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Abstract

Assuming a particular case of the Borisov–Alexeev–Borisov conjecture, we prove that finite subgroups of the automorphism group of a finitely generated field over $\mathbb{Q}$ have bounded orders. Further, we investigate which algebraic varieties have groups of birational selfmaps satisfying the Jordan property. Unless explicitly stated otherwise, all varieties are assumed to be algebraic, geometrically irreducible and defined over an *arbitrary* field $\Bbbk$ of characteristic zero.

1. Introduction

This paper is motivated by two questions of Serre [Ser09, Edi10] concerning the finite subgroups of automorphism groups of fields of characteristic zero.

Our starting point is the following.

Question 1.1 (Serre [Edi10]). Let K be a finitely generated field over $\mathbb{Q}$. Is it true that there is a constant $B = B(K)$ such that any finite subgroup $G \subset \mathrm{Aut}(K)$ has order $|G| \leqslant B$?

We will refer to the property mentioned in Question 1.1 as *boundedness of finite subgroups.*

Definition 1.2 (cf. [Pop11, Definition 2.9]). Let $\mathcal{G}$ be a family of groups. We say that $\mathcal{G}$ *has uniformly bounded finite subgroups* if there exists a constant $B = B(\mathcal{G})$ such that for any $\Gamma \in \mathcal{G}$ and for any finite subgroup $G \subset \Gamma$ one has $|G| \leqslant B$. We say that a group Γ *has bounded finite subgroups* if the family $\{\Gamma\}$ has uniformly bounded finite subgroups.

To answer Question 1.1 we will translate it into geometrical language. In some of our arguments we will rely on a particular case of the well-known Borisov–Alexeev–Borisov conjecture (see [Bor96]).

Conjecture *BAB*. Let $\bar{\Bbbk}$ be an algebraically closed field. For a given positive integer n, Fano varieties of dimension n with terminal singularities defined over $\bar{\Bbbk}$ are bounded, i.e. are contained in a finite number of algebraic families.

Remark 1.3. Note that if Conjecture BAB holds in dimension n, then it holds in any dimension $m \leqslant n$.

The first main result of our paper is as follows.

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THEOREM 1.4. *Suppose that* $\Bbbk$ *is a finitely generated field over* $\mathbb{Q}$*. Let* X *be a variety of dimension* n*. Suppose that Conjecture BAB holds in dimension* n*. Then the group* $\mathrm{Bir}(X)$ *of birational automorphisms of* X *over* $\Bbbk$ *has bounded finite subgroups.*

COROLLARY 1.5. *The answer to Question 1.1 is positive modulo Conjecture BAB (cf. Corollary 1.9 below).*

Besides boundedness of finite subgroups, there is a somewhat analogous property of groups that has recently attracted attention of algebraic geometers.

DEFINITION 1.6 (cf. [Pop11, Definition 2.1]). Let $\mathcal{G}$ be a family of groups. We say that $\mathcal{G}$ is *uniformly Jordan* if there is a constant $J = J(\mathcal{G})$ such that for any group $\Gamma \in \mathcal{G}$ and any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J. We say that a group Γ is *Jordan* if the family $\{\Gamma\}$ is uniformly Jordan.

The classically known examples of Jordan groups include $\mathrm{GL}_m(\mathbb{C})$, pointed out by C. Jordan (see e.g. [CR62, Theorem 36.13]), and thus all linear algebraic groups over an arbitrary field of characteristic zero. Serre proved that the group of birational automorphisms of the projective plane $\mathbb{P}^2$ over a field of characteristic zero is Jordan (see [Ser09, Theorem 5.3]), and asked if the same holds for groups of birational automorphisms of projective spaces $\mathbb{P}^n$ for $n \geqslant 3$ (see [Ser09, 6.1]). Recently the authors were able to establish the following theorem that deals with the case of rationally connected varieties (see e.g. [Kol96, IV.3.2]) of arbitrary dimension, and in particular answers the latter question.

THEOREM 1.7 [PS15, Theorem 1.8]. *Let* $\mathcal{G}_{\mathrm{rc}}(n)$ *be the family of groups* $\mathrm{Bir}(X)$*, where* X *varies in the set of rationally connected varieties of dimension* n*. Assume that Conjecture BAB holds in dimension* n*. Then the family* $\mathcal{G}_{\mathrm{rc}}(n)$ *is uniformly Jordan.*

Zarhin found an example of a surface X such that the group $\mathrm{Bir}(X)$ is not Jordan (see [Zar10]), and Popov classified all surfaces X such that $\mathrm{Bir}(X)$ is Jordan (see [Pop11, Theorem 2.32]). The next natural step may be to wonder whether it is possible to do something similar in higher dimensions. The second main result of our current paper is the following theorem that completely solves the question for non-uniruled varieties (see [Kol96, § IV.1.1]) and partially describes the general case. Recall that *irregularity* of a variety X is defined as $q(X) = \dim H^1(X', \mathcal{O}_{X'})$, where X' is any smooth projective variety birational to X.

THEOREM 1.8. *Let* X *be a variety of dimension* n*. Then the following assertions hold.*

(i) *The group* $\mathrm{Bir}(X)$ *has bounded finite subgroups provided that* X *is non-uniruled and has irregularity* $q(X) = 0$*.*

(ii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that* X *is non-uniruled.*

(iii) *Suppose that Conjecture BAB holds in dimension* n*. Then the group* $\mathrm{Bir}(X)$ *is Jordan provided that* X *has irregularity* $q(X) = 0$*.*

Note that Conjecture BAB is proved in dimension $n \leqslant 3$ (see [KMMT00]). Therefore, one has the following result.

COROLLARY 1.9 (cf. [PS15, Corollary 1.9]). *Theorems 1.4 and 1.8 (as well as Theorem 1.7) hold in dimension* $n \leqslant 3$ *without any additional assumptions.*

Remark 1.10. Zarhin showed in [Zar10] that the group of birational selfmaps of a variety that is isomorphic to a product of an abelian variety and a projective line over an algebraically closed field of characteristic zero violates the Jordan property. This shows that one cannot remove the conditions of non-uniruledness and vanishing irregularity from Theorem 1.8(ii), (iii). Moreover, by [Pop11, Theorem 2.32] the only surface X over an algebraically closed field of characteristic zero such that $\mathrm{Bir}(X)$ fails to have the Jordan property is the product $E \times \mathbb{P}^1$, where E is an elliptic curve. Thus (to a certain extent) we may consider Theorem 1.8(ii), (iii) to be a generalization of [Pop11, Theorem 2.32].

Besides the results listed above, in § 8 we discuss a 'solvable' analog of the Jordan property. The main result there is Proposition 8.6 which answers Question 8.3 asked by D. Allcock.

Remark 1.11. Note that the group $\mathrm{Bir}(X)$ of birational automorphisms of a variety X over $\Bbbk$ has a structure of a $\Bbbk$-scheme, although in general it is not a group scheme, and it is not a birational invariant of X (see [Han88, § 1]). However, both of these properties hold in an important particular case when X is a minimal model (see [Han87, Theorem 3.3(1)] and [Han87, Theorem 3.7(2)]). If X is non-uniruled, then the structure of $\mathrm{Bir}(X)$ is much better understood than in the general case. In particular, it is known that if X is non-uniruled, then the dimension of $\mathrm{Bir}(X)$ is at most $q(X)$ (see [Han88, Theorem 2.1(i)]). Moreover, there is an interesting conjecture that may be relevant to Theorem 1.8(i): if X is non-uniruled and satisfies some additional assumptions, then the 'discrete part' $\mathrm{Bir}(X)_{\mathrm{red}}/\mathrm{Bir}(X)^0$ is finitely generated (see [Han88, § 7.4]). Since the general structure of $\mathrm{Bir}(X)$ is not a subject of this paper, we refer the reader to [Han87], [Han88] and references therein for further information.

The plan of the paper is as follows. In § 2 we discuss the basic (group-theoretical) properties of Jordan groups and groups with bounded finite subgroups, and also collect some important examples of groups of each of these two classes. In § 3 we recall some well-known auxiliary geometrical facts. In § 4 we introduce (following a suggestion of Caucher Birkar) quasi-minimal models that are analogs of minimal models such that one does not need the full strength of the minimal model program to prove their existence. In § 5 we discuss the action of finite groups on quasi-minimal models. In § 6 we prove Proposition 6.2, which is our main auxiliary result describing the general structure of finite groups of birational automorphisms, and use it to derive Theorem 1.8. In § 7 we prove Theorem 1.4 and derive Corollary 1.5. In § 8 we discuss solvably Jordan groups. Finally, in § 9 we discuss some open questions related to the subject of this paper.

2. Basic properties

Remark 2.1. If a family $\mathcal{G}$ has uniformly bounded finite subgroups, then it is uniformly Jordan.

Lemma 2.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of groups with uniformly bounded finite subgroups. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1,$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ has uniformly bounded finite subgroups.

Proof. Straightforward. □

Lemma 2.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of groups such that $\mathcal{G}_1$ is uniformly Jordan and $\mathcal{G}_2$ has uniformly bounded finite subgroups. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1,$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ is uniformly Jordan.

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Proof. See [Pop11, Lemma 2.11]. ☐

Remark 2.4 (cf. [Pop11, Remark 2.12]). If $\mathcal{G}_1$ and $\mathcal{G}_2$ are families of groups such that $\mathcal{G}_1$ has uniformly bounded finite subgroups and $\mathcal{G}_2$ is uniformly Jordan, then a family $\mathcal{G}$ of groups G such that there exists an exact sequence

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

with $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ may fail to be uniformly Jordan. For example, this is the case if $\mathcal{G}_1$ consists of a single group $\mathbb{Z}/p\mathbb{Z}$, where p is a prime, and $\mathcal{G}_2$ consists of groups of the form $(\mathbb{Z}/p\mathbb{Z})^{2r}$ for various r.

For applications in § 6 we would like to know some additional condition that would guarantee that the extensions considered in Remark 2.4 form a uniformly Jordan family (cf. [Pop11, Corollary 2.13]). One of such conditions relies on the following auxiliary definition.

Definition 2.5. Let $\mathcal{G}$ be a family of groups. We say that $\mathcal{G}$ *has finite subgroups of uniformly bounded rank* if there exists a constant $R = R(\mathcal{G})$ such that for any $\Gamma \in \mathcal{G}$ each finite abelian subgroup $A \subset \Gamma$ is generated by at most R elements. We say that a group Γ *has finite subgroups of bounded rank* if the family $\{\Gamma\}$ has finite subgroups of uniformly bounded rank.

Remark 2.6. If a family $\mathcal{G}$ has uniformly bounded finite subgroups, then it has finite subgroups of uniformly bounded rank.

Lemma 2.7. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of groups with finite subgroups of uniformly bounded rank. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1,$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ has finite subgroups of uniformly bounded rank.

Proof. Straightforward. ☐

An advantage of Definition 2.5 is the following property explained to us by Anton Klyachko.

Lemma 2.8. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of groups such that $\mathcal{G}_1$ has uniformly bounded finite subgroups and $\mathcal{G}_2$ is uniformly Jordan and has finite subgroups of uniformly bounded rank. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1,$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ is uniformly Jordan.

Proof. It is enough to prove that if F is a finite group and A is a finite abelian group generated by r elements, then for any extension

$$1 \longrightarrow F \longrightarrow G \longrightarrow A \longrightarrow 1$$

one can bound the index $[G : Z]$ of the center Z of the group G in terms of $|F|$ and r.

Let $K \subset G$ be the commutator subgroup. Since A is abelian, one has

$$|K| \leqslant \frac{|G|}{|A|} = |F|.$$

For $x \in G$ let $K(x)$ be the set of the commutators of the form $gxg^{-1}x^{-1}$ for various $g \in G$, and $Z(x) \subset G$ be the centralizer of x. It is easy to see that

$$[G : Z(x)] = |K(x)| \leqslant |K|.$$

Now if $\{x_1, \ldots, x_N\} \subset G$ is a subset that generates G, then $Z = Z(x_1) \cap \cdots \cap Z(x_N)$, so that

$$[G : Z] \leqslant [G : Z(x_1)] \cdot \cdots \cdot [G : Z(x_N)] \leqslant |K(x_1)| \cdot \cdots \cdot |K(x_N)| \leqslant |K|^N \leqslant |F|^N.$$

It remains to notice that one can choose a set of $N \leqslant r|F|$ generators for the group G. □

Remark 2.9. Let $\mathcal{G}$ be a family of groups, and let $\tilde{\mathcal{G}}$ be the family that consists of all finite subgroups of all groups in $\mathcal{G}$. Then the family $\mathcal{G}$ has bounded finite subgroups (respectively is uniformly Jordan, has finite subgroups of uniformly bounded rank) if and only if the family $\tilde{\mathcal{G}}$ has bounded finite subgroups (respectively is uniformly Jordan, has finite subgroups of uniformly bounded rank). We will sometimes use this trivial observation without any further comments while applying Lemmas 2.2, 2.3, 2.7 and 2.8.

Now we will discuss some important examples of groups with bounded finite subgroups and of Jordan groups. We will use the following notation.

Definition 2.10. Let A be an abelian variety over $\Bbbk$. By $\mathrm{Aut}_g(A)$ we denote the group of automorphisms of A as a $\Bbbk$-variety (i.e. the group of automorphisms of the variety A that may not respect the group structure on A).

Remark 2.11. One has

$$\mathrm{Aut}_g(A) \simeq A(\Bbbk) \rtimes \Gamma,$$

where $A(\Bbbk)$ denotes the group of $\Bbbk$-points of the abelian variety A and Γ is a subgroup of $\mathrm{GL}_{2\dim(A)}(\mathbb{Z})$.

The following is a well-known theorem of H. Minkowski (see e.g. [Ser07, Theorem 5] and [Ser07, § 4.3]).

Theorem 2.12. *Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$, and m is a positive integer. Then the group $\mathrm{GL}_m(\Bbbk)$ has bounded finite subgroups.*

Corollary 2.13. *The group $\mathrm{GL}_m(\mathbb{Z})$ has bounded finite subgroups.*

Corollary 2.14. *Let N be a finitely generated abelian group. Then the group $\mathrm{Aut}(N)$ has bounded finite subgroups.*

Proof. One has an exact sequence

$$0 \longrightarrow T \longrightarrow N \longrightarrow N/T \longrightarrow 0,$$

where T is the torsion subgroup of N and N/T is a free abelian group of finite rank r. Therefore, one has an exact sequence

$$0 \longrightarrow T^r \longrightarrow \mathrm{Aut}(N) \longrightarrow \mathrm{Aut}(T) \times \mathrm{Aut}(N/T) \longrightarrow 1.$$

The group $\mathrm{Aut}(N/T) \simeq \mathrm{GL}_r(\mathbb{Z})$ has bounded finite subgroups by Corollary 2.13, while the groups T^r and $\mathrm{Aut}(T)$ are finite. Now the assertion follows by Lemma 2.2. □

Corollary 2.15. *Let $\mathcal{A}_d$ be the family of groups $\mathrm{Aut}_g(A)$, where A varies in the set of abelian varieties of dimension d over a field $\Bbbk$, while $\Bbbk$ varies in the set of all fields of characteristic zero. Then $\mathcal{A}_d$ is uniformly Jordan and has finite subgroups of uniformly bounded rank.*

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Proof. To prove that $\mathcal{A}_d$ is uniformly Jordan, apply Remark 2.11, Corollary 2.13 and Lemma 2.3.

Let us prove that $\mathcal{A}_d$ has finite subgroups of uniformly bounded rank. Let A be an abelian variety of dimension d. Then for any positive integer n the n-torsion subgroup of the group $A(\bar{\Bbbk})$, where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$, is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2d}$, which implies that any finite subgroup of $A(\Bbbk)$ is generated by at most $2d$ elements. Furthermore, any finite abelian subgroup of $\mathrm{GL}_m(\bar{\Bbbk})$ is diagonalizable, so that any finite abelian subgroup $H \subset \mathrm{GL}_{2d}(\mathbb{Z})$ is also generated by at most $2d$ elements (alternatively, one can use Corollary 2.13 to deduce that $\mathrm{GL}_{2d}(\mathbb{Z})$ has finite subgroups of bounded rank). Now the assertion follows by Remark 2.11 and Lemma 2.7. □

COROLLARY 2.16. *Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$. Let A be an abelian variety over $\Bbbk$. Then the group $\mathrm{Aut}_g(A)$ has bounded finite subgroups.*

Proof. Recall that the group $A(\Bbbk)$ of $\Bbbk$-points of A is a finitely generated abelian group by the Mordell–Weil theorem (see [Lan83, ch. 6, Theorem 1]). Thus $A(\Bbbk)$ has bounded finite subgroups, so that the assertion follows by Remark 2.11, Corollary 2.13 and Lemma 2.2. □

When $\Bbbk$ is a number field, it is expected that a stronger version of Corollary 2.16 holds. The starting point here is the following.

CONJECTURE 2.17. Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$, and d is a positive integer. Then there is a constant $M = M(d)$ such that for any abelian variety A of dimension d the order of the torsion subgroup $A(\Bbbk)_{\mathrm{tors}}$ in $A(\Bbbk)$ is less than M.

Note that Conjecture 2.17 is not universally recognized as credible (cf. [Poo12, Question 2.1]; see also [MS94, Boundedness Conjecture] and [Fak03, Corollary 2.4]). Conjecture 2.17 is proved only for dimension $d = 1$, i.e. for elliptic curves. The case of a number field was established in [Mer96], and the case of an arbitrary field $\Bbbk$ finitely generated over $\mathbb{Q}$ is derived from it in a standard way (see e.g. the remark made after Question 2.1 in [Poo12]).

Modulo Conjecture 2.17 one has the following refined version of Corollary 2.16.

COROLLARY 2.18. *Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$. Let $\mathcal{A}_d(\Bbbk)$ be the family of groups $\mathrm{Aut}_g(A)$, where A varies in the set of abelian varieties of dimension d over $\Bbbk$. Suppose that Conjecture 2.17 holds in dimension d over $\Bbbk$. Then $\mathcal{A}_d(\Bbbk)$ has uniformly bounded finite subgroups.*

3. Divisor class groups

The following notion is well known and widely used (cf. [PS15, § 4]).

LEMMA–DEFINITION 3.1. *Let X be a variety and $G \subset \mathrm{Bir}(X)$ be a finite group. There exists a normal projective variety $\tilde{X}$ with a biregular action of G and a G-equivariant birational map $\tilde{X} \dashrightarrow X$. The variety $\tilde{X}$ is called* a regularization *of G. Moreover, $\tilde{X}$ can be taken smooth and then $\tilde{X}$ is called* a smooth regularization *of G.*

Proof. By shrinking X we may assume that X is affine and G acts on X biregularly. Then the quotient $V = X/G$ is also affine, so there exists a projective completion $\tilde{V} \supset V$. Let $\tilde{X}$ be the normalization of $\tilde{V}$ in the function field $\Bbbk(X)$. Then $\tilde{X}$ is a projective variety G-birational to X, and $\tilde{X}$ admits a biregular action of G. Taking a G-equivariant resolution of singularities (see [BM08]), one can assume that $\tilde{X}$ is smooth. □

Let X be a normal projective variety. By $\mathrm{Cl}(X)$ we denote the group of Weil divisors on X modulo linear equivalence, and by $\mathrm{Cl}^0(X)$ its subgroup consisting of divisors that are algebraically

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equivalent to zero. Let $f : \tilde{X} \to X$ be a resolution of singularities. It induces a natural map

$$f_* : \mathrm{Cl}^0(\tilde{X}) \longrightarrow \mathrm{Cl}^0(X).$$

The group $\mathrm{Cl}(X)$ is canonically isomorphic to the quotient of $\mathrm{Cl}(\tilde{X})$ by the subgroup $\mathcal{E} \subset \mathrm{Cl}(\tilde{X})$ generated by f-exceptional divisors. Since prime exceptional divisors are linearly independent modulo numerical equivalence (see e.g. [Kol92, Lemma 2.19]), we have

$$\mathrm{Cl}^0(X) \cap \mathcal{E} = 0.$$

Hence $f_* : \mathrm{Cl}^0(\tilde{X}) \to \mathrm{Cl}^0(X)$ is an isomorphism. In particular, $\mathrm{Cl}^0(X)$ is a birational invariant in the category of projective varieties. Moreover, f_* induces a structure of an abelian variety on $\mathrm{Cl}^0(X)$. The group

$$\mathrm{NS}^{\mathrm{W}}(X) = \mathrm{Cl}(X)/\mathrm{Cl}^0(X)$$

is a homomorphic image of the group

$$\mathrm{NS}^{\mathrm{W}}(\tilde{X}) = \mathrm{Cl}(\tilde{X})/\mathrm{Cl}^0(\tilde{X}) \simeq \mathrm{Pic}(\tilde{X})/\mathrm{Pic}^0(\tilde{X}),$$

which is finitely generated by the Neron–Severi theorem. Therefore, $\mathrm{NS}^{\mathrm{W}}(X)$ is also finitely generated. Slightly abusing the standard terminology, we will refer to the group $\mathrm{NS}^{\mathrm{W}}(X)$ as *the Neron–Severi group* of X.

Remark 3.2. Let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. By Lemma–Definition 3.1 we can choose a regularization $\tilde{X}$ of G, so that $G \subset \mathrm{Aut}(\tilde{X})$. Since Pic^0 is a functor, the group G naturally acts on $\mathrm{Cl}^0(X) \simeq \mathrm{Pic}^0(\tilde{X})$, and this action does not depend on our choice of the resolution f.

4. Quasi-minimal models

Starting from this point we will use standard terminology and conventions of the minimal model program (see e.g. [KMM87] or [Mat02]). We note that there exist natural generalizations of the minimal model program to the cases of varieties over non-closed field and varieties with group action. Since these notions are quite standard (see e.g. [KM98, § 2.2]), we will refer to the recent results of [BCHM10] concerning the minimal model program without further comments on these different setups.

In this section we introduce the notion of quasi-minimal models, following the idea of Caucher Birkar. This is a weaker analog of a usual notion of minimal models which has an advantage that to prove its existence we do not need the full strength of the minimal model program.

Definition 4.1. An effective $\mathbb{Q}$-divisor M on a variety X is said to be $\mathbb{Q}$-*movable* if for some $n > 0$ the divisor nM is integral and generates a linear system without fixed components.

Definition 4.2. Let X be a projective variety with terminal singularities. We say that X is a *quasi-minimal model* if there exists a sequence of $\mathbb{Q}$-movable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors M_j whose limit in the Neron–Severi space $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X) = \mathrm{NS}^{\mathrm{W}}(X) \otimes \mathbb{Q}$ is K_X.

Remark 4.3. Any minimal model is a quasi-minimal model by Kleiman ampleness criterion. By [MM86, Theorem 1] any quasi-minimal model is non-uniruled.

Now we will show that the current state of art in the minimal model program allows the existence of quasi-minimal models to be proved. Recall that a (normal) variety X acted on by a finite group Γ has $\Gamma\mathbb{Q}$-*factorial* singularities, if and only if any Γ-invariant Weil divisor on X is $\mathbb{Q}$-Cartier.

Lemma 4.4. *Let X be a projective non-uniruled variety with terminal singularities, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite subgroup. Assume that X has $\Gamma\mathbb{Q}$-factorial singularities. Then there exists a quasi-minimal model X' birational to X such that $\Gamma \subset \mathrm{Aut}(X')$.*

Proof. Run a Γ-equivariant minimal model program on X. Since X is non-uniruled, we will never arrive to a non-birational contraction by [KMM87, Corollary 5-1-4]. Thus, if this Γ-equivariant minimal model program terminates, then it gives a minimal model (that is, in particular, a quasi-minimal model) X' birational to X such that $\Gamma \subset \mathrm{Aut}(X')$.

We use induction in the Picard number $\rho(X) = \dim \mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X)$. If $\rho(X) = 1$, then X is a minimal model (and, in particular, a quasi-minimal model) itself. If some step of the Γ-minimal model program is a divisorial contraction, then the Picard number drops at least by one at this step, and we proceed by induction. The only disaster that may happen is that the Γ-equivariant minimal model program ran on X does not terminate, and each of its steps is a Γ-flip. We claim that in this case X is a quasi-minimal model.

Take a very ample Γ-invariant divisor A on X and a sequence of positive numbers t_j approaching 0. According to [BCHM10] (or rather the Γ-equivariant versions of the corresponding theorems) we can run a Γ-equivariant $(K_X + t_jA)$-minimal model program on X with scaling of A to obtain a Γ-equivariant birational map

$$\psi_j : (X, t_jA) \dashrightarrow (X_j, t_jA_j).$$

Since X is not uniruled, (X_j, t_jA_j) is a log minimal model. By the construction of the minimal model program with scaling (see [BCHM10]) all extremal rays of ψ_j are A-positive. Hence, they are K-negative, and so ψ_j is a composition of Γ-flips. Since the $\mathbb{Q}$-divisor $K_{X_j} + t_jA_j$ is nef, it is a limit of $\mathbb{Q}$-movable (and even ample) $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors by Kleiman ampleness criterion. On the other hand, the varieties X_j are isomorphic in codimension one, so that the Neron–Severi spaces $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X_j)$ are naturally identified with $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X)$ (cf. § 5 below), and the divisors $K_{X_j} \in \mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X_j)$ correspond to $K_X \in \mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X)$. Therefore, the divisor K_X is also a limit of $\mathbb{Q}$-movable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, i.e. X is a quasi-minimal model. □

Remark 4.5. In this paper we will use Lemma 4.4 only for a trivial group Γ (nevertheless, this will allow us to make conclusions about certain non-trivial groups acting on X). Still we prefer to give a more general form of the lemma since we believe that it may have other applications.

Below we will establish an important property of quasi-minimal models that they share with minimal models.

Proposition 4.6 (cf. [Bir12, Proof of Theorem 4.1, Step 3]). *Let X be a quasi-minimal model and let $\chi : X \dashrightarrow X'$ be any birational map, where X' has only terminal singularities. Then χ does not contract any divisors.*

Proof. Assume that χ contracts a (prime) divisor D. Consider a common resolution

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[rr, dashed, "\chi"] & & X'
\end{tikzcd}$$

and let $D_Z \subset Z$ be the proper transform of D. Clearly, D_Z is g-exceptional.

Take a sequence of $\mathbb{Q}$-movable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors M_j whose limit in the Neron–Severi space $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(X)$ is K_X, write

$$K_Z \equiv f^*K_X + E,$$

and put $N_j = f^*M_j + E$. Since X has terminal singularities, the $\mathbb{Q}$-divisor E is effective, and thus the $\mathbb{Q}$-divisor N_j is also effective. Moreover, K_Z is the limit of the $\mathbb{Q}$-divisors N_j in the Neron–Severi space $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(Z)$, and we may assume that D_Z is not a component of N_j.

Since g is birational, by [BCHM10, Corollary 1.4.2] we can run K-minimal model program over X':

$$Z = Z_1 \dashrightarrow \cdots \dashrightarrow Z_i \overset{p_i}{\dashrightarrow} Z_{i+1} \dashrightarrow \cdots \dashrightarrow Z_n \longrightarrow X'.$$

Since X' has only terminal singularities, the map $Z_n \to X'$ is a small K-trivial contraction. We may assume that the proper transform of D is contracted by p_i. Thus p_i is a morphism whose exceptional locus $D^{(i)} \subset Z_i$ is the proper transform of D.

Let $N_j^{(i)}$ be the proper transform of N_j on Z_i. Then $D^{(i)}$ is not a component of $N_j^{(i)}$. Moreover, K_{Z_i} is a limit of $N_j^{(i)}$ in the Neron–Severi space $\mathrm{NS}^{\mathrm{W}}_{\mathbb{Q}}(Z_j)$. The divisor K_{Z_i} is strictly negative on the curves in fibers of p_i, so that $N_j^{(i)}$ is also negative on them for $j \gg 0$. Note that one can choose an algebraic family of such curves covering $D^{(i)}$. Thus $D^{(i)}$ is a component of $N_j^{(i)}$ for $j \gg 0$, which is a contradiction. □

Corollary 4.7 (cf. [Han87, Lemma 3.4]). *Let X and Y be two quasi-minimal models. Then every birational map $\chi : X \dashrightarrow Y$ is an isomorphism in codimension one.*

5. Groups acting on quasi-minimal models

Let X be a quasi-minimal model (see Definition 4.2), and $G \subset \mathrm{Bir}(X)$ be a finite group. By Corollary 4.7, any element $g \in G$ maps X to itself isomorphically in codimension one. Thus, G acts on $\mathrm{Cl}(X)$ and on $\mathrm{Cl}^0(X)$. Clearly, this induces an action of G on $\mathrm{NS}^{\mathrm{W}}(X)$, i.e. a homomorphism

$$\theta_{NS} : G \longrightarrow \mathrm{Aut}(\mathrm{NS}^{\mathrm{W}}(X)).$$

Moreover, the kernel

$$\mathrm{Ker}(\theta_{NS}) \subset G$$

acts on any algebraic equivalence class $\mathrm{Cl}_L(X) \subset \mathrm{Cl}(X)$ preserving the structure of an algebraic variety on $\mathrm{Cl}_L(X)$.

Remark 5.1. In the above notation, assume also that the field $\Bbbk$ is algebraically closed and X is a minimal model. Then, according to [Han87, Theorem 3.3(1)], the group $\mathrm{Bir}(X)$ has a natural structure of a group scheme. Using this one can define an action of the whole group $\mathrm{Bir}(X)$ on $\mathrm{Cl}(X)$ and $\mathrm{Cl}^0(X)$. Since we are interested only in finite group actions, we do not need these constructions, and take an advantage of a more elementary approach that also does not need additional assumptions on $\Bbbk$.

Lemma 5.2. *Let X be a quasi-minimal model. Let L be an ample $\mathbb{Q}$-Cartier divisor on X. Then the group $\mathrm{Bir}(X, L)$ of birational automorphisms of X that preserve the class $[L] \in \mathrm{Pic}(X)$ is finite.*

Proof. We may assume that L is a very ample Cartier divisor. Suppose that some element $\varphi \in \mathrm{Bir}(X)$ preserves the class $[L] \in \mathrm{Pic}(X) \subset \mathrm{Cl}(X)$. Since

$$X \simeq \mathrm{Proj} \bigoplus_{n \geqslant 0} H^0(X, nL),$$

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the map φ is in fact a biregular automorphism of X. Therefore, $\mathrm{Bir}(X, L)$ is a subgroup of the group of linear transformations of the projective space

$$\mathbb{P}(H^0(X, L)^\vee) \simeq \mathbb{P}^N,$$

so that

$$\mathrm{Bir}(X, L) \subset \mathrm{PGL}_{N+1}(\Bbbk).$$

If $\mathrm{Bir}(X, L)$ is not finite, then it contains a one-parameter subgroup G, where $G \simeq \mathbb{G}_{\mathrm{m}}$ or $G \simeq \mathbb{G}_{\mathrm{a}}$. In particular, for a general point $P \in X$ the orbit $G \cdot P$ must be a geometrically rational curve, so that X is uniruled. This contradicts the fact that a quasi-minimal model is non-uniruled (see Remark 4.3). □

6. Proof of Theorem 1.8

In this section we prove Proposition 6.2, which is our main auxiliary result describing the general structure of finite groups of birational automorphisms, and use it to derive Theorem 1.8.

Recall that to any variety X one can associate *the maximal rationally connected fibration*

$$\phi_{\mathrm{rc}} : X \dashrightarrow X_{\mathrm{nu}},$$

which is a canonically defined rational map with rationally connected fibers and non-uniruled base X_{nu} (see [Kol96, § IV.5] and [GHS03, Corollary 1.4]).

DEFINITION 6.1. Let X be a variety. By *a birational polarization on the base of the maximal rationally connected fibration of* X we mean an ample divisor on one of the quasi-minimal models (see Definition 4.2) of the base X_{nu} of the maximal rationally connected fibration ϕ_{rc}.

PROPOSITION 6.2. *Let X be a variety of dimension n, and let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. Choose some birational polarization L on the base of the maximal rationally connected fibration of X. Then there exist exact sequences*

$$1 \longrightarrow G_{\mathrm{rc}} \longrightarrow G \longrightarrow G_{\mathrm{nu}} \longrightarrow 1 \tag{6.3}$$

$$1 \longrightarrow G_{\mathrm{alg}} \longrightarrow G_{\mathrm{nu}} \longrightarrow G_N \longrightarrow 1 \tag{6.4}$$

$$1 \longrightarrow G_L \longrightarrow G_{\mathrm{alg}} \longrightarrow G_{\mathrm{ab}} \longrightarrow 1 \tag{6.5}$$

with the following properties:

(i) *the group G_{nu} is a subgroup of a group $\mathrm{Bir}(X_{\mathrm{nu}})$ for some quasi-minimal model X_{nu} of dimension at most n that depends only on X (but not on the subgroup G);*

(ii) *the group G_{rc} is a subgroup of a group $\mathrm{Bir}(X_{\mathrm{rc}})$ for some rationally connected variety X_{rc} of dimension at most n defined over the field $\Bbbk(X_{\mathrm{nu}})$;*

(iii) *the group G_N is a subgroup of $\mathrm{Aut}(N)$ for the finitely generated abelian group $N = \mathrm{NS}^{\mathrm{W}}(X_{\mathrm{nu}})$ that depends only on X;*

(iv) *the group G_{alg} acts (maybe not faithfully) on each of the algebraic equivalence classes of Weil divisors on X_{nu};*

(v) *the group G_{ab} is a subgroup of a group $\mathrm{Aut}_g(A)$ (see Definition 2.10), where A is an abelian variety of dimension $q(X_{\mathrm{nu}}) = q(X)$;*

(vi) *the group G_L is a subgroup of a group $\mathrm{Bir}(X_{\mathrm{nu}})$ that preserves the class L (cf. § 5).*

Proof. Let

$$\phi_{\mathrm{rc}} : X \dashrightarrow X_{\mathrm{nu}}$$

be the maximal rationally connected fibration. Choose $G_{\mathrm{rc}} \subset G$ to be the maximal subgroup acting fiberwise with respect to ϕ_{rc}. Let X_{rc} be the fiber of ϕ_{rc} over the generic scheme point of X_{nu}. Since the maximal rationally connected fibration is functorial (see [Kol96, Theorem IV.5.5]), the group G_{rc} acts by birational transformations of X_{rc}. Furthermore, the group

$$G_{\mathrm{nu}} = G/G_{\mathrm{rc}}$$

acts by birational transformations of X_{nu}. Note that X_{rc} is a rationally connected variety over the field $\Bbbk(X_{\mathrm{nu}})$, and the variety X_{nu} is non-uniruled. We may assume that X_{nu} is a quasi-minimal model (on which the group G_{nu} still acts by birational transformations), and L is an ample divisor class on X_{nu}. In particular, we have established the exact sequence (6.3) and proved properties (i) and (ii).

Consider the action of G_{nu} on the group of Weil divisors $\mathrm{Cl}(X_{\mathrm{nu}})$ and on the Neron–Severi group $\mathrm{NS}^{\mathrm{W}}(X_{\mathrm{nu}})$ (see § 5). Let $G_{\mathrm{alg}} \subset G_{\mathrm{nu}}$ be the kernel of this action. In particular, the action of G_{alg} on $\mathrm{Cl}(X_{\mathrm{nu}})$ preserves each of the algebraic equivalence classes of Weil divisors on X_{nu}. Moreover, the group

$$G_N = G_{\mathrm{nu}}/G_{\mathrm{alg}}$$

is a subgroup of the automorphism group of the finitely generated abelian group $\mathrm{NS}^{\mathrm{W}}(X_{\mathrm{nu}})$. Therefore, we have established the exact sequence (6.4) and proved properties (iii) and (iv).

Denote by

$$\mathrm{Cl}_L(X_{\mathrm{nu}}) \subset \mathrm{Cl}(X_{\mathrm{nu}})$$

the class of algebraic equivalence of the divisor $L \in \mathrm{Cl}(X_{\mathrm{nu}})$. Recall that $\mathrm{Cl}_L(X_{\mathrm{nu}})$ has a structure of an algebraic variety, so that $\mathrm{Cl}_L(X_{\mathrm{nu}})$ is a torsor over an abelian variety $\mathrm{Cl}^0(X)$. Moreover, the group G_{alg} acts by automorphisms of the variety $\mathrm{Cl}_L(X_{\mathrm{nu}})$ (see § 5). Let $G_L \subset G_{\mathrm{alg}}$ be the kernel of the action of G_{alg} on $\mathrm{Cl}_L(X_{\mathrm{nu}})$. In particular, the group G_L preserves the class L. Moreover, the group

$$G_{\mathrm{ab}} = G_{\mathrm{alg}}/G_L$$

is a subgroup of $\mathrm{Aut}_g(\mathrm{Cl}_L(X_{\mathrm{nu}}))$. Therefore, we have established the exact sequence (6.5) and proved properties (v) and (vi). $\square$

Remark 6.6. Note that in the proof of Proposition 6.2 in the case of an algebraically closed field $\Bbbk$ one can avoid using the field $\Bbbk(X_{\mathrm{nu}})$ and choose X_{rc} to be a fiber of ϕ_{rc} over a general *closed* point of X_{nu}. Still in the general case passing to the field $\Bbbk(X_{\mathrm{nu}})$ looks inevitable (at least in our approach) since the base X_{nu} may have no $\Bbbk$-points at all.

Remark 6.7. The choice of a birational polarization L on the base of the maximal rationally connected fibration of X in Proposition 6.2 is auxiliary, and the main properties of the groups we are going to consider will not depend on this choice (although the particular groups arising in the exact sequences (6.3), (6.4) and (6.5) may depend on L).

Corollary 6.8. *Let X be a variety of dimension n. Let $\mathcal{G}_{\mathrm{rc}}(X)$ and $\mathcal{G}_{\mathrm{ab}}(X)$ be the families of groups arising in Proposition 6.2 as the groups G_{rc} and G_{ab}, respectively, for various choices of finite groups $G \subset \mathrm{Bir}(X)$. Then:*

(i) *the group $\mathrm{Bir}(X)$ has bounded finite subgroups provided that $\mathcal{G}_{\mathrm{rc}}(X)$ and $\mathcal{G}_{\mathrm{ab}}(X)$ have uniformly bounded finite subgroups;*

(ii) *the group* $\mathrm{Bir}(X)$ *is Jordan provided that* $\mathcal{G}_{\mathrm{ab}}(X)$ *is uniformly Jordan and* $\mathcal{G}_{\mathrm{rc}}(X)$ *has uniformly bounded finite subgroups;*

(iii) *the group* $\mathrm{Bir}(X)$ *is Jordan provided that* $\mathcal{G}_{\mathrm{ab}}(X)$ *has uniformly bounded finite subgroups and* $\mathcal{G}_{\mathrm{rc}}(X)$ *is uniformly Jordan.*

Proof. Choose some birational polarization L on the base of the maximal rationally connected fibration of X. Let $\mathcal{G}_N(X)$ and $\mathcal{G}_L(X)$ be the families of groups arising in Proposition 6.2 as the groups G_N and G_L, respectively, for various choices of finite groups $G \subset \mathrm{Bir}(X)$. Then the family $\mathcal{G}_N$ has uniformly bounded finite subgroups by Corollary 2.14, and $\mathcal{G}_L(X)$ has uniformly bounded finite subgroups by Lemma 5.2. Therefore, assertion (i) follows from Proposition 6.2 and Lemma 2.2. Assertion (ii) follows from Proposition 6.2, Lemmas 2.7, 2.8 and 2.3 and Corollary 2.15. Finally, assertion (iii) follows from Proposition 6.2 and Lemmas 2.2 and 2.3. □

Remark 6.9 (cf. [PS15, Theorem 1.10]). Arguing as in the proof of Corollary 6.8, one can easily show that if X is non-uniruled, then the group $\mathrm{Bir}(X)$ has finite subgroups of bounded rank. Moreover, these arguments together with [PS15, Theorem 4.2] show that modulo Conjecture BAB the same assertion holds for an arbitrary variety X.

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8. Let $\mathcal{G}_{\mathrm{rc}}(X)$ and $\mathcal{G}_{\mathrm{ab}}(X)$ be the families of groups defined in Corollary 6.8. By Theorem 1.7 the family $\mathcal{G}_{\mathrm{rc}}(X)$ is uniformly Jordan. By Corollary 2.15 the family $\mathcal{G}_{\mathrm{ab}}(X)$ is uniformly Jordan. Moreover, $\mathcal{G}_{\mathrm{rc}}(X)$ consists of trivial groups (and, thus, has uniformly bounded finite subgroups) provided that the variety X is non-uniruled, and $\mathcal{G}_{\mathrm{ab}}(X)$ consists of trivial groups (and, thus, has uniformly bounded finite subgroups) provided that $q(X) = 0$. Now the assertions (i), (ii) and (iii) of Theorem 1.8 are implied by the assertions (i), (ii) and (iii) of Corollary 6.8, respectively. □

7. Proof of Theorem 1.4

In this section we use Proposition 6.2 to prove Theorem 1.4, and derive Corollary 1.5.

LEMMA 7.1. *Suppose that* $\Bbbk$ *is a finitely generated field over* $\mathbb{Q}$*, and* N *is a positive integer. Let* $\mathcal{G}$ *be the family of groups* $\mathrm{GL}_N(\mathbb{K})$*, where* $\mathbb{K}$ *varies in the set of finitely generated fields over* $\Bbbk$ *such that* $\Bbbk$ *is algebraically closed in* $\mathbb{K}$*. Then the family* $\mathcal{G}$ *has uniformly bounded finite subgroups.*

Proof. To start with, the family $\mathcal{G}$ is uniformly Jordan. Indeed, any finite subgroup of $\mathrm{GL}_N(\mathbb{K})$ is embeddable into, say, $\mathrm{GL}_N(\mathbb{C})$, so that the constants appearing in Definition 1.6 for the groups $\mathrm{GL}_N(\mathbb{K})$ are all bounded by the corresponding constant for $\mathrm{GL}_N(\mathbb{C})$. Therefore, replacing a finite subgroup $G \subset \mathrm{GL}_N(\mathbb{K})$ by its abelian subgroup of bounded index if necessary, we are left with the task to bound the order of finite abelian subgroups of $\mathrm{GL}_N(\mathbb{K})$.

Suppose that $G \subset \mathrm{GL}_N(\mathbb{K})$ is a finite abelian subgroup. Then all elements of G are simultaneously diagonalizable over the algebraic closure $\bar{\mathbb{K}}$ of the field $\mathbb{K}$. Thus, G is generated by at most N elements. Therefore, it is enough to show that the orders of the elements of G are bounded by some constant that does not depend on $\mathbb{K}$ and G.

Let $g \in \mathrm{GL}_N(\mathbb{K})$ be an element of finite order $\mathrm{ord}(g)$. We claim that $\mathrm{ord}(g)$ is bounded by a constant that depends only on the field $\Bbbk$ and the integer N, but not on the field $\mathbb{K}$. Indeed, let $F_g(u)$ be the minimal polynomial of the element $g \in \mathrm{GL}_N(\mathbb{K})$. Then $F_g(u)$ is a polynomial of degree at most N with coefficients from the field $\mathbb{K}$, and $F_g(u)$ divides $u^{\mathrm{ord}(g)} - 1$. Our goal is to prove that the roots of all possible polynomials $F_g(u)$ form a finite set of elements of $\bar{\Bbbk}$. This will imply that the set of possible eigenvalues of elements of finite order in $\mathrm{GL}_N(\mathbb{K})$ is bounded, which means boundedness of orders of such elements.

Let $\Phi_l(u)$ be the lth cyclotomic polynomial. Recall that $\Phi_l(u)$ is defined over $\mathbb{Z}$. Moreover, $\Phi_l(u)$ is either irreducible over $\mathbb{K}$, or is a product of linear polynomials over $\mathbb{K}$. As usual, one has

$$u^{\mathrm{ord}(g)} - 1 = \prod_{l \mid \mathrm{ord}(g)} \Phi_l(u). \tag{7.2}$$

Let $G(u)$ be a non-linear irreducible polynomial over $\Bbbk$ that divides $F_g(u)$. Since $\Bbbk$ is algebraically closed in $\mathbb{K}$, we see that $G(u)$ is also irreducible over $\mathbb{K}$. Note that $G(u)$ coincides with some irreducible polynomial $\Phi_l(u)$. One has

$$\varphi(l) = \deg(\Phi_l(u)) = \deg(G(u)) \leqslant \deg(F_g(u)) \leqslant N,$$

where $\varphi(l)$ is the Euler function of l. Therefore, the numbers l appearing for the *irreducible* polynomials $\Phi_l(u)$ in (7.2) are bounded in terms of N. On the other hand, if $\Phi_l(u)$ is reducible over $\mathbb{K}$, then $\mathbb{K}$ (and, thus, also $\Bbbk$) contains a primitive root of unity of degree l. Hence, the number of polynomials like this appearing in (7.2) is bounded by some constant (that depends only on $\Bbbk$) because the field $\Bbbk$ is finitely generated. Since the polynomial $u^{\mathrm{ord}(g)} - 1$, and thus also the polynomial $F_g(u)$, has no multiple roots, we conclude that $F_g(u)$ is a product of a bounded number of cyclotomic polynomials of bounded degrees. Therefore, only a finite number of elements of $\bar{\Bbbk}$ can be roots of $F_g(u)$ for various g, and the assertion of the lemma follows. $\square$

Remark 7.3. If X is a (geometrically irreducible) variety over a field $\Bbbk$, then $\Bbbk$ is algebraically closed in $\mathbb{K} = \Bbbk(X)$.

An immediate consequence of Lemma 7.1 is the following.

Corollary 7.4. *Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$, and Γ is a linear algebraic group. Let $\mathcal{G}$ be the family of groups $\Gamma(\mathbb{K})$, where $\mathbb{K}$ varies in the set of finitely generated fields over $\Bbbk$ such that $\Bbbk$ is algebraically closed in $\mathbb{K}$. Then the family $\mathcal{G}$ has uniformly bounded finite subgroups.*

Remark 7.5. Lemma 7.1 and Corollary 7.4 are also implied by [Ser09, Theorem 5]. Indeed, in the notation of [Ser09, § 4] the values of the invariant t (the invariant m, respectively) are the same for the fields $\Bbbk$ and $\mathbb{K}$ provided that $\Bbbk$ is algebraically closed in $\mathbb{K}$, so that in the notation of Lemma 7.1 and Corollary 7.4 these invariants are bounded in the family $\mathcal{G}$.

To prove Theorem 1.4 we will start with an assertion that is more or less its particular case. Recall that a G-equivariant morphism $\phi : Y \to S$ of normal varieties acted on by a finite group G is a *G-Mori fiber space*, if Y has terminal $G\mathbb{Q}$-factorial singularities, $\dim(S) < \dim(Y)$, the fibers of ϕ are connected, the anticanonical divisor $-K_Y$ is ϕ-ample, and the relative G-invariant Picard number $\rho^G(Y/S) = 1$.

Lemma 7.6. *Suppose that $\Bbbk$ is a finitely generated field over $\mathbb{Q}$. Let $\mathcal{G}^{\Bbbk}_{\mathrm{rc}}(n)$ be the family of groups $\mathrm{Bir}(X)$, where X varies in the set of rationally connected varieties of dimension n over some field $\mathbb{K}$, and $\mathbb{K}$ itself varies in the set of finitely generated fields over $\Bbbk$ such that $\Bbbk$ is algebraically closed in $\mathbb{K}$. Assume that Conjecture BAB holds in dimension n. Then the family $\mathcal{G}^{\Bbbk}_{\mathrm{rc}}(n)$ has uniformly bounded finite subgroups.*

Proof. Let X be a rationally connected variety of dimension n over a field $\mathbb{K}$, and let $G \subset \mathrm{Bir}(X)$ be a finite group. By Lemma–Definition 3.1 there exists a smooth regularization $\tilde{X}$ of G. Note that $\tilde{X}$ is rationally connected since so is X. Run a G-minimal model program on $\tilde{X}$. This is possible due to an equivariant version of [BCHM10, Corollary 1.3.3] and [MM86, Theorem 1], since rational connectedness implies uniruledness. We obtain a rationally connected variety Y birational to $\tilde{X}$ (and, thus, to X) with a faithful (regular) action of the group G and a structure $\phi : Y \to S$ of a G-Mori fiber space.

Suppose that $\dim(S) = 0$. Then Y is a Fano variety with terminal singularities. Using Conjecture BAB and arguing as in the proof of [PS15, Lemma 4.6] we see that there exists a positive integer $N = N(n)$ that does not depend on the field $\mathbb{K}$ and on the variety Y (and, thus, also on X) such that $G \subset \mathrm{PGL}_N(\mathbb{K})$. Therefore, in this case the assertion follows from Theorem 2.12.

Now suppose that $\dim(S) > 0$. Consider an exact sequence of groups

$$1 \longrightarrow G_f \longrightarrow G \longrightarrow G_b \longrightarrow 1,$$

where the action of G_f is fiberwise with respect to ϕ and G_b is the image of G in $\mathrm{Aut}(S)$. We have an embedding $G_f \subset \mathrm{Aut}(Y_\eta)$, where Y_η is the fiber of ϕ over the generic scheme point η of S (cf. the proof of Proposition 6.2). Note that S is rationally connected since it is dominated by a rationally connected variety Y. Moreover, Y_η is a Fano variety with at worst terminal singularities by [KMM87, § 5-1], so that Y_η is rationally connected by [Zha06, Theorem 1]. Note also that Y_η is defined over the field $\mathbb{K}_\eta = \mathbb{K}(S)$ that is finitely generated over $\mathbb{K}$ (and, thus, over $\Bbbk$). Since ϕ has connected fibers, the field $\mathbb{K}$ is algebraically closed in $\mathbb{K}_\eta$, so that $\Bbbk$ is algebraically closed in $\mathbb{K}_\eta$ as well.

Let $\mathcal{G}_f$ and $\mathcal{G}_b$ be the families of groups arising in the above procedure as the groups G_f and G_b, respectively, for various choices of a field $\mathbb{K}$, a variety X and a finite group $G \subset \mathrm{Bir}(X)$. Since $\dim(S) < n$ and $\dim(Y_\eta) < n$, induction in n shows that both $\mathcal{G}_b$ and $\mathcal{G}_f$ have universally bounded finite subgroups. Thus, the assertion follows by Lemma 2.2. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mathcal{G}_{\mathrm{rc}}(X)$ and $\mathcal{G}_{\mathrm{ab}}(X)$ be families of groups defined in Corollary 6.8. By Lemma 7.6 the family $\mathcal{G}_{\mathrm{rc}}(X)$ has uniformly bounded finite subgroups. By Corollary 2.16 the family $\mathcal{G}_{\mathrm{ab}}(X)$ also has uniformly bounded finite subgroups. Therefore, the assertion is implied by Corollary 6.8(i). □

Finally, we use Theorem 1.4 to derive Corollary 1.5.

Proof of Corollary 1.5. Let $\Bbbk$ be the algebraic closure of $\mathbb{Q}$ in K and let

$$\mathrm{Aut}_{\Bbbk}(K) \subset \mathrm{Aut}(K)$$

be the maximal subgroup of $\mathrm{Aut}(K)$ that acts trivially on $\Bbbk$. Then $\mathrm{Aut}_{\Bbbk}(K)$ is normal in $\mathrm{Aut}(K)$, and

$$\mathrm{Aut}(\Bbbk) \simeq \mathrm{Aut}(K)/\mathrm{Aut}_{\Bbbk}(K)$$

is a finite group. Thus, it is sufficient to show that $\mathrm{Aut}_{\Bbbk}(K)$ has bounded finite subgroups. Let $R \subset K$ be a finitely generated $\Bbbk$-subalgebra that such that the field K is the field of fractions of R. The assertion follows by Theorem 1.4 applied to the (affine) variety $X = \mathrm{Spec}(R)$. □

8. Solvably Jordan groups

Definition 8.1. Let $\mathcal{G}$ be a family of groups. We say that $\mathcal{G}$ is *uniformly solvably Jordan* if there exists a constant $J_S = J_S(\Gamma)$ such that for any group $\Gamma \in \mathcal{G}$ and any finite subgroup $G \subset \Gamma$ there exists a solvable subgroup $S \subset G$ of index at most J_S. We say that a group Γ is *solvably Jordan* if the family $\{\Gamma\}$ is uniformly solvably Jordan.

Remark 8.2. If a family $\mathcal{G}$ is uniformly Jordan, then it is uniformly solvably Jordan.

D. Allcock asked the following question.

Question 8.3. Which varieties have solvably Jordan groups of birational automorphisms?

The purpose of this section is to give an answer to Question 8.3.

Lemma 8.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be uniformly solvably Jordan families of groups. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1, \tag{8.5}$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ is uniformly solvably Jordan.

Proof. It is enough to prove the assertion assuming that G (and, thus, also G_1 and G_2) is finite. Replacing G_2 by its solvable subgroup of bounded index and replacing G by the preimage of the latter subgroup, we may assume that the group G_2 in 8.5 is solvable.

Let $S_1 \subset G_1$ be the maximal normal solvable subgroup of G_1. Then S_1 is preserved by automorphisms of G_1, and thus S_1 is a normal subgroup of the group G. Since an extension of a solvable group by a solvable group is again solvable, and the index of S_1 in G_1 is bounded, we may replace G_1 and G by G_1/S_1 and G/S_1, respectively, and assume that the order of G_1 in 8.5 is bounded.

Let $C \subset G$ be the centralizer of the subgroup G_1. Since the subgroup $G_1 \subset G$ is normal, we conclude that $C \subset G$ is also normal. Thus, the group G/C embeds into the group $\mathrm{Aut}(G_1)$. Since $|G_1|$ is bounded, we see that $|\mathrm{Aut}(G_1)|$ is bounded as well. This implies that the index $[G : C]$ is bounded. Put $H = C \cap G_1$. Then the group H is abelian. On the other hand, the group C/H embeds into G_2, so that C/H is solvable. Since an extension of a solvable group by a solvable group is solvable, we see that C is a solvable subgroup of G of bounded index. □

Proposition 8.6. *Let X be a variety of dimension n. Suppose that Conjecture BAB holds in dimension n. Then the group* $\mathrm{Bir}(X)$ *is solvably Jordan.*

Proof. Choose some birational polarization L on the base of the maximal rationally connected fibration of X. Let $\mathcal{G}_{\mathrm{rc}}(X)$, $\mathcal{G}_{\mathrm{ab}}(X)$, $\mathcal{G}_N(X)$ and $\mathcal{G}_L(X)$ be the families of groups arising in Proposition 6.2 as the groups G_{rc}, G_{ab}, G_N and G_L, respectively, for various choices of finite groups $G \subset \mathrm{Bir}(X)$. Applying Theorem 1.7, Corollary 2.15, Corollary 2.14 and Lemma 5.2 together with Remarks 2.1 and 8.2, we see that the families $\mathcal{G}_{\mathrm{rc}}(X)$, $\mathcal{G}_{\mathrm{ab}}(X)$, $\mathcal{G}_N(X)$ and $\mathcal{G}_L(X)$ are uniformly solvably Jordan. Now the assertion is implied by Proposition 6.2 and Lemma 8.4. □

Remark 8.7. Note that for non-unirational varieties the argument used in the proof of Proposition 8.6 does not rely on Conjecture BAB. Similarly, in dimension $n \leqslant 3$ Proposition 8.6 also holds without any additional assumptions (cf. Corollary 1.9).

9. Discussion

In this section we list several open questions related to the previous consideration, and mention some possible approaches to them.

Question 9.1. Can one use information on degenerate fibers of certain fibrations to establish Jordan property for automorphism (respectively birational automorphism) groups?

We note that a typical case that is not covered by Theorem 1.8 is a variety X with a structure of fibration $\phi : X \to X_{\mathrm{nu}}$ such that ϕ has rationally connected fibers and X_{nu} is a non-uniruled variety with irregularity $q(X_{\mathrm{nu}}) = q(X) > 0$. The situation when $X_{\mathrm{nu}} = A$ is an abelian variety, and ϕ is a conic bundle, is already interesting and not completely accessible on our current level of understanding the geometry of such fibrations. For example, from [Zar10] we know that if $X \simeq A \times \mathbb{P}^1$, then the group $\mathrm{Bir}(X)$ is not Jordan. On the other hand, even in dimension three we are far from being able to analyze even similar examples. For example, if ϕ is a $\mathbb{P}^1$-bundle over an abelian surface A, we do not know how to deal with the Jordan property, except for the cases that are somehow reduced to the direct product (say, we do not know if there is a Jordan example of this kind). Furthermore, if ϕ is a conic bundle with a non-trivial discriminant $\Delta \subset A$, it seems reasonable to try (but it is not yet clear for us how) to use the geometry of Δ to estimate the image of $\mathrm{Bir}(X)$ under the natural map $\mathrm{Bir}(X) \to \mathrm{Aut}(A)$. We expect that a good starting point here may be to understand the influence of ampleness of Δ on Jordan property of $\mathrm{Bir}(X)$. It is also possible that similar considerations may help to find out if the Jordan property holds for groups of automorphisms of affine varieties (cf. [Pop11, Question 2.14]).

The next thing we want to mention is the following.

Question 9.2. Can one use some canonically defined (rational) maps to provide a more geometric proof of Theorems 1.8 and 1.4?

A general observation is that if a rational map $\phi : X \dashrightarrow X'$ is equivariant with respect to $\mathrm{Bir}(X)$ (or some subgroup $G \subset \mathrm{Bir}(X)$), then one has an exact sequence associated with ϕ similar to (6.3). This observation was applied to the maximal rationally connected fibration in the proof of Proposition 6.2 and to a G-Mori fibration in the proof of Lemma 7.6. On the other hand, it is tempting to use other maps that are canonically defined and, thus, equivariant. In particular, it is possible that some information may be obtained from analysing the Albanese map

$$alb : X \dashrightarrow \mathrm{Alb}(X)$$

and making use of the fact that the target space $\mathrm{Alb}(X)$ is an abelian variety. We feel that the corresponding part of our current approach is somehow 'dual' to this. Furthermore, if X is a non-uniruled variety, one can consider a pluri-canonical map

$$\phi_{\mathrm{can}} : X \dashrightarrow X_{\mathrm{can}}$$

and make use of the properties of the fibers of ϕ_{can}. On the other hand, this approach may be hard to take since we do not really know much about the fibers of the Albanese map and about the image of the pluri-canonical map.

The last question we want to discuss is the following.

Question 9.3. Can one prove 'uniform' analogs of Theorems 1.4 and 1.8 for some natural families of varieties, i.e. show that certain families of groups of birational automorphisms are uniformly Jordan, or have uniformly bounded finite subgroups?

Of course, such result is not possible in the most general case. Indeed, for any m the symmetric group S_m acts by automorphisms of some curve C_m defined over $\mathbb{Q}$. Actually, the only (general enough) results in this direction we are aware of are Lemma 7.6 and Theorem 1.7. Another result of similar flavor is Corollary 2.18. A more reasonable version of Question 9.3 may be the following.

Question 9.4. Can one bound the constants appearing in Theorems 1.4 and 1.8 uniformly for some natural families of varieties in terms of some invariants of these varieties?

A partial answer to Question 9.4 that illustrates what we would like to know in some more wide context is a bound on the order of birational automorphism group of an n-dimensional variety of general type in terms of its canonical volume (see [HMX10]). The most general assertion that we may suggest in this direction is as follows. Suppose that $\mathcal{X}$ is a family of n-dimensional varieties such that for any $X \in \mathcal{X}$ one can choose a *very ample* birational polarization H_X on the base of the maximal rationally connected fibration $\phi_{\mathrm{rc}} : X \dashrightarrow X_{\mathrm{nu}}$ so that the volume of H_X is bounded by some constant $D = D(\mathcal{X})$. (Recall from Definition 6.1 that this polarization is supposed to be defined not on X_{nu} itself but on one of its quasi-minimal models.) Then in the assumptions of Theorem 1.8(ii) and (iii) the family $\mathcal{B}$ of groups $\mathrm{Bir}(X)$, $X \in \mathcal{X}$, is uniformly Jordan. This directly follows from an observation that for a family of (polarized) varieties of bounded degree all essential characteristics of the varieties involved in the proof of Proposition 6.2 (i.e. rank and order of torsion of Neron–Severi group and irregularity) are bounded. Similarly, in the assumptions of Theorems 1.8(i) and 1.4 the family $\mathcal{B}$ has uniformly bounded finite subgroups, provided that Conjecture 2.17 holds in dimension d that equals the maximal irregularity for varieties $X \in \mathcal{X}$. This follows by the same argument as above together with Corollary 2.18.

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Yuri Prokhorov prokhoro@gmail.com
Steklov Institute of Mathematics, 8 Gubkina Street, Moscow 119991, Russia

and

Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova Street, Moscow 117312, Russia

and

Faculty of Mathematics, Moscow State University, Moscow 117234, Russia

Constantin Shramov costya.shramov@gmail.com
Steklov Institute of Mathematics, 8 Gubkina Street, Moscow 119991, Russia

and

Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova Street, Moscow 117312, Russia

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