(f) There are no other good even values of k.

Suppose that $k = 2^r M$ with $r \geqslant 1$, M odd and assume that 3 does not divide M (because the case where 3 divides M is covered by (c), (d) above). Then $dr(M) = 1, 2, 4, 5, 7, 8$ for all of which $dr(M^6) \equiv 1 \pmod 9$. Let $n = 2^s M^5$ and consider $kn = 2^{r+s} M^6$. By (b), we can choose s so that $dr(2^{r+s})$ is 5 or 7 whence $dr(kn)$ is 5 or 7; thus k is bad unless 35 divides M, say $M = 35N$ with N odd and, as above, $dr(N^6) \equiv 1 \pmod 9$.

- If $r = 1$, we would then have $k = 70N$. Such a k is bad because $n = 14N^5$ gives $dr(kn) = dr(980N^6) = 8$ which does not divide $980N^6$.
- If $r = 2$, we would then have $k = 140N$. Such a k is bad because $n = 7N^5$ gives $dr(kn) = dr(980N^6) = 8$ which does not divide $980N^6$.

We conclude that $r \geqslant 3$, so that 280 divides k which has already been dealt with in (e).

Finally, as Max commented in our initial discussion, we can consider the same problem in other number bases.

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Feedback

On 75.12 Alan Beardon writes: Nick Lord challenged the reader to supply possible generalisations of the identity

$$(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3,$$

and asked whether any reader could find positive integers a, b, c with $b > 1$ such that

$$\left(1^a + 2^a + \ldots + n^a\right)^b = 1^c + 2^c + \ldots + n^c$$

holds for all n? In fact, this is true only in the given case, namely $(a, b, c) = (1, 2, 3)$. However, if we write

$$\sigma_k(n) = 1^k + 2^k + \ldots + n^k,$$

so that the given relation is $\sigma_1^2 = \sigma_3$, there are many other interesting relations between the quantities $\sigma_k(n)$ for various k (and all n). For example,

$$\sigma_1^3 = \tfrac{1}{4}\sigma_3 + \tfrac{3}{4}\sigma_5;$$

$$8\sigma_1^3 = -\sigma_1^2 + 9\sigma_2^2;$$

$$81\sigma_2^4 = 18\sigma_2^2\sigma_3 - \sigma_3^2 + 64\sigma_3^3;$$

$$16\sigma_3^3 = \sigma_3^2 + 6\sigma_3\sigma_5 + 9\sigma_5^2.$$

For more details, see [1].

Reference

1. Alan Beardon, Sums of powers of integers, *American Math. Monthly* **103** (1996), pp. 201-213.

On 99.03 Graham Jameson writes: This is indeed a very neat and quick proof that the partial sums of the harmonic series are not integers. It may be of interest to show how similar reasoning can be applied to sums of odd reciprocals: let

$$U_n = \sum_{r=1}^{n} \frac{1}{2r - 1}.$$

Instead of even and odd numbers, we consider multiples and non-multiples of 3. Denote by N_3 the set of non-multiples of 3. Note that if r and s belong to N_3, then so does rs.

Let A_3 be the set of rational numbers expressible as $\frac{r}{3s}$ with $r \in N_3$, and let B_3 be the set of rational numbers expressible as $\frac{r}{s}$ with $s \in N_3$. (With this notation, Nick Lord's A and B would be A_2 and B_2.) Clearly, elements of A_3 are not integers and if a is in A_3, then so is $\frac{1}{3}a$. Also, as with A_2 and B_2, the following facts are easily verified:

(i) if $a \in A_3$ and $b \in B_3$, then $a + b \in A_3$;

(ii) if b_1 and b_2 are in B_3, then so is $b_1 + b_2$.

We now prove by induction that $U_n \in A_3$ for all $n \geqslant 2$. To start, $U_2 = \frac{4}{3} \in A_3$. Now assume that $n \geqslant 2$ and $U_k \in A_3$ for $2 \leqslant k \leqslant n$: this implies that $\frac{1}{3}U_k \in A_3$ for $1 \leqslant k \leqslant n$, since $\frac{1}{3}U_1 = \frac{1}{3} \in A_3$. Among the numbers $1, 3, \dots, 2n + 1$, let the largest multiple of 3 be $3(2k - 1)$. (This slight variation from the original note avoids the need to consider two cases.) Then $U_{n+1} = a_{n+1} + b_{n+1}$, where

$$a_{n+1} = \frac{1}{3} + \frac{1}{9} + \dots + \frac{1}{3(2k - 1)} = \frac{U_k}{3},$$

so that $a_{n+1} \in A_3$, and b_{n+1} is a sum of a number of terms $\frac{1}{r}$ with $r \in N_3$, so that, by (ii), $b_{n+1} \in B_3$. So, by (i), $U_{n+1} \in A_3$.

A curious further fact about U_n emerges effortlessly from the properties of A_2 and B_2 stated in the original note, without either version of the induction proof. Clearly, $U_n \in B_2$, so if $a \in A_2$, then $U_n + a$ is in A_2, hence is not an integer. In other words, numbers of the form $U_n + \frac{r}{2s}$ with r odd (for example, $U_n + \frac{1}{2}$) are not integers. It is, at least, mildly amusing that this was obtained with less work than the fact that U_n is not an integer.

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