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Convergence of equilibria of thin elastic rods under physical growth conditions for the energy density

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We study the asymptotic behaviour of the equilibrium configurations of a nonlinearly elastic thin rod as the diameter of the cross-section tends to zero. Convergence results are established assuming physical growth conditions for the elastic energy density and suitable scalings of the applied loads that correspond at the limit to different rod models: the constrained linear theory, the analogue of the von Kármán plate theory for rods, and the linear theory.

1. Introduction and statement of the main result

A classical question in nonlinear elasticity is the derivation of lower-dimensional models for thin structures (such as plates, shells or beams) starting from the three-dimensional theory. In recent years this problem has been approached by means of Γ-convergence. This method guarantees, roughly speaking, the convergence of minimizers of the three-dimensional energy to minimizers of the deduced models. In this paper we discuss the convergence of three-dimensional stationary points, which are not necessarily minimizers, assuming physical growth conditions on the stored-energy density. In particular, we extend the recent results of [13] to the case of a three-dimensional thin beam with a cross-section of diameter h and subjected to an applied normal body force of order h^α, $\alpha > 2$. These scalings correspond at the limit to the constrained linear rod theory ($2 < \alpha < 3$), the analogue of von Kármán plate theory for rods ($\alpha = 3$), and the linear rod theory ($\alpha > 3$).

We first review the main results of the variational approach. Let $\Omega_h = (0, L) \times hS$ be the reference configuration of a thin elastic beam, where $L > 0$, $S \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and $h > 0$ is a small parameter. Without loss of generality we shall assume that the two-dimensional Lebesgue measure of S is equal to 1 and

$$\int_S x_2 \, dx_2 \, dx_3 = \int_S x_3 \, dx_2 \, dx_3 = \int_S x_2 x_3 \, dx_2 \, dx_3 = 0. \tag{1.1}$$

Let $f^h \in L^2(\Omega_h, \mathbb{R}^3)$ be an external body force applied to the beam. Given a deformation $v \in W^{1,2}(\Omega_h, \mathbb{R}^3)$ the total energy per unit cross-section associated to v is defined as

$$\mathcal{F}^h(v) = \frac{1}{h^2} \int_{\Omega_h} W(\nabla v) \, dx - \frac{1}{h^2} \int_{\Omega_h} f^h \cdot v \, dx,$$

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where the stored-energy density $W\colon \mathbb{M}^{3\times 3} \to [0,+\infty]$ is assumed to satisfy the following natural conditions:

(H1) W is of class C^1 on $\mathbb{M}^{3\times 3}_+$;

(H2) $W(F) = +\infty$ if $\det F \leqslant 0$; $W(F) \to +\infty$ if $\det F \to 0^+$;

(H3) $W(RF) = W(F)$ for every $R \in \mathrm{SO}(3)$, $F \in \mathbb{M}^{3\times 3}$ (frame indifference);

(H4) $W = 0$ on $\mathrm{SO}(3)$;

(H5) there exists $C > 0$ such that $W(F) \geqslant C\,\mathrm{dist}^2(F, \mathrm{SO}(3))$ for every $F \in \mathbb{M}^{3\times 3}$;

(H6) W is of class C^2 in a neighbourhood of $\mathrm{SO}(3)$.

Here

$$\mathbb{M}^{3\times 3}_+ = \{F \in \mathbb{M}^{3\times 3}\colon\ \det F > 0\}$$

and

$$\mathrm{SO}(3) = \{R \in \mathbb{M}^{3\times 3}\colon R^{\mathrm{T}}R = \mathrm{Id},\ \ \det R = 1\}.$$

In particular, condition (H2) is related to non-interpenetration of matter [6] and ensures local invertibility of C^1 deformations with finite energy.

The study of the asymptotic behaviour of global minimizers of $\mathcal{F}^h$ as $h \to 0$ can be performed through the analysis of the Γ-limit of $\mathcal{F}^h$ (see [7] for an introduction to Γ-convergence). To do this, it is convenient to rescale Ω_h to the domain $\Omega = (0, L)\times S$ and to rescale deformations according to this change of variables by setting

$$y(x) := v(x_1, hx_2, hx_3)$$

for every $x \in \Omega$. Assuming for simplicity that $f^h(x) = f^h(x_1)$, the energy functional can be written as

$$\mathcal{F}^h(v) = \mathcal{J}^h(y) = \int_\Omega W(\nabla_h y)\,\mathrm{d}x - \int_\Omega f^h \cdot y\,\mathrm{d}x,$$

where we have used the notation

$$\nabla_h y := \left(\partial_1 y \middle| \frac{\partial_2 y}{h} \middle| \frac{\partial_3 y}{h}\right).$$

Now let y^h be a global minimizer of $\mathcal{J}^h$ subject to the boundary condition

$$y^h(0, x_2, x_3) = (0, hx_2, hx_3) \quad \text{for every } (x_2, x_3) \in S. \tag{1.2}$$

The asymptotic behaviour of y^h as $h \to 0$ depends on the scaling of the applied load f^h in terms of h. More precisely, if f^h is of order h^α with $\alpha \geqslant 0$, then $\mathcal{J}^h(y^h) = O(h^\beta)$, where $\beta = \alpha$ for $0 \leqslant \alpha \leqslant 2$ and $\beta = 2\alpha - 2$ for $\alpha > 2$, and y^h converges in a suitable sense to a minimizer of the Γ-limit of the rescaled functionals $h^{-\beta}\mathcal{J}^h$ as $h \to 0$ [3, 9–11, 16, 17]. In particular, it has been proved in [11, 17] that if f^h is a normal force of the form $h^\alpha(f_2 e_2 + f_3 e_3)$, with $\alpha > 2$ and $f_2, f_3 \in L^2(0, L)$, then

$$y^h \to x_1 e_1 \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3).$$

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In other words, minimizers converge to the identity deformation on the mid-fibre of the rod. This suggests the introduction of the (averaged) tangential and normal displacements, respectively given by

$$u^h(x_1) := \begin{cases} \dfrac{1}{h^{\alpha-1}} \displaystyle\int_S (y_1^h - x_1)\, \mathrm{d}x_2\, \mathrm{d}x_3 & \text{if } \alpha \geqslant 3, \\ \dfrac{1}{h^{2(\alpha-2)}} \displaystyle\int_S (y_1^h - x_1)\, \mathrm{d}x_2\, \mathrm{d}x_3 & \text{if } 2 < \alpha < 3, \end{cases} \tag{1.3}$$

$$v_k^h(x_1) := \frac{1}{h^{\alpha-2}} \int_S y_k^h \, \mathrm{d}x_2\, \mathrm{d}x_3 \quad \text{for } k = 2, 3 \tag{1.4}$$

for a.e. $x_1 \in (0, L)$, and the (averaged) twist function, given by

$$w^h(x_1) := \frac{1}{\mu(S)} \frac{1}{h^{\alpha-1}} \int_S (x_2 y_3^h - x_3 y_2^h)\, \mathrm{d}x_2\, \mathrm{d}x_3 \tag{1.5}$$

for a.e. $x_1 \in (0, L)$, where

$$\mu(S) := \int_S (x_2^2 + x_3^2)\, \mathrm{d}x_2\, \mathrm{d}x_3.$$

As $h \to 0$, one has

$$\begin{aligned} u^h &\to u \quad \text{in } W^{1,2}(0, L), \\ v_k^h &\to v_k \quad \text{in } W^{1,2}(0, L) \text{ for } k = 2, 3, \\ w^h &\rightharpoonup w \quad \text{in } W^{1,2}(0, L), \end{aligned}$$

where (u, v_2, v_3, w) is a global minimizer of the functional $\mathcal{J}_\alpha$ given by the Γ-limit of $h^{-2\alpha+2}\mathcal{J}^h$. If $\alpha = 3$, the Γ-limit $\mathcal{J}_3$ corresponds to the one-dimensional analogue of the von Kármán plate functional. For $\alpha > 3$ the functional $\mathcal{J}_\alpha$ coincides with the linear rod functional, while for $2 < \alpha < 3$ the limiting energy is still linear but is subject to a nonlinear isometric constraint (see § 2 for the exact definition of the functionals $\mathcal{J}_\alpha$).

In this paper we focus on the study of the asymptotic behaviour of (possibly non-minimizing) stationary points of $\mathcal{J}^h$ as $h \to 0$. The first convergence results for stationary points have been proved in [12, 14, 15]. We also point out the recent results [1,2] concerning the dynamical case. A crucial assumption in all these papers is that the stored-energy function W is everywhere differentiable and its derivative satisfies a linear growth condition. Unfortunately, this requirement is incompatible with the physical assumption (H2). At the same time, if (H2) is satisfied, the conventional form of the Euler–Lagrange equations of $\mathcal{J}^h$ is not well defined and the extent to which minimizers of $\mathcal{J}^h$ satisfy this condition is not even clear (we refer the reader to [5, 13] for a more detailed discussion).

Following [13], we consider an alternative first-order stationarity condition, introduced by Ball in [5]. Towards this aim, we require the following additional assumption:

(H7) there exists $k > 0$ such that $|\mathrm{D}W(F)F^{\mathrm{T}}| \leqslant k(W(F) + 1)$ for every $F \in \mathbb{M}_+^{3\times 3}$.

This growth condition is compatible with (H1)–(H6) [5].

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Definition 1.1. We say that a deformation $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ is a stationary point of $\mathcal{J}^h$ if it satisfies the boundary condition $y(0, x_2, x_3) = (0, hx_2, hx_3)$ for every $(x_2, x_3) \in S$ and the equation

$$\int_\Omega DW(\nabla_h y)(\nabla_h y)^{\mathrm{T}} : [(\nabla\phi) \circ y]\, \mathrm{d}x = \int_\Omega f^h \cdot (\phi \circ y)\, \mathrm{d}x \tag{1.6}$$

for every $\phi \in C^1_b(\mathbb{R}^3, \mathbb{R}^3)$ such that $\phi(0, hx_2, hx_3) = 0$ for all $(x_2, x_3) \in S$.

In the previous definition and in the following $C^1_b(\mathbb{R}^3, \mathbb{R}^3)$ denotes the space of C^1 functions that are bounded in $\mathbb{R}^3$, with bounded first-order derivatives.

Assuming (H1)–(H7), one can show that every local minimizer y of $\mathcal{J}^h$, subject to the boundary condition $y(0, x_2, x_3) = (0, hx_2, hx_3)$ for every $(x_2, x_3) \in S$, is a stationary point of $\mathcal{J}^h$ in the sense of definition 1.1 [5, theorem 2.4]. Indeed, condition (1.6) corresponds to the requirement that the derivative of $\mathcal{J}^h$ along external variations of the form $y + \epsilon\phi \circ y$ is zero at $\epsilon = 0$. Moreover, when minimizers are invertible, (1.6) coincides with the equilibrium equation for the Cauchy stress tensor.

In [13] it has been proved that stationary points in the sense of definition 1.1 converge to stationary points of the Γ-limit $\mathcal{J}_\alpha$ in the case of a thin plate and for the scaling $\alpha \geqslant 3$ (corresponding to von Kármán and to linear plate theory). In this paper we extend this result to the range of scalings $\alpha > 2$ in the case of a thin beam. Our main result is the following.

Theorem 1.2. *Assume that W satisfies (H1)–(H7). Let $f_2, f_3 \in L^2(0, L)$. Let $\alpha > 2$ and let $\mathcal{J}_\alpha$ be the functional defined in (2.3), (2.5) and (2.6). For every $h > 0$ let y^h be a stationary point of $\mathcal{J}^h$ (according to definition 1.1) with $f^h := h^\alpha(f_2 e_2 + f_3 e_3)$. Assume there exists $C > 0$ such that*

$$\int_\Omega W(\nabla_h y^h)\, \mathrm{d}x \leqslant Ch^{2\alpha-2} \tag{1.7}$$

for every $h > 0$. Then,

$$y^h \to x_1 e_1 \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3). \tag{1.8}$$

Moreover, let u^h, v^h and w^h be the scaled displacements and twist function introduced in (1.3)–(1.5). Then, up to subsequences, we have

$$\begin{aligned} u^h &\rightharpoonup u \quad \text{in } W^{1,2}(0, L),\\ v^h_k &\to v_k \quad \text{in } W^{1,2}(0, L) \text{ for } k = 2, 3,\\ w^h &\rightharpoonup w \quad \text{in } W^{1,2}(0, L), \end{aligned}$$

where $(u, v_2, v_3, w) \in W^{1,2}(0, L) \times W^{2,2}(0, L) \times W^{2,2}(0, L) \times W^{1,2}(0, L)$ is a stationary point of $\mathcal{J}_\alpha$.

The proof of theorem 1.2 is closely related to [13] and uses as a key tool the rigidity estimate proved in [8]. The main new idea with respect to [13] is the construction of a sequence of suitable 'approximate inverse functions' of the deformations y^h (see lemma 2.7), which allows us to extend the results of [13] to the range of scalings $\alpha \in (2,3)$. This construction is based on a careful study of the asymptotic development of the deformations y^h in terms of approximate displacements and uses in a crucial way the fact that the limit space dimension is equal to 1.

2. Preliminary results

In this section we recall the expression of the Γ-limits $\mathcal{J}_\alpha$ identified in [11, 17] and we prove some preliminary results.

We start by introducing some notation. Let

$$Q_3 \colon \mathbb{M}^{3\times 3} \to [0,+\infty)$$

be the quadratic form of linearized elasticity:

$$Q_3(F) := \mathrm{D}^2W(\mathrm{Id})F \colon F \quad \text{for every } F \in \mathbb{M}^{3\times 3}.$$

We shall denote by $\mathcal{L}$ the associated linear map on $\mathbb{M}^{3\times 3}$ given by $\mathcal{L} := \mathrm{D}^2W(\mathrm{Id})$. Let

$$\mathbb{E} := \min_{a,b\in\mathbb{R}^3} Q_3(e_1|a|b), \tag{2.1}$$

and let Q_1 be the quadratic form defined on the space $\mathbb{M}^{3\times 3}_{\mathrm{skew}}$ of skew-symmetric matrices given by

$$Q_1(F) := \min_{\beta\in W^{1,2}(S,\mathbb{R}^3)} \int_S Q_3(x_2Fe_2 + x_3Fe_3|\partial_2\beta|\partial_3\beta)\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{2.2}$$

for every $F \in \mathbb{M}^{3\times 3}_{\mathrm{skew}}$. It is easy to deduce from the assumptions (H1)–(H6) that $\mathbb{E}$ is a positive constant and Q_1 is a positive definite quadratic form.

The functionals $\mathcal{J}_\alpha$ are defined on the space

$$H := W^{1,2}(0,L) \times W^{2,2}(0,L) \times W^{2,2}(0,L) \times W^{1,2}(0,L)$$

and are finite on the class $\mathcal{A}_\alpha$, which can be described as follows:

$$\begin{aligned}\mathcal{A}_\alpha := \{(u,v_2,v_3,w) \in H : u' + \tfrac{1}{2}[(v_2')^2 + (v_3')^2] = 0 \text{ in } (0,L)&\\ \text{and } u(0) = v_k(0) = v_k'(0) = w(0) = 0 \text{ for } k=2,3\}&\end{aligned}$$

for $2 < \alpha < 3$, and

$$\mathcal{A}_\alpha := \{(u,v_2,v_3,w) \in H \colon u(0) = v_k(0) = v_k'(0) = w(0) = 0 \text{ for } k=2,3\}$$

for $\alpha \geqslant 3$.

For $2 < \alpha < 3$, the functional $\mathcal{J}_\alpha$ is given by

$$\mathcal{J}_\alpha(u,v_2,v_3,w) = \frac{1}{2}\int_0^L Q_1(A')\,\mathrm{d}x_1 - \int_0^L (f_2v_2 + f_3v_3)\,\mathrm{d}x_1 \tag{2.3}$$

for every $(u,v_2,v_3,w) \in \mathcal{A}_\alpha$; $\mathcal{J}_\alpha(u,v_2,v_3,w) = +\infty$ elsewhere in H. In (2.3) the function $A \in W^{1,2}((0,L),\mathbb{M}^{3\times 3})$ is defined by

$$A(x_1) := \begin{pmatrix} 0 & -v_2'(x_1) & -v_3'(x_1)\\ v_2'(x_1) & 0 & -w(x_1)\\ v_3'(x_1) & w(x_1) & 0\end{pmatrix} \tag{2.4}$$

for a.e. $x_1 \in (0,L)$.

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For $\alpha = 3$ the Γ-limit is given by

$$\mathcal{J}_3(u, v_2, v_3, w) = \frac{1}{2}\int_0^L \mathbb{E}(u' + \tfrac{1}{2}[(v_2')^2 + (v_3')^2])^2 \, \mathrm{d}x_1 + \frac{1}{2}\int_0^L Q_1(A') \, \mathrm{d}x_1 - \int_0^L (f_2 v_2 + f_3 v_3) \, \mathrm{d}x_1 \qquad (2.5)$$

for every $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$; $\mathcal{J}_3(u, v_2, v_3, w) = +\infty$ elsewhere in H.

Finally, for $\alpha > 3$ the Γ-limit is given by

$$\mathcal{J}_\alpha(u, v_2, v_3, w) = \frac{1}{2}\int_0^L \mathbb{E}(u')^2 \, \mathrm{d}x_1 + \frac{1}{2}\int_0^L Q_1(A') \, \mathrm{d}x_1 - \int_0^L (f_2 v_2 + f_3 v_3) \, \mathrm{d}x_1 \qquad (2.6)$$

for every $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$; $\mathcal{J}_\alpha(u, v_2, v_3, w) = +\infty$ elsewhere in H.

We can now compute the Euler–Lagrange equations for the functionals $\mathcal{J}_\alpha$ introduced above. We first recall the following lemma.

Lemma 2.1. *Let $F \in \mathbb{M}^{3\times 3}_{\text{skew}}$ and let $\mathcal{G}_F \colon W^{1,2}(S, \mathbb{R}^3) \to [0, +\infty)$ be the functional*

$$\mathcal{G}_F(\beta) := \int_S Q_3(x_2 F e_2 + x_3 F e_3 | \partial_2 \beta | \partial_3 \beta) \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

for every $\beta \in W^{1,2}(S, \mathbb{R}^3)$. Then $\mathcal{G}_F$ is convex and has a unique minimizer in the class

$$\mathcal{B} := \left\{ \beta \in W^{1,2}(S, \mathbb{R}^3) \colon \int_S \beta \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \int_S \partial_2 \beta \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \int_S \partial_3 \beta \, \mathrm{d}x_2 \, \mathrm{d}x_3 = 0 \right\}.$$

Furthermore, a function $\beta \in \mathcal{B}$ is the minimizer of $\mathcal{G}_F$ if and only if the map $E : S \to \mathbb{M}^{3\times 3}$ defined by

$$E := \mathcal{L}(x_2 F e_2 + x_3 F e_3 | \partial_2 \beta | \partial_3 \beta) \qquad (2.7)$$

satisfies in a weak sense the following problem:

$$\begin{aligned} \operatorname{div}_{x_2, x_3}(Ee_2 | Ee_3) &= 0 \quad \text{in } S, \\ (Ee_2 | Ee_3)\nu_{\partial S} &= 0 \quad \text{on } \partial S, \end{aligned}$$

where $\nu_{\partial S}$ is the unit normal to ∂S. Finally, the minimizer depends linearly on F.

Proof. See [12, lemma 2.1] and [10, remark 3.4]. □

We shall use the following notation: for each $F \in L^1(\Omega, \mathbb{M}^{3\times 3})$ we define the zeroth-order moment of F as the function $\bar{F} : (0, L) \to \mathbb{M}^{3\times 3}$ given by

$$\bar{F}(x_1) := \int_S F(x) \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

for a.e. $x_1 \in (0, L)$. We also introduce the first-order moments of F as the functions $\tilde{F}, \hat{F} : (0, L) \to \mathbb{M}^{3\times 3}$ given by

$$\tilde{F}(x_1) := \int_S x_2 F(x) \, \mathrm{d}x_2 \, \mathrm{d}x_3, \qquad \hat{F}(x_1) = \int_S x_3 F(x) \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

for a.e. $x_1 \in (0, L)$.

The following proposition follows now from straightforward computations.

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Proposition 2.2. *Let $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$. For a.e. $x_1 \in (0, L)$ let $\beta(x_1, \cdot, \cdot) \in \mathcal{B}$ be the minimizer of $\mathcal{G}_{A'(x_1)}$, where A' is the derivative of the function A introduced in (2.4). Let also $E\colon \Omega \to \mathbb{M}^{3\times 3}$ be defined by*

$$E := \mathcal{L}(x_2 A' e_2 + x_3 A' e_3 | \partial_2 \beta | \partial_3 \beta),$$

and let $\tilde{E}$ and $\hat{E}$ be its first-order moments. Then we have the following.

1. *(u, v_2, v_3, w) is a stationary point of $\mathcal{J}_3$ if and only if the following equations are satisfied:*

$$u' + \tfrac{1}{2}[(v_2')^2 + (v_3')^2] = 0 \quad \text{in } (0, L), \tag{2.8}$$

$$\left.\begin{aligned} \tilde{E}_{11}'' + f_2 = 0 \quad \text{in } (0, L),\\ \tilde{E}_{11}(L) = \tilde{E}_{11}'(L) = 0, \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} \hat{E}_{11}'' + f_3 = 0 \quad \text{in } (0, L),\\ \hat{E}_{11}(L) = \hat{E}_{11}'(L) = 0, \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} \tilde{E}_{12}' = \hat{E}_{13}' \quad \text{in } (0, L),\\ \tilde{E}_{12}(L) = \hat{E}_{13}(L). \end{aligned}\right\} \tag{2.11}$$

2. *If $\alpha > 3$, then (u, v_2, v_3, w) is a stationary point of $\mathcal{J}_\alpha$ if and only if*

$$u' = 0 \quad \text{in } (0, L) \tag{2.12}$$

and (2.9)–(2.11) are satisfied.

3. *If $2 < \alpha < 3$, then (u, v_2, v_3, w) is a stationary point of $\mathcal{J}_\alpha$ if and only if (2.9)–(2.11) are satisfied.*

Remark 2.3. If $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$ and $2 < \alpha < 3$, then u is uniquely determined in terms of v_2 and v_3. Indeed, by the constraint

$$u' + \frac{(v_2')^2 + (v_3')^2}{2} = 0 \quad \text{a.e. in } (0, L)$$

and the boundary condition $u(0) = 0$, we have

$$u(x_1) = -\int_0^{x_1} \frac{(v_2'(t))^2 + (v_3'(t))^2}{2}\,\mathrm{d}t \quad \text{for a.e. } x_1 \text{ in } (0, L). \tag{2.13}$$

For $\alpha \geqslant 3$, the same conclusion holds when $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$ is a stationary point of $\mathcal{J}_\alpha$. Indeed, if $\alpha = 3$, (2.8) yields (2.13), while if $\alpha > 3$, (2.12) gives

$$u = 0 \quad \text{a.e. in } (0, L).$$

Using the previous observations and the strict convexity of Q_1, it is easy to show that, for every $\alpha > 2$, $\mathcal{J}_\alpha$ has a unique stationary point which is a minimizer.

Remark 2.4. For what concerns the three-dimensional functionals $\mathcal{J}^h$, under additional hypotheses on W (such as polyconvexity [4]) it is possible to show the existence of global minimizers, and therefore of stationary points. Furthermore, they automatically satisfy the energy estimate (1.7) [9, proof of theorem 2]. For general W, the existence of stationary points (according to definition 1.6 or to the classical formulation) is a subtle issue. We refer the reader to [5, § 2.7] for a discussion of results in this regard.

From now on we shall work with sequences of deformations $y^h \in W^{1,2}(\Omega, \mathbb{R}^3)$, satisfying the boundary condition (1.2) and the uniform energy estimate (1.7) with $\alpha > 2$. This bound, combined with the coercivity condition (H5), provides us with a control on the distance of $\nabla_h y^h$ from SO(3). This fact, together with the geometric rigidity estimate by Friesecke *et al*. [8, theorem 3.1], allows us to construct an approximating sequence of rotations (R^h), whose L^2-distance from $\nabla_h y^h$ is of the same order in terms of h of the L^2-norm of $\operatorname{dist}(\nabla_h y^h, \mathrm{SO}(3))$. More precisely, the following result holds true.

Theorem 2.5. *Assume that $W\colon \mathbb{M}^{3\times 3} \to [0, +\infty]$ is continuous and satisfies (H3)–(H6). Let $\alpha > 2$ and let (y^h) be a sequence in $W^{1,2}(\Omega, \mathbb{R}^3)$ satisfying (1.2) and (1.7) for every $h > 0$. Then there exists a sequence (R^h) in $C^\infty((0, L), \mathbb{M}^{3\times 3})$ such that*

$$R^h(x_1) \in \mathrm{SO}(3) \quad \text{for every } x_1 \in (0, L), \tag{2.14}$$

$$\|\nabla_h y^h - R^h\|_{L^2} \leqslant Ch^{\alpha-1}, \tag{2.15}$$

$$\|(R^h)'\|_{L^2} \leqslant Ch^{\alpha-2}, \tag{2.16}$$

$$\|R^h - \mathrm{Id}\,\|_{L^\infty} \leqslant Ch^{\alpha-2}. \tag{2.17}$$

We omit the proof, as it follows closely the proof of [14, proposition 4.1]. Owing to the previous approximation result, one can deduce the following compactness properties.

Theorem 2.6. *Under the assumptions of theorem 2.5, let u^h, v_2^h, v_3^h, w^h be the scaled displacements and twist function introduced in (1.3)–(1.5). Then*

$$y^h \to x_1 e_1 \quad \text{strongly in } W^{1,2}(\Omega, \mathbb{R}^3) \tag{2.18}$$

and there exists $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$ such that, up to subsequences, we have

$$u^h \to u \quad \text{strongly in } W^{1,2}(0, L) \quad \text{if } 2 < \alpha < 3, \tag{2.19}$$

$$u^h \rightharpoonup u \quad \text{weakly in } W^{1,2}(0, L) \quad \text{if } \alpha \geqslant 3, \tag{2.20}$$

$$v_k^h \to v_k \quad \text{strongly in } W^{1,2}(0, L) \quad \text{for } k = 2, 3, \tag{2.21}$$

$$w^h \rightharpoonup w \quad \text{weakly in } W^{1,2}(0, L). \tag{2.22}$$

Moreover, let $A \in W^{1,2}((0, L), \mathbb{M}^{3\times 3})$ be the function defined in (2.4). Then, if R^h is the approximating sequence of rotations given by theorem 2.5, the following convergence properties hold true:

$$\frac{\nabla_h y^h - \mathrm{Id}}{h^{\alpha-2}} \to A \quad \text{strongly in } L^2(\Omega, \mathbb{M}^{3\times 3}), \tag{2.23}$$

$$A^h := \frac{R^h - \mathrm{Id}}{h^{\alpha-2}} \rightharpoonup A \quad \text{weakly in } W^{1,2}((0, L), \mathbb{M}^{3\times 3}), \tag{2.24}$$

$$\frac{\mathrm{sym}(R^h - \mathrm{Id})}{h^{2(\alpha-2)}} \to \frac{A^2}{2} \quad \text{uniformly in } (0, L). \tag{2.25}$$

For the proof we refer the reader to [17, theorem 3.3].

We conclude this section by proving a lemma which will be crucial for extending the convergence of the equilibria result to the scalings $\alpha \in (2,3)$.

Lemma 2.7. *Under the assumptions of theorem 2.5, there exist two sequences* (ξ_k^h), $k = 2, 3$, *such that, for every* $h > 0$,

$$\xi_k^h \in C_b^1(\mathbb{R}), \quad \xi_k^h(0) = 0, \tag{2.26}$$

$$\frac{y_k^h}{h} - \frac{1}{h}\xi_k^h \circ y_1^h \to x_k \quad \textit{strongly in } L^2(\Omega), \tag{2.27}$$

$$\|\xi_k^h\|_{L^\infty} + \|(\xi_k^h)'\|_{L^\infty} \leqslant Ch^{\alpha-2}. \tag{2.28}$$

Remark 2.8. The sequences (ξ_k^h) of the previous lemma can be interpreted as follows: the functions defined by

$$\omega^h(x) = \left(x_1, \frac{x_2}{h} - \frac{\xi_2^h(x_1)}{h}, \frac{x_3}{h} - \frac{\xi_3^h(x_1)}{h}\right)$$

represent a sort of 'approximate inverse functions' of the deformations y^h, in the sense that the compositions $\omega^h \circ y^h$ converge to the identity strongly in $L^2(\Omega, \mathbb{R}^3)$ by (2.18) and (2.27).

Proof of lemma 2.7. In order to construct the functions ξ_k^h, we first study the asymptotic behaviour of the sequences (y_k^h/h), $k = 2, 3$. By the Poincaré inequality we obtain the estimate

$$\left\|\frac{y_k^h}{h} - x_k - \int_S \left(\frac{y_k^h}{h} - x_k\right) \mathrm{d}x_2\,\mathrm{d}x_3\right\|_{L^2} \leqslant C\left(\left\|\frac{\partial_k y_k^h}{h} - 1\right\|_{L^2} + \left\|\frac{\partial_j y_k^h}{h}\right\|_{L^2}\right),$$

where $k, j \in \{2, 3\}$, $k \neq j$. Therefore, by (1.4) and (2.23) we have

$$\left\|\frac{y_k^h}{h} - x_k - h^{\alpha-3} v_k^h\right\|_{L^2} \leqslant Ch^{\alpha-2}. \tag{2.29}$$

In particular, for $\alpha > 3$ it follows that $y_k^h \to x_k$ strongly in L^2, so that if $\alpha > 3$, we can simply take $\xi_k^h = 0$ for $k = 2, 3$ and every $h > 0$. If $2 < \alpha \leqslant 3$, we need to construct a suitable approximation of v_k^h. Let (R^h) be the approximating sequence of rotations associated with (y^h) (see theorems 2.5 and 2.6). By (2.17) and (2.25), we deduce the following estimates:

$$\|R_{k1}^h\|_{L^\infty} \leqslant Ch^{\alpha-2} \quad \text{for } k = 2, 3,$$
$$\|R_{11}^h - 1\|_{L^\infty} \leqslant Ch^{2(\alpha-2)}.$$

Let $r_k^h, r_1^h \in C(\mathbb{R})$ be continuous extensions of the functions R_{k1}^h and $R_{11}^h - 1$ to $\mathbb{R}$ such that, for every $h > 0$,

$$\operatorname{supp} r_k^h, \operatorname{supp} r_1^h \subset (-1, L+1), \tag{2.30}$$

$$r_k^h = R_{k1}^h \text{ in } (0, L) \quad \text{for } k = 2, 3, \tag{2.31}$$

$$r_1^h = R_{11}^h - 1 \qquad \text{in } (0, L), \tag{2.32}$$

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$$\|r_k^h\|_{L^\infty} \leqslant Ch^{\alpha-2} \qquad \text{for } k = 2, 3, \tag{2.33}$$

$$\|r_1^h\|_{L^\infty} \leqslant Ch^{2(\alpha-2)}. \tag{2.34}$$

We introduce the functions $\tilde{v}_1^h, \tilde{v}_k^h \in C_b^1(\mathbb{R})$ defined by

$$\tilde{v}_k^h(x_1) := \int_0^{x_1} r_k^h(s)\,\mathrm{d}s, \tag{2.35}$$

$$\tilde{v}_1^h(x_1) := \int_0^{x_1} r_1^h(s)\,\mathrm{d}s. \tag{2.36}$$

Using the boundary condition (1.2), the Poincaré inequality, (2.15), and (2.23), we obtain

$$\left\| \frac{y_k^h}{h} - x_k - \frac{1}{h}\tilde{v}_k^h \right\|_{L^2} \leqslant Ch^{\alpha-2} \tag{2.37}$$

and, analogously,

$$\|y_1^h - x_1 - \tilde{v}_1^h\|_{L^2} \leqslant Ch^{\alpha-1}. \tag{2.38}$$

The latter inequality, together with (2.34), implies that

$$\|y_1^h - x_1\|_{L^2} \leqslant Ch^{2(\alpha-2)} \quad \text{for } \alpha \leqslant 3. \tag{2.39}$$

We are now in a position to construct the maps ξ_k^h when $\alpha \leqslant 3$. If $\alpha = 3$, we define $\xi_k^h = \tilde{v}_k^h$. Properties (2.26) and (2.28) follow immediately. To verify (2.27) it is enough to remark that by (2.33) and (2.37) we have

$$\begin{aligned} \left\| \frac{y_k^h}{h} - x_k - \frac{\tilde{v}_k^h \circ y_1^h}{h} \right\|_{L^2} &\leqslant Ch + \frac{1}{h}\|\tilde{v}_k^h \circ y_1^h - \tilde{v}_k^h\|_{L^2} \\ &\leqslant Ch + \frac{1}{h}\|(\tilde{v}_k^h)'\|_{L^\infty}\|y_1^h - x_1\|_{L^2} \\ &\leqslant Ch. \end{aligned}$$

If $2 < \alpha < 3$, we first fix $n_0 \in \mathbb{N}$ such that

$$\alpha > 2 + \frac{1}{2n_0 + 3} \tag{2.40}$$

and we introduce a sequence of maps (ζ_n^h), $n = 1, \dots, n_0$, recursively defined as

$$\left.\begin{aligned} \zeta_{n_0}^h(x_1) &= x_1 - \tilde{v}_1^h(x_1), \\ \zeta_n^h(x_1) &= x_1 - \tilde{v}_1^h \circ \zeta_{n+1}(x_1) \quad \text{for } n = 1, \dots, n_0 - 1. \end{aligned}\right\} \tag{2.41}$$

For $k = 2, 3$ and every $h > 0$ we define

$$\xi_k^h := \tilde{v}_k^h \circ \zeta_1^h. \tag{2.42}$$

Since $\zeta_{n_0}^h(0) = 0$, we have by induction that $\zeta_n^h(0) = 0$ for each $n = 1, 2, \dots, n_0$, so that $\xi_k^h(0) = 0$. From the regularity of $\tilde{v}_1^h$ and $\tilde{v}_k^h$ it follows that (2.26) is satisfied. By (2.33) we deduce

$$\|\xi_k^h\|_{L^\infty} \leqslant \|\tilde{v}_k^h\|_{L^\infty} \leqslant (L+2)\|r_k^h\|_{L^\infty} \leqslant Ch^{\alpha-2}. \tag{2.43}$$

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To estimate $\|(\xi_k^h)'\|_{L^\infty}$, we first deduce a recursive bound for $\|(\zeta_n^h)'\|_{L^\infty}$. If h is small enough, we have

$$\|(\tilde{v}_1^h)'\|_{L^\infty} \leqslant 1.$$

By (2.41) the following inequalities hold true:

$$\|(\zeta_{n_0}^h)'\|_{L^\infty} \leqslant 1 + \|(\tilde{v}_1^h)'\|_{L^\infty} \leqslant 2, \tag{2.44}$$

$$\|(\zeta_n^h)'\|_{L^\infty} \leqslant 1 + \|(\zeta_{n+1}^h)'\|_{L^\infty} \quad \text{for } n = 1, \dots, n_0 - 1, \tag{2.45}$$

$$\|(\zeta_1^h)'\|_{L^\infty} \leqslant 1 + n_0. \tag{2.46}$$

Now by (2.46) and (2.33) we have

$$\|(\xi_k^h)'\|_{L^\infty} \leqslant \|(\tilde{v}_k^h)'\|_{L^\infty} \|(\zeta_1^h)'\|_{L^\infty} \leqslant (1 + n_0)\|r_k^h\|_{L^\infty} \leqslant Ch^{\alpha-2}. \tag{2.47}$$

Combining (2.43) and (2.47), we obtain (2.28). To conclude the proof it remains to verify (2.27). By (2.34), (2.38), and (2.39) we have

$$\begin{aligned}
\|\zeta_{n_0}^h \circ y_1^h - x_1\|_{L^2} &= \|y_1^h - \tilde{v}_1^h \circ y_1^h - x_1\|_{L^2} \\
&\leqslant \|y_1^h - \tilde{v}_1^h - x_1\|_{L^2} + \|\tilde{v}_1^h - \tilde{v}_1^h \circ y_1^h\|_{L^2} \\
&\leqslant Ch^{\alpha-1} + \|(\tilde{v}_1^h)'\|_{L^\infty} \|y_1^h - x_1\|_{L^2} \\
&\leqslant Ch^{\alpha-1} + h^{2(\alpha-2)} \|r_1^h\|_{L^\infty} \\
&\leqslant Ch^{\alpha-1} + Ch^{4(\alpha-2)}.
\end{aligned} \tag{2.48}$$

Arguing analogously for $\zeta_{n_0-1}^h$ and using (2.48), we obtain

$$\begin{aligned}
\|\zeta_{n_0-1}^h \circ y_1^h - x_1\|_{L^2} &\leqslant \|y_1^h - x_1 - \tilde{v}_1^h\|_{L^2} + \|\tilde{v}_1^h - \tilde{v}_1^h \circ \zeta_{n_0}^h \circ y_1^h\|_{L^2} \\
&\leqslant Ch^{\alpha-1} + \|(\tilde{v}_1^h)'\|_{L^\infty} \|\zeta_{n_0}^h \circ y_1^h - x_1\|_{L^2} \\
&\leqslant Ch^{\alpha-1} + Ch^{2(\alpha-2)}(h^{\alpha-1} + h^{4(\alpha-2)}) \\
&\leqslant Ch^{\alpha-1} + Ch^{6(\alpha-2)}.
\end{aligned} \tag{2.49}$$

By induction, we deduce

$$\|\zeta_n^h \circ y_1^h - x_1\|_{L^2} \leqslant Ch^{\alpha-1} + Ch^{2(n_0-n+2)(\alpha-2)}. \tag{2.50}$$

In particular, we have

$$\|\zeta_1^h \circ y_1^h - x_1\|_{L^2} \leqslant Ch^{\alpha-1} + Ch^{2(n_0+1)(\alpha-2)}. \tag{2.51}$$

We can now prove (2.27). By (2.42), (2.33) and (2.51) we obtain

$$\begin{aligned}
\frac{1}{h}\|\xi_k^h \circ y_1^h - \tilde{v}_k^h\|_{L^2} &= \frac{1}{h}\|\tilde{v}_k^h \circ \zeta_1^h \circ y_1^h - \tilde{v}_k^h\|_{L^2} \\
&\leqslant \frac{1}{h}\|(\tilde{v}_k^h)'\|_{L^\infty} \|\zeta_1^h \circ y_1^h - x_1\|_{L^2} \\
&\leqslant \frac{1}{h}\|r_k^h\|_{L^\infty}(Ch^{\alpha-1} + Ch^{2(n_0+1)(\alpha-2)}) \\
&\leqslant Ch^{\alpha-3}(Ch^{\alpha-1} + Ch^{2(n_0+1)(\alpha-2)}) \\
&\leqslant Ch^{\min\{2\alpha-4, (2n_0+3)\alpha-(4n_0+7)\}},
\end{aligned}$$

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where the last term converges to zero due to (2.40). Combining this with (2.37), we deduce (2.27). □

3. Proof of the main result

This section is devoted entirely to the proof of theorem 1.2. The proof strategy is similar to [13]. The major difference is in the analysis of the asymptotic behaviour of the first-order stress moments (steps 6 and 7), where the approximating sequences constructed in lemma 2.7 are needed to define suitable test functions in the scalings $2 < \alpha < 3$.

Proof of theorem 1.2. Let (y^h) be a sequence of deformations in $W^{1,2}(\Omega, \mathbb{R}^3)$ satisfying the energy bound (1.7), the boundary condition (1.2), and the Euler–Lagrange equations

$$\int_\Omega \mathrm{D}W(\nabla_h y^h)(\nabla_h y^h)^{\mathrm{T}} : [(\nabla\phi)\circ y^h]\,\mathrm{d}x = \int_\Omega h^\alpha [f_2(\phi_2 \circ y^h) + f_3(\phi_3 \circ y^h)]\,\mathrm{d}x \tag{3.1}$$

for every $\phi \in C^1_b(\mathbb{R}^3, \mathbb{R}^3)$ such that $\phi(0, hx_2, hx_3) = 0$ for all $(x_2, x_3) \in S$.

Convergence of the sequences (y^h), (u^h), (v^h_k) and (w^h) follows from theorem 2.6, together with the fact that $(u, v_2, v_3, w) \in \mathcal{A}_\alpha$. To conclude the proof we need to show that (u, v_2, v_3, w) is a stationary point of $\mathcal{J}_\alpha$.

The proof is split into seven steps.

Step 1 (decomposition of the deformation gradients in rotation and strain). Let (R^h) be the approximating sequence of rotations constructed in theorem 2.5 and let $A \in W^{1,2}((0, L), \mathbb{M}^{3\times3})$ be the function defined in (2.4). We introduce the strain $G^h : \Omega \to \mathbb{M}^{3\times3}$ as

$$\nabla_h y^h = R^h(\mathrm{Id} + h^{\alpha-1} G^h). \tag{3.2}$$

By (2.15) the sequence (G^h) is bounded in $L^2(\Omega, \mathbb{M}^{3\times3})$, so that there exists $G \in L^2(\Omega, \mathbb{M}^{3\times3})$ such that $G^h \rightharpoonup G$ weakly in $L^2(\Omega, \mathbb{M}^{3\times3})$. Moreover, by lemma 3.1 (see the end of this section) the symmetric part of G can be characterized as follows: there exists $\beta \in L^2(\Omega, \mathbb{R}^3)$, with zero average on S and $\partial_k\beta \in L^2(\Omega, \mathbb{R}^3)$ for $k = 2, 3$, such that, if we set

$$M(\beta) := (x_2 A' e_2 + x_3 A' e_3 | \partial_2\beta | \partial_3\beta),$$

we have

$$\operatorname{sym} G = \begin{cases} \operatorname{sym} M(\beta) + (u' + \frac{1}{2}[(v_2')^2 + (v_3')^2]) e_1 \otimes e_1 & \text{if } \alpha = 3, \\ \operatorname{sym} M(\beta) + u' e_1 \otimes e_1 & \text{if } \alpha > 3, \\ \operatorname{sym} M(\beta) + g e_1 \otimes e_1 & \text{if } 2 < \alpha < 3, \end{cases} \tag{3.3}$$

for some $g \in L^2(0, L)$. In particular, by the normalization hypotheses (1.1) on S we deduce

$$\overline{G}_{11} = \begin{cases} u' + \frac{1}{2}[(v_2')^2 + (v_3')^2] & \text{for } \alpha = 3, \\ u' & \text{for } \alpha > 3, \\ g & \text{for } 2 < \alpha < 3. \end{cases} \tag{3.4}$$

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STEP 2 (stress tensor estimate). We define the stress $E^h\colon \Omega \to \mathbb{M}^{3\times 3}$ as

$$E^h = \frac{1}{h^{\alpha-1}} \mathrm{D}W(\mathrm{Id} + h^{\alpha-1}G^h)(\mathrm{Id} + h^{\alpha-1}G^h)^{\mathrm{T}}. \tag{3.5}$$

From the frame indifference of W it follows that

$$\mathrm{D}W(F)F^{\mathrm{T}} = F(\mathrm{D}W(F))^{\mathrm{T}} \quad \text{for every } F \in \mathbb{M}^{3\times 3}_+.$$

This implies that E^h is symmetric for every $h > 0$. Moreover, the following pointwise estimate holds:

$$|E^h| \leqslant C\biggl(\frac{W(\mathrm{Id} + h^{\alpha-1}G^h)}{h^{\alpha-1}} + |G^h|\biggr). \tag{3.6}$$

Indeed, let δ be the width of the neighbourhood of SO(3), where W is of class C^2. Suppose first that $h^{\alpha-1}|G^h| \leqslant \frac{1}{2}\delta$. Then, a first-order Taylor expansion of DW around the identity, together with (H4) and (H5), yields

$$\mathrm{D}W(\mathrm{Id} + h^{\alpha-1}G^h) = h^{\alpha-1}\mathrm{D}^2W(M^h)G^h$$

for some $M^h \in \mathbb{M}^{3\times 3}$ satisfying $|M^h - \mathrm{Id}\,| \leqslant \frac{1}{2}\delta$. Since D^2W is bounded on the set $\{F \in \mathbb{M}^{3\times 3}\colon\ \mathrm{dist}(F, \mathrm{SO}(3)) \leqslant \frac{1}{2}\delta\}$, we deduce

$$|\mathrm{D}W(\mathrm{Id} + h^{\alpha-1}G^h)| \leqslant Ch^{\alpha-1}|G^h|.$$

Therefore, by (3.5) we obtain

$$|E^h| \leqslant C|G^h| + Ch^{\alpha-1}|G^h|^2 \leqslant C(1+\delta)|G^h|.$$

If instead $h^{\alpha-1}|G^h| > \frac{1}{2}\delta$, we first observe that $W(\nabla_h y^h)$ is finite a.e. in Ω by (1.7). By (H2) and by frame indifference we deduce that

$$\det \nabla_h y^h = \det(\mathrm{Id} + h^{\alpha-1}G^h) > 0 \quad \text{a.e. in } \Omega.$$

Therefore, we can use (H7), which yields

$$|E^h| \leqslant \frac{1}{h^{\alpha-1}} k(W(\mathrm{Id} + h^{\alpha-1}G^h) + 1) \leqslant k\frac{W(\mathrm{Id} + h^{\alpha-1}G^h)}{h^{\alpha-1}} + \frac{2k}{\delta}|G^h|.$$

This completes the proof of (3.6).

STEP 3 (convergence properties of the scaled stress). Arguing as in [13], some convergence properties of the stresses E^h can be deduced from (3.6). Indeed, using (1.7) and the fact that the G^h are bounded in $L^2(\Omega, \mathbb{M}^{3\times 3})$, we obtain from (3.6) that for each measurable set Λ the following estimate holds true:

$$\int_\Lambda |E^h|\,\mathrm{d}x \leqslant Ch^{\alpha-1} + C|\Lambda|^{1/2}, \tag{3.7}$$

where $|\Lambda|$ denotes the Lebesgue measure of Λ. Now let

$$B_h := \{x \in \Omega\colon h^{\alpha-1-\gamma}|G^h(x)| \leqslant 1\}, \tag{3.8}$$

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where $\gamma \in (0, \alpha - 2)$, and let χ_h be the characteristic function of B_h. By (3.7) and by the Chebyshev inequality we have

$$\int_{\Omega \setminus B_h} |E^h| \, dx \leqslant C h^{\alpha - 1 - \gamma}, \tag{3.9}$$

so that

$$(1 - \chi_h) E^h \to 0 \quad \text{strongly in } L^1(\Omega, \mathbb{M}^{3\times 3}). \tag{3.10}$$

Moreover, one can show that the remainder in the first-order Taylor expansion of $\mathrm{D}W(\mathrm{Id} + h^{\alpha-1} G^h)$ around the identity is uniformly controlled on the sets B^h, so that

$$\chi_h E^h \rightharpoonup \mathcal{L} G =: E \quad \text{in } L^2(\Omega, \mathbb{M}^{3\times 3}) \tag{3.11}$$

(see step 3 in the proof of [13, theorem 3.1] for details).

Step 4 (some consequences of the Euler–Lagrange equations). By the frame indifference of W and by (3.2) we have

$$\mathrm{D}W(\nabla_h y^h)(\nabla_h y^h)^{\mathrm{T}} = h^{\alpha-1} R^h E^h (R^h)^{\mathrm{T}}.$$

Therefore, the Euler–Lagrange equations (3.1) can be written as

$$\int_\Omega R^h E^h (R^h)^{\mathrm{T}} : [(\nabla \phi) \circ y^h] \, dx = h \int_\Omega [f_2 (\phi_2 \circ y^h) + f_3 (\phi_3 \circ y^h)] \, dx \tag{3.12}$$

for every $\phi \in C^1_b(\mathbb{R}^3, \mathbb{R}^3)$ satisfying the boundary condition $\phi(0, hx_2, hx_3) = 0$ for all $(x_2, x_3) \in S$.

Now let ϕ be a function in $C^1_b(\mathbb{R}^3, \mathbb{R}^3)$ such that $\phi(0, x_2, x_3) = 0$ for every $(x_2, x_3) \in S$. For each $h > 0$ we define

$$\phi^h(x) := h\phi\biggl(x_1, \frac{x_2}{h} - \frac{1}{h}\xi^h_2(x_1), \frac{x_3}{h} - \frac{1}{h}\xi^h_3(x_1)\biggr),$$

where ξ^h_2, ξ^h_3 are the functions constructed in lemma 2.7. By (2.26), the maps ϕ^h are admissible test functions in (3.12).

To simplify computations we introduce the following notation:

$$z^h := \biggl(y^h_1, \frac{y^h_2}{h} - \xi^h_2 \circ y^h_1, \frac{y^h_3}{h} - \frac{1}{h}\xi^h_3 \circ y^h_1\biggr). \tag{3.13}$$

From (1.8) and (2.27) it follows that

$$z^h \to x \quad \text{in } L^2(\Omega, \mathbb{R}^3). \tag{3.14}$$

Choosing ϕ^h as test function in (3.12), we obtain

$$\begin{aligned} \int_\Omega R^h E^h (R^h)^{\mathrm{T}} e_1 \cdot \biggl[h \partial_1 \phi \circ z^h - \sum_{k=2}^{3} (\partial_k \phi \circ z^h)((\xi^h_k)' \circ y^h_1)\biggr] \, dx \\ + \int_\Omega \sum_{k=2}^{3} R^h E^h (R^h)^{\mathrm{T}} e_k \cdot (\partial_k \phi \circ z^h) \, dx \\ + \int_\Omega h^2 [f_2 (\phi_2 \circ z^h) + f_3 (\phi_3 \circ z^h)] \, dx = 0. \end{aligned} \tag{3.15}$$

By (3.7) and (2.28) we have

$$\begin{aligned}\biggl|\int_\Omega R^hE^h(R^h)^{\mathrm T}e_1\cdot\biggl[h\partial_1\phi\circ z^h-\sum_{k=2}^3(\partial_k\phi\circ z^h)((\xi_k^h)'\circ y_1^h)\biggr]\,\mathrm dx\biggr|\\ \leqslant C\|E^h\|_{L^1}\biggl(\|h\partial_1\phi\|_{L^\infty}+\sum_{k=2}^3\|\partial_k\phi\|_{L^\infty}\|(\xi_k^h)'\|_{L^\infty}\biggr)\\ \leqslant C(h+h^{\alpha-2});\end{aligned}$$

therefore, the first integral in (3.15) converges to zero. Analogously, since $f_k\in L^2(0,L)$ for $k=2,3$ and $\phi_k\in C_b^1(\mathbb R)$, the last integral in (3.15) tends to zero. We deduce that the second integral in (3.15) must also converge to zero. On the other hand, this term can be written as

$$\begin{aligned}\int_\Omega\sum_{k=2}^3R^hE^h(R^h)^{\mathrm T}e_k\cdot(\partial_k\phi\circ z^h)\,\mathrm dx\\ =\int_\Omega\sum_{k=2}^3\chi_hR^hE^h(R^h)^{\mathrm T}e_k\cdot(\partial_k\phi\circ z^h)\,\mathrm dx\\ +\int_\Omega\sum_{k=2}^3(1-\chi_h)R^hE^h(R^h)^{\mathrm T}e_k\cdot(\partial_k\phi\circ z^h)\,\mathrm dx.\end{aligned}\tag{3.16}$$

By (3.14) and by the dominated convergence theorem we have

$$\partial_k\phi\circ z^h\to\partial_k\phi\quad\text{in }L^2(\Omega).\tag{3.17}$$

Thus, by (3.11) and by the fact that $R^h\to\mathrm{Id}$ in $L^\infty(0,L)$, we deduce

$$\int_\Omega\sum_{k=2}^3\chi_hR^hE^h(R^h)^{\mathrm T}e_k\cdot(\partial_k\phi\circ z^h)\,\mathrm dx\to\int_\Omega\sum_{k=2}^3Ee_k\cdot\partial_k\phi\,\mathrm dx,$$

while by (3.10) we have that the last term in (3.16) tends to zero. We conclude that

$$\int_\Omega\sum_{k=2}^3Ee_k\cdot\partial_k\phi\,\mathrm dx=0\tag{3.18}$$

for every $\phi\in C_b^1(\mathbb R^3,\mathbb R^3)$ such that $\phi(0,x_2,x_3)=0$ for all $(x_2,x_3)\in S$. Therefore, the following equations hold true a.e. in $(0,L)$:

$$\left.\begin{aligned}\mathrm{div}_{x_2,x_3}(Ee_2|Ee_3)=0\quad\text{in }S,\\ (Ee_2|Ee_3)\nu_{\partial S}=0\quad\text{on }\partial S,\end{aligned}\right\}\tag{3.19}$$

where $\nu_{\partial S}$ is the unit normal to ∂S. Moreover, for a.e. $x_1\in(0,L)$,

$$\int_S Ee_k\,\mathrm dx_2\,\mathrm dx_3=0\quad\text{for }k=2,3.\tag{3.20}$$

We conclude that $\bar Ee_2=\bar Ee_3=0$ a.e. in $(0,L)$ and, since E is symmetric,

$$\bar E=\bar E_{11}e_1\otimes e_1.$$

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STEP 5 (zeroth-order moment of the Euler–Lagrange equations). We now identify the zeroth-order moment of the limit stress E. Let ψ be a function in $C^1_b(\mathbb{R})$ such that $\psi(0) = 0$. We define

$$\phi(x) = \psi(x_1)e_1.$$

Using ϕ as a test function in the Euler–Lagrange equations (3.12) we have

$$\int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{11}(\psi' \circ y^h_1)\,\mathrm{d}x = 0. \tag{3.21}$$

To pass to the limit in the previous equation, we split Ω into the sets B_h and $\Omega \setminus B_h$, so that we obtain

$$\int_\Omega \chi_h (R^h E^h (R^h)^{\mathrm{T}})_{11}(\psi' \circ y^h_1)\,\mathrm{d}x + \int_\Omega (1-\chi_h)(R^h E^h (R^h)^{\mathrm{T}})_{11}(\psi' \circ y^h_1)\,\mathrm{d}x = 0. \tag{3.22}$$

By (1.8) and by the continuity of ψ' it follows that $\psi' \circ y^h_1$ converges to ψ' in $L^2(\Omega)$. Therefore, by (3.10) and (3.11) we can pass to the limit in (3.22) and we deduce

$$\int_0^L \bar{E}_{11}\psi'\,\mathrm{d}x_1 = \int_\Omega E_{11}\psi'\,\mathrm{d}x = 0$$

for every $\psi \in C^1_b(\mathbb{R})$ such that $\psi(0) = 0$. This implies that $\bar{E} = \bar{E}_{11} e_1 \otimes e_1 = 0$ a.e. in $(0, L)$.

Since by frame indifference $\mathcal{L}H = 0$ for every skew-symmetric $H \in \mathbb{M}^{3\times 3}$, we obtain that $\mathcal{L}\overline{\operatorname{sym} G} = \mathcal{L}\bar{G} = \bar{E} = 0$. The invertibility of $\mathcal{L}$ on the space of symmetric matrices yields that $\overline{\operatorname{sym} G} = 0$. Together with (3.4), this implies (2.8) for $\alpha = 3$, (2.12) for $\alpha > 3$, and $g = 0$ a.e. in $(0, L)$ for $2 < \alpha < 3$. Moreover, by (3.3) we deduce that

$$\operatorname{sym}\left(0 \,\middle|\, \int_S \partial_2\beta\,\mathrm{d}x_2\,\mathrm{d}x_3 \,\middle|\, \int_S \partial_3\beta\,\mathrm{d}x_2\,\mathrm{d}x_3\right) = 0,$$

so that, if we introduce $\tilde{\beta} \colon \Omega \to \mathbb{R}^3$ defined by

$$\tilde{\beta} := \left(\beta_1, \beta_2 - x_3\int_S \partial_3\beta_2\,\mathrm{d}x_2\,\mathrm{d}x_3, \beta_3 - x_2\int_S \partial_2\beta_3\,\mathrm{d}x_2\,\mathrm{d}x_3\right),$$

we have that $\tilde{\beta}(x_1, \cdot, \cdot) \in \mathcal{B}$ for a.e. $x_1 \in (0, L)$ and

$$\operatorname{sym} G = \operatorname{sym}(x_2 A' e_2 + x_3 A' e_3 | \partial_2\tilde{\beta} | \partial_3\tilde{\beta}).$$

In particular, we have the following characterization of E:

$$E = \mathcal{L}\operatorname{sym} G = \mathcal{L}(x_2 A' e_2 + x_3 A' e_3 | \partial_2\tilde{\beta} | \partial_3\tilde{\beta}).$$

Since E satisfies (3.19), we deduce from lemma 2.1 that $\tilde{\beta}$ is a minimizer of the functional

$$\mathcal{G}_{A'}(\beta) = \int_S Q_3(x_2 A' e_2 + x_3 A' e_3 | \partial_2\beta | \partial_3\beta)\,\mathrm{d}x_2\,\mathrm{d}x_3.$$

In other words, $\tilde{\beta}$ satisfies

$$Q_1(A') = \int_S Q_3(x_2 A' e_2 + x_3 A' e_3 | \partial_2\tilde{\beta} | \partial_3\tilde{\beta})\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{3.23}$$

for all $\alpha > 2$.

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STEP 6 (first-order moments of the Euler–Lagrange equations). In this step we prove that the limiting Euler–Lagrange equations (2.9) and (2.10) are satisfied. Let φ_2, φ_3 be two functions in $C^1_b(\mathbb{R})$ with $\varphi_2(0) = \varphi_3(0) = 0$. We define

$$\phi^h(x) = \left(0, \frac{\varphi_2(x_1)}{h}, \frac{\varphi_3(x_1)}{h}\right)$$

and we use ϕ^h as test function in (3.12). By (1.8) the force term can be treated as follows:

$$\lim_{h\to 0}\int_{\Omega} h[f_2(\phi^h_2\circ y^h) + f_3(\phi^h_3\circ y^h)]\,\mathrm{d}x = \lim_{h\to 0}\int_{\Omega}[f_2(\varphi_2\circ y^h_1) + f_3(\varphi_3\circ y^h_1)]\,\mathrm{d}x$$
$$= \int_0^L (f_2\varphi_2 + f_3\varphi_3)\,\mathrm{d}x_1. \tag{3.24}$$

Therefore, we have

$$\lim_{h\to 0}\int_{\Omega}\left[(R^hE^h(R^h)^{\mathrm{T}})_{21}\frac{\varphi_2'\circ y^h_1}{h} + (R^hE^h(R^h)^{\mathrm{T}})_{31}\frac{\varphi_3'\circ y^h_1}{h}\right]\mathrm{d}x$$
$$= \int_0^L (f_2\varphi_2 + f_3\varphi_3)\,\mathrm{d}x_1. \tag{3.25}$$

We shall characterize the limit on the left-hand side of (3.25) in terms of the first-order moments of the stress E. To this aim, we go back to the Euler–Lagrange equations (3.12) and we construct some ad hoc test functions with a linear behaviour in the variables x_2, x_3. Let (ω_h) be a sequence of positive numbers such that

$$h\omega_h \to +\infty, \tag{3.26}$$
$$h^{\alpha-1-\gamma}\omega_h \to 0, \tag{3.27}$$

where $\gamma \in (0, \alpha - 2)$ is the same exponent introduced in (3.8). For each $h > 0$ we consider a function $\theta^h \in C^1_b(\mathbb{R})$ which coincides with the identity in a large enough neighbourhood of the origin, that is,

$$\theta^h(t) = t \quad \text{for } |t| \leqslant \omega_h \tag{3.28}$$

and, in addition, satisfies the following properties:

$$|\theta^h(t)| \leqslant |t| \quad \text{for all } t \in \mathbb{R}, \tag{3.29}$$
$$\|\theta^h\|_{L^\infty} \leqslant 2\omega_h, \tag{3.30}$$
$$\left\|\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\right\|_{L^\infty} \leqslant 2. \tag{3.31}$$

Let η be a function in $C^1(\mathbb{R})$ with compact support and such that $\eta(0) = 0$, and let ξ^h_k, $k = 2, 3$, be the functions constructed in lemma 2.7. We consider the map

$$\phi^h(x) = \theta^h\left(\frac{x_3}{h} - \frac{1}{h}\xi^h_3(x_1)\right)\eta(x_1)e_1.$$

Choosing ϕ^h as a test function in (3.12) and using the notation introduced in (3.13), we obtain

$$\begin{aligned}\int_\Omega &(R^hE^h(R^h)^{\mathrm{T}})_{11}(\theta^h\circ z_3^h)(\eta'\circ y_1^h)\,\mathrm{d}x\\ &-\int_\Omega \frac{(R^hE^h(R^h)^{\mathrm{T}})_{11}}{h}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)[(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x\\ &\quad+\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{13}\frac{\eta\circ y_1^h}{h}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)\mathrm{d}x=0. \qquad (3.32)\end{aligned}$$

The first integral in (3.32) can be decomposed into the sum of two terms:

$$\begin{aligned}\int_\Omega &(R^hE^h(R^h)^{\mathrm{T}})_{11}(\theta^h\circ z_3^h)(\eta'\circ y_1^h)\,\mathrm{d}x\\ &=\int_\Omega \chi_h[(R^hE^h(R^h)^{\mathrm{T}})_{11}(\theta^h\circ z_3^h)(\eta'\circ y_1^h)]\,\mathrm{d}x\\ &\quad+\int_\Omega (1-\chi_h)[(R^hE^h(R^h)^{\mathrm{T}})_{11}(\theta^h\circ z_3^h)(\eta'\circ y_1^h)]\,\mathrm{d}x. \qquad (3.33)\end{aligned}$$

By (1.8), (3.14), (3.29), and by the dominated convergence theorem we deduce that

$$(\theta^h\circ z_3^h)(\eta'\circ y_1^h)\to x_3\eta' \quad \text{in } L^2(\Omega).$$

Therefore, by (3.11) we have

$$\lim_{h\to 0}\int_\Omega \chi_h(R^hE^h(R^h)^{\mathrm{T}})_{11}(\eta'\circ y_1^h)(\theta^h\circ z_3^h)\,\mathrm{d}x=\int_\Omega x_3E_{11}\eta'\,\mathrm{d}x=\int_0^L \hat{E}_{11}\eta'\,\mathrm{d}x_1.$$

The second term in (3.33) can be estimated using (3.9), as follows:

$$\begin{aligned}\int_\Omega (1-\chi_h)|(R^hE^h(R^h)^{\mathrm{T}})_{11}(\eta'\circ y_1^h)(\theta^h\circ z_3^h)|\,\mathrm{d}x &\leqslant 2\omega_h\|\eta'\|_{L^\infty(0,L)}\int_{\Omega\setminus B_h}|E^h|\,\mathrm{d}x\\ &\leqslant Ch^{\alpha-1-\gamma}\omega_h,\end{aligned}$$

and the latter is infinitesimal owing to (3.27). We conclude that

$$\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{11}(\theta^h\circ z_3^h)(\eta'\circ y_1^h)\,\mathrm{d}x\to\int_0^L \hat{E}_{11}\eta'\,\mathrm{d}x_1. \qquad (3.34)$$

As for the second integral in (3.32), we consider the following decomposition:

$$\begin{aligned}\int_\Omega &(R^hE^h(R^h)^{\mathrm{T}})_{11}\frac{1}{h}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)[(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x\\ &=\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{11}\left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)-1\right]\frac{1}{h}[(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x\\ &\quad+\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{11}\frac{1}{h}[(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x. \qquad (3.35)\end{aligned}$$

To study the first term in (3.35) we introduce the sets

$$D_h=\{x\in\Omega\colon |z_3^h(x)|\geqslant\omega_h\}. \qquad (3.36)$$

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Since (z_3^h) is uniformly bounded in $L^2(\Omega)$, by the Chebyshev inequality we deduce that

$$|D_h| \leqslant C\omega_h^{-2}. \tag{3.37}$$

Thus, by (2.28) and (3.7) we have

$$\begin{aligned}
&\left| \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{11} \left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) - 1 \right] \frac{1}{h} [(\xi_3^h)' \circ y_1^h](\eta \circ y_1^h) \,\mathrm{d}x \right| \\
&\qquad \leqslant \int_{D_h} \left| (R^h E^h (R^h)^{\mathrm{T}})_{11} \left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) - 1 \right] \frac{1}{h} [(\xi_3^h)' \circ y_1^h](\eta \circ y_1^h) \right| \mathrm{d}x \\
&\qquad \leqslant Ch^{\alpha-3} \int_{D_h} |E^h| \,\mathrm{d}x \\
&\qquad \leqslant Ch^{\alpha-3}(h^{\alpha-1} + |D_h|^{1/2}) \leqslant C\left(h^{2\alpha-4} + \frac{h^{\alpha-2}}{h\omega_h} \right),
\end{aligned} \tag{3.38}$$

where the latter term tends to zero owing to (3.26). Furthermore, we can prove that the second term in (3.35) is equal to zero. Indeed, let

$$\psi^h(x_1) := \int_0^{x_1} \frac{1}{h} (\xi_3^h)'(s) \eta(s) \,\mathrm{d}s.$$

It is easy to verify that $\psi^h \in C_b^1(\mathbb{R})$ and $\psi^h(0) = 0$ for every $h > 0$. Therefore, by (3.21) we obtain

$$\int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{11} \frac{1}{h} [(\xi_3^h)' \circ y_1^h](\eta \circ y_1^h) \,\mathrm{d}x = 0.$$

By (3.35) and (3.38) we conclude that

$$\int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{11} \frac{1}{h} \left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) [(\xi_3^h)' \circ y_1^h](\eta \circ y_1^h) \,\mathrm{d}x \to 0. \tag{3.39}$$

It remains to study the third integral in (3.32), which can be written as

$$\begin{aligned}
&\int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{13} \frac{\eta \circ y_1^h}{h} \left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) \mathrm{d}x \\
&\qquad = \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{13} \frac{\eta \circ y_1^h}{h} \left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) - 1 \right] \mathrm{d}x \\
&\qquad\quad + \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{13} \frac{\eta \circ y_1^h}{h} \,\mathrm{d}x.
\end{aligned} \tag{3.40}$$

We claim that

$$\lim_{h\to 0} \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{13} \frac{\eta \circ y_1^h}{h} \left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h \right) - 1 \right] \mathrm{d}x = 0. \tag{3.41}$$

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To prove it, we again consider the sets D_h defined in (3.36). From (3.7), (3.31), (3.37), and from the boundedness of η we obtain

$$\begin{aligned}\int_\Omega \left|(R^hE^h(R^h)^{\mathrm{T}})_{13}\frac{\eta\circ y_1^h}{h}\left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)-1\right]\right|\mathrm{d}x \\ &= \int_{D_h}\left|(R^hE^h(R^h)^{\mathrm{T}})_{13}\frac{\eta\circ y_1^h}{h}\left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)-1\right]\right|\mathrm{d}x \\ &\leqslant \frac{C}{h}\int_{D_h}|E^h|\,\mathrm{d}x \\ &\leqslant \frac{C}{h}(h^{\alpha-1}+|D_h|^{1/2}) \\ &\leqslant C\left(h^{\alpha-2}+\frac{1}{h\omega_h}\right),\end{aligned}$$

and the latter is infinitesimal owing to (3.26), so that (3.41) follows. In conclusion, combining (3.32), (3.34), and (3.39)–(3.41) we deduce that

$$\lim_{h\to 0}\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{13}\frac{\eta\circ y_1^h}{h}\,\mathrm{d}x = -\int_\Omega \hat{E}_{11}\eta'\,\mathrm{d}x \tag{3.42}$$

for every $\eta\in C^1(\mathbb{R})$ with compact support and such that $\eta(0)=0$. Choosing a test function of the form

$$\phi^h(x)=\theta^h\left(\frac{x_2}{h}-\frac{1}{h}\xi_2^h(x_1)\right)\eta(x_1)e_1,$$

one can prove analogously that

$$\lim_{h\to 0}\int_\Omega (R^hE^h(R^h)^{\mathrm{T}})_{12}\frac{\eta\circ y_1^h}{h}\,\mathrm{d}x = -\int_\Omega \tilde{E}_{11}\eta'\,\mathrm{d}x. \tag{3.43}$$

Now let $\varphi_k\in C^2(\mathbb{R})$ with compact support be such that $\varphi_k(0)=\varphi_k'(0)=0$ for $k=2,3$. We choose $\eta=\varphi_3'$ in (3.42) and $\eta=\varphi_2'$ in (3.43) and we add the two equations. Comparing the result with (3.25) and using the fact that E^h (and therefore $R^hE^h(R^h)^{\mathrm{T}}$) is symmetric, we conclude that

$$\int_0^L (\tilde{E}_{11}\varphi_2''+\hat{E}_{11}\varphi_3''+f_2\varphi_2+f_3\varphi_3)\,\mathrm{d}x_1=0$$

for every $\varphi_k\in C^2(\mathbb{R})$ with compact support and such that $\varphi_k(0)=\varphi_k'(0)=0$, $k=2,3$. By approximation we obtain (2.9) and (2.10) for all $\alpha>2$.

Step 7 (Euler–Lagrange equation for the twist function). To conclude the proof of the theorem, it remains to verify the limiting Euler–Lagrange equation (2.11). We define

$$\phi^h(x)=\left(0,-\theta^h\left(\frac{x_3}{h}-\frac{\xi_3^h(x_1)}{h}\right)\eta(x_1),\theta^h\left(\frac{x_2}{h}-\frac{\xi_2^h(x_1)}{h}\right)\eta(x_1)\right),$$

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where $\eta \in C^1(\mathbb{R})$ with compact support, $\eta(0)=0$, and θ^h is as in step 6. Using ϕ^h as test function in the Euler–Lagrange equations (3.12), we obtain

$$\begin{aligned}
&-\int_\Omega [(R^h E^h (R^h)^{\mathrm{T}})_{21}(\theta^h \circ z_3^h) - (R^h E^h (R^h)^{\mathrm{T}})_{31}(\theta^h \circ z_2^h)](\eta' \circ y_1^h)\,\mathrm{d}x \\
&\quad + \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{21}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h\right)((\xi_3^h)' \circ y_1^h)\frac{\eta \circ y_1^h}{h}\,\mathrm{d}x \\
&\quad - \int_\Omega (R^h E^h (R^h)^{\mathrm{T}})_{31}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_2^h\right)((\xi_2^h)' \circ y_1^h)\frac{\eta \circ y_1^h}{h}\,\mathrm{d}x \\
&\quad + \int_\Omega \left[(R^h E^h (R^h)^{\mathrm{T}})_{32}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_2^h\right) - (R^h E^h (R^h)^{\mathrm{T}})_{23}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h\right)\right]\frac{\eta \circ y_1^h}{h}\,\mathrm{d}x \\
&\quad + h\int_\Omega [f_2(\theta^h \circ z_3^h) - f_3(\theta^h \circ z_2^h)](\eta \circ y_1^h)\,\mathrm{d}x = 0. \qquad (3.44)
\end{aligned}$$

Arguing as in the proof of (3.34), we can show that the first integral in (3.44) satisfies

$$\begin{aligned}
\lim_{h\to 0}\int_\Omega [(R^h E^h (R^h)^{\mathrm{T}})_{21}(\theta^h \circ z_3^h) - (R^h E^h (R^h)^{\mathrm{T}})_{31}(\theta^h \circ z_2^h)](\eta' \circ y_1^h)\,\mathrm{d}x \\
= \int_0^L (-\hat{E}_{12} + \tilde{E}_{13})\eta'\,\mathrm{d}x_1.
\end{aligned}$$

The proof of (2.11) is concluded if we show that all other terms in (3.44) converge to zero as $h \to 0$. The last integral in (3.44) is infinitesimal, owing to the estimate

$$\begin{aligned}
\left| h\int_\Omega [f_2(\theta^h \circ z_3^h) - f_3(\theta^h \circ z_2^h)](\eta \circ y_1^h)\,\mathrm{d}x \right| &\leqslant Ch(\|f_2\|_{L^2}\|z_3^h\|_{L^2} + \|f_3\|_{L^2}\|z_2^h\|_{L^2}) \\
&\leqslant Ch,
\end{aligned}$$

which follows from (3.29) and (3.14).

As for the term

$$\int_\Omega \left[(R^h E^h (R^h)^{\mathrm{T}})_{32}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_2^h\right) - (R^h E^h (R^h)^{\mathrm{T}})_{23}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h\right)\right]\frac{\eta \circ y_1^h}{h}\,\mathrm{d}x,$$

we remark that by the symmetry of $R^h E^h (R^h)^{\mathrm{T}}$ it can be written as

$$\int_\Omega \frac{\eta \circ y_1^h}{h}(R^h E^h (R^h)^{\mathrm{T}})_{32}\left\{\left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_2^h\right) - 1\right] + \left[1 - \left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_3^h\right)\right]\right\}\mathrm{d}x.$$

Arguing as in the proof of (3.41), we obtain that the above expression tends to zero as $h \to 0$.

It remains to prove that

$$\lim_{h\to 0}\int_\Omega \frac{1}{h}(R^h E^h (R^h)^{\mathrm{T}})_{k1}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t} \circ z_j^h\right)[(\xi_j^h)' \circ y_1^h](\eta \circ y_1^h)\,\mathrm{d}x = 0$$

for $k, j \in \{2,3\}$, $k \neq j$. To this aim, we fix $k=2$, $j=3$ and we write the previous integral as the sum of two terms:

$$\begin{aligned}\int_\Omega \frac{1}{h}(R^h E^h (R^h)^{\mathrm{T}})_{21}\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)[(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x \\ = \int_\Omega \frac{1}{h}(R^h E^h (R^h)^{\mathrm{T}})_{21}\left[\left(\frac{\mathrm{d}\theta^h}{\mathrm{d}t}\circ z_3^h\right)-1\right][(\xi_3^h)'\circ y_1^h](\eta\circ y_1^h)\,\mathrm{d}x \\ + \int_\Omega \frac{(R^h E^h (R^h)^{\mathrm{T}})_{21}}{h^{1-\epsilon}}\frac{(\xi_3^h)'\circ y_1^h}{h^\epsilon}(\eta\circ y_1^h)\,\mathrm{d}x,\end{aligned} \tag{3.45}$$

where $0<\epsilon<\alpha-2$. Arguing as in the proof of (3.41), we obtain that the first term is infinitesimal. To study the second term, we notice that if $(\psi^h)\subset C_b^1(\mathbb{R})$ is a sequence of functions such that $\psi^h(0)=0$ and $\|\psi^h\|_{L^\infty(\mathbb{R})}\leqslant C$ for all $h>0$, then the map $\psi^h(x_1)e_j$ can be used as a test function in the Euler–Lagrange equations (3.12) for every $h>0$, and we have

$$\left|\int_\Omega \frac{(R^h E^h (R^h)^{\mathrm{T}})_{j1}}{h^{1-\epsilon}}[(\psi^h)'\circ y_1^h]\,\mathrm{d}x\right| = \left|\int_\Omega h^\epsilon f_j(\psi^h\circ y_1^h)\,\mathrm{d}x\right| \leqslant Ch^\epsilon\|f_j\|_{L^2(\Omega)}\to 0. \tag{3.46}$$

If we now choose

$$\psi^h(x_1) := \int_0^{x_1}\frac{(\xi_k^h)'(s)}{h^\epsilon}\eta(s)\,\mathrm{d}s,$$

then by (2.28) we obtain

$$\|\psi^h\|_{L^\infty}\leqslant Ch^{\alpha-2-\epsilon}\|\eta\|_{L^1}\leqslant C\quad\text{for all } h>0,$$

so that by (3.46) the last term in (3.44) is also infinitesimal as $h\to 0$. This concludes the proof of (2.11) and of the theorem. □

We conclude this section with a lemma which provides us with a characterization of the limiting strain. This result is contained in the proof of [17, theorems 4.3 and 4.4]. We present here a concise proof for the reader's convenience.

LEMMA 3.1. *Let all the assumptions of theorem 2.6 be satisfied and let (R^h) be the sequence of rotations of theorem 2.5. For every $h>0$ let $G^h\colon \Omega\to\mathbb{M}^{3\times3}$ be defined by*

$$G^h = \frac{(R^h)^{\mathrm{T}}\nabla_h y^h - \mathrm{Id}}{h^{\alpha-1}},$$

and let G be the weak limit of (G^h) in $L^2(\Omega,\mathbb{M}^{3\times3})$ (which exists, up to subsequences, by (2.15)). Then, there exist $g\in L^2(0,L)$ and $\beta\in L^2(\Omega,\mathbb{R}^3)$, with zero average on S and $\partial_k\beta\in L^2(\Omega,\mathbb{R}^3)$ for $k=2,3$, such that, if we define

$$M(\beta) := (x_2A'e_2 + x_3A'e_3|\partial_2\beta|\partial_3\beta),$$

we have

$$\operatorname{sym} G = \begin{cases}\operatorname{sym} M(\beta) + (u' + \frac{1}{2}[(v_2')^2+(v_3')^2])e_1\otimes e_1 & \text{if } \alpha=3,\\ \operatorname{sym} M(\beta) + u'e_1\otimes e_1 & \text{if } \alpha>3,\\ \operatorname{sym} M(\beta) + ge_1\otimes e_1 & \text{if } 2<\alpha<3,\end{cases} \tag{3.47}$$

where u, v_k and A are the functions introduced in theorem 2.6.

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Proof. For every $h > 0$ we consider the function $\gamma^h\colon \Omega \to \mathbb{R}^3$ defined by

$$\gamma^h(x) := \frac{1}{h^\alpha}[y^h(x) - hx_2 R^h(x_1)e_2 - hx_3 R^h(x_1)e_3]$$

for every $x \in \Omega$. By (2.17) we have that

$$\partial_k \gamma^h \rightharpoonup Ge_k \quad \text{for every } k = 2, 3. \tag{3.48}$$

Therefore, if we define $\beta^h := \gamma^h - \bar{\gamma}^h$, where $\bar{\gamma}^h$ is the average of γ^h on S, we deduce by Poincaré–Wirtinger inequality that β^h is uniformly bounded in $L^2(\Omega, \mathbb{R}^3)$. It follows that there exists $\tilde{\beta} \in L^2(\Omega, \mathbb{R}^3)$, with zero average on S, such that, up to subsequences, $\beta^h \rightharpoonup \tilde{\beta}$ in $L^2(\Omega, \mathbb{R}^3)$. Furthermore, by (3.48) we have that $\partial_k \tilde{\beta} = Ge_k$ for all $k = 2, 3$.

As for the first column of G, we remark that by (1.1) we can write

$$\begin{aligned} R^h G^h e_1 &= h\partial_1 \gamma^h + \frac{1}{h^{\alpha-2}}(x_2 (R^h)' e_2 + x_3 (R^h)' e_3) - \frac{1}{h^{\alpha-1}} R^h e_1 \\ &= h\partial_1 \beta^h + \frac{1}{h^{\alpha-2}}(x_2 (R^h)' e_2 + x_3 (R^h)' e_3) - \int_S \frac{R^h e_1 - \partial_1 y^h}{h^{\alpha-1}} \, \mathrm{d}x_2 \, \mathrm{d}x_3. \end{aligned} \tag{3.49}$$

By (2.17) we have that $R^h G^h e_1 \rightharpoonup Ge_1$ weakly in $L^2(\Omega, \mathbb{M}^{3\times 3})$. Moreover, by (2.15) there exists a function $g \in L^2((0, L), \mathbb{R}^3)$ such that

$$\int_S \frac{R^h e_1 - \partial_1 y^h}{h^{\alpha-1}} \, \mathrm{d}x_2 \, \mathrm{d}x_3 \rightharpoonup g \quad \text{weakly in } L^2((0, L), \mathbb{R}^3),$$

while (2.24) yields

$$\frac{1}{h^{\alpha-2}} x_2 (R^h)' e_2 + x_3 (R^h)' e_3 \rightharpoonup x_2 A' e_2 + x_3 A' e_3 \quad \text{weakly in } L^2((0, L), \mathbb{M}^{3\times 3}).$$

Finally, by the weak convergence of (β^h) in $L^2(\Omega, \mathbb{R}^3)$ we have that $h\partial_1 \beta^h \to 0$ in $W^{-1,2}(\Omega, \mathbb{R}^3)$; thus, passing to the limit in (3.49), we conclude that

$$G = (x_2 A' e_2 + x_3 A' e_3 + g | \partial_2 \tilde{\beta} | \partial_3 \tilde{\beta}).$$

To obtain (3.47) it is now enough to define

$$\beta := \tilde{\beta} + x_2 (g \cdot e_2) e_1 + x_3 (g \cdot e_3) e_1,$$

so that

$$\operatorname{sym} G = \operatorname{sym}(x_2 A' e_2 + x_3 A' e_3 + (g \cdot e_1) e_1 | \partial_2 \beta | \partial_3 \beta).$$

This concludes the proof for $2 < \alpha < 3$. For $\alpha \geqslant 3$ a characterization of g can be given. Indeed, one can observe that

$$\begin{aligned} \int_S \frac{\partial_1 y^h - R^h e_1}{h^{\alpha-1}} \cdot e_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 &= \int_S \frac{(\partial_1 y^h - 1) + (1 - R^h_{11})}{h^{\alpha-1}} \, \mathrm{d}x_2 \, \mathrm{d}x_3 \\ &= (u^h)' - h^{\alpha-3} \operatorname{sym}(R^h - \mathrm{Id})_{11}, \end{aligned}$$

where (u^h) is the sequence introduced in (1.3). By (2.20) and (2.25) we obtain the thesis for $\alpha \geqslant 3$. □

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