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STOCHASTIC MODEL FOR THE NUMBER OF ATOMS IN A MAGNETO-OPTICAL TRAP

ANDREW L. RUKHIN AND IONUT BEBU
National Institute of Standards and Technology
and
University of Maryland at Baltimore County
Baltimore, MD 21250
E-mail: rukhin@math.umbc.edu

In this article a Markov chain for the distribution of single atoms is suggested and studied. We explore a recursive model for the number of atoms present in a magneto-optical trap under a feedback regime with a Poisson-distributed load. Formulas for the stationary distribution of this process are derived. They can be used to adjust the loading rate of atoms to maximize the proportion of time that a single atom spends in the trap. The (approximate) optimal regime for the Poisson loading and loss processes is found.

1. INTRODUCTION

The ability to control individual atoms is crucial in nanotechnology and quantum information processing. Atom-on-demand technology will enable novel quantum computation schemes, unprecedented control over dopants in materials, and eventual realization of the ultimate goal of nanotechnology: atom-by-atom construction.

Here, we suggest a version of a recursive model for the number of atoms present in a magneto-optical trap (MOT) at any given time under a feedback regime with a Poissonian loss. Formulas for the stationary distribution of this process can be used to control the loading rate of atoms so as to maximize the proportion of time that a single atom spends in the trap. The (approximate) optimal regime for the Poisson loading and loss processes is found. More specifically, the values of loading and loss parameters maximizing the probability of exactly one atom in the trap are deter-

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mined and confirmed through Monte Carlo simulation. These results make use of the Borel–Tanner distribution (Kingman [8]), which describes the total number of customers served before a queue vanishes with random arrival of customers and a constant time needed to serve each customer.

Optical quantum cryptography is built on the use of the single photon state [2, Sect. III.A]. Practical implementation mainly relies on faint laser pulses based on standard semiconductor lasers. The Electron and Optical Physics Division of the National Institute of Standards and Technology has built a prototype device that can produce and sustain one atom of a given species, upon demand, for a useful time interval. Hill and McClelland [6] described a new accurate method of trapping neutral atoms in a vacuum chamber that does not use faint pulses. To achieve the optimal stationary distribution for single atoms in our model, one needs a weak load regime and even weaker loss processes. However, under these conditions, the convergence to the stationary distribution is the slowest.

2. MARKOV CHAIN MODEL FOR THE NUMBER OF ATOMS IN A MAGNETO-OPTICAL TRAP

Suppose the door to the MOT could be opened the instant it became empty or closed at the instant it became occupied. Then the number of atoms in the trap can be modeled as a Markov chain. Because of physical limitations, the number of atoms in the trap cannot be monitored continuously: The trap can only be checked every T seconds. During that interval T, more than one atom can sneak inside the trap before the door is closed.

Let R_n be a Poisson random variable with parameter λ, representing the load (i.e., the random number of atoms entering the trap at time n) and denote by X_n the number of atoms in the MOT at time $n, n = 1,2,\ldots$. For a fixed $X_n = x$, the random loss variable Y_n has a Poisson distribution with parameter μx. A simple recursive model for the number X_n of atoms present in the MOT at step n under a feedback regime is

$$X_n = \begin{cases} R_n & \text{if } X_{n-1} \le Y_{n-1} \\ X_{n-1} - Y_{n-1} & \text{if } X_{n-1} > Y_{n-1}, \end{cases} \tag{2.1}$$

with R_n and (X_{n-1}, Y_{n-1}) independent. As a simple example, suppose that at step n, there are three atoms in the MOT, so $X_n = 3$; according to the model, the trap is opened. If we lose two atoms, then at step $n + 1$, there is only one atom left in the trap ($X_{n+1} = 1$), but if we lose all three atoms, X_{n+1} is determined by a new load; notice that in the latter case, we only consider one step, although we have two time intervals. Our goal is to get a single atom with high probability; if we know that the trap empties, there is no point in stopping the process.

Clearly, $X_n, n = 1,2,\ldots$, forms a countable state Markov chain in discrete time. If $X_n \to X$, with the distribution of X being the steady-state (stationary) distribution for the chain, then the following equality of distributions holds:

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$$X = \begin{cases} R & \text{if } X \le Y \\ X - Y & \text{if } X > Y. \end{cases} \tag{2.2}$$

Here, R is Poisson distributed with parameter λ, and for a fixed $X = x$, the random variable Y is Poisson distributed with parameter μx.

Our goal is to find the distribution of X and to find the values of λ and μ that maximize the value $\pi_1 = P(X = 1)$. For this purpose, we determine the probability vector $\pi = (\pi_0, \pi_1, \ldots)$ with $\pi_k = P(X = k)$, $k = 0,1,2,\ldots$.

Notice that the transition probabilities of the above Markov chain are

$$\begin{aligned} P(x|y) &= P(X_n = y | X_{n-1} = x) \\ &= P(X_n = y, Y_{n-1} \ge x | X_{n-1} = x) \\ &\quad + P(X_{n-1} - Y_{n-1} = y,\ Y_{n-1} < x | X_{n-1} = x) \\ &= \sum_{k=x}^{\infty} P(R_n = y) P(Y_{n-1} = k | X_{n-1} = x) \\ &\quad + P(Y_{n-1} = x - y | X_{n-1} = x) \mathbf{1}_{\{1 \le y \le x\}} \\ &= e^{-\lambda} \frac{\lambda^y}{y!} \sum_{k=x}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} + e^{-\mu x} \frac{(\mu x)^{x-y}}{(x-y)!} \mathbf{1}_{\{1 \le y \le x\}}. \end{aligned} \tag{2.3}$$

In particular,

$$\begin{aligned} P(x|0) &= e^{-\lambda} \sum_{k=x}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} \\ &= e^{-\lambda} \left(1 - e^{-\mu x} \sum_{k=0}^{x-1} \frac{(\mu x)^{k-1}}{k!} \right) \\ &= e^{-\lambda} d_x, \end{aligned} \tag{2.4}$$

with $d_0 = 1$.

Observe that

$$P(1|1) = \lambda e^{-\lambda}(1 - e^{-\mu}) + e^{-\mu}, \tag{2.5}$$

which is maximized when $\mu = 0$ (the value of λ being immaterial.)

3. STEADY-STATE DISTRIBUTION FOR THE NUMBER OF ATOMS

THEOREM 1: *For any $\lambda, \mu > 0$, the Markov chain determined by transition probabilities given in Eq. (2.3) is ergodic. In fact, the chain is geometrically ergodic; that is, for some $R(x)$ and $\rho < 1$,*

$$|P^n(x|y) - \pi_x| \le R(x) \rho^n, \tag{3.1}$$

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$n = 1,2,\ldots$. The vector π of stationary probabilities has the form given in Eq. (3.13) with the vector $\mathbf{u}$ *satisfying Eq. (3.8).*

PROOF: We will prove that the Markov chain determined by Eq. (2.3) has a stationary distribution π by constructing it. For the stationary distribution to exist, we need

$$\sum_{n=0}^{\infty} \pi_n P(n|0) = e^{-\lambda} \sum_{n=0}^{\infty} d_n \pi_n = \pi_0, \tag{3.2}$$

so $\pi_0 \neq 0$, and for $k = 1,2,\ldots$, with $v_k = \pi_k/\pi_0$,

$$\sum_{k=1}^{\infty} d_k v_k = e^{\lambda} - 1. \tag{3.3}$$

Also, $\sum_{k=1}^{\infty} v_k = (1 - \pi_0)/\pi_0$. Introduce the lower triangular matrix $\mathbf{B}$ with the entries

$$b_{ij} = e^{-\mu i} \frac{j(\mu i)^{i-j}}{i(i-j)!}, \qquad i = j, j+1,\ldots. \tag{3.4}$$

For $\mu > 0$, we have $\sum_{i=j}^{\infty} b_{ij} \leq 1$, and for $0 < \mu \leq 1$, the elements of $\mathbf{B}$ are in fact the probabilities of the Borel–Tanner distribution, so that, with $\mathbf{e}^T = (1, 1,\ldots)$, $\mathbf{e^T B} = \mathbf{e^T}$.

According to Eq. (2.3), the reduced transition probabilities matrix $\mathbf{P}$ formed by the elements $\{P(x|y), x, y = 1,2,\ldots\}$ can be written in the form

$$\mathbf{P} = \mathbf{DBD}^{-1} + \mathbf{e}^{-\lambda}\mathbf{d}(\mathbf{D}^{-1}\ell)^{\mathbf{T}}. \tag{3.5}$$

Here, $\mathbf{D}$ is the diagonal matrix with elements $1,2,\ldots, \mathbf{d}^T = (d_1, d_2,\ldots)$, and $\ell^T = (\lambda, \lambda^2,\ldots, \lambda^j/[(j-1)!],\ldots)$. Since $P(0|y) = e^{-\lambda}\lambda^y/y!$, one has for $\mathbf{v}^T = (v_1, v_2,\ldots)$,

$$\mathbf{v}^T\mathbf{P} = \mathbf{v^T} - \mathbf{e}^{-\lambda}(\mathbf{D}^{-1}\ell)^{\mathbf{T}}. \tag{3.6}$$

On the other hand,

$$\mathbf{v}^T\mathbf{P} = \mathbf{v^T}[\mathbf{DBD}^{-1} + \mathbf{e}^{-\lambda}\mathbf{d}(\mathbf{D}^{-1}\ell)^{\mathbf{T}}] = \mathbf{v^T DBD}^{-1} + (\mathbf{1} - \mathbf{e}^{-\lambda})(\mathbf{D}^{-1}\ell)^{\mathbf{T}}. \tag{3.7}$$

Combining these formulas, one obtains for $\mathbf{u} = \mathbf{Dv}$,

$$\mathbf{u}^T[\mathbf{I} - \mathbf{B}] = \ell^{\mathbf{T}}. \tag{3.8}$$

We will prove that the norm of $\mathbf{B}$, as an operator in $\mathbf{L_2}$, is smaller than $e^{-\mu/2}$ (<1). Hence, $\mathbf{I} - \mathbf{B}$ is invertible in $\mathbf{L_2}$. To this end, we show that for any $i = 1,2,\ldots$,

$$\sum_{j=1}^{i} b_{ij} \leq b_{11} = e^{-\mu}. \tag{3.9}$$

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From definition (3.4),

$$\sum_{j=1}^{i} b_{ij} = \frac{e^{-\mu i}}{i} \sum_{k=0}^{i-1} \frac{(i-k)(\mu i)^k}{k!}; \tag{3.10}$$

so to prove Eq. (3.9), it suffices to demonstrate that

$$\sum_{k=0}^{i-1} \frac{(i-k)(\mu i)^k}{ik!} \le e^{(i-1)\mu} = \sum_{k=0}^{\infty} \frac{[\mu(i-1)]^k}{k!}. \tag{3.11}$$

However, this inequality follows directly from comparison of the coefficients $i^k - ki^{k-1} \le (i-1)^k$, $k = 1, \ldots, i-1$. Since $\sum_{i=j}^{\infty} b_{ij} \le 1$, a theorem by Schur (Halmos [4, Problem 37], with $p_i \equiv 1$, implies that the operator defined by the matrix $\mathbf{B}$ has norm in $\mathbf{L_2}$ less than $e^{-\mu/2} < 1$. Note that $b_{11} = e^{-\mu}$, so that $\|\mathbf{B}\|_2 \ge \mathbf{e}^{-\mu}$.

Since ℓ belongs to the space $\mathbf{L_2}$, the solution $\mathbf{u}$ of Eq. (3.8) is also in $\mathbf{L_2}$ and can be found from the formula

$$\mathbf{u} = \sum_{k=0}^{\infty} (\mathbf{B^T})^{\mathbf{k}} \ell. \tag{3.12}$$

Now we can prove the existence and uniqueness of the stationary distribution π for $\mu > 0$ and $\lambda > 0$. Start with $\mathbf{u}$ given by Eq. (3.12) and put $\mathbf{v} = D^{-1}\mathbf{u}$. By the Cauchy–Schwartz inequality, $\mathbf{v} \in \mathbf{L_1}$. The stationary distribution π is given by

$$\pi_0 = \frac{1}{1 + \|v\|_1}, \qquad \pi_i = \frac{v_i}{1 + \|v\|_1}, \qquad i = 1,2,\ldots. \tag{3.13}$$

The uniqueness of the stationary distribution π follows from the uniqueness of solution for Eq. (3.8). By Theorem 6.9 in Kemeny, Snell, and Knapp [7, p. 135], the Markov chain with the transition probabilities given by Eq. (2.3) is ergodic.

Our argument shows that the operator $\zeta\mathbf{I} - \mathbf{B}$ is invertible if $\zeta > e^{-\mu/2}$. It follows that for such values of ζ, one can find in $\mathbf{L}_2$ the solution $\mathbf{u}$ of the equation

$$\mathbf{u}^T[\zeta\mathbf{I} - \mathbf{B}] = \ell^{\mathbf{T}} \tag{3.14}$$

for any right-hand side ℓ in $\mathbf{L}_2$. This fact implies that one can solve the "drift" inequalities

$$\sum_{y} P(x|y)V(y) \le \zeta V(x) + b\delta_{Cx}, \tag{3.15}$$

with positive V and $C = \{0\}$. It follows from Nummelin [9, p. 90] that our chain is geometrically ergodic, which concludes the proof of the theorem. ■

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The referee suggested a proof of geometric ergodicity by checking Eq. (3.15) with $V(x) = \beta^x$ for some $\beta > 1$:

$$\sum_y P(x|y)V(y) = \sum_y \left(e^{-\lambda} \frac{\lambda^y}{y!} \sum_{k=x}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} + e^{-\mu x} \frac{(\mu x)^{x-y}}{(x-y)!} \mathbf{1}_{\{1 \le y \le x\}} \right) \beta^y$$

$$= e^{\lambda(\beta-1)} \sum_{k=x}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} + \beta^x e^{-\mu x} \sum_{1 \le y \le x} \frac{(\mu x/\beta)^{x-y}}{(x-y)!}$$

$$\le \beta^x \left[e^{-\mu(1-1/\beta)} + \beta^{-x} e^{\lambda(\beta-1)} \sum_{k=x}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} \right]. \tag{3.16}$$

Notice that as $x \to \infty$, $\beta^{-x} e^{\lambda(\beta-1)} \sum_{k=x}^{\infty} e^{-\mu x}((\mu x)^k/k!) \to 0$, and for some $0 < \zeta < 1$, $e^{-\mu(1-1/\beta)} < \zeta$. It follows that Eq. (3.15) holds for some finite x_ζ and positive β where $C = \{k : k \le x_\zeta\}$.

To get a useful approximation to π_1, we use the form of the matrix $\mathbf{B}$ and denote elements of $(\mathbf{I} - \mathbf{B})^{-1} = \sum_{\mathbf{k=0}}^{\infty} \mathbf{B^k}$ by $\alpha_{ij} = \alpha_{ij}(\mu)$, and put $\alpha_j = \sum_{k=1}^{j} \alpha_{jk} k^{-1}$.

THEOREM 2: *The probability π_1 is given by*

$$\pi_1 = \frac{\sum_{k=1}^{\infty} \alpha_{k1} \frac{\lambda^k}{(k-1)!}}{1 + \sum_{k=1}^{\infty} \alpha_k \frac{\lambda^k}{(k-1)!}} \tag{3.17}$$

with an upper bound (3.24). An approximation (3.26) is valid, and a lower bound (3.31) for π_1 holds, leading to approximation (3.36) for $0 < \mu < 1$. Small values of μ and λ are optimal.

PROOF: Using the identity $(I - B)^{-1} = I + B(I - B)^{-1}$, the α_{ij} can be found from the recurrent formula

$$\alpha_{ij} = \delta_{ij} + \sum_{k=j}^{i} \alpha_{kj} b_{ik} = \delta_{ij} + \sum_{k=j}^{i} \alpha_{ik} b_{kj} \tag{3.18}$$

for $i \ge j$. Note that

$$\alpha_{ii} = \frac{1}{1 - b_{ii}} = \frac{1}{1 - e^{-i\mu}}. \tag{3.19}$$

One has

$$u_j = \sum_{k=j}^{\infty} \alpha_{kj} \frac{\lambda^k}{(k-1)!}, \tag{3.20}$$

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so that using Eq. (3.13), we obtain Eq. (3.17). Note that

$$\alpha_j = \frac{1}{j} + \sum_{k=1}^{j} \alpha_k b_{jk} = \frac{1}{j} + \sum_{1 \le i \le k \le j} \alpha_{jk} \frac{b_{ki}}{i}. \tag{3.21}$$

It is easy to see by induction that $\sum_{k=1}^{j-1} b_{jk} k^{-1} \le (1 - b_{jj}) j^{-1}$. If for $k = 1, \ldots, j-1$, $\alpha_{k1} \le \alpha_{11}/k$, then for $j \ge 2$,

$$\alpha_{j1} = \frac{1}{1 - b_{jj}} \sum_{k=1}^{j-1} \alpha_{k1} b_{jk} \le \frac{\alpha_{11}}{1 - b_{jj}} \sum_{k=1}^{j-1} \frac{b_{jk}}{k} \le \frac{\alpha_{11}}{j}. \tag{3.22}$$

Also, as $i\alpha_{ij} \le j\alpha_{jj}$,

$$\alpha_j \le \frac{1}{j} \sum_{k=1}^{j} \alpha_{kk}. \tag{3.23}$$

From Eq. (3.22), we obtain

$$\pi_1 \le \frac{\displaystyle\sum_{k=1}^{\infty} \alpha_{k1} \frac{\lambda^k}{(k-1)!}}{1 + \displaystyle\sum_{k=1}^{\infty} \alpha_{k1} \frac{\lambda^k}{(k-1)!}} \le \frac{\alpha_{11}(e^\lambda - 1)}{1 + \alpha_{11}(e^\lambda - 1)} = \frac{e^\lambda - 1}{e^\lambda - e^{-\mu}}, \tag{3.24}$$

which shows that for this probability to be close to unity, one must have $\mu \approx 0$ or λ must be large. For

$$\pi_1^{(n)} = \frac{\displaystyle\sum_{k=1}^{n} \alpha_{k1} \frac{\lambda^k}{(k-1)!}}{1 + \displaystyle\sum_{k=1}^{n} \alpha_k \frac{\lambda^k}{(k-1)!}}, \tag{3.25}$$

it follows from Eqs. (3.22) and (3.23) that $\lim_{n\to\infty} \pi_1^{(n)} = \pi_1$ and

$$|\pi_1 - \pi_1^{(n)}| \le \frac{e^\lambda - 1}{e^\lambda - e^{-\mu}} \max\left[\gamma(\lambda, n+1), \frac{\lambda}{1 - e^{-\mu}} \gamma(\lambda, n)\right], \tag{3.26}$$

where

$$\gamma(\lambda, k) = e^\lambda - \sum_{i=0}^{k} \frac{\lambda^i}{i!} = \frac{e^\lambda}{(k-1)!} \int_0^\lambda e^{-t} t^{k-1}\, dt, \qquad k \ge 1, \tag{3.27}$$

is the incomplete gamma function.

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An argument similar to that in Eq. (3.23) can be used to demonstrate that for any fixed j, $i\alpha_{ij}$ is a nonincreasing sequence in $i = j, j+1, \ldots$, so that the limit, $\delta_j = \lim_{i\to\infty} i\alpha_{ij}$, exists. Since $\|(\mathbf{I} - \mathbf{B}^{\mathbf{T}})^{-1}\|_2 \le [1 - \|\mathbf{B}\|_2]^{-1}$ and, for a fixed l, b_{lp} is a decreasing sequence in $p \le l$, we get, for $i \ge j+2$ from Eq. (3.18),

$$(1 - b_{ii})|\alpha_{ij+1} - \alpha_{i,j}| = \left| \sum_{k=j+1}^{i-1} \alpha_{kj+1} b_{ik} - \sum_{k=j}^{i-1} \alpha_{kj} b_{ik} \right| \le \frac{(2i - 2j - 1) b_{ij}}{1 - \|\mathbf{B}\|_2}, \tag{3.28}$$

so that

$$|i\alpha_{ij+1} - i\alpha_{i,j}| < \frac{2i^2 b_{ij}}{(1 - e^{-i\mu})(1 - \|\mathbf{B}\|_2)}. \tag{3.29}$$

For $0 < \mu < 1$, $\lim_{i\to\infty} i^2 b_{ij} = 0$, and, therefore, $\delta_j = \delta_{j+1}$ for any j, demonstrating that $\delta_j \equiv \delta$. It follows that

$$u_j = \sum_{k=j}^{\infty} k\alpha_{kj} \frac{\lambda^k}{k!} \ge \delta e^{\lambda} \int_0^{\lambda} \frac{e^{-t} t^{j-1}}{(j-1)!}\, dt. \tag{3.30}$$

Using Eq. (3.23), a lower bound for π_1 is obtained,

$$\begin{aligned} \pi_1 &\ge \frac{\delta(e^{\lambda} - 1)}{1 + \sum_{k=1}^{\infty} \alpha_{kk} \sum_{j=k}^{\infty} \frac{\lambda^j}{j!}} \\ &= \frac{\delta(e^{\lambda} - 1)}{1 + \lambda e^{\lambda} + e^{\lambda} \sum_{k=1}^{\infty} \frac{e^{-k\mu}(1 - e^{-\lambda(1 - e^{-k\mu})})}{1 - e^{-k\mu}}}. \end{aligned} \tag{3.31}$$

To get a formula for δ when $0 < \mu < 1$, recall that for any positive λ,

$$e^{\lambda} - 1 = \sum_{k=1}^{\infty} \frac{d_k u_k}{k} = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=1}^{k} k\alpha_{kj} \frac{d_j}{j}. \tag{3.32}$$

As neither α_{kj} nor d_k depend on λ, it follows that for any $k = 1, 2\ldots$, $\sum_{j=1}^{k} \alpha_{kj} d_j / j = 1/k$, which implies that

$$\delta = \frac{1}{\sum_{k=1}^{\infty} \frac{d_k}{k}} = -\frac{1}{\log(1 - \mu)}. \tag{3.33}$$

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Indeed,

$$\sum_{k=1}^{\infty} \frac{d_k}{k} = \sum_{k=1}^{\infty} \int_0^{\mu k} \frac{e^{-t} t^{k-1}}{k!}\, dt = \int_0^{\mu} \sum_{k=1}^{\infty} \frac{e^{-ks} k^k s^{k-1}}{k!}\, ds$$

$$= \int_0^{\mu} \frac{ds}{1-s} = -\log(1-\mu), \tag{3.34}$$

since, as in the derivation of the expected value of the Borel–Tanner distribution (e.g., formula (16) in Haight and Breuer [3]),

$$\sum_{k=1}^{\infty} \frac{e^{-ks}(ks)^k}{k!} = \frac{s}{1-s}, \qquad 0 < s < 1. \tag{3.35}$$

These results lead to an approximation for π_1; if one replaces u_1 by $\lambda\alpha_{11} + \lambda^2\alpha_{21} + \delta(e^{\lambda} - 1 - \lambda - \lambda^2/2)$ and sets $u_2 \approx \lambda^2\alpha_{22} + \delta(e^{\lambda} - 1 - \lambda - \lambda^2/2)$, $u_j \approx \delta(e^{\lambda} - 1 - \lambda - \lambda^2/2), j \geq 3$. With $Ei(-\lambda) = \int_{\lambda}^{\infty} e^{-t}/t\, dt$ denoting the integral exponential function and C denoting Euler's constant, the approximate formula for π_1 when $0 < \mu < 1$ takes the form

$$\pi_1 = \frac{u_1}{1 + \sum_{j=1}^{\infty} \frac{u_j}{j}}$$

$$\sim \frac{\frac{\lambda}{1-e^{-\mu}} + \frac{\lambda^2 \mu e^{-2\mu}}{(1-e^{-\mu})(1-e^{-2\mu})} - \frac{e^{-\lambda} - 1 - \lambda - \lambda^2/2}{\log(1-\mu)}}{1 + \frac{\lambda}{1-e^{-\mu}} + \frac{\lambda^2}{1-e^{-2\mu}}\left[\frac{1}{2} + \frac{\mu e^{-2\mu}}{1-e^{-\mu}}\right] + \frac{\lambda(\lambda+1)}{\log(1-\mu)} - \frac{e^{\lambda}[C + \log\lambda - Ei(-\lambda)]}{\log(1-\mu)}}. \tag{3.36}$$

For small values of μ,

$$\pi_1 \sim \frac{\sum_{k=1}^{\infty} \frac{\lambda^k}{k!}}{\mu + \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=k}^{\infty} \frac{\lambda^j}{j!}} = \frac{1 - e^{-\lambda}}{\mu e^{-\lambda} + \int_0^{\lambda} \frac{1 - e^{-t}}{t}\, dt}. \tag{3.37}$$

This ratio is a decreasing function of λ; for small $\lambda, \pi_1 \sim \lambda/(\lambda + \mu)$, so that $\pi_1 \to 1$ as $\lambda \downarrow 0, \mu/\lambda \to 0$. Therefore small values of λ and $\mu = o(\lambda)$ are optimal. ■

Theorem 2 gives a method for approximating the probability of interest π_1. Given ϵ, from Eq. (3.26) we determine the level of truncation n. From Eq. (3.25), using the recurrence formulas (3.18) and (3.19), we obtain the approximate value

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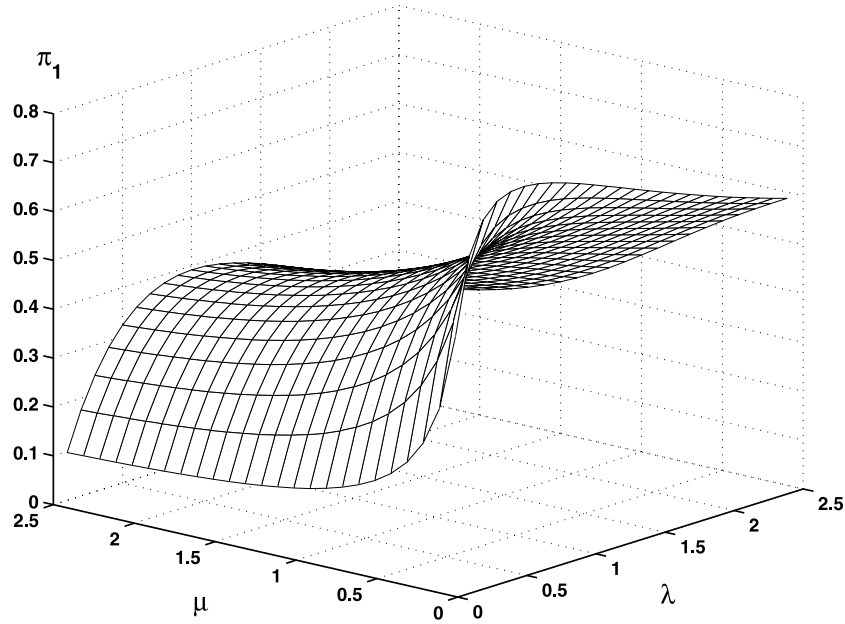

FIGURE 1. The plot of the probability π_1 as a function of λ and μ.

π_1^n of π_1 such that $|\pi_1 - \pi_1^n| \leq \epsilon$. Figure 1 depicts the probability π_1 for $\lambda, \mu \in (0, 2.5)$ using $\epsilon = 10^{-6}$.

The referee has suggested using the geometric distribution as the loss distribution. Thus, assume that $Y_n = \min(X_n, G)$, where G is a random variable with geometric distribution over nonnegative integers with parameter p. Then the inverse of $\mathbf{I} - \mathbf{B}$ can be obtained explicitly. For any load distribution R with the finite first moment, the same technique as in Theorem 1 shows that the stationary distribution is obtained from Eq. (3.13):

$$\pi_n = \frac{P(R = n) + p \sum_{k>n} P(R = k)}{1 - p + pE(R)}. \tag{3.38}$$

In particular,

$$\pi_1 = \frac{P(R = 1) + pP(R \geq 2)}{1 + p(E(R) - 1)}, \tag{3.39}$$

which is an increasing function in p for any load distribution such that $1 - P(R = 0) \geq P(R = 1)E(R)$. The latter condition holds for the Poisson load distribution. Therefore, values of p *close* to 1 and $\lambda \approx 0$ are optimal for this distribution.

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4. DISCUSSION

The bound in Eq. (3.26) suggests that the convergence to the stationary distribution (although geometric) is the slowest for small μ. For this reason, it takes more time to attain the optimal regime than for the stationary distribution associated with other parametric values. Theorem 2 gives a method for computing an approximation π_1^n as defined in Eq. (3.25) of the probability of interest π_1 with any given accuracy ϵ.

Gibson and Seneta [1] and Heyman [5] gave a method for computing a numerical approximation of the steady-state distribution $\boldsymbol{\pi}$ of a Markov chain with states $0,1,\ldots,$ and a transition probabilities matrix $\mathbf{P}$. The idea is to approximate π by π_n, the steady-state distribution of the chain obtained by truncating $\mathbf{P}$, keeping the first n states and normalizing it in a convenient way (e.g., by linear normalization or, by row normalization). The conditions (A1)–(A4) in Heyman [5] hold for our case; conditions (A2)–(A4) are easy to check and (A1) follows from Theorem 1, if its implementation is practical.

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