1}$. Then $\tau(\beta)$ has a β-expansion beginning with $\Phi_{\mathbf{S}}(0^{N'})=\mathbf{S}^-\mathbb{L}(\mathbf{S})^{N'-1}$, which is also a prefix of length N of a β_0-expansion of $\tau(\beta_0)$.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

Similarly, if $\beta \in J^{\mathbf{S}\bullet\mathbf{r}}$, then we may assume r is small enough so that $\mathbf{r}$ begins with $0^{N'-1}$. Then $\tau(\beta)$ has a β-expansion beginning with $(\mathbf{S} \bullet \mathbf{r})^-$, and therefore beginning with $\mathbf{S}^-\mathbb{L}(\mathbf{S})^{N'-1}$, and we conclude as above.

Case VI. $\beta_0 \in E_\infty$. Then there exists a sequence $(\mathbf{s}_k)$ of Farey words such that

$$\{\beta\} = \bigcap_{k=1}^{\infty} J^{\mathbf{S}_k},$$

where $\mathbf{S}_k := \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k$. Note that $J^{\mathbf{S}_k} \supset J^{\mathbf{S}_{k+1}}$ for all $k \geq 1$, and $|\mathbf{S}_k| \to \infty$ as $k \to \infty$.

Let $N \in \mathbb{N}$ be given, and choose k so large that $|\mathbf{S}_k| > N$. Choose $r > 0$ sufficiently small so that $(\beta_0 - r, \beta_0 + r) \subset J^{\mathbf{S}_k}$. Take $\beta \in (\beta_0 - r, \beta_0 + r)$. Then $\beta \in J^{\mathbf{S}_k}$, so by equations (7.1)–(7.4), it follows that $\tau(\beta)$ has a β-expansion beginning with $\mathbf{S}_k^-$, which is also a prefix (of length at least N) of a β_0-expansion of $\tau(\beta_0)$. Hence, we obtain $(*)$.

Finally, we consider $\beta_0 = \beta_r^{\mathbf{S}}$ for $\mathbf{S} \in \Lambda$. The left continuity at β_0 (that is, the analog of $(*)$ for $\beta \in (\beta_0 - r, \beta_0)$) follows just as in Case III. The jump at β_0 (that is, equation (1.6)) can be seen as follows. Since $\tau(\beta_0) = (\mathbf{S}0^\infty)_{\beta_0} < (\mathbf{S}^\infty)_{\beta_0}$ by Proposition 6.2 (or rather, Case II of its proof), it suffices to show that

$$\lim_{\beta \searrow \beta_0} \tau(\beta) = (\mathbf{S}^\infty)_{\beta_0}. \tag{7.12}$$

First assume $\mathbf{S} \in \Omega_F^*$. Then $\delta(\beta_0) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty$, and by Lemma 2.3, it follows that $\mathbf{S}^\infty = 0\delta_2(\beta_0)\delta_3(\beta_0)\ldots$. So, by the same argument as in Case II, we obtain equation (7.12).

Next suppose $\mathbf{S} = \mathbf{S}' \bullet \mathbf{r}$ for some $\mathbf{S}' \in \Lambda$ and $\mathbf{r} \in \Omega_F^*$. Then

$$\delta(\beta_0) = \mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty = (\mathbf{S}' \bullet \mathbb{L}(\mathbf{r}))^+(\mathbf{S}' \bullet \mathbf{r})^\infty = \Phi_{\mathbf{S}'}(\mathbb{L}(\mathbf{r})^+\mathbf{r}^\infty) = \Phi_{\mathbf{S}'}(\delta(\hat{\beta}_0)),$$

where $\hat{\beta}_0 = \Psi_{\mathbf{S}'}^{-1}(\beta_0) \in E$. This implies that

$$\Phi_{\mathbf{S}'}(0\delta_2(\hat{\beta}_0)\delta_3(\hat{\beta}_0)\ldots) = \Phi_{\mathbf{S}'}(\mathbf{r}^\infty) = (\mathbf{S}' \bullet \mathbf{r})^\infty = \mathbf{S}^\infty.$$

The same argument as in Case III then yields equation (7.12). □

Proof of Theorem 1. The theorem follows by Proposition 1.9, and Theorems 2 and 3. □

Acknowledgements. The authors thank the anonymous referee for several helpful comments. P.A. is partially supported by Simons Foundation grant # 709869. D.K. thanks Charlene Kalle for some useful discussions when she visited Chongqing University in November 2019. He is supported by NSFC No. 11971079 and the Fundamental and Frontier Research Project of Chongqing No. cstc2019jcyj-msxmX0338 and No. cx2019067. Derong Kong is the corresponding author.

References

[1] P. C. Allaart. An algebraic approach to entropy plateaus in non-integer base expansions. *Discrete Contin. Dyn. Syst.* **39**(11) (2019), 6507–6522.

[2] P. C. Allaart and D. Kong. Relative bifurcation sets and the local dimension of univoque bases. *Ergod. Th. & Dynam. Sys.* **41** (2021), 2241–2273.

https://doi.org/10.1017/etds.2022.24 Published online by Cambridge University Press

[3] J.-P. Allouche and M. Cosnard. The Komornik–Loreti constant is transcendental. *Amer. Math. Monthly* **107**(5) (2000), 448–449.
[4] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet–Thue–Morse sequence. *Sequences and Their Applications (Singapore, 1998) (Springer Series in Discrete Mathematics and Theoretical Computer Science)*. Eds. C. Ding, T. Helleseth and H. Niederreiter. Springer, London, 1999, pp. 1–16.
[5] J.-P. Allouche and J. Shallit. *Automatic Sequences. Theory, Applications, Generalizations*. Cambridge University Press, Cambridge, 2003.
[6] C. Baiocchi and V. Komornik. Greedy and quasi-greedy expansions in non-integer bases. *Preprint*, 2007, arXiv:0710.3001v1.
[7] L. A. Bunimovich and A. Yurchenko. Where to place a hole to achieve a maximal escape rate. *Israel J. Math.* **182** (2011), 229–252.
[8] C. Carminati, S. Isola and G. Tiozzo. Continued fractions with SL $(2, Z)$-branches: combinatorics and entropy. *Trans. Amer. Math. Soc.* **370**(7) (2018), 4927–4973.
[9] C. Carminati and G. Tiozzo. The local Hölder exponent for the dimension of invariant subsets of the circle. *Ergod. Th. & Dynam. Sys.* **37**(6) (2017), 1825–1840.
[10] L. Clark. The β-transformation with a hole. *Discrete Contin. Dyn. Syst.* **36**(3) (2016), 1249–1269.
[11] Z. Daróczy and I. Kátai. Univoque sequences. *Publ. Math. Debrecen* **42**(3–4) (1993), 397–407.
[12] M. de Vries and V. Komornik. A two-dimensional univoque set. *Fund. Math.* **212**(2) (2011), 175–189.
[13] M. Demers, P. Wright and L.-S. Young. Escape rates and physically relevant measures for billiards with small holes. *Comm. Math. Phys.* **294**(2) (2010), 353–388.
[14] M. F. Demers. Markov extensions for dynamical systems with holes: an application to expanding maps of the interval. *Israel J. Math.* **146** (2005), 189–221.
[15] M. F. Demers and L.-S. Young. Escape rates and conditionally invariant measures. *Nonlinearity* **19**(2) (2006), 377–397.
[16] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons Ltd., Chichester, 1990.
[17] K. Falconer. *Techniques in Fractal Geometry*. John Wiley & Sons Ltd., Chichester, 1997.
[18] P. Glendinning and N. Sidorov. The doubling map with asymmetrical holes. *Ergod. Th. & Dynam. Sys.* **35**(4) (2015), 1208–1228.
[19] E. Hille. A problem in 'factorisatio numerorum'. *Acta Arith.* **2** (1936), 134–144.
[20] C. Kalle, D. Kong, N. Langeveld and W. Li. The β-transformation with a hole at 0. *Ergod. Th. & Dynam. Sys.* **40**(9) (2020), 2482–2514.
[21] V. Komornik and P. Loreti. Unique developments in noninteger bases. *Amer. Math. Monthly* **105** (1998), 636–639.
[22] D. Kong and W. Li. Critical base for the unique codings of fat Sierpinski gasket. *Nonlinearity* **33**(9) (2020), 4484–4511.
[23] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[24] M. Lothaire. *Algebraic Combinatorics on Words (Encyclopedia of Mathematics and Its Applications, 90)*. Cambridge University Press, Cambridge, 2002.
[25] K. Mahler. *Lectures on Transcendental Numbers (Lecture Notes in Mathematics, 546)*. Springer-Verlag, Berlin, 1976.
[26] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.
[27] G. Pianigiani and J. A. Yorke. Expanding maps on sets which are almost invariant. Decay and chaos. *Trans. Amer. Math. Soc.* **252** (1979), 351–366.
[28] P. Raith. Continuity of the Hausdorff dimension for invariant subsets of interval maps. *Acta Math. Univ. Comenian. (N.S.)* **63**(1) (1994), 39–53.
[29] M. Urbański. On Hausdorff dimension of invariant sets for expanding maps of a circle. *Ergod. Th. & Dynam. Sys.* **6**(2) (1986), 295–309.
[30] M. Urbański. Invariant subsets of expanding mappings of the circle. *Ergod. Th. & Dynam. Sys.* **7**(4) (1987), 627–645.