Finitely many physical measures for sectional-hyperbolic attracting sets and statistical stability

VITOR ARAUJO

Instituto de Matemática e Estatística, Universidade Federal da Bahia,
Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil
(e-mail: vitor.araujo.im.ufba@gmail.com)

Abstract. We show that a sectional-hyperbolic attracting set for a Hölder-C^1 vector field admits finitely many physical/SRB measures whose ergodic basins cover Lebesgue almost all points of the basin of topological attraction. In addition, these physical measures depend continuously on the flow in the C^1 topology, that is, sectional-hyperbolic attracting sets are statistically stable. To prove these results we show that each central-unstable disk in a neighborhood of this class of attracting sets is eventually expanded to contain a ball whose inner radius is uniformly bounded away from zero.

Contents

1. *Introduction and statements of the results*

The term *statistical properties* of a dynamical system refers to the statistical behavior of *typical* trajectories of the system. It is well known that this relates to the properties of the evolution of measures by the dynamics. Statistical properties are often a better object of study than pointwise behavior. In fact, the future behavior of initial data can be unpredictable, but statistical properties are often regular and their description simpler.

Arguably one of the most influential concepts in the theory of dynamical systems has been the notion of physical (or Sinai–Ruelle–Bowen, SRB) measure. We say that an invariant probability measure μ for a flow ϕ_t is *physical* if the set

$$B(\mu) = \left\{ z \in M : \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi(\phi_s(z)) \, ds = \int \psi \, d\mu, \ \forall \psi \in C^0(M, \mathbb{R}) \right\}$$

has non-zero volume, with respect to any volume form on the ambient compact manifold M. The set $B(\mu)$ is by definition the *basin* of μ. It is assumed that time averages of these orbits are observable if the flow models a physical phenomenon.

The study of the existence of these special measures and their statistical properties for uniformly hyperbolic diffeomorphisms and flows has a long and rich history, starting with the works of Sinai, Ruelle and Bowen [17, 18, 47, 48, 52]. Some classes of systems that do not satisfy all the basic assumptions of uniform hyperbolicity have much more recently been shown to possess physical measures: sectional hyperbolicity is a generalization of Smale's notion of Axiom A [53] that allows for the inclusion of equilibria (also known as singularities or steady states) and incorporates the classical Lorenz attractor [29] as well as the geometric Lorenz attractors of [1, 24]. For three-dimensional flows, sectional-hyperbolic attractors are precisely those that are robustly transitive, and they reduce to Axiom A attractors when there are no equilibria [38].

For arbitrary dimensions this notion was established first in [32] and the first concrete example provided by [15]. Sectional-hyperbolic attractors are those robustly transitive attracting sets for which the flow is a star flow in the trapping region, that is, there are no bifurcations of singularities or periodic orbits for all nearby dynamics (also known as 'strongly homogeneous flow'). Again these sets reduce to Axiom A attractors if there are no equilibria.

Sectional-hyperbolic attractors in 3-manifolds were shown to have a unique physical measure in [7, 8] and sectional-hyperbolic attracting sets have finitely many ergodic physical measures whose basins cover a full volume subset of a neighborhood of the attracting set; see [9, 51]. The study of statistical properties of these measures is well developed; among recent works are [3–6, 10, 12, 23, 25, 30, 50].

The existence of a unique physical measure for sectional-hyperbolic attractors for flows in manifolds with any finite dimension was recently shown in [28] using the thermodynamical formalism and assuming certain properties of a stable foliation in a neighborhood of the attracting set, common to the above mentioned works in the three-dimensional setting; see also [33] for a different proof using stochastic stability of such attractors.

Various issues regarding the existence and smoothness of the stable foliation in a neighborhood of sectional-hyperbolic attracting sets are clarified in [4]; a topological foliation always exists, and an analytic proof of smoothness of the foliation for the

classical Lorenz attractor (and nearby attractors) is given in [**4**, **6**]. In [**5**] sufficient conditions are provided for these foliations to have absolutely continuous holonomy maps, a crucial technical feature to obtain many statistical properties in dynamics. For higher differentiability properties of these foliations for geometric Lorenz attractors, see [**54**].

Here we pave the way to further study of statistical properties of sectional-hyperbolic attracting sets. We solve the *basin problem* for sectional-hyperbolic attracting sets, that is, we show that an open dense and full measure subset of points in a neighborhood of these sets is exponentially asymptotic to some orbit inside the set. More precisely, given a neighborhood U of an invariant sectional-hyperbolic attracting set Λ of a smooth flow ϕ_t, there exist $K, \lambda > 0$ and an open and dense subset $W \subset U$ with full Lebesgue measure (Leb$(U \setminus W) = 0$) such that for any given $y \in W$ there exists $x \in \Lambda$ satisfying $d(\phi_t y, \phi_t x) \leq K e^{-\lambda t}$ for all $t > 0$.

Moreover, coupled with recent results from [**20**] on weak limits of time averages for almost all orbits in partially hyperbolic sets with applications to sectional-hyperbolic attracting sets, we complement [**28**] proving the existence of finitely many ergodic physical measures for sectional-hyperbolic attracting sets in any dimension. In addition, the basins of these measures cover a full Lebesgue measure subset of a neighborhood of the sectional-hyperbolic attracting set.

With this in hand, we use recent results from [**40**] on robust entropy expansiveness for sectional-hyperbolic attracting sets to prove that the physical measures depend continuously on the flow, showing that asymptotic time averages for Lebesgue almost all points in a neighborhood of such attracting sets are robust under small perturbations of the dynamics. This is known as *statistical stability* and our proof provides a far-reaching extension of the results already obtained for the 3-flows having geometric Lorenz attractors in [**2**] and the classical Lorenz attractor in [**11**].

1.1. *Preliminary definitions.* Let M be a compact Riemannian manifold with induced distance d and volume form Leb. Let $\mathcal{X}^1(M)$ be the set of C^1 vector fields on M and denote by ϕ_t^G the flow generated by $G \in \mathcal{X}^1(M)$. We say that G is Hölder-C^1 if on any local chart the derivative DG is α-Hölder for some fixed $0 < \alpha < 1$. We write $\mathcal{X}^{1+}(M)$ for the vector space of all Hölder-C^1 vector fields over M.

Given a compact invariant set Λ for $G \in \mathcal{X}^1(M)$, we say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t \in \mathcal{R}} \phi_t(U)$. If U can be chosen so that

$$\text{Closure} (\varphi_t(U)) \subset U \quad \text{for all } t > 0,$$

then we say that Λ is an *attracting set*.

A compact invariant set Λ is *partially hyperbolic* if the tangent bundle over Λ can be written as a continuous $D\phi_t$-invariant sum $T_\Lambda M = E^s \oplus E^{cu}$, where $d_s = \dim E_x^s \geq 1$ and $d_{cu} = \dim E_x^{cu} \geq 2$ for $x \in \Lambda$, and there exist constants $C > 0$, $\lambda \in (0, 1)$ such that for all $x \in \Lambda$, $t \geq 0$, we have:

- *uniform contraction along E^s* ($\|D\phi_t|E_x^s\| \leq C\lambda^t$); and
- *domination of the splitting* ($\|D\phi_t|E_x^s\| \cdot \|D\phi_{-t}|E_{\phi_t x}^{cu}\| \leq C\lambda^t$).

We say that E^s is the *stable bundle* and E^{cu} the *center-unstable bundle*. A *partially hyperbolic attracting set* is a partially hyperbolic set that is also an attracting set.

We say that the center-unstable bundle E^{cu} is *sectional expanding* if for every two-dimensional subspace $P_x \subset E_x^{cu}$,

$$|\det(D\phi_t(x) \mid P_x)| \geq Ke^{\theta t} \quad \text{for all } x \in \Lambda, t \geq 0. \tag{1.1}$$

If $\sigma \in M$ and $G(\sigma) = 0$, then σ is called an *equilibrium* or *singularity* in what follows, and we denote by $\text{Sing}(G)$ the family of all such points. An invariant set is *non-trivial* if it is neither a periodic orbit nor an equilibrium.

We say that a compact invariant set Λ is a *sectional-hyperbolic set* if Λ is partially hyperbolic with sectional expanding center-unstable bundle and all equilibria in Λ are hyperbolic. A sectional-hyperbolic set which is also an attracting set is called a *sectional-hyperbolic attracting set*.

A *singular-hyperbolic set* is a compact invariant set Λ which is partially hyperbolic with volume expanding center-unstable subbundle and all equilibria within the set are hyperbolic. A sectional-hyperbolic set is singular-hyperbolic and both notions coincide if, and only if, $d_{cu} = 2$.

Remark 1.1

(1)   A sectional-hyperbolic set with no equilibria is necessarily a *hyperbolic set*, that is, the center-unstable subbundle admits a splitting $E_x^{cu} = \mathbb{R}\{G(x)\} \oplus E_x^u$ for all $x \in \Lambda$ where E_x^u is uniformly contracting under the time reversed flow; see, for example, [7].

(2)   A sectional-hyperbolic attracting set cannot contain isolated periodic orbits. For otherwise such orbit must be a periodic sink, contradicting volume expansion.

We recall that a subset $\Lambda \subset M$ is *transitive* if it has a full dense orbit, that is, there exists $x \in \Lambda$ such that $\text{Closure}\{\phi_t x : t \geq 0\} = \Lambda = \text{Closure}\{\phi_t x : t \leq 0\}$.

A non-trivial transitive sectional-hyperbolic attracting set is a *sectional-hyperbolic attractor*. For more details on these notions, see, for example, [7] and references therein.

1.2. *Statement of the results.*   The definition of singular hyperbolicity ensures that every invariant probability measure supported in a singular-hyperbolic set is a hyperbolic measure. Moreover, if the vector field is smooth (at least Hölder-C^1), from the proof of [8, Theorem B, §4] or explicitly from [51, Theorem 1.5] we get that *every singular-hyperbolic attracting set admits finitely many* $\mu_1, \ldots, \mu_k$ *ergodic physical/SRB invariant measures; and the union of the ergodic basins of these measures covers a full Lebesgue measure subset of the topological basin of attraction of* Λ *(i.e. Leb* $(U \setminus \bigcup_{i=1}^{k} B(\mu_i)) = 0$*).*

We show here that the same result is true in higher dimensions for sectional-hyperbolic attracting sets.

THEOREM A. *Every sectional-hyperbolic attracting set for a Hölder-C^1 vector field admits finitely many* $\mu_1, \ldots, \mu_k$ *ergodic physical/SRB invariant probability measures. Moreover, the union of the ergodic basins of these measures covers a full Lebesgue measure subset of the topological basin of attraction of* Λ.

In [28] existence and uniqueness of the physical measure were obtained for sectional-hyperbolic *attractors* of C^2 vector fields. We extend the argument from [28], avoiding the use of a dense orbit, taking advantage of the recent results from [20] which hold in the C^{1+} topology.

By robustness of partial hyperbolicity and sectional expansion, given a sectional-hyperbolic attracting set $\Lambda_G(U) = \bigcap_{t>0} \phi_t(U)$ with trapping region U, there exists a neighborhood $\mathcal{U} \subset \mathcal{X}^{1+}(M)$ of G such that U is a trapping region and $\Lambda_Y(U)$ is sectional-hyperbolic for all $Y \in \mathcal{U}$. It is then natural to study the stability of the physical measures under small perturbation of the vector field G.

THEOREM B. *Let $G \in \mathcal{X}^{1+}(M)$ be a vector field with a trapping region U whose attracting set $\Lambda_G(U) = \bigcap_{t>0} \phi_t(U)$ is sectional-hyperbolic. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{X}^{1+}(M)$ of G such that, for each choice of $G_n \in \mathcal{U}$ and μ_n physical measures for G_n supported in U such that $\|G_n - G\|_{C^1} \to 0$ when $n \nearrow \infty$, each weak* accumulation point μ of $(\mu_n)_{n\geq 1}$ is a linear convex combination of the ergodic physical measures of Λ_G provided in Theorem A:*

$$\mu \in \Phi(G) = \left\{ \sum_{i=1}^{k} t_i \mu_i : t_i \geq 0 \text{ and } \sum_{i=1}^{k} t_i = 1 \right\}.$$

In other words, the convex hull $\Phi(G)$ of the ergodic physical measures of a sectional-hyperbolic attracting set depends continuously on the vector field, with respect to the C^1 topology of vector fields and weak* topology of probability measures on a manifold.

Statistical stability means that time averages $\tilde{\psi}^G = \lim_{t \to \infty} (1/t) \int_0^t \psi \circ \phi_s^G \, ds$ of continuous observables $\varphi : U \to \mathbb{R}$, in a neighborhood of the sectional-hyperbolic attracting sets, are well defined Lebesgue almost everywhere in U, depend continuously on the vector field G generating the flow ϕ_t^G, so that we can ensure that $|\tilde{\psi}^G - \tilde{\psi}^{G'}|$ is small as long has $\|G - G'\|_{C^1}$ is small enough.

Theorem B improves both [2] and [11] since, although not dealing with the density of the invariant probability of the quotient map along stable leaves on a global cross-section of the geometric Lorenz attractor, its statement and proof apply to a much larger family of sectional-hyperbolic attracting sets.

In particular, the attracting sets appearing as small perturbations of singular-hyperbolic attractors as in Morales [36], which must have a singular component, are statistical stable whatever the number of singularities involved.

We note that there are many examples of singular-hyperbolic attracting sets, non-transitive and containing non-Lorenz-like singularities; see Figure 1 for an example obtained by conveniently modifying the geometric Lorenz construction, and many others in [37]. Statistical stability follows for all these examples.

Moreover, Theorem B applies to the multidimensional Lorenz attractor described in [15] without further ado.

In addition, the open families of Lorenz-like attractors obtained after bifurcating saddle connections by many authors [21, 27, 34, 35, 39, 44–46, 49] are automatically endowed with statistical stability after Theorem B, that is, in the (generic) unfolding of double
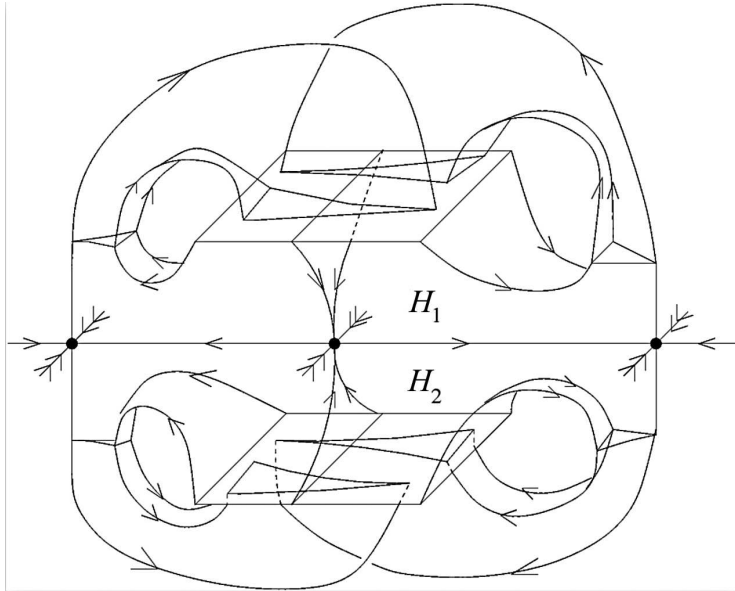
FIGURE 1. Example of a singular-hyperbolic attracting set, non-transitive (in fact, it is the union of two transitive sets indicated by H_1, H_2 above) and containing non-Lorenz like singularities.

(resonant) homoclinic cycle or saddle connections, the physical measure for the ensuing Lorenz-like attractors depends continuously on the parameters.

We mention that in the preprint [33] the authors prove stochastic stability for C^2 *transitive* sectional-hyperbolic attracting sets which, in particular, provides an alternative proof of the existence of a unique SRB measure for C^2 sectional-hyperbolic *attractors*. This is a different kind of stability which is in general unrelated to statistical stability. Whereas statistical stability compares the SRB measures of close vector fields, stochastic stability considers random perturbations of a given vector field, usually though a diffusion, and checks whether the stationary measure for the randomly perturbed vector field converges to some special invariant measure for the original unperturbed vector field when the size of the diffusion vanishes. The results strongly depend on the type of random perturbation chosen; see, for example, [14, Appendix D].

The proofs of Theorems A and B use a construction of adapted cross-sections, generalizing that presented in the 3-flow setting in [8] and in the codimension 2 setting in [5], which has been used to prove many delicate statistical properties of these flows; a similar construction (but built in a different way) of higher-dimensional adapted cross-sections was recently proposed in [19]. This enables us to solve the *basin problem* as follows; see, for example, [13] for a similar but more delicate instance in a highly non-uniformly hyperbolic setting.

For a periodic point p of G we write $\mathcal{O}(p)$ for its compact orbit $\{\phi_t p : t \in \mathbb{R}\} = \{\phi_t p : t \in [0, T_p]\}$ where the minimal $T_p > 0$ satisfying this is the period of any point $q \in \mathcal{O}(p)$. Moreover, we write

$$W_x^s = \{y \in M : d(\phi_t y, \phi_t x) \xrightarrow[t \to +\infty]{} 0\}$$

for the *stable manifold* of $x \in M$, and for a given $\varepsilon_0 > 0$ we write

$$W_x^u(\varepsilon_0) = \{y \in M : d(\phi_{-t}y, \phi_{-t}x) \le \varepsilon_0, \ \forall t \ge 0\}$$

for the *local unstable manifold* of x of size ε_0. There are analogous and dual notions of local stable manifolds and unstable manifolds.

We say that a periodic point p is hyperbolic if $D\phi_{T_p}(p) : T_pM \circlearrowleft$ admits three $D\phi_{T_p}$-invariant subspaces forming a splitting $T_pM = E_p^s \oplus \langle G \rangle \oplus E_p^u$, where $\langle G \rangle = \mathbb{R} \cdot G$ is an eigenspace with eigenvalue 1, E_p^s is contracted and E_p^u expanded. The (un)stable manifold theorem ensures that $W_q^u(\varepsilon_0)$ is an embedded manifold with $T_q W_q^u(\varepsilon_0) = E_q^u$ and W_q^s is an immersed submanifold with $T_q W_q^s = E_q^s$, for each $q \in \mathcal{O}(p)$. We write $W_{\mathcal{O}(p_i)}^u(\varepsilon_0)$ for the union $\bigcup \{W_q^u(\varepsilon_0) : q \in \mathcal{O}(p)\}$ and analogously $W_{\mathcal{O}(p_i)}^s = \bigcup \{W_q^s(\varepsilon_0) : q \in \mathcal{O}(p)\}$. For more details on these notions from hyperbolic dynamics, see, for example, [41].

THEOREM C. *Let $G \in \mathfrak{X}^1(M)$ be a vector field with a trapping region U whose attracting set $\Lambda = \bigcap_{t>0} \phi_t(U)$ is sectional-hyperbolic. Then there are $\varepsilon_0 > 0$ and finitely many (hyperbolic) periodic points $p_1, \ldots, p_l$ of Λ such that*

$$\mathcal{W}^{cs} = \{W_x^s : x \in W_{\mathcal{O}(p_i)}^u(\varepsilon_0); \ i = 1, \ldots, l\}$$

is open and dense in $\mathcal{U} = \{y \in M : d(\phi_t y, \Lambda) \xrightarrow[t \to +\infty]{} 0\} \supset U$. In particular, $\bigcup_i W_{\mathcal{O}(p_i)}^s$ is dense in $\mathcal{U}$.

In addition, if $G \in \mathfrak{X}^{1+}(M)$, then $\mathcal{W}^{cs}$ contains the basin of any physical probability measure supported in Λ and has full volume: $\mathrm{Leb}(\mathcal{U} \backslash \mathcal{W}^{cs}) = 0$.

Recently [56] obtained a similar structure for C^2 sectional-hyperbolic attractors, showing that they are homoclinic classes.

We conjecture that in this setting $\mathcal{W}^{sc}$ is the entire basin, as follows.

Conjecture 1. For any sectional-hyperbolic attracting set Λ the topological basin of attraction coincides with the family of stable manifolds through the points of local unstable leaves of finitely many periodic orbits, that is, $\mathcal{U} = \mathcal{W}^{cs}$.

The following example shows that partially hyperbolic attracting sets which are not sectional-hyperbolic do not necessarily satisfy the conclusions of Theorem C.

Example 1. (Bowen's example flow; see [55] for the not very clear reason for the name) This is a folklore example showing that Birkhoff averages need not exist almost everywhere. Indeed, in the system pictured in Figure 2 time averages only exist for the sources s_3, s_4 and for the set of separatrixes and saddle equilibria $\Lambda = W_1 \cup W_2 \cup W_3 \cup W_4 \cup \{s_1, s_2\}$, which is an attracting set.

The orbit under this flow ϕ_t of every point $z \in \mathbb{S}^1 \times [-1, 1] = M$ not in $\Lambda \cup \{s_3, s_4\}$ accumulates on either side of the separatrixes, as suggested in the figure, if we impose the condition $\lambda_1^- \lambda_2^- > \lambda_1^+ \lambda_2^+$ on the eigenvalues of the saddle fixed points s_1 and s_2; for more specifics on this, see, for example, [55] and references therein.
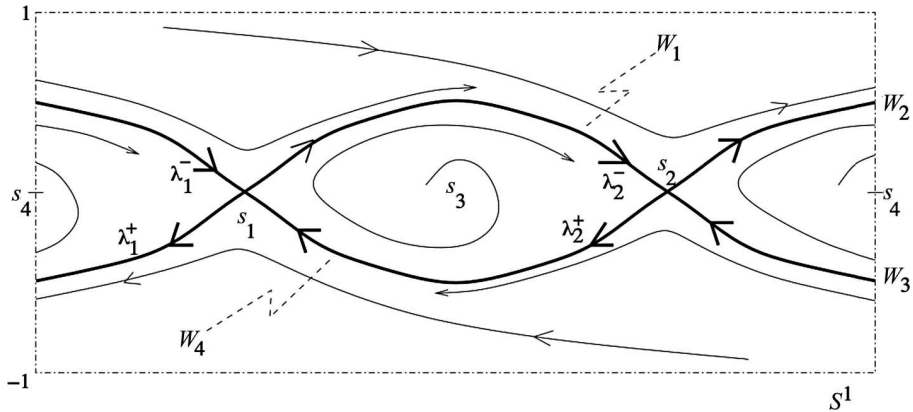
FIGURE 2. A sketch of Bowen's example flow.

Due to the very long sojourn times around s_1 and s_2, future time averages of continuous functions $\varphi : M \to \mathbb{R}$ with $\varphi(s_1) \neq \varphi(s_2)$ do not exist; see, for example, [26]. However, time averages do exist for points in $\Lambda \cup \{s_3, s_4\}$.

Hence, no point in $M \setminus (\Lambda \cup \{s_3, s_4\})$ belongs to the stable manifold of any point of Λ. That is, we have that $W^s(\Lambda) = M \setminus (\Lambda \cup \{s_3, s_4\})$ but the union of the stable manifolds of the points of Λ is simply Λ.

To obtain an example with Lorenz-like singularities, just multiply this system by the north–south flow on $\mathbb{S}^1$ so that the derivative of the flow at the sink in the south pole S has an eigenvalue $\lambda < 0$ so that $\lambda_i^+ + \lambda > 0$, $i = 1, 2$; and then take the attracting set $\Lambda \times \{S\}$ in $M \times \mathbb{S}^1$.

1.3. *Organization of the text.* In §2 we present precise statements of the main properties of sectional-hyperbolic attracting sets together with a precise description of the construction of a family of adapted cross-sections Ξ and a corresponding piecewise smooth and uniformly hyperbolic global Poincaré return map (with singularities) on a subset Ξ'' of Ξ, which might be of independent interest for further work on statistical properties of these systems.

In §3 we consider the basin of a C^1 sectional-hyperbolic attracting set Λ proving the topological part of the statement of Theorem C as a consequence of showing that *every center-unstable disk contains subdisks which are sent, by arbitrarily large iterates of the Poincaré map, to center-unstable disks with inner radius uniformly bounded away from zero, accumulating the local unstable manifold of a hyperbolic periodic orbit,* from a finite subset of such orbits in the attracting set. In §4.1 we obtain as a consequence of the previous result that *every positively invariant subset of Λ containing* Leb-*almost every (a.e.) point of a center-unstable disk must contain a center-unstable disk with uniform inner radius.*

This enables us to complete the proof of Theorem A in §4.2 by using and completing the relevant steps presented in [28] together with the results from §4.1 and the more recent results from [20], under the assumption that the vector field is Hölder-C^1. Following this, a proof of the measure-theoretic part of the statement of Theorem C is presented in §4.3.

Finally, we present a proof of Theorem B on statistical stability in §5, coupling the previous results with robust entropy expansiveness of sectional-hyperbolic attracting sets obtained in [40].

2. *Preliminary results on sectional-hyperbolic attracting sets*

Let G be a C^1 vector field admitting a singular-hyperbolic attracting set Λ with isolating neighborhood U. Given $x \in M$, we denote the *omega-limit* set

$$\omega(x) = \omega_G(x) = \left\{ y \in M : \exists t_n \nearrow \infty \text{ s.t. } \phi_{t_n} x \xrightarrow[n \to \infty]{} y \right\}$$

and the *alpha-limit* set $\alpha(x) = \omega_{-G}(x)$ which are non-empty on a compact ambient space M.

2.1. *Lorenz-like singularities.*

We first recall some properties of sectional-hyperbolic attracting sets, extending some results from [4, 5] which hold for $d_{cu} \geq 2$.

PROPOSITION 2.1. *Let Λ be a sectional-hyperbolic attracting set and let $\sigma \in \Lambda$ be an equilibrium. If there exists $x \in \Lambda \backslash \{\sigma\}$ such that $\sigma \in \omega(x) \cup \alpha(x)$, then σ is generalized Lorenz-like: that is, $DG(\sigma)|E_\sigma^{cu}$ has a real eigenvalue λ^s and $\lambda^u = \inf\{\text{Re}(\lambda) : \lambda \in \text{sp}(DG(\sigma)), \text{Re}(\lambda) \geq 0\}$ satisfies $-\lambda^u < \lambda^s < 0 < \lambda^u$ and so the index of σ is $\dim E_\sigma^s = d_s + 1$.*

Remark 2.2
(1) If $\sigma \in \text{Sing}(G) \cap \Lambda$ is a generalized Lorenz-like singularity and γ_σ^s is its local stable manifold, then $T_w \gamma_\sigma^s = E_w^{cs} = E_w^s \oplus \mathbb{R} \cdot \{G(w)\}$ at $w \in \gamma_\sigma^s \backslash \{\sigma\}$ since $T\gamma_\sigma^s$ is $D\phi_t$-invariant and contains $G(w)$ (because γ_σ^s is ϕ_t-invariant) and the dimensions coincide.
(2) If an equilibrium $\sigma \in \text{Sing}(G) \cap \Lambda$ is not generalized Lorenz-like, then σ is not in the limit set of $\Lambda \backslash \{\sigma\}$, that is, there is no $x \in \Lambda \backslash \{\sigma\}$ such that $\sigma \in \alpha(x) \cup \omega(x)$. An example is provided by the pair of equilibria of the Lorenz system of equations away from the origin: these are saddles with an expanding complex eigenvalue which belong to the attracting set of the trapping ellipsoid already known to E. Lorenz; see, for example, [7, §3.3] and references therein.

Proof of Proposition 2.1. It follows from sectional hyperbolicity that σ is a hyperbolic saddle and that at most d_{cu} eigenvalues have positive real part. If there are only $d_{cu} - 1$ such eigenvalues, then the constraints on λ^s and λ^u follow from sectional expansion.

Let γ be the local stable manifold for σ. If $\sigma \in \omega(x) \cap \alpha(x)$ for some $x \in \Lambda \backslash \{x\}$, it remains to rule out the case $\dim \gamma = d - d_{cu} = d_s$.

In this case, $T_p \gamma = E_p^s$ for all $p \in \gamma \cap \Lambda$ and, in particular, $G(p) \in E_p^s$. On the one hand, $G(p) \in E_p^{cu}$ (see, for example, [7, Lemma 6.1]), so we deduce that $G(p) = 0$ for all $p \in \gamma \cap \Lambda$ and so $\gamma \cap \Lambda = \{\sigma\}$.

On the other hand, if $\sigma \in \omega(x)$ (the case $\sigma \in \alpha(x)$ is analogous), then by the local behavior of orbits near hyperbolic saddles, there exists $p \in (\gamma \backslash \{\sigma\}) \cap \omega(x) \subset (\gamma \backslash \{\sigma\}) \cap \Lambda$ which, as we have seen, is impossible. $\square$

2.2. *Extension of the stable bundle and center-unstable cone fields.* Let $\mathcal{D}^k$ denote the k-dimensional open unit disk and let $\mathrm{Emb}^r(\mathcal{D}^k, M)$ denote the set of C^r embeddings $\psi : \mathcal{D}^k \to M$ endowed with the C^r distance. We say that *the image of any such embedding is a C^r k-dimensional disk.*

PROPOSITION 2.3 [**4**, Proposition 3.2, Theorem 4.2 and Lemma 4.8]. *Let Λ be a partially hyperbolic attracting set.*

(1) *The stable bundle E^s over Λ extends to a continuous uniformly contracting $D\phi_t$-invariant bundle E^s on an open positively invariant neighborhood U_0 of Λ.*

(2) *There exists a constant $\lambda \in (0, 1)$, such that the following statements hold.*

   (a) *For every point $x \in U_0$ there is a C^r embedded d_s-dimensional disk $W_x^s \subset M$, with $x \in W_x^s$, such that $T_x W_x^s = E_x^s$, $\phi_t(W_x^s) \subset W_{\phi_t x}^s$ and $d(\phi_t x, \phi_t y) \leq \lambda^t d(x, y)$ for all $y \in W_x^s$, $t \geq 0$ and $n \geq 1$.*

   (b) *The disks W_x^s depend continuously on x in the C^0 topology: there is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathcal{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathcal{D}^{d_s}) = W_x^s$. Moreover, there exists $L > 0$ such that $\mathrm{Lip}\, \gamma(x) \leq L$ for all $x \in U_0$.*

   (c) *The family of disks $\mathcal{F}^s = \{W_x^s : x \in U_0\}$ defines a topological foliation $\mathcal{W}^s$ of U_0: every $x_0 \in U_0$ admits a neighborhood $V \subset U_0$ and a homeomorphism $\psi : V \to \mathbb{R}^{d_s} \times \mathbb{R}^{d_{cu}}$ so that $\psi(W_x^s) = \pi_s^{-1}\{\pi_s(\psi(x))\}$ where $\pi^s : \mathbb{R}^{d_s} \times \mathbb{R}^{d_{cu}} \to \mathbb{R}^{d_s}$ is the canonical projection.*

Remark 2.4. For any two close enough d_{cu}-disks D_1, D_2 contained in U_0 and transverse to $\mathcal{F}^s$ there exists an open subset $\hat{D}_1$ of D_1 such that $W_x^s \cap D_2$ is a singleton. This defines the *holonomy map* $h : \hat{D}_1 \to D_2$, $\hat{D}_1 \ni x \mapsto W_x^s \cap D_2$ and Proposition 2.3 ensures that h is continuous.

The splitting $T_\Lambda M = E^s \oplus E^{cu}$ extends continuously to a splitting $T_{U_0} M = E^s \oplus E^{cu}$ where E^s is the invariant uniformly contracting bundle in Proposition 2.3 (however, E^{cu} is not invariant in general). Given $a > 0$ and $x \in U_0$, we define the center-unstable cone field as $\mathcal{C}_x^{cu}(a) = \{v = v^s + v^{cu} \in E_x^s \oplus E_x^{cu} : \|v^s\| \leq a\|v^{cu}\|\}$.

PROPOSITION 2.5. *Let Λ be a partially hyperbolic attracting set.*

(1) *There exists $T_0 > 0$ such that for any $a > 0$, after possibly shrinking U_0,*
   $D\phi_t \cdot \mathcal{C}_x^{cu}(a) \subset \mathcal{C}_{\phi_t x}^{cu}(a)$ *for all $t \geq T_0$, $x \in U_0$.*

(2) *Let $\lambda_1 \in (0, 1)$ be given. After possibly increasing T_0 and shrinking U_0, there exist constants $K, \theta > 0$ such that $|\det(D\phi_t|P_x)| \geq K\, e^{\theta t}$ for each two-dimensional subspace $P_x \subset E_x^{cu}$ and all $x \in U_0$, $t \geq 0$.*

Proof. For item (1) see [**4**, Proposition 3.1]. Item (2) follows from the robustness of sectional expansion; see [**5**, Proposition 2.10] with straightforward adaptation to area expansion along any two-dimensional subspace of E_x^{cu}. $\square$

2.3. *Global Poincaré map on adapted cross-sections.* We assume that Λ is a partially hyperbolic attracting set and recall how to construct a piecewise smooth Poincaré map $f : \Xi \to \Xi$ preserving a contracting stable foliation $\mathcal{W}^s(\Xi)$. This largely follows [**8**] (see also [**7**, Ch. 6]) and [**5**, §3] with slight modifications to account for the higher-dimensional setup.

We write $\rho_0 > 0$ for the injectivity radius of the exponential map $\exp_z : T_z M \to M$ for all $z \in U$, so that $\exp_z \mid B_z(0, \rho_0) : B_z(0, \rho_0) \to M$, $v \mapsto \exp_z v$ is a diffeomorphism with $B_z(0, \rho_0) = \{v \in T_z M : \|v\| \le \rho_0\}$ and $D \exp_z(0) = \mathrm{Id}$ and also $d(z, \exp_z(v)) = \|v\|$ for all $v \in B_z(0, \rho_0)$.

2.3.1. *Construction of a global adapted cross-section.* Let $y \in \Lambda$ be a regular point $(G(y) \ne \vec{0})$. Then there exists an open flow box $V_y \subset U_0$ containing y. That is, if we fix $\varepsilon_0 \in (0, 1)$ small, then we can find a diffeomorphism $\chi : \mathcal{D}^{d-1} \times (-\varepsilon_0, \varepsilon_0) \to V_y$ with $\chi(0, 0) = y$ such that $\chi^{-1} \circ \phi_t \circ \chi(z, s) = (z, s + t)$. Define the cross-section $\Sigma_y = \chi(\mathcal{D}^{d-1} \times \{0\})$.

Remark 2.6. We assume that $\Sigma_y \subset \exp_y(B_y(0, \rho_0/3) \cap G(z)^\perp)$ and $\|D(\exp_y)_x^{-1}\| \le 2$ for all $x \in \Sigma_y$ without loss of generality.

For each $x \in \Sigma_y$, let $W_x^s(\Sigma_y) = \bigcup_{|t| < \varepsilon_0} \phi_t(W_x^s) \cap \Sigma_y$. This defines a topological foliation $\mathcal{W}^s(\Sigma_y)$ of Σ_y. We can also assume that Σ_y is diffeomorphic to $\mathcal{D}^{d_{cu}-1} \times \mathcal{D}^{d_s}$ by reducing the size of the Σ_y if needed. The stable boundary $\partial^s \Sigma_y \cong \partial \mathcal{D}^{d_{cu}-1} \times \mathcal{D}^{d_s} \cong \mathbb{S}^{d_{cu}-2} \times \mathcal{D}^{d_s}$ is a regular topological manifold homeomorphic to a cylinder of stable leaves, since $\mathcal{W}^s$ is a topological foliation; that is, $\cong$ denotes only the existence of a homeomorphism, and the subspace topology of $\partial^s \Sigma_y$ induced by M coincides with the manifold topology.

Let $\mathcal{D}_a^{d_s}$ denote the open disk of radius $a \in (0, 1]$ in $\mathcal{R}^{d_s}$. Define the *sub-cross-section* $\Sigma_y(a) \cong \mathcal{D}^{d_{cu}-1} \times \mathcal{D}_a^{d_s}$, and the corresponding sub-flow box $V_y(a) \cong \Sigma_y(a) \times (-\varepsilon_0, \varepsilon_0)$ consisting of trajectories in V_y which pass through $\Sigma_y(a)$. In what follows we fix $a_0 = 3/4$.

For each equilibrium $\sigma \in \Lambda$, we let V_σ be an open neighborhood of σ on which the flow is locally conjugated by a homeomorphism to a linear flow (Hartman–Grobman theorem). Let γ_σ^s and γ_σ^u denote the local stable and unstable manifolds of σ within V_σ; trajectories starting in V_σ remain in V_σ for all future time if and only if they lie in γ_σ^s.

Define $V_0 = \bigcup_{\sigma \in \mathrm{Sing}(G) \cap U} V_\sigma$. We shrink the neighborhoods V_σ so that they are disjoint, $\Lambda \not\subset V_0$, and $\gamma_\sigma^u \cap \partial V_\sigma \subset V_y(a_0)$ for some regular point $y = y(\sigma)$.

By compactness of Λ, there exist $\ell \in \mathbb{Z}^+$ and regular points $y_1, \ldots, y_\ell \in \Lambda$ such that $\Lambda \setminus V_0 \subset \bigcup_{j=1}^\ell V_{y_j}(a_0)$. We enlarge the set $\{y_j\}$ to include the points $y(\sigma)$ mentioned above, adjust the positions of the cross-sections Σ_{y_j} if necessary to ensure that they are disjoint, and define the global cross-section $\Xi = \bigcup_{j=1}^\ell \Sigma_{y_j}$ and its smaller version $\Xi(a) = \bigcup_{j=1}^\ell \Sigma_{y_j}(a)$ for each $a \in (0, 1)$.

In what follows we modify the choices of U_0 and T_0. However, V_{y_j}, Σ_{y_j} and Ξ remain unchanged from now on and correspond to our current choice of U_0 and T_0. All subsequent choices will be labeled $U_1 \subset U_0$ and $T_1 \ge T_0$. In particular, $U_1 \subset V_0 \cup \bigcup_{j=1}^\ell V_{y_j}(a_0)$. We set $\delta_0 = d(\partial \Xi, \partial \Xi(a_0)) > 0$ where $\partial \Xi(a)$ is the boundary of the submanifold $\Xi(a)$ of M, $a \in (0, 1]$, and $\Xi = \Xi(1)$.

2.3.2. *The Poincaré map.* By Proposition 2.3, for any $\delta > 0$ we can choose $T_1 \ge T_0$ such that $\mathrm{diam}\,\phi_t(W_x^s(\Sigma_{y_j})) < \delta$, for all $x \in \Sigma_{y_j}$, $j = 1, \ldots, \ell$ and $t > T_1$. We fix

$T_1 = T_1(\delta_0)$ for $\delta = \delta_0 = d(\partial \Xi, \partial \Xi(a_0))$ in what follows and define $\Gamma_0 = \{x \in \Xi : \phi_{T_1+1}(x) \in \bigcup_{\sigma \in \mathrm{Sing}(G) \cap U_0}(\gamma^s_\sigma \setminus \{\sigma\})\}$ and $\Xi' = \Xi \setminus \Gamma_0$. If $x \in \Xi'$, then $\phi_{T_1+1}(x)$ cannot remain inside V_0 so there exist $t > T_1 + 1$ and $j = 1, \ldots, \ell$ such that $\phi_t x \in V_{y_j}(a_0)$. Since $\varepsilon_0 < 1$, there exists $t > T_1$ such that $\phi_t x \in \Sigma_{y_j}(a_0)$.

For each $\Sigma = \Sigma_{y_j} \in \Xi$, we choose a center-unstable disk W_Σ which crosses Σ and is transversal to $\mathcal{W}^s(\Sigma)$, that is, every stable leaf $W^s_x(\Sigma)$ intersects W_Σ transversely at only one point, for each $x \in \Sigma$. Then, *for every given $x \in W_\Sigma \cap \Xi'$*, we define

$$f(x) = \phi_{\tau(x)}(x) \quad \text{where } \tau(x) = \inf \left\{ t > T_1 : \phi_t x \in \bigcup_{j=1}^{\ell} \text{Closure } \Sigma_{y_j}(a_0) \right\}. \quad (2.1)$$

We note that by the choice of $T_1 = T_1(\delta_0)$ we have diam $\phi_{\tau(x)}(W^s_x(\Sigma)) < \delta_0$ and so the disk $\phi_{\tau(x)}(W^s_x(\Sigma))$, although not necessarily contained in any Σ_{y_j}, is certainly contained in some V_{y_j} by construction. Thus we can define

$$f(y) = \phi_{\tau(y)}(y) \quad \text{where } \tau(y) = \inf \left\{ t > T_1 : \phi_t y \in \bigcup_{j=1}^{\ell} \Sigma_{y_j} \right\} \quad (2.2)$$

for each $y \in \mathcal{W}^s_x(\Sigma)$. This defines τ and f on Ξ'.

We define the topological foliation $\mathcal{W}^s(\Xi) = \bigcup_{j=1}^{\ell} \mathcal{W}^s(\Sigma_{y_j})$ of Ξ with leaves $W^s_x(\Xi)$ passing through each $x \in \Xi$. From the uniform contraction of stable leaves together with the choice of δ_0, $T_1(\delta_0)$ and the definition of $\mathcal{W}^s(\Xi)$ and flow invariance of $\mathcal{W}^s$ we obtain the following proposition.

PROPOSITION 2.7 [**5**, Proposition 3.4]. *For large enough $T_1 > T_0$, $f(W^s_x(\Xi)) \subset W^s_{fx}(\Xi)$ for all $x \in \Xi'$.*

Remark 2.8 The definition of τ in [**5**] is pointwise, namely, $\tau(x) = \inf\{t > T_1 : \phi_t x \in \bigcup_{j=1}^{\ell} \text{Closure } \Sigma_{y_j}(a_0)\}$, and this allows discontinuities of the return time and return map along stable leaves, that is, a pair of indexes $i \neq j$ and of close points $w, z \in W^s_x$ so that $\phi_{\tau(w)}(w) \in \text{Closure } \Sigma_{y_j}(a_0)$ and $\phi_{\tau(z)}(z) \in \text{Closure } \Sigma_{y_i}(a_0)$. So the statement of [**5**, Proposition 3.4] (which is analogous to Proposition 2.7 with $d_{cu} = 2$) does not make sense in this setup, since we would have $f(W^s_x)$ with elements in distinct cross-sections.

The definition of τ first on the points of a fixed collection of center-unstable leaves (2.1), and then the smooth extension to the stable leaves through each of these points (2.2), provides for the crucial property stated in Proposition 2.7.

The rest of the features of the global Poincaré map f stated and used in [**4**, **5**, **9**] remain valid with the same proofs.

In this way we obtain a piecewise C^r global Poincaré map $f : \Xi' \to \Xi = \bigcup_{j=1}^{\ell} \Sigma_j(a_0)$ with piecewise C^r roof function $\tau : \Xi' \to [T_1, \infty)$, and deduce the following standard result.

LEMMA 2.9 [**5**, Lemma 3.2]. *If Σ_y contains no equilibria (i.e. $\Gamma_0 \cap \Sigma_y = \emptyset$), then $\tau \mid \Sigma_y \leq T_1 + 2$. In general, there is $C > 0$ such that $\tau(x) \leq -C \log \text{dist}(x, \Gamma_0)$ for all $x \in \Xi'$; in particular, $\tau(x) \to \infty$ as $\text{dist}(x, \Gamma_0) \to 0$.*

We define $\partial^s \Xi(a_0) = \bigcup_{j=1}^{\ell} \partial^s \Sigma_{y_j}(a_0)$ and $\Gamma_1 = \{x \in \Xi' : fx \in \partial^s \Xi(a_0)\}$ and then set $\Gamma = \Gamma_0 \cup \Gamma_1$. Clearly $\Gamma_0 \cap \Gamma_1 = \emptyset$.

LEMMA 2.10

(1) Γ_0 *is a d_s-submanifold of Ξ given by a finite union of stable leaves $W_{s_i}(\Xi)$, $i = 1, \ldots, k$; and*

(2) Γ_1 *is a regular embedded $(d-2)$-topological submanifold foliated by stable leaves from $\mathcal{W}^s(\Xi)$ with finitely many connected components.*

Remark 2.11. Note that Γ_0 is a (smooth) submanifold of Ξ with codimension $d_{cu} - 1$, so it separates Ξ only if $d_{cu} = 2$; while Γ_1 is a regular topological codimension 1 submanifold of Ξ and so it separates Ξ.

Proof. It is clear that $W_x^s(\Xi) \subset \Gamma$ for all $x \in \Gamma$, so Γ is foliated by stable leaves. We claim that Γ is precisely the set of those points of Ξ which are sent to the boundary of Ξ or never visit Ξ in the future.

Indeed, if $x_0 \in \Xi' \backslash \Gamma_1$, then $fx_0 = \phi_{\tau(x_0)}(x_0) \in \Sigma'$ for some $\Sigma' \in \Xi(a_0) = \{\Sigma_{y_j}(a_0)\}$. For x close to x_0, it follows from continuity of the flow that $fx \in \Sigma'$ (with $\tau(x)$ close to $\tau(x_0)$). Hence $x \in \Xi' \backslash \Gamma_1$ and, since $\Xi' = \Xi \backslash \Gamma_0$, the claim is proved and, moreover, Γ is closed.

For item (1), we note that $\Gamma_0 \subset \Xi \cap \phi_{[0,T_1+1]}^{-1}(\bigcup_\sigma \gamma_\sigma^s)$ and we may assume without loss of generality that the above union comprises only generalized Lorenz-like equilibria; cf. Remark 2.2(2). Hence $T_w \gamma_\sigma^s = E_w^{cs}$ for $w \in \gamma_\sigma^s \backslash \{\sigma\}$; see Remark 2.2(1). Thus Γ_0 is contained in the transversal intersection between a compact $(d_s + 1)$-submanifold and a compact $(d-1)$-manifold, so Γ_0 is a compact differentiable d_s-submanifold of M and Ξ. In addition, since Γ_0 is foliated by stable leaves which are d_s-dimensional, Γ_0 has only finitely many connected components in Ξ.

For item (2), note that for each $x \in \Gamma_1$ we have that $fx \in \partial \Sigma_j(a_0) \subset \Sigma_j$. Thus there exists a neighborhood W_x of x in Ξ and V_{fx} of fx in Σ_j such that $f \mid W_x : W_x \to V_{fx}$ is a diffeomorphism. Hence $\Gamma_1 \cap W_x = (f \mid W_x)^{-1}(V_x \cap \partial^s \Sigma(a_0))$ is homeomorphic to a $(d_{cu} - 2 + d_s)$-dimensional disk. Moreover, this shows that the topology of Γ_1 is the same as the subspace topology induced by the topology of Ξ. We conclude that Γ_1 is a regular topological $(d-2)$-dimensional submanifold.

It remains to rule out the possibility of existence of infinitely many connected components Γ_1^m, $m \in \mathbb{Z}^+$, of Γ_1 in Ξ. Since Ξ contains finitely many sections only, there exist cross-sections Σ_j, Σ_i in Ξ and, taking a subsequence if necessary, an accumulation set $\tilde{\Gamma} = \lim_m \Gamma_1^m$ within $\text{Closure}(\Sigma_j)$ so that $f(\Gamma_1^m) \subset \partial^s \Sigma_i(a_0)$ for all $m \geq 1$. By the continuity of the stable foliation, $\tilde{\Gamma}$ is a union of stable leaves.

We claim that the Poincaré times $\tau(x_m)$ for $x_m \in \Gamma_1^m$, $m \geq 1$, are uniformly bounded from above. For otherwise the trajectory $\phi_{[0,\tau(x_m)]}(x_m)$ intersects V_σ for some $\sigma \in \text{Sing}(G) \cap U$ and accumulates σ. Hence, by the local behavior of trajectories near saddles and the choice of the cross-sections near V_σ, we get that $\tilde{\Gamma} \subset \Sigma_i(a_0)$ is not contained in the boundary of the cross-section. This contradiction proves the claim. Let T be an upper bound for $\tau(x_m)$.

Then, for an accumulation point $x \in \tilde{\Gamma}$ of $(x_m)_{m \geq 1}$, we have that the trajectories $\phi_{[0,T]}(x_m)$ converge in the C^1 topology (taking a subsequence if necessary) to a limit curve $\phi_{[0,T]}(x)$ and so $fx = \phi_{\tau(x)}(x) \in \partial^s \Sigma_i(a_0)$. Thus we can find neighborhoods W_x of x and V_{fx} of fx in Ξ such that for arbitrarily large m we have that $f \mid W_x : W_x \to V_{fx}$ is a diffeomorphism and $\Gamma_1 \cap W_x = (f \mid W_x)^{-1}(V_x \cap \partial^s \Sigma_i(a_0))$, which contradicts the regularity of Γ_1 as topological submanifold.

This concludes the proof of item (2) and the lemma. $\qquad \square$

Let us set $\Xi'' = \Xi(a_0) \backslash \Gamma$ from now on. Then $\Xi'' = S_1 \cup \cdots \cup S_m$ for some $m \geq 1$, where each S_i is a connected *smooth strip*, homeomorphic to either (i) $\mathcal{D}^{d_{cu}} \times \mathcal{D}^{d_s}$ if $\Gamma_0 \cap \text{Closure } S_i = \emptyset$; or (ii) $\mathcal{D}^{d_{cu}} \times (\mathcal{D}^{d_s} \backslash \{0\})$ otherwise. The latter are *singular (smooth) strips*.

We note that $f \mid S_i : S_i \to \Xi(a_0)$ is a diffeomorphism onto its image, $\tau \mid S_i : S_i \to [T_1, \infty)$ is smooth for each i, and $\tau \mid S_i \leq T_1 + 2$ on non-singular strips S_i and also on a neighborhood of $\partial^s(S_i \cup \Gamma_0)$ for singular strips S_i. The foliation $\mathcal{W}^s(\Xi)$ restricts to a foliation $\mathcal{W}^s(S_i)$ on each S_i.

Remark 2.12. In what follows it may be necessary to increase T_1, leading to changes to f, τ, Γ and $\{S_i\}$ (and the constant C in Lemma 2.9). However, the global cross-section $\Xi = \bigcup \Sigma_{y_j}$ is fixed throughout the argument.

Remark 2.13. Since f sends Ξ'' into $\Xi = \Xi(1)$, there are *smooth extensions* $\widetilde{f}_i : \widetilde{S}_i \to \Xi$ of $f \mid S_i : S_i \to \Xi$, where $\widetilde{S}_i \supset \text{Closure}(S_i) \backslash \Gamma_0$.

2.4. *Hyperbolicity of the global Poincaré map.* We assume from now on that Λ is a sectional-hyperbolic attracting set with $d_{cu} > 2$ and proceed to show that, for large enough $T_1 > 1$, the global Poincaré map $f : \Xi'' \to \Xi$ is *piecewise* uniformly hyperbolic (with discontinuities and singularities).

2.4.1. *Hyperbolicity at each smooth strip.* Let $S \in \{S_i\}$ be one of the smooth strips. Then there are cross-sections $\Sigma, \widetilde{\Sigma} \in \Xi$ such that $S \subset \Sigma$ and $f(\Sigma) \subset \widetilde{\Sigma}$. The splitting $T_{U_0}M = E^s \oplus E^{cu}$ induces the continuous splitting $T\Sigma = E^s(\Sigma) \oplus E^u(\Sigma)$, where $E_x^s(\Sigma) = (E_x^s \oplus \mathbb{R}\{G(x)\}) \cap T_x\Sigma$ and $E_x^u(\Sigma) = E_x^{cu} \cap T_x\Sigma$ for $x \in \Sigma$; and analogous definitions apply to $\widetilde{\Sigma}$.

PROPOSITION 2.14. *The splitting $T\Sigma = E^s(\Sigma) \oplus E^u(\Sigma)$ is invariant, that is, $Df \cdot E_x^s(\Sigma) = E_{fx}^s(\widetilde{\Sigma})$ for all $x \in S$, and $Df \cdot E_x^u(\Sigma) = E_{fx}^u(\widetilde{\Sigma})$ for all $x \in \Lambda \cap S$. It is also uniformly hyperbolic, that is, for each given $\lambda_1 \in (0, 1)$ there exists $T_1 > 0$ such that if $\inf \tau > T_1$, then $\|Df \mid E_x^s(\Sigma)\| \leq \lambda_1$ for each $x \in S$; and $\|(Df \mid E_x^u(\Sigma))^{-1}\| \geq \lambda_1^{-1}$ for all $x \in S \cap \Lambda$.*

Moreover, there exists $0 < \widetilde{\lambda}_1 < \lambda_1$ such that, for all x on a non-singular strip S, or for x on a neighborhood of $\partial^s(S \cup \Gamma_0)$ of a singular strip S, we have $\widetilde{\lambda}_1 < \|(Df \mid E_x^s(\Sigma))^{-1}\|$ and $\|Df \mid E_x^u(\Sigma)\| < \widetilde{\lambda}_1^{-1}$.

Proof. See [**5**, Proposition 4.1], with straightforward adaptation to area expansion along each two-dimensional within $E_x^u(\Sigma)$ in order to obtain uniform expansion; cf. [**7**, Lemma 8.25]. The last statement follows from the boundedness of τ on the designated domains; cf. Lemma 2.9. □

2.4.2. *Hyperbolicity of the extensions of the Poincaré maps at smooth strips.* For a given $a > 0$, $x \in \Sigma$ and $\Sigma \in \Xi$, we define the *unstable cone field* at x as $\mathcal{C}_x^u(\Sigma, a) = \{w = w^s + w^u \in E_x^s(\Sigma) \oplus E_x^u(\Sigma) : \|w^s\| \leq a\|w^u\|\}$.

Remark 2.15. We assume that $\mathcal{C}_x^u(\Sigma_y, a) \subset D(\exp_y)_{\exp_y^{-1} x} \cdot \mathcal{C}_y^u(\Sigma_y, 2a)$ for all $x \in \Sigma_y$ and each $\Sigma_y \in \Xi$ without loss of generality; recall Remark 2.6. Consequently, letting $\pi^u : E_y^s(\Sigma_y) \oplus E_y^u(\Sigma_y) \to E_y^u(\Sigma_y)$ be the canonical projection, we get $\|\pi^u w\|/\|w\| \in (1 - 2a, 1 + 2a)$ for all $w \in \mathcal{C}_x^u(\Sigma_y, a)$, where we implicitly identify $\mathcal{C}_x^u(\Sigma_y, a)$ with a subcone of $\mathcal{C}_y^u(\Sigma_y, 2a)$, for $x \in \Sigma_y$ and $\Sigma_y \in \Xi$.

PROPOSITION 2.16. *For any $a > 0$, $\lambda_1 \in (0, 1)$, we can increase T_1 and shrink U_1 such that, if $\inf \tau > T_1$ and $x \in S$ and $S, S' \in \{S_i\}$ so that $f x \in S'$, then:*
- $Df(x) \cdot \mathcal{C}_x^u(S, a) \subset \mathcal{C}_{fx}^u(S', a);$ *and*
- $\|Df(x)w\| \geq \|\pi^u Df(x)w\| \geq \lambda_1^{-1}\|w\|$ *for all $w \in \mathcal{C}_x^u(S, a)$.*

Moreover, $\|Df(x)w\| \leq \widetilde{\lambda_1}^{-1}\|w\|$ for x in a non-singular S or x in a neighborhood of $\partial^s(S \cup \Gamma_0)$ for a singular S.

Proof. See [**5**, Proposition 4.2]; use $\widetilde{\lambda_1}$ from Proposition 2.14 and the estimate on $\|\pi^u\|$ from Remark 2.15. □

Considering the union of the smooth strips S, the previous results shows that we obtain a global continuous uniformly hyperbolic splitting $T\Xi'' = E^s(\Xi) \oplus E^u(\Xi)$ in the following sense.

THEOREM 2.17. *For given $a > 0$ and $\lambda_1 \in (0, 1)$, we obtain a global Poincaré map f so that the stable bundle $E^s(\Xi)$ and the restricted splitting $T_\Lambda \Xi'' = E_\Lambda^s(\Xi) \oplus E_\Lambda^u(\Xi)$ are Df-invariant; and $Df \cdot \mathcal{C}_x^u(\Xi, a) \subset \mathcal{C}_{fx}^u(\Xi, a)$ and $\|\pi_u Df(x)w\| \geq \lambda_1^{-1}\|w\|$ for all $x \in \Xi''$ and $w \in \mathcal{C}_x^u(\Xi, a)$.*

Remark 2.18. The extensions $\widetilde{f_i} : \widetilde{S_i} \to \Xi$ of $f \mid S_i : S_i \to \Xi(a_0)$ mentioned in Remark 2.13 are such that on $\widetilde{S_i} \supset \text{Closure}(S_i) \backslash \Gamma_0$ the map $\widetilde{f_i}$ behaves as f in Propositions 2.14 and 2.16. In particular, $\delta_1 = d(S_i, \partial\widetilde{S_i}) \geq \widetilde{\lambda_1} \cdot d(\Xi(a_0), \Xi) = \widetilde{\lambda_1}\delta_0$.

3. *The basin of sectional-hyperbolic attracting sets*

Here we prove the topological part of the statement of Theorem C using first the following technical result. The measure-theoretic part is dealt with in the next section; see §4.3.

THEOREM 3.1. *There are finitely many (hyperbolic) periodic points $p_1, \ldots, p_l$ of Λ in Ξ such that $\mathcal{W}^s = \{W_x^s(\Xi) : x \in W_{p_i}^u(\Xi); i = 1, \ldots, l\}$ is open and dense in Ξ and, in particular, $\bigcup_i W_{p_i}^u(\Xi)$ is dense in Ξ.*

This is enough to deduce the topological part of Theorem C, since it implies that $\mathcal{W}^{cs}$ is open and dense in $\mathcal{U}$. In the rest of the section we prove Theorem 3.1.

3.1. *Denseness of stable leaves of Λ on U*

Proof of Theorem 3.1. In what follows we say that a $C^1 (d_{cu} - 1)$-dimensional disk $D \subset \Sigma$ such that $T_x D \subset \mathcal{C}^u_x(\Sigma, a)$ for all $x \in D$ is a *center-unstable disk*, or just a *cu-disk*. A *cu*-disk D is an *unstable disk*, or just a *u-disk*, if for any given $x, y \in D$ there exists a sequence $\widetilde{f}_i : \tilde{S}_i \to \Xi$ of smooth extensions of $f_i = f \mid S_i$ together with a subsequence i_k and $x_k, y_k \in \Xi$ such that $g_k = \widetilde{f}_{i_k} \circ \widetilde{f}_{i_k - 1} \circ \cdots \circ \widetilde{f}_2 \circ \widetilde{f}_1$ satisfies $g_k x_k = x$, $g_k y_k = y$ and $d(x_k, y_k) \leq \lambda_1^{i_k} d(x, y)$ for all $k \geq 1$ (note that an unstable disk is necessarily contained in the attracting set Λ).

From Remark 2.15, if $S \subset \Sigma_y$ for some $\Sigma_y \in \Xi$, then $\tilde{D} = \exp_y^{-1}(D) = \text{Graph}(g : D_u \to E^s_y(\Sigma))$ where $D_u = \pi_u \tilde{D} \subset E^u_y(\Sigma)$ is a open subset of $E^u_y(\Sigma_y)$ and g is a C^1 map such that $\|Dg\| \leq 2a$. Indeed, D is transverse to $\mathcal{W}^s(\Sigma_y)$ and each $W^s_x(\Sigma_y)$ is the graph of $\varphi_x : B(0, \rho) \cap E^s_y(\Sigma_y) \to E^u_y(\Sigma_y)$ which is C^1 and depends continuously on x in the C^1 topology; and the tangent space at any point of $\tilde{D}$ is contained in $\mathcal{C}^u_y(\Sigma_y, 2a)$.

We define $\rho(D) = \sup\{r > 0 : B(x, r) \subset D_u, x \in E^u_y(\Sigma_y)\}$ as the *inner radius* of any given *cu*-disk D.

We use uniform expansion along center-unstable cones by the extension of f to obtain the following proposition.

PROPOSITION 3.2. *There exist $\rho_0 > 0$ and finitely many periodic points $p_1, \ldots, p_l$ of Λ in Ξ such that any given center-unstable disk D_0 contains a nested sequence $D_0 \supset \hat{D}_1 \supset \hat{D}_2 \supset \ldots$ of disks admitting:*
- *a sequence $\widetilde{f}_i : \tilde{S}_i \to \Xi$ of smooth extensions; and*
- *a subsequence i_k such that $g_k = \widetilde{f}_{i_k} \circ \widetilde{f}_{i_k - 1} \circ \cdots \circ \widetilde{f}_2 \circ \widetilde{f}_1$ satisfies*

$$g_k \mid \hat{D}_k : \hat{D}_k \to D_k = g_k \hat{D}_k \subset \Xi \text{ is a diffeomorphism for each } k \geq 1.$$

Moreover, $(D_k)_{k \geq 1}$ accumulates a u-disk D in the C^1 topology which:
- *contains the local unstable manifold $W^u_q(\Xi)$ with respect to f of a point of q of the orbit of p_i for some $i \in \{1, \ldots, l\}$; and*
- *whose inner radius is uniformly bounded away from zero, $\rho(D) \geq \rho_0$.* □

We prove Proposition 3.2 in the next subsection. Since $\mathcal{W}^s(\Xi)$ is transversal to any *cu*-disk and the nested disks $\hat{D}_k$ with vanishing diameter intersect at a unique point $r \in D_0 \cap \Lambda$, this shows that *every center-unstable disk in any smooth strip contains the transversal intersection of the stable manifold of all points in the local unstable manifold of a periodic point of Λ*, completing the proof of Theorem 3.1. □

3.2. *Local uniform expansion of cu-disks.* Here we fix a *cu*-disk D_0 in $S \in \{S_i\}$ and prove Proposition 3.2.

We obtain by induction a sequence of disks D_n, $n \geq 0$ in Ξ as follows. First, the inner radius of any cu-disk contained in a smooth strip $\tilde{S}$ is uniformly expanded by the global Poincaré map.

LEMMA 3.3. *If $\lambda_2 \in (0, 1)$ satisfies $2\lambda_1 < \lambda_2(1 - 2a)$ and D is a cu-disk contained in some extension $\tilde{S}$ of a smooth strip $S \in \{S_i\}$, then*

$$\rho(\tilde{f}D) \geq \lambda_2^{-1}\rho(D) \quad and \quad (1 - 2a)\rho(\tilde{f}D) \leq 2\mathrm{diam}(\tilde{f}D) \leq (1 + 2a)\rho(\tilde{f}D),$$

where $\tilde{f} : \tilde{S} \to \Xi$ is the extension of $f \mid S : S \to \Xi(a_0)$.

Proof. Let $S \subset \Sigma_y \in \Xi$. From Remark 2.18, $\tilde{f}D$ is a cu-disk contained in some $\Sigma_{y'} \in \Xi$ and we can write $\exp_{y'}^{-1}(\tilde{f}D) = \mathrm{Graph}(g : D_u^1 \to E_{y'}^s(\Sigma_{y'}))$ where $D_u^1 \subset E_{y'}^u(\Sigma_{y'})$ is an open subset. Then for a ball $B(x', r) \subset D_u^1$ and C^1 curve $\gamma_1 : (I, 0, 1) \to$ (Closure $B(x', r), x', \partial B(x', r))$ there exists a unique curve $\gamma : I \to D_u = \pi_u \exp_y^{-1} D$ such that $\gamma_1(s) = \pi_u \tilde{f} \exp_y(\gamma(s) + g_1\gamma(s))$, where $s \in I = [0, 1]$. By Theorem 2.17 and Remark 2.18, together with the choice of $\Sigma_y, \Sigma_{y'}$ in Remark 2.6,

$$\|\dot{\gamma}_1(s)\| = \|\pi_u D\tilde{f} \cdot D\exp_y(\dot{\gamma}(s) + Dg_1(\gamma(s)) \cdot \dot{\gamma}(s))\|$$

$$\geq \lambda_1^{-1}\|D\exp_y(\dot{\gamma}(s) + Dg_1(\gamma(s)) \cdot \dot{\gamma}(s))\| \geq \frac{\lambda_1^{-1}}{2} \cdot (1 - 2a)\|\dot{\gamma}(s)\|.$$

Then the bound on the inner radius follows by the choice of λ_2, since γ_1 is any curve joining $\gamma_1(0) = x'$ to the boundary $\gamma_1(1) \in \partial B(x', r)$ inside D_u^1. For the diameter, note that $\|u - v\|(1 - 2a) \leq \|u + g_1 u - (v + g_1 v)\| \leq (1 + 2a)\|u - v\|$ for all $u, v \in D_u^1$ and take account of the effect of $\exp_{y'}$ on distances; cf. Remark 2.6. $\square$

We let λ_2 be as in the statement of Lemma 3.3 in what follows; fix $\lambda_2 < a_1 < 1$ and assume without loss of generality that $a_1\lambda_2^{-1} > 5$. We assume that cu-disks $D_0, \ldots, D_n$ have already been obtained so that there are smooth strips $S_0, \ldots, S_n$ satisfying $D_i \subset \tilde{S}_i \subset \Sigma_{y_i}$ and $D_{i+1} \subset \tilde{f}_i D_i$, $i = 0, \ldots, n - 1$.

Letting $D_n = \exp_{y_n} \mathrm{Graph}(g_n)$, we consider the balls $\mathcal{B} = \{B(x, a_1\rho(D_n)) \subset \pi_u \exp_{y_n}^{-1} D_n\}$ and corresponding disks $\mathcal{D} = \{D = \exp_{y_n} \mathrm{Graph}(g_n \mid B), B \in \mathcal{B}\}$. We set $\hat{\mathcal{D}} = \{D \in \mathcal{D} : \exists S, D \cap \partial^s S \neq \emptyset\}$ and $\hat{\mathcal{D}}_\sigma = \{D \in \mathcal{D} : D \cap \Gamma_0 \neq \emptyset\}$. Then we have the following cases.

(1) If $\mathcal{D} \not\subseteq \hat{\mathcal{D}} \cup \hat{\mathcal{D}}_\sigma$, then we choose some $D \in \mathcal{D} \backslash (\hat{\mathcal{D}} \cup \hat{\mathcal{D}}_\sigma)$. There exists a smooth section S such that $D \subset S$ and we reset $D_n = D$ and define $D_{n+1} = fD_n = (f \mid S)(D_n) \subset \Xi(a_0)$.

(2) Otherwise: either $\hat{\mathcal{D}} \neq \emptyset$ or $\hat{\mathcal{D}}_\sigma \neq \emptyset$.

    (a) If $\hat{\mathcal{D}}_\sigma \neq \emptyset$, then we choose $D \in \hat{\mathcal{D}}_\sigma$ and $B \subset \pi_u \exp_{y_n}^{-1} D$ a ball of radius $a_1\rho(D_n)/4$ so that, resetting $D_n = \exp_{y_n} B$, we have

$$d(D_n, \Gamma_0) > (1 - 2a)\rho(D_n) \quad and$$

$$\rho(D_n) = \rho(D)/4 > a_1\lambda_2^{-1}\rho(D_{n-1})/4 > (5/4)\rho(D_{n-1}).$$

    We then define $D_{n+1} = fD_n$.

(b) Otherwise, we have $\hat{\mathcal{D}} \neq \emptyset = \hat{\mathcal{D}}_\sigma$ and consider the subfamily $\widetilde{\mathcal{D}} = \{D \in \hat{\mathcal{D}} : \exists S, D \cap \partial^s S \neq \emptyset \neq D \cap \partial^s \tilde{S}\}$ of those disks which intersect both $\partial^s S$ and $\partial^s \tilde{S}$ for some S.

(i) If $\widetilde{\mathcal{D}} = \emptyset$, then we choose some $D \in \hat{\mathcal{D}}$ and S so that $D \cap \partial^s S \neq \emptyset$, reset $D_n = D$, and define $D_{n+1} = \tilde{f} D_n$, where $\tilde{f}$ is the extension of $f \mid S$ to $\tilde{S}$.

(ii) Otherwise, we choose $D \in \widetilde{\mathcal{D}}$. There exists a subdisk $\hat{D} \subset D$ such that $\hat{D} \subset \tilde{S}$ and $\rho(\hat{D}) \geq a_0 \delta_1/2$ by Remark 2.18 and the definition of δ_1 and $\hat{D}$. We reset $D_n = \hat{D}$ and define $D_{n+1} = \tilde{f} D_n$, with $\tilde{f}$ denoting the extension of $f \mid S$ to $\tilde{S}$.

This completes the inductive step of the construction of a sequence $(D_n)_{n \geq 0}$ of *cu*-disks in Ξ. Lemma 3.3 ensures that $\rho(D_{n+1}) \geq (a_2 \lambda_2^{-1}/4)\rho(D_n)$ and $a_1 \lambda_2^{-1}/4 > 5/4 > 1$ by the choice of a_1.

Since diam $S <$ diam $\tilde{S}$ is bounded by a uniform constant for all smooth strips $S \in \{S_i\}$, the expansion of the inner radius implies that the induction cannot go consecutively through cases (1), (2a) or (2b)(i) above infinitely many times. *Each time we are in case* (2b)(ii) *we restart the algorithm, choosing a subdisk of D_{n+1} with half the inner radius.*

We conclude that, starting with any disk D_0 as above, we obtain a subsequence $n_k \nearrow \infty$ so that D_{n_k} is in case (2b)(ii) and $\rho(D_{n_k}) > a_0 \delta_1/2$ for all $k \geq 1$. Moreover, there exist a subdisk $D'_{n_{k-1}}$ with $2\rho(D'_{n_{k-1}}) = \rho(D_{n_k})$ and an iterate f^{m_k} such that $f^{m_k} \mid D'_{n_{k-1}} : D'_{n_{k-1}} \to D_{n_k}$ is a diffeomorphism. In addition, the uniform bound on the diameter also ensures that $m_k = n_k - n_{k-1}$ is bounded: $m_k \leq \bar{m}$.

Finally, since Ξ contains finitely many cross-sections, we can assume without loss of generality that $D_{n_k} \subset \Sigma_y \in \Xi$ for (possibly a subsequence of) all k. This is a sequence of graphs of C^1 functions with uniformly bounded derivative and domains given by balls with radius uniformly bounded away from zero. It follows that there exists a subsequence of such disks uniformly converging to a *cu*-disk D in the C^1-topology.

In particular, for large enough k, we have that the stable holonomy map $h : D'_{n_{k-1}} \to D_{n_k}$ within Σ is well defined by the choice of $D'_{n_{k-1}}$ with half of the inner radius of D_{n_k} and, moreover, is a continuous map; see Remark 2.4. Hence $g = h \circ (f^{m_k} \mid D_{n_k})^{-1} : D_{n_k} \to D_{n_k}$ has a fixed point by Brouwer's fixed point theorem, that is, there exists a stable leaf satisfying $f^{m_k} W_x^s(\Xi) \subset W_x^s(\Xi)$. The contraction of stable leaves implies the existence of a fixed point p of $f^{m_k} \mid \Sigma$ which is a periodic point for the flow whose stable manifold crosses D_{n_k}. Since we can take k as large as needed, we obtain that $W_p^s(\Sigma)$ transversely crosses D also.

To complete the proof, since $D_{n+1} \subset \tilde{f}_n D_n$ by construction, if we set $g_n = \tilde{f}_n \circ \cdots \circ \tilde{f}_1$, $n \geq 1$, then we can find $\hat{D}_{n+1} \subset D_0$ such that $g_n \hat{D}_n = D_n$ and $D_{n+1} \subset D_n$, $n \geq 1$. Since $D_{n_k} \to D$ uniformly as graphs, for $x, y \in D$ there are $\tilde{x}_k, \tilde{y}_k \in \hat{D}_{n_k}$ such that $(g_{n_k} \tilde{x}_k, g_{n_k} \tilde{y}_k) \to (x, y)$. By uniform expansion on *cu*-disks, for any given $i \geq 1$ we get $d(g_{n_k-i} \tilde{x}_k, g_{n_k-i} \tilde{y}_k) \leq \lambda_1^i d(g_{n_k} \tilde{x}_k, g_{n_k} \tilde{y}_k)$ for all $k \geq 1$. Thus, for an accumulation pair (x_i, y_i) of $(g_{n_k-i} \tilde{x}_k, g_{n_k-i} \tilde{y}_k)$ and sequence $g^i = \tilde{f}_i \circ \cdots \circ \tilde{f}_0$ of $\widetilde{f_{n_k-i} \circ \cdots \circ f_{n_k}}$ as $k \nearrow \infty$, we get $(g^i x_i, g^i y_i) = (x, y)$ and $d(x_i, y_i) \leq \lambda_1^i d(x, y)$. Hence D is a *u*-disk as claimed.

In addition, by the inclination lemma (λ-lemma of [**41**, Ch. 2, §7]), we can assume without loss of generality that n_k are multiples of m_k for all large enough k and that $D \subset W_p^u$ is a piece of the local unstable manifold of the hyperbolic periodic orbit p of f, whose period is a divisor of m_k. In particular, p is a hyperbolic periodic orbit for the flow whose period τ_p is bounded: $\tau_p \leq T = T(\bar{m})$.

This ensures the uniform size of the unstable manifold of the periodic orbits obtained by the previous algorithm and, moreover, since their period is bounded, that the possible periodic orbits are finite in number, by hyperbolicity and compactness.

This completes the proof of Proposition 3.2.

4. *Finitely many ergodic physical measures for sectional-hyperbolic attracting sets*

Here we prove Theorem A. We first obtain an auxiliary result which is a consequence of the previous arguments on cu-disks contained in adapted cross-sections.

4.1. *Uniformly center-unstable size of invariant subsets.*

We prepare the proof of Theorem A by obtaining a result on uniform size of positively flow-invariant subsets in the center-unstable direction.

We say that a d_{cu}-dimensional C^1 disk $D_0 \subset U$ is a cu-disk if $T_x D_0 \subset \mathcal{C}_x^{cu}(a)$ for all $x \in D$ (observe that such D is not contained in any cross-section $\Sigma \in \Xi$).

PROPOSITION 4.1. *For a sectional-hyperbolic attracting set Λ of a C^1 vector field G, there exists $\delta > 0$ so that, given a positively G-invariant subset $E \subset \Lambda$ having a cu-disk D such that $D \cap E$ has full Lebesgue induced measure in D, there exists a cu-disk $\tilde{D}$ whose inner radius is larger than δ and such that $\tilde{D} \cap E$ has full Lebesgue induced measure in $\tilde{D}$. Moreover, there exists $i \in \{1, \ldots, l\}$ so that for any $\varepsilon > 0$ we can find $\tilde{D}$ as above ε- C^1-close to the local unstable manifold of $\mathcal{O}(p_i)$, where the periodic point p_i is given by Proposition 3.2.*

Proof. This is a consequence of Proposition 3.2. Indeed, if $E \subset \Lambda$ and D are as stated, then we project D into D_0 through the flow to the nearest cross-section, that is, for any $x \in D$ we consider $t(x) = \inf\{t > 0 : \phi_t x \in \Xi(a_0)\}$ and $p(x) = \phi_{t(x)} x$, $x \in D$.

We claim that $p(D)$ contains a cu-disk D_0 inside some $\Sigma \in \Xi$ and, moreover, $E \cap D_0$ has full Lebesgue induced measure in D_0.

Assuming this claim, then $\hat{D}_k \cap E$ also has full Lebesgue induced measure in $\hat{D}_k$ for each of the disks $\hat{D}_k \subset D_0$ provided by Proposition 3.2. Moreover, since the Poincaré map f is a piecewise C^1 diffeomorphism as well as its extensions, $D_k = g_k \hat{D}_k$ is such that $D_k \cap E$ also has full Lebesgue induced measure in D_k by invariance of E under all transformations ϕ_t, $t \in \mathbb{R}$. The statement of Proposition 4.1 follows since, by construction, (i) the cu-disks D_k have inner radius larger than some $\delta > 0$ inside Σ; (ii) fixing some $k \geq 1$, we have that $\tilde{D} = \phi_{[-\delta,\delta]}(D_k)$ is a d_{cu}-dimensional center-unstable disk for the flow of G with inner radius bounded away from zero; and (iii) by smoothness of the flow and invariance of E we have that $\tilde{D} \cap E = \phi_{[-\delta,\delta]}(D_k \cap E)$ also has full Lebesgue induced measure inside $\phi_{[-\delta,\delta]}(D_k)$. Moreover, the C^1-closeness to the local unstable manifold of one of the hyperbolic periodic orbits provided by Proposition 3.2 follows,

using the transversal intersection of the stable manifold of this orbit with $\tilde{D}$ together with the inclination lemma.

It remains to prove the claim. Since $D \subset U$ we have $t(x) < \infty$ for all $x \in D$ and we fix $x_0 \in D$ and $y_0 = p(x_0) \in \Sigma$, for some adapted cross-section $\Sigma \in \Xi(a_0)$ in what follows, where we assume without loss of generality that x_0 is not a singularity.

We take a cross-section S to G at x_0 and note that, since D is a cu-disk for the flow, there exists a neighborhood V of x_0 in M such that (i) $p(V) \subset \Sigma$ and (ii) S is transversal to $D \cap V$. So $D_S = S \cap D \cap V$ is a submanifold of M of codimension $1 + d_s$. Hence, D_S is a submanifold of S of dimension $d_{cu} - 1$ and a cu-disk inside S, that is, $T_x D_S \subset \mathcal{C}_x^{cu}(a, S)$ according to the definition of the induced center-unstable cone fields on a cross-section S. Consequently, $p(D_S)$ is a cu-disk inside Σ and contained in $p(D)$. It remains to show that E has full Lebesgue induced measure in $p(D_S)$.

We now conveniently choose coordinates on a local chart of M at V so that $S = \mathbb{R}^{d-1} \times \{0\}$, $G(x_0) = (0, \ldots, 0, 1)$ and $E^s = \mathbb{R}^{d_s} \times \{0^{d_{cu}}\}$, $E_{x_0}^{cu} = \{0^{d_s}\} \times \mathbb{R}^{d_{cu}}$, and also $D \cap V$ is the graph of a C^1 map $\varphi : \mathbb{R}^{d_{cu}} \to \mathbb{R}^{d_s}$. Since $\Phi : \{0^{d_s}\} \times \mathbb{R}^{d_{cu}} \to D \cap V$, $u \mapsto (\varphi u, u)$ is a C^1 diffeomorphism and $E \cap D \cap V$ has full Lebesgue induced measure in $D \cap V$, it follows that $\tilde{E} = \Phi^{-1}(E \cap D \cap V)$ has full Lebesgue measure in $\{0^{d_s}\} \times \mathbb{R}^{d_{cu}}$.

However, $D_S = \Phi(\{0^{d_s}\} \times \{\mathbb{R}^{d_{cu}-1} \times \{0\})$ does not necessarily intersect E in a full Lebesgue induced measure subset. But Fubini's theorem ensures that $\tilde{E} \cap \{0^{d_s}\} \times \mathbb{R}^{d_{cu}-1} \times \{t\}$ has full Lebesgue measure for Lebesgue almost every $t \in \mathbb{R}$.

Thus we can choose t as close to 0 as needed so that $S_t = \mathbb{R}^{d-1} \times \{t\}$ is a cross-section to G; $D_t = S_t \cap D \cap V$ is a cu-disk inside S_t and $E \cap D_t$ has full Lebesgue induced measure in D_t. Moreover, we also have that $p(D_t) \subset p(D) \subset \Sigma$ is a cu-disk inside Σ and $p(D_t \cap E)$ has full Lebesgue induced measure in $p(D_t)$, since $p \mid D_t : D_t \to p(D_t)$ is a diffeomorphism as smooth as G.

This completes the proof of the claim with $D_0 = p(D_t)$ and Proposition 4.1 follows. $\square$

4.2. *Uniform volume of ergodic basis of physical measures.* We now extend the steps presented in [28] together with Proposition 4.1 and the following result.

THEOREM 4.2 [20, Appendix: Corollary B.1 and Theorem I]. *A C^1 vector field with a sectional-hyperbolic attracting set Λ supports an SRB measure. More precisely, for Lebesgue almost every point x in the trapping region of Λ, any weak* limit measure of the family $(T^{-1} \int_0^T \delta_{\phi_t x} \, dt)_{T>0}$ is an SRB measure. Moreover, if the vector field is Hölder-C^1, then each limit measure is a physical measure.*

The above result states that any weak* accumulation point μ of the empirical measures along the orbit of a Lebesgue generic point in U is an equilibrium state for the logarithm of the center-unstable Jacobian, that is,

$$h_\mu(\phi_1) = \int \log |\det D\phi_1 \mid E^{cu}| \, d\mu > 0, \tag{4.1}$$

the positiveness being a consequence of sectional hyperbolicity.

Moreover, *if the flow is Hölder-C^1, then this SRB measure is also a physical measure* since its support contains the (Pesin) unstable manifold through μ-a.e. point and the stable

foliation is absolutely continuous (this is a consequence of the partial hyperbolicity of the attracting set, the fact that the vector field is Hölder-C^1, and Proposition 2.3; this can be seen by adapting known arguments from [42, 43]), following standard geometric and ergodic arguments; see, for example, [28, §§2 and 3] and the proof of [20, Theorem I]. In particular, the center-unstable manifold W_x^{cu} through μ-a.e. x is a cu-disk contained in the attracting set Λ.

Remark 4.3. From Proposition 4.1, since the support of any SRB measure μ is a forward invariant closed subset and contains a cu-disk, there exists a periodic orbit $\mathcal{O}(p_i)$ contained in supp μ for some $i \in \{1, \ldots, l\}$. The stable leaves through the points of the local unstable manifold $W_{\mathcal{O}(p_i)}^{cu}(\varepsilon_0)$, for all small enough $\varepsilon_0 > 0$, intersect each center-unstable manifold through μ-a.e. point in an open subset, which contains μ-generic points by the absolute continuity property of the stable foliation. In particular, this shows that no such periodic orbit can be shared by distinct SRB measures of a sectional-hyperbolic attracting subset.

Proof of Theorem A. From Theorem 4.2 we have that any sectional-hyperbolic attracting set for a C^1 flow admits some physical/SRB probability measure μ which we can assume, without loss of generality, to be ergodic. Indeed, using ergodic decomposition, by Ruelle's inequality [31] we have $h_\mu(\phi_1) \le \int \log |\det D\phi_1 \mid E^{cu}| \, d\mu$ and so if μ satisfies (4.1), then each ergodic component of μ also satisfies (4.1)

Now we use that the ergodic basin $B(\mu)$ of μ contains a full Lebesgue measure subset of some center-unstable disk D_0 inside the sectional-hyperbolic attracting set together with Proposition 4.1.

COROLLARY 4.4. *Every sectional-hyperbolic attracting set Λ for a Hölder-C^1 vector field admits $\varepsilon_0 > 0$ so that the volume of the ergodic basin $B(\mu)$ of any ergodic SRB measure μ supported in Λ is uniformly bounded away from zero:* $\mathrm{Leb}(B(\mu)) \ge \varepsilon_0$.

Proof. By assumption, μ is an ergodic SRB measure and, as explained above, in our setting the stable holonomies are absolutely continuous. Then by [28, Lemma 3.2] we have that there exists a open subset V of the basin of attraction of Λ so that Leb-a.e. $x \in V$ is μ-generic, that is, $\mathrm{Leb}(V \setminus B(\mu)) = 0$.

Hence there exists a cu-disk $D_0 \subset V$ such that $D_0 \cap B(\mu)$ has full Lebesgue induced measure in D_0. Proposition 4.1 implies that the positively invariant subset $B(\mu)$ contains a cu-disk D with $\rho(D) \ge \delta$ for some uniform $\delta > 0$ depending only on Λ. The same proof of [28, Lemma 3.2], using the uniform size of local stable leaves of $\mathcal{W}^s$ and the angle between E_x^s and E_x^{cu} at $x \in D$ uniformly bounded away from zero (due to domination), implies that the set $W = \bigcup \{W_x^s : x \in D\}$ is open, diffeomorphic to a cylinder $D \times \mathcal{D}^{d_s}$ of uniform height. So $\mathrm{Leb}(W) \ge \varepsilon_0$ for some uniform $\varepsilon_0 > 0$. In addition, Leb-a.e. $x \in W$ belongs to $B(\mu)$ by the absolute continuity of the stable foliation.   □

We are now ready to complete the proof of Theorem A. Let U be a trapping region for Λ. If $\mathrm{Leb}(U \setminus B(\mu)) = 0$, then μ is the unique physical/SRB measure supported in Λ. Otherwise, let $\mu_1 = \mu$ and since $U_1 = U \setminus B(\mu_1)$ is such that $\mathrm{Leb}(U_1) > 0$ we can use [20, Theorem I] to ensure that Leb-a.e. $x \in U_1$ belongs to the ergodic basin of some SRB

measure $\mu_2 \neq \mu_1$. This measure μ_2 is a physical measure, satisfies $\mathrm{Leb}(B(\mu_2)) > \delta > 0$ by Corollary 4.4, and $B(\mu_1) \cap B(\mu_2) = \emptyset$ and $B(\mu_1) \cup B(\mu_2) \subset U$.

Again, if $\mathrm{Leb}(U \backslash (B(\mu_1) \cup B(\mu_2))) = 0$, then Λ supports exactly the pair μ_1, μ_2 of ergodic physical measures whose ergodic basins cover the topological basin of Λ except perhaps a Lebesgue zero subset. Otherwise $U_2 = U \backslash (B(\mu_1) \cup B(\mu_2))$ is such that $\mathrm{Leb}(U_2) > 0$ and we can repeat the argument.

Since the ergodic basins of distinct ergodic physical probability measures are disjoint subsets of the trapping region U which has finite volume, and each ergodic basin has a minimum volume bounded away from zero, this inductive process stops with finitely many $\mu_1, \dots, \mu_k$ ergodic physical/SRB measures supported on Λ, whose basins cover the trapping region U, Leb mod 0. This completes the proof of Theorem A. $\qquad \square$

4.3. *Full volume of stable leaves in the topological basin.*   Here we prove the measure-theoretic part of the statement of Theorem C.

Let μ be an ergodic SRB measure supported in Λ for a Hölder-C^1 vector field, that is, a physical measure. Let $\mathcal{O}(p_i)$ be the hyperbolic periodic orbit contained in supp μ; see Remark 4.3. Let V be an open neighborhood of $\mathcal{O}(p)$ and $\varphi : M \to \mathbb{R}$ a non-negative continuous function supported in V, so that $\mu(\varphi) = \int \varphi \, d\mu > 0$. Hence

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(\phi_t x) \, dt = \mu(\varphi) > 0 \quad \text{for each } x \in B(\mu).$$

Thus $\varphi(\phi_t x) > 0$ for some $t > 0$ and so $\phi_t x \in V$. This ensures that $x \in W_y^s$ for some $y \in W_{\mathcal{O}(p_i)}^{cu}$, that is, $x \in \mathcal{W}^{cs}$ for each $x \in B(\mu)$.

Finally, from Theorem A, there are finitely many ergodic SRB measures whose basins cover Leb-a.e. point of $\mathcal{U}$. This ensures that $\mathrm{Leb}(\mathcal{U} \backslash \mathcal{W}^{cs}) = 0$, and completes the proof of the measure-theoretic part of the statement of Theorem C.

5. *Statistical stability of sectional-hyperbolic attracting sets*
Statistical stability is essentially a consequence of the existence of finitely many physical measures whose basins cover Leb-a.e points of the trapping region together with recent results from [**40**] on robust entropy expansiveness of sectional-hyperbolic attractors on their trapping regions. We recall some relevant notions in what follows so as to be able to present a proof of Theorem B in §5.4.

5.1. *Entropy expansiveness.*   Let $g : M \to M$ be a continuous map and K a not necessarily invariant subset of M. For $\varepsilon > 0$ and $n \geq 1$, the (ε, n)-dynamical ball around $x \in M$ is $B(x, \varepsilon, n) = \{y \in M : d(g^j x, g^j y) < \varepsilon, \ \forall 0 \leq j < n\}$. A subset $E \subset M$ is an (n, ε)-generator for K if, given $x \in K$, there exists $y \in E$ so that $d(g^i x, g^i y) < \varepsilon$ for each $0 \leq i < n$. Equivalently, the dynamical ball $\{B(y, \varepsilon, n) : y \in E\}$ is an open cover of K.

Let $r_n(K, \varepsilon)$ be the cardinality of the smallest (n, ε)-generator for K and $r(K, \varepsilon) = \limsup_{n \to \infty} (1/n) \log r_n(K, \varepsilon)$. The *topological entropy of g on K* is given by

$$h_{\mathrm{top}}(g, K) = \lim_{\varepsilon \to 0} r(K, \varepsilon),$$

and the *topological entropy of g* is defined by $h_{\mathrm{top}}(g) = h_{\mathrm{top}}(g, M)$.

For $x \in M$ and $\varepsilon > 0$ we define the *two-sided ε-dynamical ball at x* as $B(x, \varepsilon, \infty) = \{y : d(g^n x, g^n y) < \varepsilon, \ \forall n \in \mathbb{Z}\}$ and say that *g is ε-entropy expansive* if all these infinite dynamical balls have zero topological entropy, that is, $\sup_{x \in M} h_{\text{top}}(g, B(x, \varepsilon, \infty)) = 0$.

5.2. *Upper semicontinuity of metric entropy.* Let μ be a g-invariant measure and $\mathcal{P}$ a finite μ mod 0 measurable partition. The *metric entropy of μ with respect to the partition* $\mathcal{P}$ is given by

$$h_\mu(g, \mathcal{P}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\mathcal{P}^n) \quad \text{where } H_\mu(\mathcal{P}^n) = \sum_{B \in \mathcal{P}^{n-1}} -\mu(B) \log \mu(B)$$

and $\mathcal{P}^n$ is the *nth dynamical refinement* of $\mathcal{P}$: $\mathcal{P}^n = \mathcal{P} \vee g^{-1}\mathcal{P} \vee \cdots \vee g^{-(n-1)}\mathcal{P}$. The *metric entropy of μ* is $h_\mu(g) = \sup_{\mathcal{P}} h_\mu(g, \mathcal{P})$ and the supremum is taken over all finite measurable partitions.

If g is ε-entropy expansive, then *every finite partition $\mathcal{P}$ with* diam $\mathcal{P} < \varepsilon$ *is generating*, that is, it satisfies $h_\mu(g) = h_\mu(g, \mathcal{P})$ for all $\mu \in \mathcal{M}_1^g$, where $\mathcal{M}_1^g$ is the family of all g-invariant probability measures; see, for example, [16].

The metric entropy of a vector field is the metric entropy of the time 1 map of its induced flow. *A vector field is ε-entropy expansive* if the time-one map of its induced flow is ε-entropy expansive.

Entropy expansiveness is a sufficient condition to ensure upper semicontinuity of the entropy map $\mu \in \mathcal{M}_1^g \mapsto h_\mu(g)$, as follows.

LEMMA 5.1 [16]. *If G is entropy expansive, then the metric entropy function is upper semicontinuous.*

5.3. *Equilibrium states and physical measures.* Since the family $\mathcal{M}_1^G$ of G-invariant probability measures is compact in the weak* topology, for entropy expansive vector fields there exists some measure which maximizes the function $\mu \in \mathcal{M}_1^G \mapsto h_\mu(G) + \int \psi \, d\mu$ for any given continuous function $\psi : M \to \mathbb{R}$, known as an *equilibrium state for ψ, G*.

In order to use equilibrium states to obtain statistical stability, we relate equilibrium states for the potential $\psi = \log |\det D\phi_1 \mid E^{cu}|$ with physical measures in the same way as for hyperbolic attracting sets; see, for example, [18].

THEOREM 5.2. *Let Λ be a sectional-hyperbolic attracting set for a Hölder-C^1 vector field G with the open subset U as trapping region.*

(1) *For each G-invariant ergodic probability measure μ supported in Λ the following are equivalent:*

    (a) $h_\mu(\phi_1) = \int \psi \, d\mu > 0$;

    (b) *μ is an SRB measure, that is, it admits an absolutely continuous disintegration along unstable manifolds;*

    (c) *μ is a physical measure, that is, its basin $B(\mu)$ has positive Lebesgue measure.*

(2) *In addition, the family $\mathbb{E}$ of all G-invariant probability measures which satisfy item (a) above is the convex hull $\mathbb{E} = \{\sum_{i=1}^{k} t_i \mu_i : \sum_i t_i = 1; 0 \leq t_i \leq 1, i = 1, \ldots, k\}$.*

We recall that, from sectional hyperbolicity together with Ruelle's inequality [**48**], $h_\nu(\phi_1) \leq \int \psi \, d\nu$ for all $\nu \in \mathcal{M}_1^G$. Hence, the set $\mathbb{E}$ defined above is formed by equilibrium states for $-\psi, G$. The proof of Theorem 5.2 can be found in [**9**, §2.3] where the same properties were stated and proved in the $d_{cu} = 2$ setting (singular-hyperbolic attracting sets). However, the proof presented there also holds in the present setting without change.

5.4. *Statistical stability.* Here we prove Theorem B.

We consider vector fields G on a subset $\mathcal{U}$ of $\mathfrak{X}^{1+}(M)$ with a trapping region U of a sectional-hyperbolic attracting set $\Lambda_G = \Lambda_G(U) = \bigcap_{t>0} \phi_t^G(U)$ so that each $G \in \mathcal{U}$ is ε-entropy expansive. Then the map $\mathcal{U} \to \mathcal{K}(U)$, $G \mapsto \Lambda_G(U)$ is continuous, where $\mathcal{K}(U)$ is the family of compact subsets of U with the Hausdorff distance between compact subsets $K, L \subset U$ of a metric space given by (see, for example, [**22**])

$$d_H(K, L) = \inf\{r > 0 : K \subset B(L, r) \text{ and } L \subset B(K, r)\}.$$

LEMMA 5.3 [**7**, Lemma 2.3]. *For every $\varepsilon > 0$ there is a neighborhood $\mathcal{V}$ of G in $\mathfrak{X}^1(M)$ such that $\Lambda_Y(U) \subset B(\Lambda_G(U), \varepsilon)$ and $\Lambda_G(U) \subset B(\Lambda_Y(U), \varepsilon)$ for all $Y \in \mathcal{V}$.*

Moreover, the map $\nu \in \mathcal{M} \mapsto \text{supp}\, \nu \in \mathcal{K}(M)$ is also continuous, where $\mathcal{M}$ is the family of probability measures in M with the weak * topology. In addition, the domination of the splitting $E_\Lambda^s \oplus E_\Lambda^{cu}$ implies its continuity for nearby vector fields; see, for example, [**14**, Appendix B.1].

For any fixed $G \in \mathcal{U}$ and any sequence $G_n \in \mathcal{U}$ such that $\|G_n - G\|_{C^1} \to 0$ when $n \nearrow \infty$, we let $\mu_n \in \mathcal{M}_1^{G_n}$ be equilibrium states for $\psi_n, G_n, n \geq 1$, where $\psi_n = \psi_{G_n} = \log |\det D\phi_1^{G_n} \mid E_{\Lambda_{G_n}(U)}^{cu}|$, and μ be a weak* limit point of $(\mu_n)_{n \geq 1}$. We assume that $\mu = \lim \mu_n$, restricting to a subsequence if necessary. Since the splitting $E_{\Lambda_{G_n}(U)}^s \oplus E_{\Lambda_{G_n}(U)}^{cu}$ is continuous, we can deal with its continuous extension $E_n^s \oplus E_n^{cu}$ to define ψ_n on the whole of M.

The continuity of dominated splittings for nearby vector fields means that for each $\xi > 0$ there exist $N \geq 1$ and a neighborhood V of $\text{supp}\, \mu$ such that

$$\text{supp}\, \mu_n \subset V \quad \text{and} \quad \text{dist}(E_{n,x}^*; E_{\Lambda_G(U),x}^*) < \xi, \quad x \in V, \ * = s, cu, \ \text{for all } n > N;$$

where the distance $\text{dist}(E, F)$ between two subspaces E, F of $T_x M$ is defined to be

$$\text{dist}(E, F) := \max \left\{ \sup_{\|v\|=1, v \in E} \text{dist}(v, F), \sup_{\|v\|=1, v \in F} \text{dist}(v, E) \right\},$$

and $\text{dist}(v, H) := \min_{w \in E} \|v - w\|$ for each subspace H of $T_x M$ and any $x \in M$.

Moreover, since $D\phi_1^{G_n}(x)$ converges to $D\phi_1^G(x)$ uniformly in x when $n \nearrow \infty$, we have $\psi_n \to \psi = \psi_G$ uniformly by definition of the C^1 topology, in the following sense: for any given $\xi > 0$ there exist $N \geq 1$ and a neighborhood V of $\text{supp}\, \mu$ so that $|\psi_n(x) - \psi(x)| < \xi$ for all $x \in V$ and each $n > N$.

Proof of Theorem B. Using the compactness of the manifold M, we construct a finite open cover $\{B(x_i, \delta) : i = 1, \ldots, k\}$ for some $2\delta < \varepsilon$ such that $\mu(\partial B(x, \delta)) = 0, i = 1, \ldots, k$, and obtain the partition $\mathcal{P} = \bigvee_{i=1}^k B(x_i, \varepsilon/2)$ with diameter smaller than ε and the

boundaries of each atom with zero μ-measure. Hence, for each $k \geq 1$, we have that $\mu(\partial \mathcal{P}^k) = 0$ since by continuity we have

$$\partial \mathcal{P}^n \subset \partial \mathcal{P} \cup \partial(\phi_{-1}\mathcal{P}) \cup \cdots \cup \partial(\phi_{-k+1}\mathcal{P}) \subset \partial \mathcal{P} \cup \phi_{-1}\partial \mathcal{P} \cup \cdots \cup \phi_{-k+1}\partial \mathcal{P}.$$

Now for each fixed $k \geq 1$ we find that

$$0 = \limsup_n \left(h_{\mu_n}(G_n) + \int \psi_n \, d\mu_n \right) \leq \limsup_n \left(\frac{1}{k} H_{\mu_n}(\mathcal{P}_n^k) + \int \psi_n \, d\mu_n \right)$$

where $\mathcal{P}_n^k = \bigvee_{i=0}^{k-1} \phi_{-i}^{G_n} \mathcal{P}$ and $(\phi_t^{G_n})_t$ is the flow induced by G_n.

LEMMA 5.4. *For each fixed* $k \geq 1$ *we have* $\limsup_n (1/k) H_{\mu_n}(\mathcal{P}_n^k) \leq (1/k) H_\mu(\mathcal{P}^k)$ *where* $\mathcal{P}^k = \bigvee_{i=0}^{k-1} \phi_{-i}^G \mathcal{P}$.

Assuming the lemma, since $k \geq 1$ is arbitrary and (possibly taking a subsequence) we have $\mu_n \to \mu$ in the weak* topology, then

$$\left| \int \psi_n \, d\mu_n - \int \psi \, d\mu \right| \leq \left| \int (\psi_n - \psi) \, d\mu_n \right| + \left| \int \psi \, d\mu_n - \int \psi \, d\mu \right| \xrightarrow[n \to \infty]{} 0.$$

Consequently, we deduce that

$$0 \leq \inf_{k \geq 1} \left(\frac{1}{k} H_\mu(\mathcal{P}^k) - \int \psi \, d\mu \right) = h_\mu(G) - \int \psi \, d\mu \leq 0$$

and so μ achieves the maximum of $\mu \in \mathcal{M}_1^G \mapsto h_\mu(G) - \int \psi \, d\mu$. From Theorem 5.2 we have that μ is a convex linear combination of the finitely many ergodic physical measures supported in $\Lambda_G(U)$ provided by Theorem A.

To complete the proof of Theorem B we present the proof of the lemma.

Proof of Lemma 5.4. Observe that

$$\sup_{|t| < k} d(\phi_t^{G_n}(x), \phi_t^G(x)) \xrightarrow[n \to \infty]{u} 0$$

for all fixed $k \geq 1$ and uniformly in $x \in M$. Moreover, we may assume without loss of generality that each $P \in \mathcal{P}$ has non-empty interior by construction.

Thus for each $\delta > 0$ and atom $Q \in \mathcal{P}$ there exists $N = N(\delta, Q) \in \mathbb{Z}^+$ such that for all $n \geq N$ and $0 \leq t \leq k$,
- $\phi_{-t}^G(Q) \cap \phi_{-t}^{G_n}(Q) \neq \emptyset$ and $\phi_{-t}^{G_n}(Q) \subset B_\delta(\phi_{-t}^G(Q))$, and
- $\mu(\partial B_\delta(Q)) = 0$,

where $B_\delta(Q) = \bigcup_{x \in Q} B(x, \delta)$ is the δ-neighborhood of the set Q. Let $N(\delta, \mathcal{P}^k) = \max_{Q \in \mathcal{P}^k} N(\delta, Q)$ be chosen to satisfy the previous relations simultaneously for all $Q \in \mathcal{P}^k$.

For $\omega > 0$, let $\zeta > 0$ be such that

$$|t_i - s_i| < \zeta, t_i, s_i \in \mathbb{R}, i = 1, \ldots, k \implies \sum_{i=1}^k -x_i \log x_i < \omega;$$

and, for each $\delta > 0$, let $L = L(\zeta, \delta, \mathcal{P}^k)$ be such that $\mu(\partial B(Q, \delta)) = 0$ for all $Q \in \mathcal{P}^k$ and

$$n \geq L, Q \in \mathcal{P}^k \implies \mu_n(B_\delta(Q)) \leq \mu(B_\delta(Q)) + \frac{\zeta}{2}.$$

Since $\mu(\partial \mathcal{P}^k) = 0$, let δ_0 be such that $\mu(B_\delta(Q)) \leq \mu(Q) + \zeta/2$ for all $Q \in \mathcal{P}^k$.

We now take $0 < \delta < \delta_0$ in the previous choices, and for $n \geq L(\zeta, \delta, \mathcal{P}^k) + N(\delta, \mathcal{P}^k)$ we have for each $Q_n \in \mathcal{P}_n^k$ that there exists $Q \in \mathcal{P}^k$ so that

$$Q_n \subset B_\delta(Q) \quad \text{and} \quad \mu_n(Q_n) \leq \mu_n(B_\delta(Q)) \leq \mu(B_\delta(Q)) + \frac{\zeta}{2} \leq \mu(Q) + \zeta,$$

which ensures by the choice of the pair (ζ, ω) that

$$\frac{1}{k} H_{\mu_n}(\mathcal{P}_n^k) \leq \frac{1}{k}(H_\mu(\mathcal{P}^k) + \omega) \leq \frac{1}{k} H_\mu(\mathcal{P}^k) + \frac{\omega}{k}$$

for all large enough n depending on ω. Since $\omega > 0$ is arbitrary, this shows that

$$\limsup_{n \to \infty} \frac{1}{k} H_{\mu_n}(\mathcal{P}_n^k) \leq \frac{1}{k} H_\mu(\mathcal{P}^k)$$

and completes the proof of the lemma. □

REFERENCES

[1] V. S. Afraimovich, V. V. Bykov and L. P. Shil'nikov. On the appearance and structure of the Lorenz attractor. *Dokl. Acad. Sci. USSR* **234** (1977), 336–339.

[2] J. Alves and M. Soufi. Statistical stability of geometric Lorenz attractors. *Fund. Math.* **224**(3) (2014), 219–231.

[3] V. Araujo, S. Galatolo and M. J. Pacifico. Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors. *Math. Z.* **276**(3–4) (2014), 1001–1048.

[4] V. Araujo and I. Melbourne. Existence and smoothness of the stable foliation for sectional hyperbolic attractors. *Bull. Lond. Math. Soc.* **49**(2) (2017), 351–367.

[5] V. Araujo and I. Melbourne. Mixing properties and statistical limit theorems for singular hyperbolic flows without a smooth stable foliation. *Adv. Math.* **349** (2019), 212–245.

[6] V. Araujo, I. Melbourne and P. Varandas. Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps. *Comm. Math. Phys.* **340**(3) (2015), 901–938.

[7] V. Araujo and M. J. Pacifico. *Three-dimensional flows. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge (A Series of Modern Surveys in Mathematics, 53)*. Springer, Berlin, 2010. With a foreword by Marcelo Viana.

[8] V. Araujo, M. J. Pacifico, E. R. Pujals and M. Viana. Singular-hyperbolic attractors are chaotic. *Trans. Amer. Math. Soc.* **361** (2009), 2431–2485.

[9] V. Araujo, A. Souza and E. Trindade. Upper large deviations bound for singular-hyperbolic attracting sets. *J. Dynam. Differential Equations* **31**(2) (2019), 601–652.

[10] V. Araujo and P. Varandas. Robust exponential decay of correlations for singular-flows. *Comm. Math. Phys.* **311** (2012), 215–246.

[11] W. Bahsoun and M. Ruziboev. On the statistical stability of Lorenz attractors with a $c^{1+\alpha}$ stable foliation. *Ergod. Th. & Dynam. Sys.* **39**(12) (2018), 3169–3184.

[12] P. Bálint and I. Melbourne. Statistical properties for flows with unbounded roof function, including the Lorenz attractor. *J. Stat. Phys.* **172**(4) (2018), 1101–1126.

[13] M. Benedicks and M. Viana. Solution of the basin problem for Hénon-like attractors. *Invent. Math.* **143**(2) (2001), 375–434.

[14] C. Bonatti, L. J. Díaz and M. Viana. *Dynamics beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective (Encyclopaedia of Mathematical Sciences, 102)*. Springer, Berlin, 2005.

[15] C. Bonatti, A. Pumariño and M. Viana. Lorenz attractors with arbitrary expanding dimension. *C. R. Acad. Sci. Paris Sér. I Math.* **325**(8) (1997), 883–888.

[16] R. Bowen. Entropy-expansive maps. *Trans. Amer. Math. Soc.* **164** (1972), 323–331.

[17] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, Berlin, 1975.

[18] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), 181–202.

[19] S. Crovisier and D. Yang. Robust transitivity of singular hyperbolic attractors. *Math. Z.* (2020), doi:10.1007/s00209-020-02618-1.

[20] S. Crovisier, D. Yang and J. Zhang. Empirical measures of partially hyperbolic attractors. *Comm. Math. Phys.* **375**(1) (2020), 725–764.

[21] F. Dumortier, H. Kokubu and H. Oka. A degenerate singularity generating geometric Lorenz attractors. *Ergodic Th. & Dynam. Sys.* **15**(5) (1995), 833–856.

[22] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, New York, 1990.

[23] S. Galatolo and M. J. Pacifico. Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence. *Ergod. Th. & Dynam. Sys.* **30** (2010), 703–1737.

[24] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 59–72.

[25] M. Holland and I. Melbourne. Central limit theorems and invariance principles for Lorenz attractors. *J. Lond. Math. Soc. (2)* **76**(2) (2007), 345–364.

[26] T. Jordan, V. Naudot and T. Young. Higher order Birkhoff averages. *Dyn. Syst.* **24**(3) (2009), 299–313.

[27] M. Kisaka, H. Kokubu and H. Oka. Supplement to homoclinic doubling bifurcation in vector fields. *Dynamical Systems, Santiago, 1990 (Pitman Research Notes in Mathematics Series 285)*. Longman Scientific & Technical, Harlow, 1993, pp. 92–116.

[28] R. Leplaideur and D. Yang. SRB measure for higher dimensional singular partially hyperbolic flows. *Ann. Inst. Fourier* **67**(2) (2017), 2703–2717.

[29] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmosph. Sci.* **20** (1963), 130–141.

[30] S. Luzzatto, I. Melbourne and F. Paccaut. The Lorenz attractor is mixing. *Comm. Math. Phys.* **260**(2) (2005), 393–401.

[31] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, New York, 1987.

[32] R. Metzger and C. Morales. Sectional-hyperbolic systems. *Ergod. Th. & Dynam. Sys.* **28** (2008), 1587–1597.

[33] R. J. Metzger and C. A. Morales. Stochastic stability of sectional-anosov flows. *Preprint,* 2015, arXiv:1505.01761.

[34] C. Morales. Lorenz attractor through saddle-node bifurcations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996), 589–617.

[35] C. Morales and E. Pujals. Singular strange attractors on the boundary of Morse-Smale systems. *Ann. Sci. Éc. Norm. Supér.* **30** (1997), 693–717.

[36] C. A. Morales. The explosion of singular-hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **24**(2) (2004), 577–591.

[37] C. A. Morales. Examples of singular-hyperbolic attracting sets. *Dyn. Syst.* **22**(3) (2007), 339–349.

[38] C. A. Morales, M. J. Pacifico and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. Math. (2)* **160**(2) (2004), 375–432.

[39] C. A. Morales, M. J. Pacifico and B. S. Martin. Expanding Lorenz attractors through resonant double homoclinic loops. *SIAM J. Math. Anal.* **36**(6) (2005), 1836–1861.

[40] M. J. Pacifico, F. Yang, and J. Yang. Entropy theory for sectional hyperbolic flows. *Preprint*, 2019, arXiv:1901.07436.

[41] J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems*. Springer, New York, 1982.

[42] Y. B. Pesin. *Lectures on Partial Hyperbolicity and Stable Ergodicity (Zurich Lectures in Advanced Mathematics, 1)*. European Mathematical Society, Zurich, 2004.

[43] C. Pugh and M. Shub. Ergodicity of Anosov actions. *Invent. Math.* **15** (1972), 1–23.

[44] C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. *Nonlinearity* **2**(4) (1989), 495–518.

[45] C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. II. *SIAM J. Math. Anal.* **23**(5) (1992), 1255–1268.

[46] C. Robinson. Nonsymmetric Lorenz attractors from a homoclinic bifurcation. *SIAM J. Math. Anal.*, **32**(1) (2000), 119–141.

[47] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.* **98** (1976), 619–654.

[48] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Bras. Mat.* **9** (1978), 83–87.

[49] M. R. Rychlik. Lorenz attractors through Šil'nikov-type bifurcation. I. *Ergod. Th. & Dynam. Sys.* **10**(4) (1990), 793–821.

[50] E. A. Sataev. Some properties of singular hyperbolic attractors. *Sb. Math.* **200**(1) (2009), 35.

[51] E. A. Sataev. Invariant measures for singular hyperbolic attractors. *Sb. Math.* **201**(3) (2010) 419.

[52] Y. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys* **27** (1972), 21–69.

[53] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[54] D. Smania and J. Vidarte. Existence of c^k-invariant foliations for Lorenz-type maps. *J. Dynam. Differential Equations* **30**(1) (2018), 227–255.

[55] F. Takens. Heteroclinic attractors: time averages and moduli of topological conjugacy. *Bull. Braz. Math. Soc.* **25** (1995), 107–120.

[56] D. Yang. On the historical behavior of singular hyperbolic attractors. *Proc. Amer. Math. Soc.* **148**(4) (2019), 1641–1644.