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Spiraling solutions of nonlinear Schrödinger equations

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We study a new family of sign-changing solutions to the stationary nonlinear Schrödinger equation

$$-\Delta v + qv = |v|^{p-2}v, \qquad \text{in } \mathbb{R}^3,$$

with $2 < p < \infty$ and $q \geqslant 0$. These solutions are spiraling in the sense that they are not axially symmetric but invariant under screw motion, i.e., they share the symmetry properties of a helicoid. In addition to existence results, we provide information on the shape of spiraling solutions, which depends on the parameter value representing the rotational slope of the underlying screw motion. Our results complement a related analysis of Del Pino, Musso and Pacard in their study (2012, *Manuscripta Math.*, 138, 273–286) for the Allen–Cahn equation, whereas the nature of results and the underlying variational structure are completely different.

Keywords: elliptic equations; sign-changing solutions; screw motion invariance; asymptoyic analysis; variational methods

1. Introduction

The present paper is concerned with a new class of solutions to the stationary nonlinear Schrödinger equation

$$-\Delta v + qv = |v|^{p-2}v \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 2$ and $q \geqslant 0$ is a constant. Since the case $q > 0$ is equivalent to $q = 1$ by rescaling, we only consider the cases $q = 1$ and $q = 0$ in the following.

For subcritical exponents p (i.e., $p < \frac{2N}{N-2}$, if $N \geqslant 3$) and $q = 1$, there is a vast literature on solutions of (1.1) in $H^1(\mathbb{R}^N)$, which decay expontially at infinity, see e.g. the monographs [**1**, **18**, **24**, **25**, **28**] and the references therein.

In the present paper, we focus on solutions with only partial decay. These solutions are less understood but have attracted considerable attention in recent years.

To be more precise, let us write $\bar{x} = (x, t) \in \mathbb{R}^N$ with $x \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$. We shall consider solutions $v : \mathbb{R}^N \to \mathbb{R}$ satisfying the condition

$$\lim_{|x|\to\infty} v(x, t) = 0 \quad \text{uniformly in } t. \tag{1.2}$$

A trivial class of solutions satisfying (1.2) is the class of solutions that are axially symmetric with respect to the axis $\{(0_{\mathbb{R}^{N-1}}, t) : \ t \in \mathbb{R}\} \subset \mathbb{R}^N$ and that in addition are t-invariant, i.e., solutions having the form $v(x, t) = \tilde{v}(x)$, where $\tilde{v}$ is a radial solution of (1.1) in $\mathbb{R}^{N-1}$ satisfying $\tilde{v}(x) \to 0$ as $|x| \to \infty$. Here and in the following, axial symmetry is always understood with respect to the t-axis.

In a seminal paper, Dancer [**11**] constructed, for $q = 1$, nontrivial, t-periodic axially symmetric solutions of (1.1) by means of bifurcation theory. The solutions found in [**11**] are positive, and they bifurcate from the unique family of t-invariant axially symmetric positive solutions of (1.1).

It is natural to ask whether, for a given *positive* solution of (1.1), the decay property (1.2) enforces axial symmetry up to translations. As shown in the following theorem by Farina, Malchiodi and Rizzi in [**15**], this is true for positive solutions which are periodic in t.

THEOREM 1.1 [**15**, Special case of Theorem 2]. *Let $p > 2$, $q = 1$, and let $v \in C^2(\mathbb{R}^N)$ be a bounded positive solution of* (1.1) *satisfying the uniform decay property* (1.2). *Suppose moreover that v is periodic in t, i.e., there exists $\tau \in \mathbb{R}$ with*

$$v(x, t + \tau) = v(x, t) \qquad \textit{for all } (x, t) \in \mathbb{R}^N \textit{ with some constant } \tau > 0.$$

Then, up to translations in the x-variable, v is axially symmetric.

Let us also briefly discuss the case $q = 0$ in (1.1). In this case, for subcritical p, it is known that (1.1) does not admit positive solutions (see [**16**, Theorem 1.1]), and it also does not admit solutions of any sign in $H^1(\mathbb{R}^N)$ (by Pohozaev's identity, see e.g. [**28**, Appendix B]). The latter property is related to the fact that, in this case, equation (1.1) remains invariant under the rescaling transformation $v \mapsto \kappa^{\frac{2}{p-2}} v(\kappa\,\cdot)$.

In the present paper, we discuss solutions of (1.1)–(1.2) with periodicity in t, but without axial symmetry. By Theorem 1.1 and the remarks above, such solutions have to change sign. As far as we know, solutions of this type have not been studied yet with the exception of the t-independent case where $v(x, t) = \tilde{v}(x)$ for some nonradial sign-changing solution $\tilde{v}$ of (1.1) in $\mathbb{R}^{N-1}$ with $\tilde{v}(x) \to 0$ as $|x| \to \infty$.

In this context, we briefly recall some existence results on nonradial sign-changing solutions of (1.1) in $\mathbb{R}^N$ for $q = 1$ with exponential decay in all variables. In the work of Bartsch and Willem [**3**], solutions of this type were found for $N = 4$ or $N \geqslant 6$ by a careful application of the Fountain Theorem within the space of functions in $H^1(\mathbb{R}^N)$ that are invariant under the action of the group $O(m) \times O(m) \times O(N - 2m)$, with $N \geqslant 2m + 1$. The case $N = 5$ was considered subsequently by a related argument in [**21**]. More recently, in [**2**, **22**], nonradial sign-changing solutions to (1.1) with no symmetry and with dihedral symmetry, respectively, have been constructed with the Lyapunov–Schmidt reduction method in any dimension $N \geqslant 2$.

In the following, we restrict our attention to the case $N = 3$ and consider the special class of *spiraling solutions* of the nonlinear Schrödinger equation

$$- \Delta v + qv = |v|^{p-2}v \qquad \text{in} \mathbb{R}^3, \tag{1.3}$$

i.e., solutions that are invariant under the action of a screw motion.

To be more precise, let $\lambda > 0$. We call a function $v : \mathbb{R}^3 \to \mathbb{R}$ λ-spiraling if for any $\theta \in \mathbb{R}$,

$$v(R_\theta x, t + \lambda\theta) = v(x, t) \qquad \text{for}\, x \in \mathbb{R}^2, t \in \mathbb{R}, \tag{1.4}$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation with angle θ in $\mathbb{R}^2$. Notice that λ-spiraling functions are $2\lambda\pi$-periodic in t. Hence, the parameter λ represents the rotational slope of the underlying screw motion, and $2\lambda\pi$ is the associated turn-around shift.

Our work is partly inspired by the papers [**12**] resp. [**9**], where spiraling solutions have been constructed for the classical and fractional Allen–Cahn equation, respectively. Without going into detail, we mention the well-known fact that, despite its similar-looking form, the Allen–Cahn equation $-\Delta u = u - u^3$ differs significantly from the nonlinear Schrödinger equation (1.3) with regard to the variational framework and the shape of solutions.

In cylindrical coordinates $(x, t) = (r \cos\varphi, r \sin\varphi, t)$ with $(r, \varphi, t) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$, λ-spiraling functions have the form

$$v(r, \varphi, t) = u\left(r, \varphi - \frac{t}{\lambda}\right)$$

with a function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ which is 2π-periodic in the second variable. Also, in these coordinates the equation (1.3) reads as

$$-v_{rr} - \frac{v_r}{r} - \frac{v_{\varphi\varphi}}{r^2} - v_{tt} + q\, v = |v|^{p-2}v$$

so that the equation for u has the form

$$- u_{rr} - \frac{u_r}{r} - \left(\frac{1}{\lambda^2} + \frac{1}{r^2}\right)u_{\theta\theta} + q\, u = |u|^{p-2}u. \tag{1.5}$$

It is convenient to transform equation (1.5) further to planar euclidean coordinates $x = (x_1, x_2)$, where $r = |x|$ and $\theta = \arcsin \frac{x_2}{|x|}$. This leads to the problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + q\, u = |u|^{p-2}u & \text{on } \mathbb{R}^2, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.6}$$

Observe that radial solutions of (1.6) correspond to axially symmetric and t-invariant solutions of (1.3). By Theorem 1.1, every positive solution of (1.6) is radial. On the other hand, nonradial solutions of (1.6) correspond to solutions of (1.3) which are $2\lambda\pi$-periodic in t but neither axially symmetric nor t-invariant. We, therefore, restrict our attention to nodal (i.e., sign changing) solutions of (1.6).

We study problem (1.6) using variational methods, and hence we first introduce some notation related to its variational structure.

We write $\partial_\theta := x_1\partial_{x_2} - x_2\partial_{x_1}$ for the angular derivative and consider the space

$$H := \left\{ u \in H^1(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} |\partial_\theta u|^2 \mathrm{d}x < \infty \right\}. \tag{1.7}$$

For $\lambda > 0$, we endow H with the λ-dependent scalar product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + uv \Big)\mathrm{d}x \tag{1.8}$$

and consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_\lambda)$.

Let $E_\lambda : H \to \mathbb{R}$ be the energy functional associated to (1.6) in the case $q = 1$, defined by

$$E_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + |u|^2 \Big)\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \,\mathrm{d}x. \tag{1.9}$$

By standard arguments, E_λ is of class C^1, and critical points of E_λ are weak solutions of (1.6).

By definition, a least energy nodal solution of (1.6) is a minimizer of E_λ within the class of sign-changing solutions of (1.6). Our first main result is concerned with the least energy nodal solutions and reads as follows.

THEOREM 1.2. *Let $p > 2$ and $q = 1$. For every $\lambda > 0$ there exists a least energy nodal solution of* (1.6). *Furthermore, there exist $0 < \lambda_0 \leqslant \Lambda_0 < \infty$ with the following properties:*

(i) *For $\lambda < \lambda_0$, every least energy nodal solution of* (1.6) *is radial.*

(ii) *For $\lambda > \Lambda_0$, every least energy nodal solution of* (1.6) *is nonradial.*

Theorem 1.2 establishes a symmetry breaking phenomenon for least energy nodal solutions, which occurs within a finite range of parameters $\lambda \in [\lambda_0, \Lambda_0]$. We are not aware of any other setting where such a transition from radiality to nonradiality has been observed for least energy nodal solutions. The main difficulty when dealing with least energy radial nodal solutions of the equation $-\Delta u + u = |u|^{p-2}u$ in $\mathbb{R}^2$ is given by the fact that so far neither uniqueness (up to sign) nor nondegeneracy is known. Hence, in order to prove the first part of Theorem 1.2, we have to follow an approach, which does not rely on these properties. In fact, a more general radiality result for solutions of (1.6) with small $\lambda > 0$ can be obtained by combining uniform elliptic L^∞-estimates with Poincaré type inequalities in the angular variable. More precisely, we have the following.

THEOREM 1.3. *Let $p > 2$ and $q = 1$.*

i. *If $u \in H$ is a nontrivial weak solution of* (1.6) *for some $\lambda > 0$ satisfying $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$, then u is a radial function.*

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ii. *For every $c > 0$, there exists $\lambda_c > 0$ with the property that every weak solution $u \in H$ of (1.6) for some $\lambda \in (0, \lambda_c)$ with $E_\lambda(u) \leqslant c$ is radial.*

The first part of Theorem 1.2 turns out to be a consequence of Theorem 1.3(ii) and uniform (in λ) energy estimates for least energy nodal solutions of (1.6) in the case $p > 2$, $q = 1$, see § 5 below.

While least energy nodal solutions are particularly interesting from a variational point of view, Theorem 1.2(i) and Theorem 1.3(ii) show that, in order to detect nonradial sign-changing solutions of (1.6) for small values $\lambda > 0$, we have to pass to higher energy levels. A natural class of nonradial nodal solutions of (1.6) is the class of odd solutions with respect to a hyperplane reflection.

If we consider the hyperplane $\{x_1 = 0\}$, then any such solution corresponds to a solution of the boundary value problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + q\,u = |u|^{p-2}u & \text{on } \mathbb{R}^2_+, \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{1.10}$$

in the half space $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \,:\, x_1 > 0\}$. Moreover, by odd reflection and transformation of coordinates, any such solution u gives rise to a λ-spiraling nodal solution $v : \mathbb{R}^3 \to \mathbb{R}$ of (1.3) with the property that

$$v(0,t) = 0 = v(R_t(0, x_2), \lambda t) \qquad \text{for all } t, x_2 \in \mathbb{R}.$$

Consequently, v vanishes on a helicoid, i.e. the condition $u = 0$ on $\partial\mathbb{R}^2_+$ implies that v is zero on the set $\{(x\sin t, x\cos t, \lambda t) : t, x \in \mathbb{R}\}$.

Weak solutions of (1.10) correspond to critical points of the C^1-functional $E^+_\lambda : H^+ \to \mathbb{R}$ defined by

$$E^+_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + q u^2\right)\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^2_+} |u|^p \,\mathrm{d}x, \tag{1.11}$$

where

$$H^+ := \left\{ u \in H^1_0(\mathbb{R}^2_+) \,:\, \int_{\mathbb{R}^2_+} |\partial_\theta u|^2 \mathrm{d}x < \infty \right\}. \tag{1.12}$$

By trivial extension, we regard H^+ as a closed subspace of H, see § 3 below for details.

Our main result for (1.10) reads as follows.

Theorem 1.4. *Let $p > 2$, $q \in \{0, 1\}$ and $\lambda > 0$.*

(i) *(Existence) Problem (1.10) admits a positive least energy solution.*

(ii) *(Symmetry) Any positive solution u of (1.10) is symmetric with respect to reflection at the x_1-axis and decreasing in the angle $|\theta|$ from the x_1-axis. In particular, u takes its maximum on the x_1-axis.*

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(iii) *(Asymptotics) If $q = 1$ and $\lambda_k \geqslant 1$ are given with $\lambda_k \to +\infty$ as $k \to \infty$ and u_k is a positive least energy solution of (1.10) with $\lambda = \lambda_k$, then, after passing to a subsequence, there exists a sequence of numbers $\tau_k > 0$ with*

$$\tau_k \to +\infty, \qquad \frac{\tau_k}{\lambda_k} \to 0 \qquad as\, k \to \infty$$

such that the translated functions $w_k \in H^1(\mathbb{R}^2)$, $w_k(x) = u_k(x_1 + \tau_k, x_2)$ satisfy

$$w_k \to w_\infty \qquad strongly \;\; in\, H^1(\mathbb{R}^2),$$

where w_∞ is the unique positive radial solution of

$$-\Delta w_\infty + w_\infty = |w_\infty|^{p-2} w_\infty, \qquad w_\infty \in H^1(\mathbb{R}^2). \tag{1.13}$$

Similarly, as defined for the equation (1.6), a least energy solution of (1.10) is, by definition, an energy minimizer within the class of nontrivial solutions of (1.10). More specifically, least energy solutions will be characterized as minimizers of E_λ^+ w.r.t. the associated Nehari manifold and attain the mountain pass level

$$c_\lambda = \inf_{u \in H^+ \setminus \{0\}} \sup_{t \geqslant 0} E_\lambda^+(tu), \tag{1.14}$$

see § 3 below. We also point out that the uniqueness of a positive radial solution to (1.13) was shown by Kwong [**17**].

Remark 1.5.

(i) Let $p > 2$ and $q = 1$. As a consequence of Theorem 1.4, the energy of the least energy nodal solution of (1.6), as considered in Theorem 1.2, tends to $2c_\infty$ as $\lambda \to \infty$, where c_∞ is the least energy of nontrivial solutions of the limit problem (1.13). This fact is the key ingredient in the proof of Theorem 1.2(ii).

(ii) The existence result for (1.10) for $p > 2$ and $q \in \{0, 1\}$ relies on compact embeddings. More precisely, we will prove in § 2 below that the space H is compactly embedded into $L^\rho(\mathbb{R}^2)$ for $\rho \in (2, \infty)$, which readily implies that the space H^+ is compactly embedded in $L^\rho(\mathbb{R}^2_+)$ for $\rho \in (2, \infty)$. With the help of these embeddings and by applying the symmetric mountain pass theorem (see Theorem 6.5 in [**24**]), we may also prove, for any $\lambda > 0$, the existence of a sequence of pairs of solutions $\pm u_j$ whose sequence of energies is unbounded.

The existence and symmetry parts of Theorem 1.4 extend to a larger class of semilinear equations, see § 3 below. Next, we shall see that the case $q = 0$ in (1.10) arises naturally when considering the asymptotics of positive least energy solutions of (1.10) in the case $q = 1$ when $\lambda \to 0$. We shall see that these solutions concentrate at the origin as $\lambda \to 0$. More precisely, we have the following.

Theorem 1.6. *Let $(\lambda_k)_k$ be sequence of numbers $\lambda_k \leqslant 1$ such that $\lambda_k \to 0$ as $k \to \infty$. Moreover, let $u_k \in H^+$ be a positive least energy solution of (1.10) with $q = 1$, and let $v_k \in H^+$ be defined by $v_k(x) = \lambda_k^{\frac{2}{p-2}} u_k(\lambda_k x)$.*

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Then, after passing to a subsequence, we have $v_k \to v^*$ in H^+, where v is a positive least energy solution of the problem

$$\begin{cases} -\Delta v^* - [x_1\partial_{x_2} - x_2\partial_{x_1}]^2 v^* = |v^*|^{p-2}v^* & \text{on } \mathbb{R}^2_+, \\ v = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{1.15}$$

REMARK 1.7. The statements given in Theorems 1.4(i) and 1.6 remain valid when the underlying half space $\mathbb{R}^2_+$ is replaced by the cone

$$C_\alpha := \{x \in \mathbb{R}^2 \ : \ x_1 > 0, \quad \arcsin \frac{x_2}{|x|} < \alpha\}.$$

In particular, in the case where $\alpha = \frac{\pi}{2j}$ for some positive integer j, successive reflection yields solutions with precisely $2j$ nodal domains.

The paper is organized as follows. Section 2 sets up the functional analytic framework and provides some preliminary results. In particular, we shall prove the compactness of the embedding $H \hookrightarrow L^\rho(\mathbb{R}^2)$ for $\rho \in (2, \infty)$, and we establish the existence of least energy nodal solutions for problem (1.6). In § 3, we study the symmetry and existence of ground state solutions for a generalization of problem (1.10). In § 4 we discuss the asymptotics of least energy solutions to (1.10) as $\lambda \to \infty$ and as $\lambda \to 0$ and prove Theorems 1.4 and 1.6. Finally, § 5 is devoted to the proofs of Theorems 1.2 and 1.3. In the appendix, we prove a result on uniform L^∞-bounds for weak solutions of (1.6) in the case $q = 1$.

2. Preliminary results

In the following, all functions are assumed to be real-valued. We consider the space H defined in (1.7) with the λ-dependent scalar product defined in (1.8) with $\|\cdot\|_\lambda$ denoting the corresponding norm. The space $(H, \langle\cdot,\cdot\rangle_\lambda)$ is a Hilbert space and clearly, all the norms $\|\cdot\|_\lambda$, $\lambda > 0$, are equivalent.

For easier distinction from the norms on H, for $\rho \in [1, \infty]$, we will use the notation $|\cdot|_\rho$ to denote the standard norm on $L^\rho(\mathbb{R}^2)$.

Recall also that we have set $\partial_\theta := [x_1\partial_{x_2} - x_2\partial_{x_1}]$ for the angular derivative operator. We first note the following.

LEMMA 2.1. *For any $\lambda > 0$, the space $C_c^\infty(\mathbb{R}^2)$ of test functions is dense in $(H, \langle\cdot,\cdot\rangle_\lambda)$.*

Proof. The argument is essentially the same as the one proving the density of $C_c^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$, see e.g. the proof of Theorem 9.2 in [7]. We only sketch it briefly. Let W denote the subspace of functions in H which vanish outside a bounded subset of $\mathbb{R}^2$. By a straightforward cut-off argument, W is dense in H. Moreover, for a given function $u \in W$, it is well known that a sequence of mollifications $u_n \in C_c^\infty(\mathbb{R}^2)$ of u converges to u in the H^1-norm. Moreover, since there is a compact set $K \subset \mathbb{R}^2$ with the property that every u_n, $n \in \mathbb{N}$ vanishes in $\mathbb{R}^2 \setminus K$, the convergence in the H^1-norm also implies convergence in $\|\cdot\|_\lambda$. This shows the claim. □

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Next, we consider the radial averaging operator

$$L^1_{loc}(\mathbb{R}^2) \to L^1_{loc}(\mathbb{R}^2), \quad u \mapsto u^\#, \qquad \text{with}$$
$$u^\#(x) := \frac{1}{2\pi}\int_{S^1} u(|x|\omega)\,\mathrm{d}\omega \quad \text{for a.e.}\, x \in \mathbb{R}^2. \tag{2.1}$$

We note that, as a consequence of Jensen's inequality, the averaging operator extends to a continuous linear map $L^\rho(\mathbb{R}^2) \to L^\rho(\mathbb{R}^2)$ for every $\rho \in [1,\infty]$ with

$$|u^\#|_\rho \leqslant |u|_\rho \qquad \text{for every}\, u \in L^\rho(\mathbb{R}^2). \tag{2.2}$$

Moreover, since $u^\# \in C^1_c(\mathbb{R}^2)$ for $u \in C^1_c(\mathbb{R}^2)$ and

$$\|u^\#\|_\lambda = \|u^\#\|_{H^1(\mathbb{R}^2)} \leqslant \|u\|_{H^1(\mathbb{R}^2)} \leqslant \|u\|_\lambda \quad \text{for}\, \lambda > 0,$$

the operator $u \mapsto u^\#$ extends to a continuous linear map $H \to H$.

We need the following angular Poincaré type estimates.

Lemma 2.2.

(i) *For any $u \in H$,*

$$|u|_2^2 \leqslant |\partial_\theta u|_2^2 + |u^\#|_2^2.$$

In particular, any $u \in H$ with $u^\# \equiv 0$ satisfies $|u|_2^2 \leqslant |\partial_\theta u|_2^2$.

(ii) *Let $\theta_0 \in (0,\pi)$, and consider the cone*

$$C_{\theta_0} := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,:\, r > 0, |\theta| < \theta_0\}.$$

If $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$, then we have

$$|u|_2 \leqslant \frac{2\theta_0}{\pi}|\partial_\theta u|_2.$$

Proof. (i) By lemma 2.1, it suffices to prove the claim for $u \in C^\infty_c(\mathbb{R}^2)$.

We first assume that $u^\# \equiv 0$. In this case, we have, in polar coordinates,

$$|u|_2^2 = \int_0^\infty r\int_0^{2\pi} |u(r,\theta)|^2\,\mathrm{d}\theta\,\mathrm{d}r,$$

where the function $\theta \mapsto u(r,\theta)$ is 2π-periodic and satisfies $\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta = 0$ for every $r > 0$. Consequently, by *Wirtinger's inequality* for periodic functions,

$$\int_0^{2\pi} |u(r,\theta)|^2\,\mathrm{d}\theta \leqslant \int_0^{2\pi} |\partial_\theta u(r,\theta)|^2\,\mathrm{d}\theta \qquad \text{for every}\, r > 0,$$

which implies that

$$|u|_2^2 \leqslant \int_0^\infty r\int_0^{2\pi} |\partial_\theta u(r,\theta)|^2\,\mathrm{d}\theta\,\mathrm{d}r = |\partial_\theta u|_2^2.$$

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If $u \in C_c^\infty(\mathbb{R}^2)$ is arbitrary, we may apply the above argument to the function $u - u^\#$. Since $(u - u^\#)^\# = 0$ and $\langle u - u^\#, u^\# \rangle_{L^2(\mathbb{R}^2)} = 0$, we get that

$$|u|_2^2 - |u^\#|_2^2 = |u - u^\#|_2^2 \leqslant |\partial_\theta (u - u^\#)|_2^2 = |\partial_\theta u|_2^2,$$

as claimed.

(ii) Let $u \in H$ with $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$. By lemma 2.1, there exists a sequence $(u_n)_n$ in $C_c^\infty(\mathbb{R}^2)$ with $u_n \to u$.

We fix $r_0 > 0$ and we let $\rho \in C^\infty([0,\infty))$ be a function with $0 \leqslant \rho \leqslant 1$, $\rho \equiv 0$ on $[0, r_0]$ and $\rho \equiv 1$ on $[2r_0, \infty)$. Moreover, we let $\theta' \in (\theta_0, \pi)$ and $\psi \in C_c^\infty(\mathbb{R})$ be a function with $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in $[-\theta_0, \theta_0]$ and $\psi \equiv 0$ in $\mathbb{R} \setminus [-\theta', \theta']$. Next we define, in polar coordinates,

$$\varphi_0, \varphi_1 \in L^\infty(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2), \qquad \varphi_0(r,\theta) = \rho(r), \quad \varphi_1(r,\theta) = \rho(r)\psi(\theta).$$

Setting $v_n := u_n \varphi_1$ for $n \in \mathbb{N}$, it is then easy to see that

$$v_n \to u\varphi_1 = u\varphi_0 \qquad \text{in } H, \tag{2.3}$$

where the last equality follows since $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$. Moreover, we have, in polar coordinates,

$$|v_n|_2^2 = \int_0^\infty r \int_{-\pi}^{\pi} |v_n(r,\theta)|^2 \, \mathrm{d}\theta \mathrm{d}r,$$

where the function $\theta \mapsto v_n(r,\theta)$ is of class C^1 and satisfies $v_n(r,\theta) = 0$ for $\theta \notin [-\theta', \theta']$, $r > 0$. Using again a classical Wirtinger type inequality (see § 1.7 in [**13**]),

$$\int_{-\pi}^{\pi} |v_n(r,\theta)|^2 \, \mathrm{d}\theta \leqslant \left(\frac{2\theta'}{\pi}\right)^2 \int_{-\pi}^{\pi} |\partial_\theta v_n|^2(r,\theta) \, \mathrm{d}\theta \qquad \text{for every } r > 0,$$

which implies that

$$|v_n|_2^2 \leqslant \left(\frac{2\theta'}{\pi}\right)^2 \int_0^\infty r \int_{-\pi}^{\pi} |\partial_\theta v_n|^2(r,\theta) \, \mathrm{d}\theta \mathrm{d}r = \left(\frac{2\theta'}{\pi}\right)^2 |\partial_\theta v_n|_2^2 \tag{2.4}$$

for every $n \in \mathbb{N}$.

Using (2.3), we may thus pass to the limit in (2.4) to obtain the inequality

$$|u\varphi_0|_2^2 \leqslant \left(\frac{2\theta'}{\pi}\right)^2 |\varphi_0 \partial_\theta u|_2^2,$$

which yields that

$$\|u\|_{L^2(\mathbb{R}^2 \setminus B_{2r_0}(0))} \leqslant \frac{2\theta'}{\pi} \|\partial_\theta u\|_{L^2(\mathbb{R}^2)}.$$

Since $r_0 > 0$ and $\theta' > \theta_0$ were chosen arbitrarily, the claim follows. □

Next, we note embedding properties of the space H.

Lemma 2.3. *For every $\lambda > 0$, $(H, \langle \cdot, \cdot \rangle_\lambda)$ is a Hilbert space canonically embedded in $H^1(\mathbb{R}^2)$. Moreover, H is compactly embedded in $L^\rho(\mathbb{R}^2)$ for all $\rho \in (2, \infty)$.*

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Proof. We have

$$\|u\|_{H^1(\mathbb{R}^2)} \leqslant \|u\|_\lambda \qquad \text{for all}\, \lambda > 0, v \in H,$$

which implies that H is a Hilbert space contained in $H^1(\mathbb{R}^2)$. By standard Sobolev embeddings, H is thus embedded in $L^\rho(\mathbb{R}^2)$ for all $\rho \in [2, \infty)$. It remains to show that these embeddings are compact for $\rho > 2$.

Let $(u_n)_n$ be a sequence in H with $u_n \rightharpoonup 0$ in H, and suppose by contradiction that $u_n \not\to 0$ in $L^\rho(\mathbb{R}^2)$ for some $\rho > 2$.

Since, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$, it follows from Lions' Lemma [19, Lemma I.1] and Rellich's Theorem that, after passing to a subsequence, there exists a sequence $x^n \in \mathbb{R}^2$ with $|x^n| \to \infty$ and such that

$$v_n \rightharpoonup v \neq 0 \qquad \text{in}\, H^1(\mathbb{R}^2) \tag{2.5}$$

for the functions $v_n \in H^1(\mathbb{R}^2)$, $v_n = u_n(\cdot + x^n)$.

Let $r_n := |x^n|$. Passing to a subsequence, we may assume that the limits

$$a := \lim_{n\to\infty} \frac{x_1^n}{r_n}, \qquad b := \lim_{n\to\infty} \frac{x_2^n}{r_n}$$

exist, whereas $a^2 + b^2 = 1$. For every $R > 0$, we then have

$$\begin{aligned}
\lambda^2 \|u_n\|_\lambda^2 &\geqslant \int_{\mathbb{R}^2_+} |x_1 \partial_{x_2} u_n - x_2 \partial_{x_1} u_n|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} |(x_1 + x_1^n)\partial_{x_2} v_n - (x_2 + x_2^n)\partial_{x_1} v_n|^2 \mathrm{d}x \\
&\geqslant \int_{B_R(0)} |(x_1 + x_1^n)\partial_{x_2} v_n - (x_2 + x_2^n)\partial_{x_1} v_n|^2 \, \mathrm{d}x \\
&= r_n^2 \int_{B_R(0)} \left| \frac{x_1 + x_1^n}{r_n} \partial_{x_2} v_n - \frac{x_2 + x_2^n}{r_n} \partial_{x_1} v_n \right|^2 \mathrm{d}x \\
&\geqslant r_n^2 \Bigg(\int_{B_R(0)} |a\partial_{x_2} v_n - b \partial_{x_1} v_n|^2 \mathrm{d}x \\
&\qquad - \sup_{x \in B_R(0)} \left| \frac{x_1 + x_1^n}{r_n} - a \right| \|\partial_{x_2} v_n\|_{L^2(B_R(0))}^2 \\
&\qquad - \sup_{x \in B_R(0)} \left| \frac{x_2 + x_1^n}{r_n} - b \right| \|\partial_{x_1} v_n\|_{L^2(B_R(0))}^2 \Bigg) \\
&\geqslant r_n^2 \left(\int_{B_R(0)} |a\partial_{x_2} v_n - b \partial_{x_1} v_n|^2 \mathrm{d}x + o(1) \right) \\
&\geqslant r_n^2 \left(\int_{B_R(0)} |a\partial_{x_2} v - b \partial_{x_1} v|^2 \, \mathrm{d}x + o(1) \right),
\end{aligned}$$

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where in the last step we used the fact that

$$a\partial_{x_2}v_n - b\partial_{x_1}v_n \rightharpoonup a\partial_{x_2}v - b\partial_{x_1}v \qquad \text{in } L^2(B_R(0))$$

and the weak lower semicontinuity of the L^2-norm. The boundedness of $(u_n)_n$ in H now implies that

$$\int_{B_R(0)} [a\partial_{x_2}v - b\partial_{x_1}v]^2 \mathrm{d}x = 0 \qquad \text{for every } R > 0,$$

and thus

$$\int_{\mathbb{R}^2} |a\partial_{x_2}v - b\partial_{x_1}v|^2 \mathrm{d}x = 0. \tag{2.6}$$

Since $a^2 + b^2 = 1$, if we had $a = 0$ or $b = 0$ it would follow that

$$\int_{\mathbb{R}^2} |\partial_{x_2}v|^2 \,\mathrm{d}x = 0 \quad \text{or} \quad \int_{\mathbb{R}^2} |\partial_{x_1}v|^2 \,\mathrm{d}x.$$

The fact that $v \in L^2(\mathbb{R}^2)$ would imply $v \equiv 0$, contradicting (2.5). If, on the other hand, $a, b \neq 0$, (2.6) implies that $\partial_{x_1}v = \frac{a}{b}\partial_{x_2}v$ in $L^2(\mathbb{R}^2)$. Thus v satisfies $\partial_\beta v = 0$ with $\beta = (1, -\frac{a}{b})$, which again implies $v \equiv 0$ and thus contradicts (2.5). The proof is finished. □

LEMMA 2.4. *The embedding $H \hookrightarrow L^2(\mathbb{R}^2)$ is not compact.*

Proof. Let $\psi \in C_c^\infty((1,2)) \setminus \{0\}$. After trivially extending ψ to $\mathbb{R}$, for $n \in \mathbb{N}$ consider the functions

$$u_n(r,s) = \frac{1}{\sqrt{r}}\psi(r-n)$$

so that

$$\operatorname{supp} u_n \subset \left\{x \in \mathbb{R}^2_+ : n+1 < |x| < n+2\right\}.$$

Clearly, $u_n \rightharpoonup 0$ in H, but

$$|u_n|_2^2 = 2\pi \int_0^\infty \psi(r-n)^2 \,\mathrm{d}r = 2\pi \int_0^\infty \psi(r)^2 \,\mathrm{d}r > 0$$

so $u_n \not\to 0$ in $L^2(\mathbb{R}^2)$. □

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In the following, we fix $p > 2$ and $q = 1$ in (1.6), i.e., we consider the equation

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + u = |u|^{p-2}u & \text{on } \mathbb{R}^2, \\ \qquad\qquad u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{2.7}$$

Here and in what follows, for a given $\lambda > 0$, a function $u \in H$ will be called a weak solution of (2.7) if

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^2} |u|^{p-2}uv \,\mathrm{d}x \qquad \text{for all } v \in H.$$

As a consequence of lemma 2.3 and standard arguments in the calculus of variations, we see that for $\lambda > 0$, the energy functional

$$E_\lambda : H \to \mathbb{R}, \quad E_\lambda(u) := \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \,\mathrm{d}x$$

is of class C^1 and critical points of E_λ are weak solutions of (2.7).

We note the following uniform boundedness property of weak solutions of (1.6).

LEMMA 2.5. *Fix $\lambda > 0$ and let $u \in H$ be a weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad \mathit{in}\, \mathbb{R}^2. \tag{2.8}$$

Then $u \in L^\infty(\mathbb{R}^2)$. Moreover, there exist constants $\sigma, C > 0$, depending on $p > 2$ but not on u and λ, such that

$$|u|_\infty \leqslant C\|u\|_{H^1(\mathbb{R}^2)}^\sigma. \tag{2.9}$$

The fact that the constants C and σ in (2.9) do not depend on λ is of key importance in the proofs of Theorems 1.2(i) and Theorem 1.3(ii). The proof of lemma 2.5 follows by a Moser iteration scheme based on uniform estimates which do not depend on $\lambda > 0$. We include the details in the appendix, see lemma A.1 below.

REMARK 2.6. If $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(0) = 0$ and $u \in H \cap L^\infty(\mathbb{R}^2)$, it is easy to see that also $f(u) = f \circ u \in H \cap L^\infty(\mathbb{R}^2)$ with $\nabla f(u) = f'(u)\nabla u$ and $\partial_\theta f(u) = f'(u)\partial_\theta u$.

By lemma 2.5, this observation applies, in particular, to weak solutions $u \in H$ of (2.8).

Next we note that every nontrivial solution of (2.7) is contained in the Nehari manifold

$$\mathcal{N}_\lambda := \{u \in H \setminus \{0\} : E_\lambda'(u)u = 0\}.$$

Let

$$\alpha_\lambda := \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) > 0, \tag{2.10}$$

then every minimizer is a critical point and hence a solution (cf. [**26**] and Theorem 3.5 below). It is easy to see that such a minimizer is positive and thus radial by Theorem 1.1. Therefore, $\alpha = \alpha_\lambda$ does not depend on λ.

Hence we now focus on sign-changing solutions. Consider

$$\begin{aligned}\mathcal{M}_\lambda &:= \{u \in H : u^+ \not\equiv 0, u^- \not\equiv 0,\ E'_\lambda(u)u^+ = E'_\lambda(u)u^- = 0\}\\ &= \{u \in H \setminus \{0\} : u^+, u^- \in \mathcal{N}_\lambda\}\end{aligned}$$

and set

$$\beta_\lambda := \inf_{u \in \mathcal{M}_\lambda} E_\lambda(u). \tag{2.11}$$

Proposition 2.7. *The value β_λ is positive. Moreover, every minimizer $u \in \mathcal{M}_\lambda$ of (2.11) is a critical point of E_λ and hence a sign-changing solution of (2.7).*

The proof of proposition 2.7 follows the same argument as in the proof of proposition 3.1 in [**4**].

We also remark that $\beta_\lambda \geqslant 2\alpha > 0$ in view of (2.10) and the fact that for any $u \in H$,

$$E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \quad \text{and} \quad E'_\lambda(u)u = E'_\lambda(u^+)u^+ + E'_\lambda(u^-)u^-.$$

We say that a function $u \in H$ is a *least energy nodal solution* of (2.7) if u is a sign-changing solution of (2.7) such that $E_\lambda(u) = \beta_\lambda$. The following lemma yields the existence of a least energy nodal solution.

Lemma 2.8. *There exists $u \in \mathcal{M}_\lambda$ such that $E_\lambda(u) = \beta_\lambda$.*

Proof. We proceed similarly as in [**8**]. Let $(u_n)_n \subset \mathcal{M}_\lambda$ be a minimizing sequence. Note that for any $u \in \mathcal{M}_\lambda$ we have

$$E_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + u^2\right) \mathrm{d}x,$$

which implies that E_λ is coercive on $\mathcal{M}_\lambda$. This yields that $(u_n)_n$ is bounded and we may, therefore, pass to a subsequence such that

$$u_n \rightharpoonup u \quad \text{in}\, H.$$

We then also have $u_n^\pm \rightharpoonup u^\pm$ in H, and the compact embedding $H \hookrightarrow L^p$ implies

$$\int_{\mathbb{R}^2} |u^\pm|^p \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n^\pm|^p \,\mathrm{d}x = C\|u_n^\pm\|_\lambda^2 \geqslant C' > 0.$$

Hence $u^\pm \not\equiv 0$. Next, we show that $u_n^\pm \to u^\pm$ in H. Arguing by contradiction, assume first that $\|u^+\|_\lambda^2 < \liminf_{n\to\infty} \|u_n^+\|_\lambda^2$. Then

$$E'_\lambda(u^+)u^+ = \|u^+\|_\lambda^2 - \|u^+\|_p^p < \liminf_{n\to\infty} \left(\|u_n^+\|_\lambda^2 - \|u_n^+\|_p^p\right) = 0.$$

Hence the characterization of $\mathcal{N}_\lambda$ yields the existence of $a \in (0,1)$ such that $au^+ \in \mathcal{N}_\lambda$. A similar argument yields $bu^- \in \mathcal{N}_\lambda$ for some $0 < b \leqslant 1$. Thus, $au^+ + bu^- \in$

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$\mathcal{M}_\lambda$ and we estimate

$$\begin{aligned}\beta_\lambda &\leqslant E_\lambda(au^+ + bu^-) < \liminf_{n\to\infty} E_\lambda(au_n^+ + bu_n^-) = \liminf_{n\to\infty}\big(E_\lambda(au_n^+) + E_\lambda(bu_n^-)\big)\\ &\leqslant \liminf_{n\to\infty}(E_\lambda(u_n^+) + E_\lambda(u_n^-)) = \liminf_{n\to\infty} E_\lambda(u_n) = \beta_\lambda,\end{aligned}$$

which is a contradiction. Thus, after passing to a subsequence if necessary and using the uniform convexity of $(H, \|\cdot\|_\lambda)$, we conclude that $u_n^+ \to u^+$ strongly in H. In particular, $u^+ \in \mathcal{N}_\lambda$. Proceeding similarly, we prove that $u_n^- \to u^-$ strongly in H and that $u^- \in \mathcal{N}_\lambda$ and consequently, $u \in \mathcal{M}_\lambda$ with $E_\lambda(u) = \beta_\lambda$. □

Summarizing the previous results, we have the following.

Corollary 2.9. *Let $p > 2$. For every $\lambda > 0$ there exists a least energy nodal solution to (2.7), i.e. a sign-changing solution $u \in H$ such that $E_\lambda(u) = \beta_\lambda$.*

Remark 2.10. We may also consider the more general equation

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + u = f(u) & \text{on } \mathbb{R}^2,\\ \qquad\qquad\qquad\qquad u(x) \to 0 & \text{for } |x| \to \infty,\end{cases} \tag{2.12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In order to extend our results, consider the following conditions:

(A_1) There exists $C > 0$ such that $|f(t)| \leqslant C(|t| + |t|^p)$ for $t \in \mathbb{R}$

(A_2) $t \mapsto \dfrac{f(t)}{t}$ is strictly increasing on $\mathbb{R}\backslash\{0\}$ and $\lim_{t\to 0} \dfrac{f(t)}{t} \leqslant 0$, $\lim_{t\to\pm\infty} \dfrac{f(t)}{t} = \infty$.

Under these assumptions, it can be shown that the results of this section, concerned with problem (2.7), continue to hold true for (2.12).

3. Existence and symmetry of odd solutions

This section is devoted to the study of solutions of the problem (1.10), which correspond, by odd reflection, to solutions of (1.6) with hyperplane antisymmetry. In particular, we shall prove Parts (i) and (ii) of Theorem 1.4.

Consider the space H^+ defined in (1.12). For fixed $\lambda > 0$ and $q \in \{0, 1\}$, we endow H^+ with the λ-dependent scalar product

$$\langle u, v\rangle_{\lambda,q} \mapsto \int_{\mathbb{R}^2_+}\Big(\nabla u\cdot\nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + q\,uv\Big)\mathrm{d}x,$$

and we let $\|\cdot\|_{\lambda,q}$ denote the corresponding norm. Observe that any $u \in H^+$ can be extended to an element of H either trivially or by odd reflection. Therefore, lemmas 2.2 and 2.3 immediately yield the following.

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Corollary 3.1.

(i) *Any $u \in H^+$ satisfies*

$$|u|_2^2 \leqslant \int_{\mathbb{R}^2_+} |\partial_\theta u|^2\,dx. \tag{3.1}$$

In particular, the norms $\|\cdot\|_{\lambda,0}$ and $\|\cdot\|_{\lambda,1}$ are equivalent on H^+, and H^+ is a Hilbert space with either of these norms. Moreover, we have a continuous embedding $H^+ \hookrightarrow H^1(\mathbb{R}^2_+)$.

(ii) *The space H^+ is compactly embedded in $L^\rho(\mathbb{R}^2_+)$ for $\rho > 2$.*

Remark 3.2. (i) Similar statements are also true, when the underlying space is the cone C_{θ_0} described in lemma 2.2. (ii) As in lemma 2.4, we see that the embedding $H^+ \hookrightarrow L^2(\mathbb{R}^2_+)$ is not compact.

First, we establish the symmetry of positive weak solutions of (1.10) as a consequence of the following.

Theorem 3.3. *Let $\lambda > 0$, and let $f \in C^1([0,\infty))$ satisfy*

$$f'(t) \leqslant C\Big(t^{\sigma_1} + t^{\sigma_2}\Big) \quad \textit{for} \quad t \geqslant 0 \tag{3.2}$$

with constants $\sigma_1, \sigma_2 > 0$. Moreover, let $u \in H^+ \cap L^\infty(\mathbb{R}^2)$ be a positive weak solution of the problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u = f(u) & \text{on } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.3}$$

Then u is symmetric with respect to the x_1-axis and decreasing with respect to the angle $|\theta|$ from the x_1-axis.

Remark 3.4. Theorem 3.3 in particular applies in the case where the nonlinearity f is given by $f(t) = -qt + |t|^{p-2}t$ for some $p \in (2,\infty)$, $q \in \{0,1\}$. In this case, lemma 2.5 and remark A.2 below imply that every weak solution $u \in H^+$ of (3.2) is bounded. Hence we deduce the statement of Theorem 1.4(ii).

Proof of Theorem 3.3. For simplicity, we assume $\lambda = 1$. We shall argue by the method of rotating planes. For $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, set $e_\theta := (\cos\theta, \sin\theta)$,

$$T_\theta := \{x \in \mathbb{R}^2 : x \cdot e_\theta = 0\} \quad \text{and} \quad \Sigma_\theta := \{x \in \mathbb{R}^2_+ : x \cdot e_\theta < 0\}.$$

Given a positive solution $u \in H^+ \cap L^\infty(\mathbb{R}^2_+)$ of (3.3), consider the functions $u_\theta, w_\theta : \Sigma_\theta \to \mathbb{R}$ defined by

$$u_\theta(x) = u(x - 2(x \cdot e_\theta)e_\theta) \quad \text{and} \quad w_\theta := u_\theta - u$$

and extend them trivially outside Σ_θ. A direct calculation shows that w_θ satisfies

$$\begin{aligned} -\Delta w_\theta - \partial_\theta^2 w_\theta = c_\theta(x) w_\theta \qquad & \text{in}\, \Sigma_\theta \\ w_\theta = 0 \qquad & \text{on}\, T_\theta \\ w_\theta > 0 \qquad & \text{on}\, \partial\Sigma_\theta \setminus T_\theta, \end{aligned} \tag{3.4}$$

where

$$c_\theta(x) = \int_0^1 f'((1-t)u(x) + t u_\theta(x))\, dt.$$

Consider the set

$$\Theta^+ := \left\{\theta \in \left(0, \frac{\pi}{2}\right) : w_\theta \geqslant 0 \text{ in}\, \Sigma_\theta\right\}$$

which is clearly a closed set in $(0, \frac{\pi}{2})$.

We claim that Θ^+ is non-empty. To prove this claim, we proceed as follows. Observe first that $w_\theta^- := \min\{w_\theta, 0\} \in H^+$. Moreover, using (3.2), we have that for $x \in \Sigma_\theta$ with $w_\theta^-(x) < 0$,

$$\begin{aligned} c_\theta(x) \leqslant & C \int_0^1 \Big[((1-t)u(x) + t u_\theta(x))^{\sigma_1} + ((1-t)u(x) + t u_\theta(x))^{\sigma_2}\Big] dt \\ \leqslant & C\Big[u^{\sigma_1}(x) + u^{\sigma_2}(x)\Big]. \end{aligned} \tag{3.5}$$

Also, the boundary conditions imply $w_\theta^- \equiv 0$ on $\partial\Sigma_\theta$, and testing the equation (3.4) against w_θ^- yields

$$\begin{aligned} |\nabla w_\theta^-|_2^2 + |\partial_\theta w_\theta^-|_2^2 = & \int_{\mathbb{R}^2} c_\theta(x)(w_\theta^-)^2\, \mathrm{d}x \\ \leqslant & C \int_{\mathbb{R}^2} \Big[u^{\sigma_1} + u^{\sigma_2}\Big] (w_\theta^-)^2\, \mathrm{d}x \\ \leqslant & C_0 |w_\theta^-|_2^2 \end{aligned} \tag{3.6}$$

with $C_0 = C(|u|_\infty^{\sigma_1} + |u|_\infty^{\sigma_2})$. Therefore, by lemma 2.2(ii),

$$\frac{\pi}{2\theta} |w_\theta^-|_2 \leqslant |\partial_\theta w_\theta^-|_2 \leqslant \sqrt{C_0} |w_\theta^-|_2.$$

Consequently, $w_\theta^- \equiv 0$ provided that $0 < |\theta| < \frac{\sqrt{C_0}\pi}{2}$ and this proves the claim.

Next, we claim that Θ^+ is also open in $(0, \frac{\pi}{2})$. To see this, let $\theta_0 \in \Theta^+$. Since $w_{\theta_0} \not\equiv 0$ by (3.4), the strong maximum principle implies that $w_{\theta_0} > 0$ in Σ_{θ_0}.

Fix $\rho > 2$ such that $\tau_i := \frac{\sigma_i \rho}{\rho - 2} > 2$ for $i = 1, 2$. By lemma 2.3, there exists $\kappa_\rho > 0$ such that

$$|w|_\rho^2 \leqslant \kappa_\rho \Big(|\nabla w|_2^2 + |\partial_\theta w|_2^2\Big) \qquad \text{for all}\, w \in H^+.$$

Moreover, we may choose a compact set $D \subset \Sigma_{\theta_0}$ such that

$$\|u\|_{L^{\tau_1}(\Sigma_{\theta_0} \setminus D)}^{\sigma_1} + \|u\|_{L^{\tau_2}(\Sigma_{\theta_0} \setminus D)}^{\sigma_2} < \frac{1}{2\kappa_\rho C},$$

where $C > 0$ is the constant in (3.5).

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On the other hand, by continuity of the family w_θ w.r.t. θ there exists a neighborhood $N \subset (0, \frac{\pi}{2})$ of θ_0 with the property that for all $\theta \in N$,

$$w_\theta > 0 \quad \text{in } D \quad \text{and} \quad \|u\|^{\sigma_1}_{L^{\tau_1}(\Sigma_\theta \setminus D)} + \|u\|^{\sigma_2}_{L^{\tau_2}(\Sigma_\theta \setminus D)} < \frac{1}{2\kappa_\rho C}.$$

From (3.6) and Hölder's inequality, it follows that

$$\begin{aligned} |w_\theta^-|_\rho^2 &\leqslant \kappa_\rho \Big(|\nabla w_\theta^-|_2^2 + |\partial_\theta w_\theta^-|_2^2 \Big) \leqslant \kappa_\rho C \int_{\mathbb{R}^2} \Big[u^{\sigma_1} + u^{\sigma_2} \Big] (w_\theta^-)^2 \,\mathrm{d}x \\ &\leqslant \kappa_\rho C \Big(\|u\|^{\sigma_1}_{L^{\tau_1}(\Sigma_{\theta_0} \setminus D)} + \|u\|^{\sigma_2}_{L^{\tau_2}(\Sigma_{\theta_0} \setminus D)} \Big) |w_\theta^-|_\rho^2 \leqslant \frac{1}{2} |w_\theta^-|_\rho^2 \end{aligned}$$

for any $\theta \in N$.

Consequently, $w_\theta^- \equiv 0$ for $\theta \in N$ and this proves the claim.

Since Θ^+ is an open, closed and nonempty subset of $(0, \frac{\pi}{2})$, we conclude that $\Theta^+ = (0, \frac{\pi}{2})$. In the same manner, we see that

$$\Theta^- := \Big\{ \theta \in \Big(-\frac{\pi}{2}, 0 \Big) : w_\theta \geqslant 0 \text{ in } \Sigma_\theta \Big\} = (-\frac{\pi}{2}, 0)$$

Consequently, u is decreasing with respect to the angle $|\theta|$ from the x_1-axis.

Finally, a continuity argument also shows that $w_\theta \geqslant 0$ in Σ_θ for $\theta \in \{\pm \frac{\pi}{2}\}$, which, in particular, forces the symmetry of u with respect to reflection at the x_1-axis. □

Next, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1) and (A_2) as in remark 2.10 and set $F(t) = \int_0^t f(s)\, ds$. We consider the energy functional

$$E_\lambda^+ : H^+ \to \mathbb{R}, \quad E_\lambda^+(u) := \frac{1}{2} \|u\|^2_{\lambda,0} - \int_{\mathbb{R}^2_+} F(u) \,\mathrm{d}x$$

Again, standard arguments in the calculus of variations show that E_λ^+ is of class C^1, and critical points of E_λ^+ are solutions of the associated Euler–Lagrange equation

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2} \partial_\theta^2 u = f(u) & \text{on } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial \mathbb{R}^2_+. \end{cases} \tag{3.7}$$

As in § 2 we consider the associated Nehari manifold

$$\mathcal{N}_\lambda^+ := \big\{ u \in H^+ \setminus \{0\} : [E_\lambda^+]'(u) u = 0 \big\}$$

and set

$$c_\lambda := \inf_{u \in \mathcal{N}^+} E_\lambda^+(u). \tag{3.8}$$

This is the ground state energy in the sense that $E_\lambda^+(u) \geqslant c_\lambda$ for every nontrivial solution of (3.7).

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Theorem 3.5. *Let $p > 2$, $\lambda > 0$, and assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumptions (A_1) and (A_2) listed in remark 2.10. Then*

$$c_\lambda = \inf_{u \in H^+ \setminus \{0\}} \sup_{t \geqslant 0} E_\lambda^+(tu). \tag{3.9}$$

Moreover, problem (3.7) admits a ground state solution, i.e., a solution $v \in H^+ \setminus \{0\}$ such that $E_\lambda^+(v) = c_\lambda$.

Proof. The proof essentially follows the lines of the proof of [**26**, Theorem 20], see also [**20**, Section 4]. We note here that (A_1) and (A_2) ensure that the assumptions in [**26**, Theorem 20] are satisfied. Indeed, (A_2) implies that for any $R > 0$ there exists $t_R > 0$ such that $f(t) \geqslant Rt$ for $t \geqslant t_R$. Thus

$$F(t) = \int_0^t f(s)\, ds \geqslant \int_{t_R}^t Rs\, ds = \frac{R}{2}(t^2 - t_R^2)$$

for $t \geqslant t_R$. It follows that

$$\lim_{t\to\infty} \frac{F(t)}{t^2} = \infty,$$

i.e. assumption (iv) in [**26**, Theorem 20] is satisfied. Consequently, the proof given there can be carried through similarly, with some simplifications because the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$ replaces arguments based on compactness modulo translations in the periodic setting of [**26**, Theorem 20]. □

Remark 3.6. (i) The statement of Theorem 1.4(i) is a special case of Theorem 3.5, since the nonlinearity $t \mapsto f(t) = -qt + |t|^{p-2}t$ satisfies conditions (A_1) and (A_2) if $q \in \{0, 1\}$ and $p \in (2, \infty)$.

(ii) Under the assumptions of Theorem 3.5, it can be shown that ground state solutions cannot change sign, see [**26**, Remark 17].

4. Asymptotics of least energy odd solutions

In this section we fix $p \in (2, \infty)$, $q = 1$, and we study the asymptotics of least energy solutions to (1.10) in the case $q = 1$ as $\lambda \to \infty$ and as $\lambda \to 0$. In particular, we shall complete the proofs of Theorem 1.4(iii) and of Theorem 1.6. We will use the notation introduced in the previous section in the special case of the nonlinearity $t \mapsto f(t) = -t + |t|^{p-2}t$, which satisfies conditions (A_1) and (A_2). By the definition of the mountain pass value in (3.8) and the fact that $E^+_{\lambda_1} \geqslant E^+_{\lambda_2}$ for $0 < \lambda_1 < \lambda_2 < \infty$, we infer that the function

$$(0, \infty) \to (0, \infty), \qquad \lambda \mapsto c_\lambda$$

is decreasing, and therefore, the limits

$$c_0 := \lim_{\lambda\to 0} c_\lambda \qquad \text{and} \qquad c_\infty := \lim_{\lambda\to\infty} c_\lambda \tag{4.1}$$

exist in $[0, \infty]$. Next we note that

$$\sup_{t \geqslant 0} E_\lambda^+(tv) = E_\lambda^+(t_v^\lambda v) = \Big(\frac{1}{2} - \frac{1}{p}\Big) \frac{\|v\|_{\lambda,1}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} \quad \text{for every } v \in H^+ \setminus \{0\} \tag{4.2}$$

with

$$t_v^\lambda = \left(\frac{\|v\|_{\lambda,1}^2}{|v|_p^p}\right)^{\frac{1}{p-2}}.$$

We start by considering the asymptotics of least energy solutions to (1.10) as $\lambda \to \infty$.

4.1. The limit $\lambda \to \infty$

Consider the limit energy functional

$$E_* : H^1(\mathbb{R}^2) \to \mathbb{R}, \qquad E_*(v) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + v^2\right) \mathrm{d}x - \frac{1}{p} \int_{\mathbb{R}^2} |v|^p \, \mathrm{d}x.$$

Similarly as in (4.2), for $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ we have

$$\sup_{t \geqslant 0} E_*(tv) = E_*(t_v v) = \Big(\frac{1}{2} - \frac{1}{p}\Big) \frac{\|v\|_{H^1(\mathbb{R}^2)}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} \tag{4.3}$$

with $t_v = (\frac{\|v\|^2_{H^1(\mathbb{R}^2)}}{|v|_p^p})^{\frac{1}{p-2}}$.

Observe that for every $v \in H^1(\mathbb{R}^2)$ with $E_*'(v)v = 0$ we have $t_v = 1$ and hence

$$\sup_{t \geqslant 0} E_*(tv) = E_*(v).$$

Define

$$\hat{c}_\infty := \inf_{v \in H^1(\mathbb{R}^2) \setminus \{0\}} \sup_{t \geqslant 0} E_*(tv) \tag{4.4}$$

and let w_∞ denote the unique positive radial solution (see [**17**]) of the problem

$$-\Delta w_\infty + w_\infty = |w_\infty|^{p-2} w_\infty, \qquad w_\infty \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2). \tag{4.5}$$

Since $E_*'(w_\infty) w_\infty = 0$, $t_{w_\infty} = 1$ and hence

$$\sup_{t \geqslant 0} E_*(t w_\infty) = E_*(w_\infty). \tag{4.6}$$

The following result provides a variational characterization of the limit c_∞, defined in (4.1), in terms of $\hat{c}_\infty$ and w_∞.

Lemma 4.1.

$$c_\infty = \hat{c}_\infty = E_*(w_\infty). \tag{4.7}$$

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Proof. We first prove the second equality in (4.7). Since the proof is standard, we only sketch the argument. By (4.6), we have $\hat{c}_\infty \leqslant E_*(w_\infty)$. On the other hand, using Schwarz symmetrization and (4.3), it is easy to see that

$$\hat{c}_\infty = \inf_{v\in H^1_{rad}(\mathbb{R}^2)\setminus\{0\}} \sup_{t\geqslant 0} E_*(tv).$$

Proceeding as in Theorem 20 and remark 17 in [**26**] and using the compactness of the embedding $H^1_{rad}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, one can prove that $\hat{c}_\infty$ is attained at a positive radial solution of (4.5). By uniqueness, we then deduce that $\hat{c}_\infty = E_*(w_\infty)$.

Next, we prove the first equality in (4.7). Identifying $v \in H^+$ with its trivial extension in H, we see that $E^+_\lambda(v) = E_\lambda(v) \geqslant E_*(v)$ for any $v \in H^+$ and any $\lambda > 0$. Hence $c_\lambda \geqslant \hat{c}_\infty$ for any $\lambda > 0$ by (3.9) and (4.4). Taking the limit as $\lambda \to \infty$, we obtain that $c_\infty \geqslant \hat{c}_\infty$.

To see the opposite inequality, we let $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ be arbitrary. Let $t_v > 0$ be as in (4.3), which implies that

$$0 = \frac{\partial_t\big|_{t_v} E_*(tv)}{t_v} = \|v\|^2_{H^1(\mathbb{R}^2)} - t_v^{p-2}\int_{\mathbb{R}^2} |v|^p \,\mathrm{d}x.$$

From this, we find that

$$\|v\|^2_{H^1(\mathbb{R}^2)} < (2t_v)^{p-2}\int_{\mathbb{R}^2} |v|^p \,\mathrm{d}x.$$

Since $C^\infty_c(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, there exists a sequence $\psi_n \in C^\infty_c(\mathbb{R}^2)$ such that $\|v - \psi_n\|_{H^1(\mathbb{R}^2)} \to 0$ as $n \to \infty$, and

$$\|\psi_n\|^2_{H^1(\mathbb{R}^2)} < (2t_v)^{p-2}\int_{\mathbb{R}^2} |\psi_n|^p \,\mathrm{d}x \qquad \text{for all}\, n \in \mathbb{N}.$$

This implies that

$$\sup_{t\geqslant 0} E_*(t\psi_n) = \sup_{0\leqslant t\leqslant 2t_v} E_*(t\psi_n) \to \sup_{0\leqslant t\leqslant 2t_v} E_*(tv) = E_*(t_v v) \qquad \text{as}\, n \to \infty. \quad (4.8)$$

Next, we fix $n \in \mathbb{N}$ and choose $y_n \in \mathbb{R}^2$ such that $\tilde{\psi}_n \in C^\infty_c(\mathbb{R}^2_+) \subset H^+$ for the function $\tilde{\psi}_n : \mathbb{R}^2_+ \to \mathbb{R}$, $\tilde{\psi}_n(x) = \psi_n(x - y_n)$. Then there exists $t_n > 2t_v$ such that

$$\|\psi_n\|^2_{\lambda,1} = \|\psi_n\|^2_{H^1(\mathbb{R}^2_+)} + \frac{1}{\lambda^2}\|\partial_\theta\psi_n\|^2_{L^2(\mathbb{R}^2_+)} < (2t_n)^{p-2}\int_{\mathbb{R}^2} |\psi_n|^p \,\mathrm{d}x \quad \text{for all}\ \lambda \geqslant 1.$$

Using the fact that

$$\frac{t^2}{\lambda^2}\int_{\mathbb{R}^2_+} |\partial_\theta\psi_n|^2 \,\mathrm{d}x \to 0 \quad \text{as}\, \lambda \to \infty uniformly in t \in [0, t_n],$$

we find that

$$\begin{aligned} c_\infty &= \lim_{\lambda\to\infty} c_\lambda \leqslant \lim_{\lambda\to\infty} \sup_{t\geqslant 0} E^+_\lambda(t\tilde{\psi}_n) = \lim_{\lambda\to\infty} \sup_{0\leqslant t\leqslant t_n} E^+_\lambda(t\tilde{\psi}_n) \\ &= \sup_{0\leqslant t\leqslant t_n} E_*(t\tilde{\psi}_n) = \sup_{t\geqslant 0} E_*(t\tilde{\psi}_n) = \sup_{t\geqslant 0} E_*(t\psi_n), \end{aligned} \quad (4.9)$$

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Combining (4.8) and (4.9), it follows that

$$c_\infty \leqslant E_*(t_v v) = \sup_{t \geqslant 0} E_*(tv).$$

Since $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ was arbitrary, we conclude that $c_\infty \leqslant \hat{c}_\infty$. This completes the proof of the theorem. □

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. The existence statement in (i) is a direct consequence of Theorem 3.5, whereas the symmetry property stated in Theorem 1.4 (ii) is a special case of Theorem 3.3.

Next, we prove the asymptotics in (iii). In what follows, the functions in H^+ are extended trivially outside $\mathbb{R}^2_+$. Assume that $1 \leqslant \lambda_k \to \infty$ and, for every $k \in \mathbb{N}$, let $u_k \in H^+$ denote a positive least energy solution of (1.10) for $\lambda = \lambda_k$. Observe that for $k \in \mathbb{N}$,

$$\|u_k\|^2_{\lambda_k,1} = |u_k|_p^p$$

and

$$c_1 \geqslant c_{\lambda_k} = E^+_{\lambda_k}(u_k) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_k\|^2_{\lambda_k,1} = \left(\frac{1}{2} - \frac{1}{p}\right)|u_k|_p^p \geqslant c_\infty > 0.$$

Since

$$\|u_k\|^2_{H^1_0(\mathbb{R}^2_+)} \leqslant \|u_k\|^2_{\lambda_k,1} \qquad \text{for every } k \in \mathbb{N},$$

we conclude that $(u_k)_k$ is bounded in $H^1_0(\mathbb{R}^2_+) \subset H^1(\mathbb{R}^2)$. Moreover, $|u_k|_p$ remains bounded away from zero. From Lions' Lemma [**19**, Lemma I.1] and Theorem 3.3, it thus follows that, after passing to a subsequence, there exists a sequence of numbers $\tau_k \in (0,\infty)$ such that $w_k \rightharpoonup w \neq 0$ in $H^1(\mathbb{R}^2)$ for the functions $w_k := u_k(\cdot + (\tau_k, 0))$. Observe that $w \geqslant 0$ a.e. in $\mathbb{R}^2$.

We first claim that

$$\tau_k \to \infty \qquad \text{as } k \to \infty. \tag{4.10}$$

Indeed, suppose by contradiction that $(\tau_k)_k$ contains a bounded subsequence. Then we may again pass to a subsequence with the property that

$$u_k \rightharpoonup u \neq 0 \qquad \text{in } H^1_0(\mathbb{R}^2_+),$$

where $u \geqslant 0$ a.e. in $\mathbb{R}^2_+$. For $\varphi \in C^\infty_c(\mathbb{R}^2_+)$ and $R > 0$ with $\operatorname{supp}\varphi \subset B_R(0)$ we then have

$$\frac{1}{\lambda_k^2}\int_{\mathbb{R}^2_+}(\partial_\theta u_k)(\partial_\theta \varphi)\mathrm{d}x \leqslant \frac{R^2}{\lambda_k^2}\|\nabla u_k\|_{L^2(R^2_+)}\|\nabla\varphi\|_{L^2(R^2_+)} \to 0 \qquad \text{as } k \to \infty$$

and thus

$$\int_{\mathbb{R}^2_+}\left(\nabla u \cdot \nabla\varphi + u\varphi - u^{p-1}\varphi\right)\mathrm{d}x = \lim_{k\to\infty}\left(\langle u_k, \varphi\rangle_{\lambda_k,1} - \int_{\mathbb{R}^2_+} u_k^{p-1}\varphi\,\mathrm{d}x\right) = 0.$$

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Hence $u \in H_0^1(\mathbb{R}^2_+)$ is a nontrivial nonnegative weak solution of the problem

$$-\Delta u + u = u^{p-1} \quad \text{in}\, \mathbb{R}^2_+, \qquad u = 0 \quad \text{on}\, \partial\mathbb{R}^2_+$$

which contradicts a classical nonexistence result of Esteban and Lions in [**14**]. Thus (4.10) is true.

We now claim that

$$\frac{\tau_k}{\lambda_k} \to 0 \qquad \text{as}\, k \to \infty. \tag{4.11}$$

Before proving the claim, observe that by weak lower semicontinuity,

$$\begin{aligned} \tau_k^{-2} \int_{\mathbb{R}^2_+} |\partial_\theta u_k|^2 \mathrm{d}x &= \tau_k^{-2} \int_{\mathbb{R}^2_+} |x_1 \partial_{x_2} u_k - x_2 \partial_{x_1} u_k|^2 \mathrm{d}x \\ &= \tau_k^{-2} \int_{\mathbb{R}^2} |(x_1 + \tau_k)\partial_{x_2} w_k - x_2 \partial_{x_1} w_k|^2 \mathrm{d}x \\ &\geqslant \int_{B_R(0)} |\frac{x_1 + \tau_k}{\tau_k} \partial_{x_2} w_k - \frac{x_2}{\tau_k} \partial_{x_1} w_k|^2 \mathrm{d}x \\ &\geqslant \int_{B_R(0)} |\partial_{x_2} w|^2 \, \mathrm{d}x + o(1) \qquad \text{for every } R > 0, \end{aligned} \tag{4.12}$$

whereas for $R > 0$ large enough,

$$\int_{B_R(0)} |\partial_{x_2} w|^2 \mathrm{d}x > 0$$

since $w \in H_0^1(\mathbb{R}^2_+)$ is not identically zero.

Now, in order to prove (4.11), assume by contradiction that, passing to a subsequence,

$$\frac{\tau_k}{\lambda_k} \to d \in (0, \infty] \qquad \text{as}\, k \to \infty.$$

In the case where $d = \infty$ the estimate (4.12) implies that

$$\frac{1}{\lambda_k^2} \int_{\mathbb{R}^2_+} |\partial_\theta u_k|^2 \mathrm{d}x \to \infty \qquad \text{as}\, k \to \infty$$

and therefore

$$\|u_k\|_{\lambda_k, 1} \to \infty \qquad \text{as}\, k \to \infty$$

which contradicts the fact that $\|u_k\|_{\lambda_k,1}$ is bounded in k.

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Therefore, we have $d < \infty$ and from (4.12),

$$\liminf_{k\to\infty} \frac{1}{\lambda_k^2} \int_{\mathbb{R}^2_+} |\partial_\theta u_k|^2 \mathrm{d}x \geqslant d^2 \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 \mathrm{d}x. \tag{4.13}$$

Notice that in this case, $w \in H^1(\mathbb{R}^2)$ is a weak solution of

$$-\Delta w + d^2 \partial_{x_2 x_2} w + w = w^{p-1} \qquad \text{on } \mathbb{R}^2. \tag{4.14}$$

Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^2)$ and let $\varphi_k \in C_c^\infty(\mathbb{R}^2_+)$ be defined by

$$\varphi_k(x_1, x_2) = \varphi(x_1 - \tau_k, x_2)$$

for k sufficiently large. We then have

$$\begin{aligned}
&\frac{1}{\lambda_k^2} \int_{\mathbb{R}^2_+} (\partial_\theta u_k)(\partial_\theta \varphi_k) \mathrm{d}x \\
&\quad = \frac{(d^2 + o(1))}{\tau_k^2} \int_{\mathbb{R}^2_+} (x_1 \partial_{x_2} u_k - x_2 \partial_{x_1} u_k)(x_1 \partial_{x_2} \varphi_k - x_2 \partial_{x_1} \varphi_k) \mathrm{d}x \\
&\quad = (d^2 + o(1)) \int_{\mathbb{R}^2} \Big(\frac{x_1 + \tau_k}{\tau_k} \partial_{x_2} w_k - \frac{x_2}{\tau_k} \partial_{x_1} w_k \Big) \Big(\frac{x_1 + \tau_k}{\tau_k} \partial_{x_2} \varphi - \frac{x_2}{\tau_k} \partial_{x_1} \varphi \Big) \mathrm{d}x \\
&\quad = d^2 \int_{\mathbb{R}^2} \partial_{x_2} w \partial_{x_2} \varphi \mathrm{d}x + o(1) \qquad \text{as } k \to \infty
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} \Big(\nabla w \cdot \nabla \varphi + d^2 \partial_{x_2} w \partial_{x_2} \varphi + w\varphi - w^{p-1} \varphi \Big) \mathrm{d}x \\
&\quad = \lim_{k\to\infty} \int_{\mathbb{R}^2_+} \Big(\nabla u_k \cdot \nabla \varphi_k + \frac{1}{\lambda_k^2} (\partial_\theta u_k)(\partial_\theta \varphi_k) + u_k \varphi_k - u_k^{p-1} \varphi_k \Big) \mathrm{d}x \\
&\quad = \lim_{k\to\infty} \Big(\langle u_k, \varphi \rangle_{\lambda_k, 1} - \int_{\mathbb{R}^2_+} u_k^{p-1} \varphi_k \mathrm{d}x \Big) = 0.
\end{aligned}$$

Hence w satisfies (4.14) in this case. By (4.13) and weak lower semicontinuity, this implies that

$$\begin{aligned}
&\sup_{t\geqslant 0} \Big(E_*(tw) + \frac{t^2 d^2}{2} \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 \mathrm{d}x \Big) = \Big(\frac{1}{2} - \frac{1}{p} \Big) \Big(\|w\|^2_{H^1(\mathbb{R}^2)} + d^2 \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 \mathrm{d}x \Big) \\
&\quad \leqslant \Big(\frac{1}{2} - \frac{1}{p} \Big) \lim_{k\to\infty} \|u_k\|^2_{\lambda_k, 1} = \lim_{k\to\infty} E_{\lambda_k}(u_k) = \lim_{k\to\infty} c_{\lambda_k, 1} = c_\infty.
\end{aligned}$$

On the other hand, we have

$$c_\infty \leqslant \sup_{t\geqslant 0} E_*(tw) < \sup_{t\geqslant 0} \Big(E_*(tw) + t^2 d^2 \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 \mathrm{d}x \Big).$$

Combining these inequalities yields a contradiction. Hence (4.11) holds.

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The same argument as above with $d = 0$ yields that $w \geqslant 0$ is a solution of the limit problem

$$-\Delta w + w = w^{p-1} \qquad \text{in } \mathbb{R}^2$$

and by uniqueness, we have $w = w_\infty$ after adding a finite translation to the sequence τ_k if necessary.

We finish the proof by showing that $w_k \to w$ strongly in $H^1(\mathbb{R}^2)$. Indeed, by weak lower semicontinuity,

$$\begin{aligned} c_\infty &= \left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^2_{H^1(\mathbb{R}^2)} \leqslant \left(\frac{1}{2} - \frac{1}{p}\right)\liminf_{k\to\infty}\|w_k\|^2_{H^1(\mathbb{R}^2)} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\liminf_{k\to\infty}\|u_k\|^2_{H^1(\mathbb{R}^2_+)} \leqslant \left(\frac{1}{2} - \frac{1}{p}\right)\lim_{k\to\infty}\left(\|u_k\|^2_{\lambda_k,1}\right) \\ &= \lim_{k\to\infty} c_{\lambda_k} = c_\infty. \end{aligned}$$

Hence equality holds in all steps. Since $H^1(\mathbb{R}^2)$ is uniformly convex, this shows that $w_k \to w$ strongly in $H^1(\mathbb{R}^2)$, as claimed and this completes the proof of the theorem. □

4.2. The limit $\lambda \to 0$

Next we consider the asymptotics of least energy solutions to (1.10) in the case $q = 1$ as $\lambda \to 0$. To find a suitable limit problem, we consider the transformed Dirichlet problem

$$\begin{cases} -\Delta v - \partial_\theta^2 v + \lambda^2 v = |v|^{p-2}v & \text{in } \mathbb{R}^2_+, \\ v = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{4.15}$$

Weak solutions $v \in H^+$ of (4.15) are critical points of the associated energy functional given by

$$J_\lambda : H^+ \to \mathbb{R}, \quad J_\lambda(v) = \frac{1}{2}\left(|\nabla v|_2^2 + |\partial_\theta v|_2^2 + \lambda^2|v|_2^2\right) - \frac{1}{p}\|v\|_p^p.$$

These notions can be related to the original problem as follows: For $\lambda > 0$, consider the transformation

$$H^+ \ni u \mapsto v \in H^+, \quad v(x) = \lambda^{\frac{2}{p-2}}u(\lambda x)$$

so that

$$J_\lambda(v) = \lambda^{\frac{4}{p-2}}E^+_\lambda(u). \tag{4.16}$$

Moreover, u is a (least energy) solution of (1.10) if and only if v is a (least energy) solution of (4.15).

In order to prove Theorem 1.6, let $(\lambda_k)_k$ be sequence of numbers $\lambda_k \leqslant 1$ such that $\lambda_k \to 0$ as $k \to \infty$ and let $u_k \in H^+$ be positive least energy solutions of (1.10) for $\lambda = \lambda_k$.

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For any $k \in \mathbb{N}$, set

$$v_k(x) = \lambda_k^{\frac{2}{p-2}} u_k(\lambda_k x), \quad v_k \in H^+.$$

LEMMA 4.2. *The sequence $(v_k)_k$ is bounded in H^+.*

Proof. By Corollary 3.1, it suffices to show that there exists $C > 0$ such that

$$\|v_k\|_{1,0} \leqslant C \qquad \text{for all} k \in \mathbb{N}.$$

By the remarks above, v_k is a least energy solution of the transformed problem (4.15) with $\lambda = \lambda_k$. Multiplying this equation with v_k and integrating by parts yields

$$\|v_k\|_{1,0}^2 + \lambda_k^2 |v_k|_2^2 = |v_k|_p^p \qquad \text{for all}\, k \in \mathbb{N}. \tag{4.17}$$

Moreover, we have

$$J_{\lambda_k}(v_k) = \inf_{v \in H^+ \setminus \{0\}} \sup_{t \geqslant 0} J_{\lambda_k}(tv).$$

Fix $\varphi \in C_c^\infty(\mathbb{R}_+^2) \setminus \{0\}$. Since v_k is a least energy solution of (4.15) for $\lambda = \lambda_k \leqslant 1$, we have

$$J_{\lambda_k}(v_k) \leqslant \sup_{t \geqslant 0} J_{\lambda_k}(t\varphi) \leqslant \sup_{t \geqslant 0} J_1(t\varphi) =: C_0$$

where, clearly, C_0 is independent of k. We can then use (4.17) to get

$$J_{\lambda_k}(v_k) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|v_k\|_{1,0}^2 + \lambda_k^2 |v_k|_2^2\right) \geqslant \left(\frac{1}{2} - \frac{1}{p}\right) \|v_k\|_{1,0}^2$$

and hence

$$\|v_k\|_{1,0}^2 \leqslant \frac{C_0}{\frac{1}{2} - \frac{1}{p}} \qquad \text{for all}\, k \in \mathbb{N}.$$

□

As a consequence of lemma 4.2, we can pass to a subsequence and assume

$$v_k \rightharpoonup v^* \quad \text{in}\, H^+.$$

LEMMA 4.3. *The weak limit v^* is a nontrivial weak solution of (1.15).*

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Proof. Since every v_k is a weak solutions of (1.10), for any test function $\varphi \in C_c^\infty(\mathbb{R}^2_+)$ we have

$$\int_{\mathbb{R}^2_+} (\nabla v_k \cdot \nabla \varphi + \partial_\theta v_k \partial_\theta \varphi)\,\mathrm{d}x = \int_{\mathbb{R}^2_+} |v_k|^{p-2} v_k \varphi\,\mathrm{d}x - \lambda_k^2 \int_{\mathbb{R}^2_+} v_k \varphi\,\mathrm{d}x.$$

Besides, since $v_k \rightharpoonup v^*$ weakly in H^+ and $\lambda_k \to 0^+$ as $k \to \infty$,

$$\int_{\mathbb{R}^2_+} (\nabla v_k \cdot \nabla \varphi + \partial_\theta v_k \partial_\theta \varphi)\,\mathrm{d}x - \lambda_k^2 \int_{\mathbb{R}^2_+} v_k \varphi\,\mathrm{d}x \to \int_{\mathbb{R}^2_+} (\nabla v^* \cdot \nabla \varphi + \partial_\theta v^* \partial_\theta \varphi)\,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^2_+} |v_k|^{p-2} v_k \varphi\,\mathrm{d}x \to \int_{\mathbb{R}^2+} |v^*|^{p-2} v^* \varphi\,\mathrm{d}x$$

as a consequence of the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$. It then follows that $v^* \in H^+$ is a weak solution of

$$-\Delta v^* - \partial_\theta^2 v^* = |v^*|^{p-2} v^* \quad \text{in}\, \mathbb{R}^2_+.$$

Next, we prove that $v^* \not\equiv 0$. To do so, first, observe that the embedding $H^+ \hookrightarrow L^p$ yields

$$C := \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}}{|u|_p} \in (0, \infty).$$

Thus, the above comments, together with the fact that $|u|_2^2 \leqslant |\partial_\theta u|_2^2 \leqslant \|u\|_{1,0}^2$ for $u \in H^+$ (see Corollary 3.1), imply that

$$C^2 = \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2}{|u|_p^2} \leqslant \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2}{|u|_p^2} \leqslant 2 \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2}{|u|_p^2} = 2C^2.$$

Recalling also that

$$J_{\lambda_k}(v_k) = \inf_{u \in H^+ \setminus \{0\}} \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2}{|u|_p^2}\right)^{\frac{p}{p-2}},$$

we thus have

$$\left(\frac{1}{2} - \frac{1}{p}\right) C^{\frac{2p}{p-2}} \leqslant J_{\lambda_k}(v_k) \leqslant \left(\frac{1}{2} - \frac{1}{p}\right) (2C^2)^{\frac{p}{p-2}} \qquad \text{for all}\, k \in \mathbb{N}. \tag{4.18}$$

Now assume by contradiction that $v^* = 0$, i.e., $v_k \rightharpoonup 0$ weakly in H^+. The compact embedding $H^+ \hookrightarrow L^p$ implies $v_k \to 0$ in L^p, and therefore $\|v_k\|_{1,0} \to 0$ by (4.17). Hence also $|v_k|_2 \to 0$ by Corollary 3.1. We then deduce that

$$J_{\lambda_k}(v_k) = \left(\frac{1}{2} - \frac{1}{p}\right) (\|v_k\|_{1,0}^2 + \lambda_k^2 |v_k|_2^2) \to 0,$$

which contradicts (4.18). We conclude that $v^* \neq 0$, as claimed. □

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We will now use Γ-convergence to finish the proof of Theorem 1.6:

Proof of Theorem 1.6. It remains to prove that v^* is a least energy solution of (1.15), and that $v_k \to v^*$ strongly in H^+ as $k \to \infty$.

To deduce these properties from Γ-convergence theory, we consider the space $X := H^+ \setminus \{0\}$ endowed with the weak topology (induced by $\|\cdot\|_{1,0}$). Consider the functionals $F_k, F : X \to [0,\infty]$ defined by

$$F_k(u) := \frac{(\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2)^{\frac{p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}} \quad \text{and} \quad F(u) := \frac{\|u\|_{1,0}^{\frac{2p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}}.$$

Then we have

$$F(u) \leqslant F_k(u) \qquad \text{for every } k \in \mathbb{N} \text{ and } u \in H^+.$$

Let $(\tilde{u}_k)_k \subset X$ be an arbitrary sequence such that $\tilde{u}_k \to \tilde{u}$ in X (recall that X has the weak topology of H^+). The compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$ and the weak lower semicontinuity of $\|\cdot\|_{1,0}$ imply

$$F(\tilde{u}) \leqslant \liminf_{k\to\infty} F(\tilde{u}_k) \leqslant \liminf_{k\to\infty} F_k(\tilde{u}_k).$$

On the other hand, for any $\tilde{u} \in X$, the constant sequence $\tilde{u}_k := \tilde{u}$ satisfies that $\tilde{u}_k \to \tilde{u}$ in X and

$$F(\tilde{u}) = \lim_{k\to\infty} F_k(\tilde{u}_k).$$

We conclude that $F_k \xrightarrow{\Gamma} F$. Since,

$$F_k(v_k) = \inf_{u\in X} F_k(u)$$

and $v_k \to v$ in X, it follows from [**10**, Corollary 7.20] that

$$F(v) = \inf_{u\in X} F(u) = \lim_{k\to\infty} F_k(v_k). \tag{4.19}$$

Consequently,

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p}\right) \frac{\|v\|_{1,0}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} &= \inf_{u\in H^+\setminus\{0\}} \left(\frac{1}{2} - \frac{1}{p}\right) \frac{\|u\|_{1,0}^{\frac{2p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}} \\ &= \inf_{u\in H^+\setminus\{0\}} \sup_{t\geqslant 0} \left(\frac{t^2}{2}\|u\|_{1,0}^2 - \frac{t^p}{p}|u|_p^p\right),\end{aligned}$$

and this implies that v is a least energy solution of (1.15). Moreover, since $v_k \to v$ in $L^p(\mathbb{R}^2_+)$ by the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$, it follows from (4.19) and

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the definition of the functionals F_k and F that

$$\|v\|_{1,0}^2 = \lim_{k\to\infty}\left(\|v_k\|_{1,0}^2 + \lambda_k^2|v_k|_2^2\right) \geqslant \limsup_{k\to\infty}\|v_k\|_{1,0}^2 \geqslant \liminf_{k\to\infty}\|v_k\|_{1,0}^2 \geqslant \|v\|_{1,0}^2.$$

Consequently, we have

$$\|v_k\|_{1,0} \to \|v\|_{1,0} \qquad \text{as}\, k\to\infty,$$

and the uniform convexity of $(H^+, \|\cdot\|_{1,0})$ implies that $v_k \to v$ strongly in H^+ as $k\to\infty$. □

5. Radial versus nonradial least energy nodal solutions

In this section we complete the proofs of Theorem 1.2 and Theorem 1.3. Given the assumptions of Theorem 1.2, the existence of a least energy nodal solution of (1.6) for every $\lambda > 0$ is a direct consequence of Corollary 2.9.

We will now first prove Theorem 1.2(ii), which will be a consequence of lemma 4.1 and a result in [**27**].

We recall that, as in § 4.1 and § 2, the energy functionals $E_*, E_\lambda : H \to \mathbb{R}$ are defined by

$$E_*(v) := \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2 + |v|^2\right)\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^2}|v|^p\mathrm{d}x$$

and

$$E_\lambda(v) = E_*(v) + \frac{1}{\lambda^2}\int_{\mathbb{R}^2}|\partial_\theta v|^2\mathrm{d}x$$

for $v \in H$. Moreover, as in § 2, we consider the λ-dependent scalar product $\langle\cdot,\cdot\rangle_\lambda$ defined in (1.8) on H and the corresponding norm $\|\cdot\|_\lambda$. In particular, we shall use $\|\cdot\|_1$ given by

$$\|u\|_1^2 = \int_{\mathbb{R}^2}(|\nabla u|^2 + |\partial_\theta u|^2 + |u|^2)\,\mathrm{d}x \qquad \text{for}\, u\in H.$$

Proposition 5.1. *There exists $\varepsilon_* > 0$ such that for every $\lambda > 0$ and every radial nodal solution $u \in H$ of (1.6) we have*

$$E_*(u) = E_\lambda(u) > 2c_\infty + \varepsilon_*,$$

where c_∞ is given in (4.1).

Proof. First observe that $E_*(u) = E_\lambda(u)$ for every radial function $u \in H$. Moreover, if u is a radial nodal solution of (1.6), then u also solves the limit problem (1.13). By [**27**, Theorem 1.5], and the variational characterization of c_∞ given (4.4) and (4.7), there exists $\varepsilon_* > 0$ with the property that $E_*(u) > 2c_\infty + \varepsilon_*$ for every nodal solution of (1.13). This proves the claim. □

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Proof of Theorem 1.2(ii) (completed). Let $\varepsilon_* > 0$ be given by proposition 5.1. By (4.1), there exists $\Lambda_0 > 0$ with the property that

$$c_\lambda < c_\infty + \frac{\varepsilon_*}{2} \qquad \text{for every } \lambda > \Lambda_0.$$

Consequently, for $\lambda > \Lambda_0$, problem (1.10) admits a nontrivial solution $u \in H^+$ with $E^+_\lambda(u) < c_\infty + \frac{\varepsilon_*}{2}$. By odd reflection, we may extend u to a nodal solution of (1.6) with $E_\lambda(u) < 2c_\infty + \varepsilon_*$. Proposition 5.1, therefore, implies that the least energy nodal solutions of (1.6) cannot be radial. □

Next, we complete the proof of Theorem 1.3, which we restate here for the reader's convenience.

THEOREM 5.2. *Let $p > 2$.*

(i) *If $u \in H$ is a nontrivial weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad \mathit{in}\, \mathbb{R}^2 \tag{5.1}$$

for some $\lambda > 0$ satisfying $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$, then u is a radial function.

(ii) *For every $c > 0$, there exists $\lambda_c > 0$ with the property that every weak solution $u \in H$ of (5.1) for some $\lambda \in (0, \lambda_c)$ with $E_\lambda(u) \leqslant c$ is radial.*

Proof. (i) Let $u \in H$ be a nontrivial weak solution of (5.1) for some $\lambda > 0$, and let, as before, $u^\#$ denote the radial average of u as defined in (2.1). It is easy to see that, for every $k \in \mathbb{N}$, the function $u^\# \in H$ is a weak solution of

$$-\Delta u^\# + u^\# = \left(|u|^{p-2}u\right)^\# \quad \text{in}\, \mathbb{R}^2.$$

Consequently we have, in weak sense,

$$-\Delta(u - u^\#) - \frac{1}{\lambda^2}\partial_\theta^2(u - u^\#) + (u - u^\#) = |u|^{p-2}u - \left(|u|^{p-2}u\right)^\# \quad \text{in } \mathbb{R}^2.$$

Testing this equation against $u - u^\#$ yields

$$\begin{aligned}
\frac{1}{\lambda^2}|\partial_\theta u|_2^2 = \frac{1}{\lambda^2}|\partial_\theta(u - u^\#)|_2^2 &\leqslant |\nabla(u - u^\#)|_2^2 + \frac{1}{\lambda^2}|\partial_\theta(u - u^\#)|_2^2 + |u - u^\#|_2^2 \\
&= \int_{\mathbb{R}^2} \left(|u|^{p-2}u - \left(|u|^{p-2}u\right)^\#\right)(u - u^\#)\,\mathrm{d}x \\
&\leqslant \left||u|^{p-2}u - \left(|u|^{p-2}u\right)^\#\right|_2 |u - u^\#|_2 \\
&\leqslant \left||u|^{p-2}u - \left(|u|^{p-2}u\right)^\#\right|_2 |\partial_\theta u|_2,
\end{aligned} \tag{5.2}$$

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where we used lemma 2.2 in the last step. Moreover, $|u|^{p-2}u \in H$ by remark 2.6, and therefore

$$\begin{aligned} \left| |u|^{p-2}u - \left(|u|^{p-2}u\right)^{\#} \right|_2 &\leqslant |\partial_\theta(|u|^{p-2}u)|_2 = (p-1)||u|^{p-2}\partial_\theta u|_2 \\ &\leqslant (p-1)|u|_\infty^{p-2}|\partial_\theta u|_2, \end{aligned} \tag{5.3}$$

again by lemma 2.2. Combining (5.2) and (5.3), we obtain that

$$\frac{1}{\lambda^2}|\partial_\theta u|_2^2 \leqslant (p-1)|u|_\infty^{p-2}|\partial_\theta u|_2^2$$

which implies that $\partial_\theta u \equiv 0$ if $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$. The proof of (i) is thus finished.

(ii) Let $c > 0$ be given, and let $u \in H$ be a nontrivial weak solution of (5.1) for some $\lambda > 0$ with $E_\lambda(u) \leqslant c$. Since $E_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_\lambda^2$, it then follows that

$$\|u\|_{H^1(\mathbb{R}^2)}^2 \leqslant \|u\|_\lambda^2 = \frac{2p}{p-2}E_\lambda(u) \leqslant \frac{2pc}{p-2}$$

and therefore

$$|u|_\infty \leqslant C\|u\|_{H^1(\mathbb{R}^2)}^\sigma \leqslant C\Big(\frac{2pc}{p-2}\Big)^{\frac{\sigma}{2}} =: \mu_c$$

by lemma 2.5 with the constants $C, \sigma > 0$ given there. Hence, if

$$\lambda < \lambda_c := \Big(\frac{1}{(p-1)\mu_c^{p-2}}\Big)^{\frac{1}{2}},$$

then also $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$ and therefore, u is radial by (i). The proof is finished. □

Next we provide uniform energy estimates for least energy nodal solutions of (5.1).

Lemma 5.3. *Let $p > 2$. There exist constants $c, C > 0$ with the property that*

$$c \leqslant E_\lambda(u) \leqslant C \tag{5.4}$$

for every $\lambda > 0$ and every least energy nodal solution $u \in H$ of (5.1).

Proof. The lower bound is obtained by choosing $c = \hat{c}_\infty$ as defined in (4.4), since

$$E_\lambda(u) = \sup_{t \geqslant 0} E_\lambda(tu) \geqslant \sup_{t \geqslant 0} E_*(tu) \geqslant \hat{c}_\infty$$

for every $\lambda > 0$ and every nontrivial solution $u \in H$ of (5.1).

For the upper bound, we first remark that the existence of radial nodal solutions of (1.13) is well known, see for instance Theorems 4 and 5 in [**23**]. Let $\hat{u} \in H^1(\mathbb{R}^2)$

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be a fixed radial nodal solution of (1.13) and set $C = E_*(\hat{u})$. For every $\lambda > 0$, the function $\hat{u} \in H$ is then also a nodal solution of (5.1), and therefore

$$E_\lambda(u) \leqslant E_\lambda(\hat{u}) = E_*(\hat{u}) = C$$

for every least energy nodal solution $u \in H$ of (5.1). □

The proof of Theorem 1.2 is now completed by deriving Part (i) of this theorem as follows:

Let $C > 0$ be given by lemma 5.3, and let $u \in H$ be a least energy solution of (5.1) for some $\lambda > 0$. Then we have $E_\lambda(u) \leqslant C$. Applying Theorem 5.2 with $c = C$ and considering $\lambda_0 := \min\{\lambda_c, \Lambda_0\}$ with $\Lambda_0 > 0$ given as in Theorem 1.2(ii), we then deduce that $0 < \lambda_0 \leqslant \Lambda_0$, and u is radial if $\lambda < \lambda_0$. The proof of Theorem 1.2(i) is thus finished.

Appendix A.

We give the proof of lemma 2.5, which we restate here for the reader's convenience.

Lemma A.1. *Let $\lambda > 0$ and let $u \in H$ be a weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad in\,\mathbb{R}^2. \tag{A.1}$$

Then $u \in L^\infty(\mathbb{R}^2)$. Furthermore, there exist constants $C, \sigma > 0$, depending on $p > 2$ but not on u and λ, such that

$$|u|_\infty \leqslant C\|u\|^\sigma_{H^1(\mathbb{R}^2)}. \tag{A.2}$$

Proof. The proof is based on Moser iteration, cf. Appendix B in [**24**] and the references therein. We fix $L, s \geqslant 2$ and consider auxiliary functions $h, g \in C^1([0,\infty))$ defined by

$$h(t) := s\int_0^t \min\{\tau^{s-1}, L^{s-1}\}\,\mathrm{d}\tau \qquad \text{and} \qquad g(t) := \int_0^t [h'(\tau)]^2\,\mathrm{d}\tau$$

We note that

$$h(t) = t^s \quad \text{for}\, t \leqslant L \qquad \text{and} \qquad g(t) \leqslant tg'(t) = t(h'(t))^2 \quad \text{for}\, t \geqslant 0, \tag{A.3}$$

since the function $t \mapsto h'(t) = s\min\{t^{s-1}, L^{s-1}\}$ is nondecreasing. We shall now show that $w := u^+ \in L^\infty(\mathbb{R}^2)$, and that $\|w\|_\infty$ is bounded by the r.h.s. of (A.2). Since we may replace u with $-u$, the claim will then follow. We note that $w \in H$

and $\varphi := g(w) \in H$ with

$$\nabla w = \mathbf{1}_{\{u>0\}} \nabla u, \quad \nabla \varphi = g'(w) \nabla w, \quad \partial_\theta w = \mathbf{1}_{\{u>0\}} \partial_\theta u, \quad \partial_\theta \varphi = g'(w) \partial_\theta w.$$

This follows from the boundedness of g' and the estimate $g(t) \leqslant s^2 t^{2s-1}$ for $t \geqslant 0$. Testing (A.1) with φ gives

$$\int_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla \varphi + \frac{1}{\lambda^2} (\partial_\theta u \; \partial_\theta \varphi) + u\varphi \Big) \mathrm{d}x = \int_{\mathbb{R}^2} |u|^{p-2} u \varphi \, \mathrm{d}x,$$

from where we estimate,

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big(|\nabla h(w)|^2 + \frac{1}{\lambda^2} (\partial_\theta h(w))^2 + w g(w) \Big) \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^2} \Big(g'(w) \Big(|\nabla w|^2 + \frac{1}{\lambda^2} (\partial_\theta w)^2 \Big) + u g(w) \Big) \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^2} |u|^{p-2} u g(w) \, \mathrm{d}x \\ &\quad \leqslant \int_{\mathbb{R}^2} w^p (h'(w))^2 \, \mathrm{d}x. \end{aligned} \tag{A.4}$$

Here we used (A.3) in the last step. We now fix $r > 1$ with $\frac{(p-2)r}{r-1} \geqslant 2$ and $q > 4r$. Combining (A.4) with Sobolev embeddings, we obtain the inequality

$$\frac{1}{c_0} |h(w)|_q^2 - |h(w)|_2^2 + \int_{\mathbb{R}^2} w g(w) \, \mathrm{d}x \leqslant \int_{\mathbb{R}^2} w^p (h'(w))^2 \, \mathrm{d}x \tag{A.5}$$

with a constant $c_0 = c_0(q) > 0$. Since

$$\begin{aligned} &h(t) = t^s, \quad h'(t) = s t^{s-1} \quad \text{and} \\ &g(t) = s^2 \int_0^t \tau^{2s-2} \, \mathrm{d}\tau = \frac{s^2}{2s-1} t^{2s-1} \qquad \text{for } t \leqslant L, \end{aligned}$$

we may let $L \to \infty$ in (A.5) and apply Lebesgue's theorem to obtain

$$\frac{1}{c_0} |w^s|_q^2 + \Big(\frac{s^2}{2s-1} - 1 \Big) |w^s|_2^2 \leqslant s^2 \int_{\mathbb{R}^2} w^{p+2s-2} \, \mathrm{d}x \leqslant s^2 |w|^{p-2}_{\frac{(p-2)r}{r-1}} |w|_{2rs}^{2s}$$

Since $s \geqslant 2$, we have $\frac{s^2}{2s-1} \geqslant 1$, and we thus obtain the inequality

$$|w|_{sq} \leqslant (c_1 s)^{\frac{1}{s}} |w|_{2rs} \qquad \text{with } c_1 := \Big(c_0 |w|^{p-2}_{\frac{r(p-2)}{r-1}} \Big)^{\frac{1}{2}}. \tag{A.6}$$

Next we note that the choice of r and q only depends on p but not on $s \geqslant 2$. We may, therefore, consider $s = s_n = \rho^n$ for $n \in \mathbb{N}$ with $\rho := \frac{q}{2r} > 2$, so that

$$2 s_1 r = q \qquad \text{and} \qquad 2 s_{n+1} r = q s_n \quad \text{for } n \in \mathbb{N}.$$

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Iteration of (A.6) then gives

$$|w|_{\rho^n q} = |w|_{s_n q} \leqslant |w|_q \prod_{j=1}^{n} (c_1 \rho^j)^{\rho^{-j}} \leqslant c_1^{\frac{\rho}{\rho-1}} c_2 |w|_q \quad \text{for all } n \quad \text{with}$$

$$c_2 := \rho^{\sum_{j=1}^{\infty} j\rho^{-j}} < \infty.$$

It follows that

$$|w|_\infty = \lim_{n\to\infty} |w|_{\rho^n q} \leqslant c_1^{\frac{\rho}{\rho-1}} c_2 |w|_q. \tag{A.7}$$

Moreover, by Sobolev embeddings, we have

$$c_1 \leqslant c_1' \|w\|_{H^1(\mathbb{R}^2)}^{\frac{p-2}{2}} \leqslant c_1' \|u\|_{H^1(\mathbb{R}^2)}^{\frac{p-2}{2}} \qquad \text{and} \qquad |w|_q \leqslant \tilde{c}\|w\|_{H^1} \leqslant \tilde{c}\|u\|_{H^1(\mathbb{R}^2)}$$

with constants $c_1', \tilde{c} > 0$ depending only on p, r and q. It thus follows from (A.7) that

$$|w|_\infty \leqslant C\|u\|_{H^1(\mathbb{R}^2)}^{\frac{(p-2)\rho}{2(\rho-1)}+1} \qquad \text{with} \qquad C := c_2 (c_1')^{\frac{\rho}{\rho-1}} \tilde{c}.$$

The proof is thus finished. □

REMARK A.2. Let $\lambda > 0$ and $p \in (2, \infty)$. By a variant of the Moser iteration argument given above, we can also show that every weak solution $u \in H^+$ of

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2_+, \qquad u = 0 \quad \text{on } \partial\mathbb{R}^2_+ \tag{A.8}$$

satisfies $u \in L^\infty(\mathbb{R}^2_+)$. To see this, we replace, with the help of Corollary 3.1 and (A.8), the inequalities (A.4) and (A.5) by

$$\frac{1}{c}|h(w)|_q^2 \leqslant \|h(w)\|_{\lambda,0}^2 = \int_{\mathbb{R}^2_+} |u|^{p-2}ug(w)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^2_+} w^p (h'(w))^2\,\mathrm{d}x$$

with a constant $c > 0$ depending on q and λ. We can then complete the argument as above, noting that in this case, the constants depend on $\lambda > 0$.

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