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Global attractors for nonlinear beam equations

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This paper is concerned with the dynamics of nonlinear one-dimensional beam equations. We consider nonlinear beam equations with viscosity or with a lower-order damping term instead of the viscosity, and we establish the existence of global attractors for both systems.

1. Introduction

In this paper we investigate the existence of global attractors for the nonlinear one-dimensional beam equations arising from the study of mechanical movements of shape memory alloys of constant mass density ρ (assumed to be normalized to unity, i.e. $\rho = 1$). We consider equations either with viscosity or without viscosity but with a lower-order damping term. For both cases, our general aim, roughly stated, is to show that the equations possess global attractors in the corresponding complete metric spaces.

Let $\Omega = (0, 1)$ and, for any $t > 0$, $\Omega_t = \Omega \times (0, t)$. For the system with viscosity, the nonlinear partial differential equation we are studying is

$$u_{tt} - \nu u_{xxt} - f(u_x)_x + Ru_{xxxx} = 0 \tag{1.1}$$

with u, f and g being the displacement, stress and density of distributed loads, respectively, and subject to the boundary conditions

$$u|_{x=0,1} = u_{xx}|_{x=0,1} = 0 \tag{1.2}$$

and the initial conditions

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1. \tag{1.3}$$

For the system without viscosity, the equation we are studying is

$$u_{tt} + \mu u_t - f(u_x)_x + Ru_{xxxx} = 0 \tag{1.4}$$

subject to the same boundary conditions (1.2) and initial conditions (1.3).

To study the thermomechanics of shape memory alloys in one space dimension, Falk [5, 6] proposed a Ginzburg–Landau theory, using the strain $\varepsilon = u_x$ as order

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parameter and assuming that the Helmholtz free energy density F is a potential of Ginzburg–Landau form, i.e.

$$F = F(u_x, u_{xx}, \theta) \tag{1.5}$$

where θ is the absolute temperature. Here the beam equations studied in our paper with positive constant temperature can be taken as the special case of [5, 6]. The simplest form for the free energy density F that accounts quite well for the experimental behaviour and takes couple stresses into account is

$$F(u_x, u_{xx}) = F_1(u_x) + \tfrac{1}{2} R u_{xx}^2, \tag{1.6}$$

where

$$F_1(u_x) = \tfrac{1}{6}\alpha_1 u_x^6 - \tfrac{1}{4}\alpha_2 u_x^4 - \tfrac{1}{2}\alpha_3 u_x^2 \tag{1.7}$$

with positive constants α_i and R.

The stress $f = f(u_x)$ in (1.1) or (1.4) is given by

$$f(u_x) = F_1'(u_x) = \alpha_1 u_x^5 - \alpha_2 u_x^3 - \alpha_3 u_x, \tag{1.8}$$

where ν and μ are positive constants.

The physical meaning of the boundary conditions is that both ends of the rod are hinged, respectively.

Before stating and proving our results, let us first recall some related results in the literature.

Ball [1] proved the existence of weak solutions to the nonlinear beam equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left[\beta + \kappa \int_0^l u_\xi(\xi, t)^2 \, \mathrm{d}\xi\right] \frac{\partial^2 u}{\partial x^2} = 0$$

subject to clamped or hinged boundary conditions. Later, Ball [2] proved the stability of an extensile beam equation as time tends to infinity. Eden and Milani [4] proved the existence of a compact attractor, and also an exponential attractor to the equations of the type

$$\varepsilon u_{tt} + u_t + \alpha \Delta^2 u = \left(\kappa \int_\Omega |\nabla u|^2 - \beta\right) \Delta u + f. \tag{1.9}$$

They also proved, in the special case where damping is large (i.e. ε is small), that the exponential attractor contains the global attractor.

For the non-isothermal case, i.e. the coupled partial differential equations, which consist of a nonlinear beam equation with respect to the displacement u and a second-order parabolic equation with respect to the temperature. Shang [10] proved the existence of a global attractor to the one-dimensional thermoviscoelastic system arising from the study of phase transitions in shape memory alloys with hinged boundary conditions in closed subspaces. Motivated by [10], equation (1.1) can be taken as the special case of [10] with constant temperature, but we can prove the existence of a global attractor in the whole Sobolev space H. For the same model as in [10], but with stress-free boundary conditions at one or both ends of the rod, Sprekels and Zheng [12] obtained the existence of a global attractor for the Ginzburg–Landau form for shape memory alloys. Shang [10] and Sprekels and Zheng [12] studied the systems whose free energy density F was a potential of Ginzburg–Landau form, i.e. $R > 0$. For the case $R = 0$, $\nu > 0$, Racke and Zheng [9] obtained the global existence and asymptotic behaviour of the solution to the nonlinear thermoviscoelastic system with stress-free conditions at least at one end of the rod. For the system with clamped boundary conditions, Chen and Hoffmann [3] proved the global existence and uniqueness of the smooth solution. Shen *et al* [13] obtained the global existence and asymptotic behaviour of the weak solution, and they established a new approach to derive *a priori* estimates on the L^∞-norm of the strain u independent of the length of time. Recently, Qin *et al* [8] obtained the existence of a global attractor for the same system as in [13].

In this paper, we consider problems (1.1)–(1.3) and (1.2)–(1.4). By deriving delicate uniform *a priori* estimates independent of T and the initial data for both cases, we obtain the results on the existence of global attractors.

First, we study the problem (1.1)–(1.3). Let

$$H := \{(u, u_t) \in H^4 \times H^2 \colon u|_{x=0,1} = u_{xx}|_{x=0,1} = 0\}.$$

Our main result in this case reads as follows.

Theorem 1.1. *Suppose $u_0 \in H^4$, $u_1 \in H^2$ are given functions that satisfy the compatibility conditions $u_0|_{x=0,1} = u_{0xx}|_{x=0,1} = 0$. Then, for problem (1.1)–(1.3), the following results hold.*

(i) *The problem admits a unique global solution (u, u_t) satisfying*

$$u \in C([0, +\infty); H^4) \cap C^1([0, +\infty); H^2) \cap L^2([0, +\infty); H^5); \tag{1.10}$$

$$u_t \in C([0, +\infty); H^2) \cap L^2([0, +\infty); H^3). \tag{1.11}$$

(ii) *An orbit starting from H will reenter itself after finite time, and stay there forever. Moreover, it possesses in H a global attractor A which is compact.*

Remark 1.2. Note that, in the proof of theorem 1.1 (proof of lemma 2.6), we require that the coefficients α_i, R and ν satisfy

$$\frac{\alpha_2^2}{2\alpha_1} + \alpha_3 \leqslant \min\{\tfrac{1}{4}\nu^2, \tfrac{1}{2}R\}. \tag{1.12}$$

This is actually no restriction, since we may assume (1.12) without loss of generality using the following scaling argument.

For $\varepsilon > 0$, let ω be defined by

$$u(t, x) = \varepsilon\omega\left(\frac{t}{\varepsilon^{1/4}}, \frac{x}{\varepsilon^{1/8}}\right) \equiv \varepsilon\omega(s, y).$$

Then ω satisfies the 'same' differential equation as u, with α_i replaced by $\alpha_{i,\varepsilon}$:

$$\omega_{ss} - \nu\omega_{yys} - (\alpha_{1,\varepsilon}\omega_y^5 - \alpha_{2,\varepsilon}\omega_y^3 - \alpha_{3,\varepsilon}\omega_y)_y + R\omega_{yyyy} = 0, \tag{1.13}$$

where

$$\alpha_{1,\varepsilon} := \varepsilon^{15/4}\alpha_1, \qquad \alpha_{2,\varepsilon} := \varepsilon^2\alpha_2, \qquad \alpha_{3,\varepsilon} := \varepsilon^{1/4}\alpha_3.$$

The condition (1.12) for the equation (1.1) for u then turns into the following condition for the equation (1.13) for ω:

$$\frac{\alpha_{2,\varepsilon}^2}{2\alpha_{1,\varepsilon}} + \alpha_{3,\varepsilon} \leqslant \min\{\tfrac{1}{4}\nu^2, \tfrac{1}{2}R\},$$

or, equivalently,

$$\varepsilon^{1/4}\left(\frac{\alpha_2^2}{2\alpha_1} + \alpha_3\right) \leqslant \min\{\tfrac{1}{4}\nu^2, \tfrac{1}{2}R\},$$

which can be fulfilled for sufficiently small ε.

Second, for the problem (1.2)–(1.4), our result is the following.

Theorem 1.3. *Suppose $u_0 \in H^4$, $u_1 \in H^2$ are given functions that satisfy the compatibility conditions $u_0|_{x=0,1} = u_{0xx}|_{x=0,1} = 0$. Then, for the problem (1.2)–(1.4), the following results hold.*

(i) *The problem admits a unique global solution (u, u_t) satisfying*

$$u \in C([0,+\infty); H^4) \cap C^1([0,+\infty); H^2) \cap L^2([0,+\infty); H^5);$$
$$u_t \in C([0,+\infty); H^2) \cap L^2([0,+\infty); H^3).$$

(ii) *For $\beta > 0$, we define the space*

$$H_\beta := \left\{(u, u_t) \in H, \int_0^1 (\tfrac{1}{2}u_t^2 + \tfrac{1}{2}Ru_{xx}^2 + F_2(u_x))\,\mathrm{d}x \leqslant \beta\right\}.$$

Then an orbit starting from H_β will reenter itself after finite time, and stay there forever. Moreover, it possesses in H_β a global attractor A_β which is compact.

In what follows, we explain some mathematical difficulties that appear in this paper.

First, in the course of deriving the existence of an absorbing set in H or H_β, the estimates obtained in the proof of global existence are not sufficient, and we should derive uniform estimates of $\|u\|_{H^4}$, $\|u_t\|_{H^2}$ independent of the initial data and t. It turns out that more delicate estimates are needed due to the higher degree of nonlinearity inherent in the system and to the higher-order derivative arising for $R > 0$.

Second, we recall the results obtained in Eden and Milani [4], which followed a procedure similar to that of Hale [7], but replaced the role of the Lyapunov functions with different types of energy norms. Using the method of α-contractions, [4] proved the existence of a compact, finite fractal dimensional invariant set towards which all solutions converged exponentially in time. However, the existence of a global attractor, i.e. the boundedness of the attractor in the corresponding norm, could only be obtained when the damping is large, i.e. ε is small in (1.9). In contrast with [4], in order to establish the existence of a global attractor, we shall apply theorem 6.4.1 of Zheng [15]. The crucial step is to show the existence of an absorbing set and the uniform compactness of the orbits starting from any bounded set. In a similar manner to [15], we can obtain the existence of bounded, invariant absorbing set B_0 or B_β for both cases. However, in the proof of uniform compactness, we can see that problem (1.2)–(1.4), i.e. the system without viscosity, seems to be totally different from the problem (1.1)–(1.3). The uniform compactness of the solution to problem (1.2)–(1.4) cannot be derived directly like problem (1.1)–(1.3), since the term μu_t in (1.4) is not as good as $-\nu u_{xxt}$ in (1.1). In order to overcome this difficulty, we should instead consider the dynamics in closed subspaces defined by the parameter β, i.e. H_β in our paper. We shall show that the constraint in the definition of H_β is invariant under $S(t)$. We shall prove that the orbit starting from H_β will reenter itself after a finite time and stay there forever.

This paper is organized as follows. In § 2 we prove the existence of a global attractor for the problem (1.1)–(1.3) in the Sobolev space H. In § 3 we prove the existence of a global attractor for the problem (1.2)–(1.4) in the closed subspace H_β.

The notation used in this paper will be as follows. L^p, $W^{m,p}$, $1 \leqslant p \leqslant \infty$, $m \in N$, $H^1 \equiv W^{1,2}$ and $H_0^1 \equiv W_0^{1,2}$, respectively, denote the usual Lebesgue and Sobolev space on $(0,1)$. We use the abbreviation $\|\cdot\| := \|\cdot\|_{L^2}$, and $C^k(I,B)$, $k \in N_0$, to denote the space of k-times continuously differentiable functions from $I \in R$ into a Banach space B. The spaces $L^p(I,B)$, $1 \leqslant p \leqslant \infty$, are defined analogously. Finally, ∂_t or a subscript t, and, likewise, ∂_x or a subscript x, denote the partial derivations with respect to t and x, respectively.

2. The existence of a global attractor for the system with viscosity

We consider the initial boundary-value problem (1.1)–(1.3). In this section, we shall prove the existence of a global attractor for this system in the whole Sobolev space H.

We first establish a local existence and uniqueness result for this problem.

Lemma 2.1. *Suppose that $u_0 \in H^4$ and $u_1 \in H^2$ are given functions that satisfy the compatibility conditions $u_0|_{x=0,1} = u_{0xx}|_{x=0,1} = 0$. Then there exists $t^* > 0$ depending only on $\|u_0\|_{H^4(\Omega)}$, $\|u_1\|_{H^2(\Omega)}$ such that problem (1.1)–(1.3) admits a unique solution (u,u_t) in $\bar{\Omega} \times [0,t^*]$ such that*

$$u \in C([0,t^*];H^4) \cap C^1([0,t^*];H^2) \cap L^2([0,t^*];H^5),$$
$$u_t \in C([0,t^*];H^2) \cap L^2([0,t^*];H^3).$$

Proof. We use the contraction mapping theorem to prove the local existence and uniqueness. Since the proof is essentially the same as in Shang [10], we can omit the details here. □

In the following, we prove theorem 1.1.

2.1. Proof of theorem 1.1(i)

In order to prove the global existence, we have to establish *a priori* estimates for $\|u\|_{H^4}$, $\|u_t\|_{H^2}$. In fact, we can derive uniform *a priori* estimates independent of t, which is crucial for the proof of uniform compactness of the orbits. In this proof, C denotes a universal positive constant that may depend on the norm of the initial data, but not on t.

LEMMA 2.2. *For any $t > 0$, the following estimates hold:*

$$\|u_t\| \leqslant C, \qquad \|u_{xx}\| \leqslant C, \qquad \|u_x\|_{L^\infty} \leqslant C, \tag{2.1}$$

$$\int_0^t \int_0^1 u_{xt}^2 \,\mathrm{d}x \,\mathrm{d}\tau \leqslant C, \qquad \int_0^t \|u_t\|^2 \,\mathrm{d}\tau \leqslant C, \qquad \int_0^t \|u_t\|_{L^\infty}^2 \,\mathrm{d}\tau \leqslant C. \tag{2.2}$$

Proof. Multiplying (1.1) with u_t and integrating with respect to x and t yields

$$\frac{1}{2}\int_0^1 u_t^2 \,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2 \,\mathrm{d}x + \int_0^1 F_1(u_x) \,\mathrm{d}x + \nu\int_0^t \int_0^1 u_{xt}^2 \,\mathrm{d}x \,\mathrm{d}\tau \leqslant C. \tag{2.3}$$

Here $F_1'(x) = f(x)$, and applying Young's inequality, we have

$$F_1(u_x) \geqslant Cu_x^6 - C. \tag{2.4}$$

Combining (2.3) with (2.4), we obtain the estimates (2.1). The estimates (2.2) can be derived form (2.1) and the boundary conditions (1.2) immediately. The proof is complete. □

LEMMA 2.3. *For any $t > 0$, the following estimates hold:*

$$\|u_{tt}\| \leqslant C, \qquad \|u_{xxt}\| \leqslant C, \qquad \int_0^t \|u_{xtt}\|^2 \,\mathrm{d}\tau \leqslant C, \qquad \int_0^t \|u_{xxxt}\|^2 \,\mathrm{d}\tau \leqslant C. \tag{2.5}$$

Proof. We differentiate (1.1) with respect to t, multiply the result by u_{tt} and integrate with respect to x over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}^2 \,\mathrm{d}x + \nu\int_0^1 u_{xtt}^2 \,\mathrm{d}x + \int_0^1 f(u_x)_t u_{xtt} \,\mathrm{d}x + \tfrac{1}{2}R\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2 \,\mathrm{d}x = 0. \tag{2.6}$$

Since

$$\begin{aligned}\int_0^1 f(u_x)_t u_{xtt} \,\mathrm{d}x &\leqslant \tfrac{1}{2}\nu\int_0^1 u_{xtt}^2 \,\mathrm{d}x + C\int_0^1 |f'(u_x)u_{xt}|^2 \,\mathrm{d}x \\ &\leqslant \tfrac{1}{2}\nu\|u_{xtt}\|^2 + C\|u_{xt}\|^2,\end{aligned} \tag{2.7}$$

using (2.2) and integrating (2.6) with respect to t yields

$$\|u_{tt}\| \leqslant C, \qquad \|u_{xxt}\| \leqslant C, \qquad \int_0^t \|u_{xtt}\|^2 \,\mathrm{d}\tau \leqslant C. \tag{2.8}$$

Then, we differentiate (1.1) with respect to t, multiply the result by $-u_{xxt}$ and integrate with respect to x over Ω to obtain

$$\begin{aligned}\tfrac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2 \,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}u_{xxt} \,\mathrm{d}x - \int_0^1 u_{xtt}^2 \,\mathrm{d}x \\ + R\int_0^1 u_{xxxt}^2 \,\mathrm{d}x - \int_0^1 f(u_x)_t u_{xxxt} \,\mathrm{d}x = 0.\end{aligned} \tag{2.9}$$

Using the estimates we obtain in (2.8), we have

$$\int_0^1 f(u_x)_t u_{xxxt}\,\mathrm{d}x \leqslant \tfrac{1}{2}R\int_0^1 u_{xxxt}^2\,\mathrm{d}x + C\int_0^1 |f(u_x)_t|^2\,\mathrm{d}t$$
$$\leqslant \tfrac{1}{2}R\|u_{xxxt}\|^2\,\mathrm{d}x + C. \tag{2.10}$$

Combining (2.9) with (2.10), we finally have

$$\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau \leqslant C. \tag{2.11}$$

The proof is complete. □

Having established uniform *a priori* estimates, the global existence and uniqueness follows from the continuation argument. In what follows, we will prove the compactness of the orbit for $t>0$ in $H^4\times H^2$. For the time being, we assume that the initial data are so smooth that the solution will have enough smoothness to carry out the following argument. If the initial data just belong to $H^4\times H^2$, we can approximate them by smooth functions and then pass to the limit.

LEMMA 2.4. *For any $\mu>0$, the triple (u,u_t) is bounded in $C([\mu,+\infty);H^5\times H^3)$.*

Proof. First, we differentiate (1.1) with respect to t, multiply the result by $-u_{xxtt}$ and integrate with respect to x over Ω to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tfrac{1}{2}Ru_{xxxt}^2 + \tfrac{1}{2}u_{xtt}^2)\,\mathrm{d}x + \tfrac{1}{2}\nu\int_0^1 u_{xxtt}^2\,\mathrm{d}x \leqslant C\int_0^1 |f(u_x)_{xt}|^2\,\mathrm{d}x. \tag{2.12}$$

Multiplying (2.12) by t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(tR\|u_{xxxt}\|^2 + t\|u_{xtt}\|^2) + \nu t\|u_{xxtt}\|^2 \leqslant (R\|u_{xxxt}\|^2 + \|u_{xtt}\|^2) + Ct\|f(u_x)_{xt}\|^2. \tag{2.13}$$

Then, since

$$\int_0^t \|f(u_x)_{xt}\|^2\,\mathrm{d}\tau = \int_0^t (\|f'(u_x)u_{xxt}\|^2 + \|f''(u_x)u_{xx}u_{xt}\|^2)\,\mathrm{d}\tau,$$

using Nirenberg's inequality, we have

$$\|u_{xxt}\| \leqslant C\|u_{xxxt}\|^{1/2}\|u_{xt}\|^{1/2}$$

and Young's inequality gives

$$\|u_{xxt}\|^2 \leqslant C\|u_{xxxt}\|\|u_{xt}\| \leqslant \tfrac{1}{2}C\|u_{xxxt}\|^2 + \tfrac{1}{2}C\|u_{xt}\|^2.$$

Combining with the estimates in lemma 2.3 yields

$$\int_0^t \|u_{xxt}\|^2\,\mathrm{d}\tau \leqslant C.$$

Similarly,

$$\int_0^t \|u_{xx}u_{xt}\|^2\,\mathrm{d}\tau \leqslant \int_0^t \|u_{xt}\|_{L^\infty}^2\|u_{xx}\|^2\,\mathrm{d}\tau \leqslant C\int_0^t \|u_{xt}\|_{L^\infty}^2\,\mathrm{d}\tau$$

and

$$\|u_{xt}\|_{L^\infty}^2 \leqslant C\|u_{xxxt}\|^{1/2}\|u_{xt}\|^{3/2} \leqslant \tfrac{1}{2}C\|u_{xxxt}\|^2 + \tfrac{1}{2}C\|u_{xt}\|^2.$$

Thus,

$$\int_0^t \|u_{xt}\|_{L^\infty}^2 \,\mathrm{d}\tau \leqslant C.$$

Finally, we obtain

$$\int_0^t \|f(u_x)_{xt}\|^2 \,\mathrm{d}\tau \leqslant C.$$

Thus, we can obtain from (2.13) that

$$R\|u_{xxxt}\|^2 + \|u_{xtt}\|^2 \leqslant \tilde{C}t^{-1} + C \tag{2.14}$$

with $\tilde{C} = \tilde{C}(\|u_0\|_{H^4}, \|u_1\|_{H^2})$. The proof is complete. □

The compactness of the orbit in $H^4 \times H^2$ follows from this lemma. In what follows, we shall prove part (ii) of theorem 1.1, i.e. the existence of a global attractor in H.

2.2. Proof of theorem 1.1(ii)

In order to prove the existence of a global attractor, we shall apply [14, theorem I.1.1], which was rephrased in [11] as follows.

THEOREM 2.5. *Suppose that*

(a) *the mapping $S(t)$, $t \geqslant 0$, defined by the solution to problem (1.1)–(1.3) is a nonlinear continuous semigroup from H into itself and is uniformly compact for t large;*

(b) *there exists a bounded set B in H such that B is absorbing in H.*

Then the ω-limit set of B is a global attractor which is compact and attracts the bounded sets of H.

Concerning (a), we proved the global existence of the solution in theorem 1.1(i). It is clear from the proof that the family of operators $S(t)$, $t \geqslant 0$, defined by the solution, are continuous operators from H to H and they enjoy the usual semigroup properties. The uniform compactness of the orbit was proven in lemma 2.4. Hence, it remains to verify condition (b). In the following, the letters C, C_i denote positive constants independent of the initial data and the time t.

Let $B_0 = \{(u, u_t) \in H,\ \|u\|_{H^4} \leqslant \bar{C}_1,\ \|u_t\|_{H^2} \leqslant \bar{C}_2\}$, where $\bar{C}_1$, $\bar{C}_2$ are also positive constants independent of the initial data and t, which will be specified later. Then we have the following.

LEMMA 2.6. *B_0 is an absorbing set in H, i.e. for any bounded set B in H, there exists some time $t_2 = t_2(B) > 0$, such that, when $t \geqslant t_2(B)$, $S(t)B \subset B_0$.*

Proof. Multiplying (1.1) with u and integrating with respect to x yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 uu_t\,\mathrm{d}x - \int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_x^2\,\mathrm{d}x + \int_0^1 (\alpha_1 u_x^6 - \alpha_2 u_x^4 - \alpha_3 u_x^2)\,\mathrm{d}x + R\int_0^1 u_{xx}^2\,\mathrm{d}x = 0. \tag{2.15}$$

By Young's inequality, we obtain

$$\int_0^1 \alpha_2 u_x^4\,\mathrm{d}x \leqslant \frac{\alpha_1}{2}\int_0^1 u_x^6\,\mathrm{d}x + \frac{\alpha_2^2}{2\alpha_1}\int_0^1 u_x^2\,\mathrm{d}x. \tag{2.16}$$

Using Poincaré's inequality and the boundary conditions (1.2), we have

$$\|u_x\|_{L^2} \leqslant \|u_x\|_{L^\infty} \leqslant \|u_{xx}\|_{L^2}.$$

Due to (1.12) (see remark 1.2), we know that $\alpha_2^2/2\alpha_1 + \alpha_3 \leqslant \frac{1}{2}R$ holds. Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 uu_t\,\mathrm{d}x - \int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_x^2\,\mathrm{d}x + \tfrac{1}{2}\alpha_1\int_0^1 u_x^6\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2\,\mathrm{d}x \leqslant 0. \tag{2.17}$$

Next, multiplying (1.1) with u_t and integrating with respect to x yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}R\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xx}^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tfrac{1}{6}\alpha_1 u_x^6 - \tfrac{1}{4}\alpha_2 u_x^4 - \tfrac{1}{2}\alpha_3 u_x^2)\,\mathrm{d}x + \nu\int_0^1 u_{xt}^2\,\mathrm{d}x = 0. \tag{2.18}$$

Now, we multiply (2.17) by $\frac{1}{2}\nu$ and add the result to (2.18) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\tfrac{1}{2}\nu\int_0^1 uu_t\,\mathrm{d}x + \tfrac{1}{4}\nu^2\int_0^1 u_x^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2\,\mathrm{d}x + \tfrac{1}{6}\alpha_1\int_0^1 u_x^6\,\mathrm{d}x - \tfrac{1}{4}\alpha_2\int_0^1 u_x^4\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_x^2\,\mathrm{d}x\biggr) + \tfrac{1}{4}\nu\alpha_1\int_0^1 u_x^6\,\mathrm{d}x + \tfrac{1}{4}\nu R\int_0^1 u_{xx}^2\,\mathrm{d}x + \tfrac{1}{2}\nu\int_0^1 u_{xt}^2\,\mathrm{d}x = 0. \tag{2.19}$$

In a similar way to the above estimates, if we define

$$E_1(t) := \tfrac{1}{2}\nu\int_0^1 uu_t\,\mathrm{d}x + \tfrac{1}{4}\nu^2\int_0^1 u_x^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2\,\mathrm{d}x + \tfrac{1}{6}\alpha_1\int_0^1 u_x^6\,\mathrm{d}x - \tfrac{1}{4}\alpha_2\int_0^1 u_x^4\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_x^2\,\mathrm{d}x, \tag{2.20}$$

we have

$$E_1(t) \geqslant \tfrac{1}{2}\nu \int_0^1 u u_t \,\mathrm{d}x + \frac{1}{2}\int_0^1 u_t^2 \,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2 \,\mathrm{d}x + \tfrac{1}{24}\alpha_1 \int_0^1 u_x^6 \,\mathrm{d}x + \tfrac{1}{8}\nu^2 \int_0^1 u_x^2 \,\mathrm{d}x \tag{2.21}$$

provided $\alpha_2^2/4\alpha_1 + \alpha_3 \leqslant \frac{1}{4}\nu^2$, which can be derived from (1.12) easily. Let

$$E_2(t) := \tfrac{1}{4}\nu\alpha_1 \int_0^1 u_x^6 \,\mathrm{d}x + \tfrac{1}{4}\nu R \int_0^1 u_{xx}^2 \,\mathrm{d}x + \tfrac{1}{2}\nu \int_0^1 u_{xt}^2 \,\mathrm{d}x.$$

We can see that

$$E_1(t) \leqslant C E_2(t).$$

Thus, we have

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} + C_1 E_1(t) \leqslant C_2,$$

which leads to

$$E_1(t) \leqslant E_1(0)\mathrm{e}^{-C_1 t} + \frac{C_2}{C_1}. \tag{2.22}$$

We can see from (2.22) that, for any initial data, starting from any bounded set B of H, there exists $t_1(B)$ such that, when $t \geqslant t_1(B)$,

$$E_1(t) \leqslant \frac{2C_2}{C_1}. \tag{2.23}$$

In what follows, we consider the solution in $[t_1(B), +\infty)$. From (2.23), we have

$$\|u_t\|^2 \leqslant \frac{2C_2}{C_1}, \quad \|u_{xx}\|^2 \leqslant \frac{2C_2}{C_1} \quad \text{for any } t \geqslant t_1(B) \tag{2.24}$$

and

$$\|u_x\|_{L^\infty}^{n+2} \leqslant \|u_{xx}\|_{L^2}^{n+2} \leqslant \left(\frac{2C_2}{C_1}\right)^{(n+2)/2}. \tag{2.25}$$

Differentiating (1.1) with respect to t, multiplying the result by u_{tt}, and integrating with respect to x over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}^2 \,\mathrm{d}x + \nu \int_0^1 u_{xtt}^2 \,\mathrm{d}x + \frac{R}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2 \,\mathrm{d}x = -\int_0^1 f(u_x)_t u_{xtt} \,\mathrm{d}x. \tag{2.26}$$

Differentiating (1.1) with respect to t, multiplying the result by $-u_{xxt}$ and integrating with respect to x over Ω yields

$$\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2 \,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}u_{xxt} \,\mathrm{d}x - \int_0^1 u_{xtt}^2 \,\mathrm{d}x + R\int_0^1 u_{xxxt}^2 \,\mathrm{d}x = \int_0^1 f(u_x)_t u_{xxxt} \,\mathrm{d}x. \tag{2.27}$$

In the following, we estimate the right-hand side of (2.26), (2.27):

$$\int_0^1 f(u_x)_t u_{xtt}\,\mathrm{d}x \leqslant \tfrac{1}{4}\nu \int_0^1 u_{xtt}^2\,\mathrm{d}x + C\int_0^1 f(u_x)_t^2\,\mathrm{d}x. \tag{2.28}$$

Observe that

$$\int_0^1 f(u_x)_t^2\,\mathrm{d}x = \int_0^1 |f'(u_x)u_{xt}|^2\,\mathrm{d}x \leqslant C\int_0^1 u_x^8 u_{xt}^2\,\mathrm{d}x + C\int_0^1 u_{xt}^2\,\mathrm{d}x. \tag{2.29}$$

By virtue of the previous estimates,

$$\int_0^1 u_x^8 u_{xt}^2\,\mathrm{d}x \leqslant \|u_x\|_{L^\infty}^8 \|u_{xt}\|_{L^2}^2 \leqslant \left(\frac{2C_2}{C_1}\right)^4 \|u_{xt}\|_{L^2}^2$$

and

$$\|u_{xt}\|_{L^2}^2 \leqslant C\|u_{xxxt}\|_{L^2}^{2/3}\|u_t\|_{L^2}^{4/3} \leqslant \delta\|u_{xxxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^2, \tag{2.30}$$

with δ being a positive constant. Thus,

$$\int_0^1 u_x^8 u_{xt}^2\,\mathrm{d}x \leqslant \delta\|u_{xxxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^2. \tag{2.31}$$

Similarly, we have

$$\int_0^1 f(u_x)_t u_{xxxt}\,\mathrm{d}x \leqslant \delta\|u_{xxxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^2. \tag{2.32}$$

Multiplying (2.27) by η and adding the result to (2.26) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 u_{tt}^2\,\mathrm{d}x + (\tfrac{1}{2}R + \tfrac{1}{2}\nu\eta)\int_0^1 u_{xxt}^2\,\mathrm{d}x - \eta\int_0^1 u_{tt}u_{xxt}\,\mathrm{d}x\biggr)&\\ + (\nu-\eta)\int_0^1 u_{xtt}^2\,\mathrm{d}x + R\eta\int_0^1 u_{xxxt}^2\,\mathrm{d}x \leqslant \delta\|u_{xxxt}\|^2 + C_\delta\|u_t\|^2.&\end{aligned} \tag{2.33}$$

We can choose sufficiently small η, δ to ensure the positivity of the coefficients on the left-hand side of (2.33). Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 u_{xxt}^2\,\mathrm{d}x\biggr) + C_3\biggl(\int_0^1 u_{xtt}^2\,\mathrm{d}x + \int_0^1 u_{xxxt}^2\,\mathrm{d}x\biggr) \leqslant C_4.$$

Combining with (2.19), we finally have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 uu_t\,\mathrm{d}x + \int_0^1 u_x^2\,\mathrm{d}x + \int_0^1 u_t^2\,\mathrm{d}x + \int_0^1 u_{xx}^2\,\mathrm{d}x&\\ + \int_0^1 u_x^6\,\mathrm{d}x + \int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 u_{xxt}^2\,\mathrm{d}x\biggr)&\\ + C_5\biggl(\int_0^1 u_x^6\,\mathrm{d}x + \int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 u_{xt}^2\,\mathrm{d}x + \int_0^1 u_{xtt}^2\,\mathrm{d}x + \int_0^1 u_{xxxt}^2\,\mathrm{d}x\biggr) \leqslant C_6.&\end{aligned} \tag{2.34}$$

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If we define

$$E_3(t) := \int_0^1 uu_t\,\mathrm{d}x + \int_0^1 u_x^2\,\mathrm{d}x + \int_0^1 u_t^2\,\mathrm{d}x + \int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 u_x^6\,\mathrm{d}x + \int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 u_{xxt}^2\,\mathrm{d}x$$

and

$$E_4(t) := \int_0^1 u_x^6\,\mathrm{d}x + \int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 u_{xt}^2\,\mathrm{d}x + \int_0^1 u_{xtt}^2\,\mathrm{d}x + \int_0^1 u_{xxxt}^2\,\mathrm{d}x,$$

then, using Poincaré's inequality and the boundary condition (1.2), we have

$$E_3(t) \leqslant CE_4(t).$$

In a similar way to the estimates of $E_1(t)$, we have

$$\frac{\mathrm{d}E_3(t)}{\mathrm{d}t} + C_7E_3(t) \leqslant C_8 \quad \text{for any } t \geqslant t_1(B), \tag{2.35}$$

which immediately leads to

$$E_3(t) \leqslant E_3(0)\mathrm{e}^{-C_7t} + \frac{C_8}{C_7} \quad \text{for any } t \geqslant t_1(B). \tag{2.36}$$

For the initial data, starting from the bounded set B mentioned above, there exists $t_2(B) \geqslant t_1(B)$ such that, when $t \geqslant t_2(B)$, we have

$$E_3(t) \leqslant \frac{2C_8}{C_7}. \tag{2.37}$$

From (2.37), we can see that if we choose $\bar{C}_1 = \bar{C}_2 = 2C_8/C_7$ in the definition of B_0, the existence of absorbing set B_0 follows. The proof is complete. □

3. The existence of a global attractor for the system without viscosity

We consider the initial boundary-value problem (1.2)–(1.4). In this section, we shall prove the existence of a global attractor for this system in the closed subspace H_β.

Here we define H_β as

$$H_\beta := \left\{(u, u_t) \in H, \int_0^1 (\tfrac{1}{2}u_t^2 + \tfrac{1}{2}Ru_{xx}^2 + F_1(u_x))\,\mathrm{d}x \leqslant \beta\right\}.$$

We establish the local existence and uniqueness results in a similar way to § 2.

Lemma 3.1. *Under the same assumption as in theorem 1.3, there exists $t^* > 0$ depending only on $\|u_0\|_{H^4(\Omega)}$, $\|u_1\|_{H^2(\Omega)}$, such that problem (1.2)–(1.4) admits a unique solution (u, u_t) in $\bar{\Omega} \times [0, t^*]$ such that*

$$u \in C([0,t^*]; H^4) \cap C^1([0,t^*]; H^2) \cap L^2([0,t^*]; H^5),$$
$$u_t \in C([0,t^*]; H^2) \cap L^2([0,t^*]; H^3).$$

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3.1. Proof of theorem 1.3(i)

We can only obtain *a priori* estimates depending on T. In what follows, the letter C_T denotes a positive constant which may depend on the initial data and the time T.

LEMMA 3.2. *For any $t \in [0,T]$, the following estimates hold.*

$$\|u_t\| \leqslant C_T, \qquad \|u_{xx}\| \leqslant C_T, \qquad \|u_x\|_{L^\infty} \leqslant C_T, \qquad \int_0^t \|u_t\|^2 \,\mathrm{d}\tau \leqslant C_T. \tag{3.1}$$

Proof. Multiplying (1.4) by u_t and integrating with respect to x yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^1 u_t^2 \,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2 \,\mathrm{d}x + \int_0^1 F(u_x)\,\mathrm{d}x\right) + \mu\int_0^1 u_t^2 \,\mathrm{d}x = 0. \tag{3.2}$$

From (3.2), the estimates of (3.1) follow immediately. □

LEMMA 3.3. *For any $t \in [0,T]$, the following estimates hold.*

$$\|u_{tt}\| \leqslant C_T, \qquad \|u_{xxt}\| \leqslant C_T. \tag{3.3}$$

Proof. We differentiate (1.4) with respect to t, multiply the result by u_{tt} and integrate with respect to x over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}^2 \,\mathrm{d}x + \mu\int_0^1 u_{tt}^2 \,\mathrm{d}x + \int_0^1 f(u_x)_t u_{xtt} \,\mathrm{d}x + \tfrac{1}{2}R\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2 \,\mathrm{d}x = 0. \tag{3.4}$$

Since

$$\int_0^1 f(u_x)_{xt} u_{tt} \,\mathrm{d}x \leqslant \tfrac{1}{2}\mu\int_0^1 u_{tt}^2 \,\mathrm{d}x + C_\mu\int_0^1 |f(u_x)_{xt}|^2 \,\mathrm{d}x \tag{3.5}$$

and

$$\begin{aligned}\int_0^1 |f(u_x)_{xt}|^2 \,\mathrm{d}x &= \int_0^1 |f''(u_x)u_{xx}u_{xt}|^2 \,\mathrm{d}x + \int_0^1 |f'(u_x)u_{xxt}|^2 \,\mathrm{d}x \\ &\leqslant C\|u_{xt}\|^2 + C\|u_{xxt}\|^2 \\ &\leqslant C\|u_{xxt}\|^2 + C,\end{aligned} \tag{3.6}$$

here,

$$\|u_{xt}\|^2 \leqslant C\|u_{xxt}\|^2 + C\|u_t\|^2.$$

Applying Gronwall's inequality, we can obtain

$$\|u_{tt}\| \leqslant C_T, \qquad \|u_{xxt}\| \leqslant C_T.$$

The proof is complete. □

Combining lemma 3.2 with equation (1.4), we can obtain the boundedness of $\|u\|_{H^4}$, $\|u_t\|_{H^2}$, then the global existence and uniqueness follows.

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3.2. Proof of theorem 1.3(ii)

First, we prove the existence of an absorbing set in H_β. In the following, C and C_i denote positive constants depending only on β.

Let

$$B_\beta = \{(u, u_t) \in H_\beta, \|u\|_{H^4} \leqslant \bar{C}_1, \|u_t\|_{H^2} \leqslant \bar{C}_2\},$$

where $\bar{C}_1$, $\bar{C}_2$ are positive constants that may depend on β, but not on the initial data and t, and they will be specified later. Then we have the following.

LEMMA 3.4. *B_β is an absorbing set in H_β, i.e. for any bounded set B in H_β, there exists some time $t = t_0(B) > 0$ such that, when $t \geqslant t_0(B)$, $S(t)B \subset B_\beta$.*

Proof. From now on, we assume that the initial data $(u_0, u_1) \in B \subset H_\beta$.

First, we multiply (1.4) by u_t and integrate with respect to x to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^1 u_t^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 F(u_x)\,\mathrm{d}x\right) + \mu\int_0^1 u_t^2\,\mathrm{d}x = 0. \tag{3.7}$$

Then we have

$$\begin{aligned}\frac{1}{2}\int_0^1 u_t^2\,\mathrm{d}x &+ \tfrac{1}{2}R\int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 F(u_x)\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\int_0^1 u_1^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 D^2u_0^2 + \int_0^1 F(Du_0)\,\mathrm{d}x \\ &\leqslant \beta.\end{aligned} \tag{3.8}$$

From (3.8) we can see that $S(t)$ maps (u, u_t) from H_β into itself and stays there forever. Moreover, we obtain

$$\|u_t\| \leqslant C, \qquad \|u_{xx}\| \leqslant C, \qquad \|u_x\|_{L^\infty} \leqslant C \tag{3.9}$$

and

$$\int_0^t \|u_t\|^2\,\mathrm{d}\tau \leqslant C, \quad \int_0^t \|u_t\|^{n+2}\,\mathrm{d}\tau \leqslant C, \quad \forall n > 0. \tag{3.10}$$

Second, we differentiate (1.4) with respect to t, multiply the result by u_{tt} and integrate with respect to x over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}^2\,\mathrm{d}x + \mu\int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 f(u_x)_t u_{xtt}\,\mathrm{d}x + \frac{R}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xxt}^2\,\mathrm{d}x = 0. \tag{3.11}$$

Here,

$$\begin{aligned}\int_0^1 f(u_x)_t u_{xtt}\,\mathrm{d}x = &\int_0^1 5\alpha_1 u_x^4 u_{xt} u_{xtt}\,\mathrm{d}x \\ &- \int_0^1 3\alpha_2 u_x^2 u_{xt} u_{xtt}\,\mathrm{d}x - \int_0^1 \alpha_3 u_{xt} u_{xtt}\,\mathrm{d}x.\end{aligned} \tag{3.12}$$

In what follows, we estimate the right-hand side of (3.12).

Since

$$\int_0^1 5\alpha_1 u_x^4 u_{xt} u_{xtt}\,\mathrm{d}x = \frac{1}{2}\biggl(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 5\alpha_1 u_x^4 u_{xt}^2\,\mathrm{d}x - \int_0^1 20\alpha_1 u_x^3 u_{xt}^3\,\mathrm{d}x\biggr), \tag{3.13}$$

and from the estimates in (3.9), we have

$$\biggl|\int_0^1 u_x^3 u_{xt}^3\,\mathrm{d}x\biggr| \leqslant C\int_0^1 |u_{xt}|^3\,\mathrm{d}x.$$

Using Nirenberg's inequality yields

$$\|u_{xt}\|_{L^3}^3 \leqslant C\|u_{xxt}\|_{L^2}^{7/4}\|u_t\|_{L^2}^{5/4} \leqslant \delta\|u_{xxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^{10}$$

with δ being a positive constant again.

In a similar manner, we have

$$\int_0^1 3\alpha_2 u_x^2 u_{xt} u_{xtt}\,\mathrm{d}x = \frac{1}{2}\biggl(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 3\alpha_2 u_x^2 u_{xt}^2\,\mathrm{d}x - \int_0^1 6\alpha_2 u_x u_{xt}^3\,\mathrm{d}x\biggr)$$

and

$$\biggl|\int_0^1 u_x u_{xt}^3\,\mathrm{d}x\biggr| \leqslant C\int_0^1 |u_{xt}|^3\,\mathrm{d}x \leqslant \delta\|u_{xxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^{10}.$$

Therefore, we infer from (3.11) and the above estimates that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xxt}^2\,\mathrm{d}x + \tfrac{5}{2}\alpha_1\int_0^1 u_x^4 u_{xt}^2\,\mathrm{d}x \\ - \tfrac{3}{2}\alpha_2\int_0^1 u_x^2 u_{xt}^2\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_{xt}^2\,\mathrm{d}x\biggr) \\ + \mu\int_0^1 u_{tt}^2\,\mathrm{d}x \leqslant \delta\|u_{xxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^{10}.\end{aligned} \tag{3.14}$$

Finally, we differentiate (1.4) with respect to t, multiply the result by u_t and integrate with respect to x over Ω to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t u_{tt}\,\mathrm{d}x + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x + R\int_0^1 u_{xxt}^2\,\mathrm{d}x \\ - \int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 f(u_x)_t u_{xt}\,\mathrm{d}x = 0.\end{aligned} \tag{3.15}$$

Here,

$$\begin{aligned}\biggl|\int_0^1 f(u_x)_t u_{xt}\,\mathrm{d}x\biggr| &= \biggl|\int_0^1 f'(u_x) u_{xt}^2\,\mathrm{d}x\biggr| \\ &\leqslant C\|u_{xt}\|_{L^2}^2 \\ &\leqslant \delta\|u_{xxt}\|_{L^2}^2 + C_\delta\|u_t\|_{L^2}^2.\end{aligned}$$

Now we multiply (3.15) by $\frac{1}{2}\mu$ and add the result to (3.14) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xxt}^2\,\mathrm{d}x + \tfrac{5}{2}\alpha_1\int_0^1 u_x^4u_{xt}^2\,\mathrm{d}x - \tfrac{3}{2}\alpha_2\int_0^1 u_x^2u_{xt}^2\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_{xt}^2\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_tu_{tt}\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_t^2\,\mathrm{d}x\biggr) + \tfrac{1}{4}\mu^2\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{2}\mu R\int_0^1 u_{xxt}^2\,\mathrm{d}x \leqslant \delta\|u_{xxt}\|^2 + C_\delta\|u_t\|^2. \tag{3.16}$$

Choosing sufficiently small δ, we finally have

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xxt}^2\,\mathrm{d}x + \tfrac{5}{2}\alpha_1\int_0^1 u_x^4u_{xt}^2\,\mathrm{d}x - \tfrac{3}{2}\alpha_2\int_0^1 u_x^2u_{xt}^2\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_{xt}^2\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_tu_{tt}\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_t^2\,\mathrm{d}x\biggr) + \tfrac{1}{4}\mu^2\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{4}\mu R\int_0^1 u_{xxt}^2\,\mathrm{d}x \leqslant C. \tag{3.17}$$

If we define

$$E_1(t) := \frac{1}{2}\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{2}R\int_0^1 u_{xxt}^2\,\mathrm{d}x + \tfrac{5}{2}\alpha_1\int_0^1 u_x^4u_{xt}^2\,\mathrm{d}x - \tfrac{3}{2}\alpha_2\int_0^1 u_x^2u_{xt}^2\,\mathrm{d}x - \tfrac{1}{2}\alpha_3\int_0^1 u_{xt}^2\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_tu_{tt}\,\mathrm{d}x + \tfrac{1}{4}\mu^2\int_0^1 u_t^2\,\mathrm{d}x$$

and

$$E_2(t) := \tfrac{1}{2}\mu\int_0^1 u_{tt}^2\,\mathrm{d}x + \tfrac{1}{4}\mu R\int_0^1 u_{xxt}^2\,\mathrm{d}x.$$

Combining the estimates obtained in (3.9), (3.10) with equation (1.4), we get

$$E_1(t) \sim \|u\|_{H^4}^2 + \|u_t\|_{H^2}^2 \quad\text{and}\quad E_1(t) \leqslant CE_2(t).$$

Therefore,

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} + C_1E_1(t) \leqslant C_2,$$

then it immediately leads to

$$E_1(t) \leqslant E_1(0)\mathrm{e}^{-C_1t} + \frac{C_2}{C_1}. \tag{3.18}$$

It is clear that, here, C_1 and C_2 are positive constants depending only on β. Then we have, for any initial data starting from any bounded set B of H_β, that there exists some time $t_0(B)$ such that, when $t \geqslant t_0(B)$,

$$E_1(t) \leqslant \frac{2C_2}{C_1}. \tag{3.19}$$

The existence of an absorbing set follows. The proof is complete. □

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Next, we focus on proving the uniform compactness of the orbits. For this we have to estimate higher-order derivatives. From now on, we assume that the initial data belong to a bounded set B contained in H_β, and we use C, $\tilde{C}$ to denote positive constants depending on B and β, i.e. $\|u_0\|_{H^4}$, $\|u_1\|_{H^2}$ and β.

Lemma 3.5. *There exists some time $t_1 = t_1(B) > 0$, such that (u, u_t) is bounded in $C([t_1, +\infty); H^5 \times H^3)$.*

Proof. First, we differentiate (1.4) with respect to t, multiply the result by $-u_{xxtt}$ and integrate with respect to x over Ω to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tfrac{1}{2}Ru_{xxxt}^2 + \tfrac{1}{2}u_{xtt}^2)\,\mathrm{d}x + \tfrac{1}{2}\mu\int_0^1 u_{xtt}^2\,\mathrm{d}x + \int_0^1 f(u_x)_{xt}u_{xxtt}\,\mathrm{d}x = 0. \tag{3.20}$$

Multiplying (3.20) by t yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}t\|u_{xtt}\|^2 + \tfrac{1}{2}Rt\|u_{xxxt}\|^2) + \tfrac{1}{2}\mu t\int_0^1 u_{xtt}^2\,\mathrm{d}x \\ = \tfrac{1}{2}\|u_{xtt}\|^2 + \tfrac{1}{2}R\|u_{xxxt}\|^2 + t\int_0^1 f(u_x)_{xxt}u_{xtt}\,\mathrm{d}x.\end{aligned} \tag{3.21}$$

Next, we differentiate (1.4) with respect to t, multiply the result by $-u_{xxt}$ and integrate with respect to x over Ω to obtain

$$\begin{aligned}\tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xt}^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}u_{xxt}\,\mathrm{d}x \\ + R\int_0^1 u_{xxxt}^2\,\mathrm{d}x - \int_0^1 u_{xtt}^2\,\mathrm{d}x + \int_0^1 f(u_x)_{xt}u_{xxt}\,\mathrm{d}x = 0.\end{aligned} \tag{3.22}$$

Observe that if we integrate (3.17) with respect to t, we arrive at

$$\|u_{tt}\| \leqslant C, \qquad \|u_{xxt}\| \leqslant C, \qquad \int_0^t \|u_{tt}\|^2\,\mathrm{d}\tau \leqslant C, \qquad \int_0^t \|u_{xxt}\|^2\,\mathrm{d}\tau \leqslant C \tag{3.23}$$

with $C = C(\|u_0\|_{H^4}, \|u_1\|_{H^2})$. From (3.10), we also have

$$\int_0^t \|u_t\|^2\,\mathrm{d}\tau \leqslant C.$$

Using Nirenberg's inequality and equation (1.4), we have

$$\int_0^t \|u_{xt}\|^2\,\mathrm{d}\tau \leqslant C, \qquad \|u_{xxxx}\| \leqslant C, \qquad \|u_{xxx}\| \leqslant C. \tag{3.24}$$

Then we integrate (3.22) with respect to t to arrive at

$$\begin{aligned}R\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau + \int_0^t\int_0^1 f(u_x)_{xt}u_{xxt}\,\mathrm{d}x\,\mathrm{d}\tau \\ + \tfrac{1}{2}\mu\int_0^1 u_{xt}^2\,\mathrm{d}x - \tfrac{1}{2}\mu\int_0^1 u_{xt}^2|_{t=0}\,\mathrm{d}x \\ = \int_0^1 u_{tt}u_{xxt}\,\mathrm{d}x - \int_0^1 u_{tt}u_{xxt}|_{t=0}\,\mathrm{d}x + \int_0^t \|u_{xtt}\|^2\,\mathrm{d}\tau.\end{aligned} \tag{3.25}$$

Combining the estimates obtained in lemma 3.4 and (3.23), we have

$$\begin{aligned}\int_0^t\int_0^1 f(u_x)_{xt}u_{xxt}\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t\int_0^1(5\alpha_1u_x^4-30\alpha_2u_x^2-\alpha_3)u_{xxt}^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+\int_0^t\int_0^1(20\alpha_1u_x^3u_{xt}u_{xx}-6\alpha_2u_xu_{xt}u_{xx})u_{xxt}\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant C\int_0^t\|u_{xxt}\|^2\,\mathrm{d}\tau+C\\ &\leqslant C. \end{aligned}\tag{3.26}$$

Thus, it follows from (3.25) that

$$\int_0^t\|u_{xxxt}\|^2\,\mathrm{d}\tau\leqslant C\int_0^t\|u_{xtt}\|^2\,\mathrm{d}\tau+C.\tag{3.27}$$

Similarly, we also have

$$\int_0^t\|u_{xtt}\|^2\,\mathrm{d}\tau\leqslant C\int_0^t\|u_{xxxt}\|^2\,\mathrm{d}\tau+C.\tag{3.28}$$

In what follows, we estimate the last term on the right-hand side of (3.21).

Since

$$f(u_x)_{xt}=20\alpha_1u_x^3u_{xt}u_{xx}+5\alpha_1u_x^4u_{xxt}-3\alpha_2u_x^2u_{xxt}-6\alpha_2u_xu_{xt}u_{xx}-\alpha_3u_{xxt},\tag{3.29}$$

here,

$$\begin{aligned}&\left|\int_0^t\int_0^1(20\alpha_1u_x^3u_{xt}u_{xx})_xu_{xtt}\,\mathrm{d}x\,\mathrm{d}\tau\right|\\ &\qquad\leqslant\delta\int_0^t\|u_{xtt}\|^2\,\mathrm{d}\tau+C_\delta\int_0^t\int_0^1(u_x^3u_{xt}u_{xx})_x^2\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}\int_0^t\int_0^1(u_x^3u_{xt}u_{xx})_x^2\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t\int_0^1(3u_x^2u_{xx}^2u_{xt}+u_x^3u_{xt}u_{xxx}+u_x^3u_{xx}u_{xxt})^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant C\int_0^t\int_0^1(u_{xt}^2+u_{xxt}^2)\,\mathrm{d}x\,\mathrm{d}\tau\leqslant C.\end{aligned}\tag{3.30}$$

Thus, we have

$$\left|\int_0^t\int_0^1(20\alpha_1u_x^3u_{xt}u_{xx})_xu_{xtt}\,\mathrm{d}x\,\mathrm{d}\tau\right|\leqslant\delta\int_0^t\|u_{xtt}\|^2\,\mathrm{d}\tau+C_\delta.$$

In a similar manner to (3.30), we have

$$\left|\int_0^t\int_0^1(6\alpha_2u_xu_{xt}u_{xx})_xu_{xtt}\,\mathrm{d}x\,\mathrm{d}\tau\right|\leqslant\delta\int_0^t\|u_{xtt}\|^2\,\mathrm{d}\tau+C_\delta\tag{3.31}$$

and

$$\int_0^1 (5\alpha_1 u_x^4 u_{xxt})_x u_{xtt}\,\mathrm{d}x = -\int_0^1 (5\alpha_1 u_x^4 u_{xxt}) u_{xxtt}\,\mathrm{d}x$$
$$= -\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 5\alpha_1 u_x^4 u_{xxt}^2\,\mathrm{d}x - \int_0^1 20\alpha_1 u_x^3 u_{xxt}^3\,\mathrm{d}x\right). \tag{3.32}$$

By Nirenberg's inequality and Young's inequality, we find that

$$\left|\int_0^t \int_0^1 20\alpha_1 u_x^3 u_{xxt}^3\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant C\int_0^t \int_0^1 |u_{xxt}|^3\,\mathrm{d}x\,\mathrm{d}\tau \tag{3.33}$$

and

$$\|u_{xxt}\|_{L^3} \leqslant C\|u_{xxxt}\|_{L^2}^{1/6}\|u_{xxt}\|_{L^2}^{5/6}, \tag{3.34}$$

$$\|u_{xxt}\|_{L^3}^3 \leqslant C\|u_{xxxt}\|_{L^2}^{1/2}\|u_{xxt}\|_{L^2}^{5/2}$$
$$\leqslant \delta\|u_{xxxt}\|_{L^2}^2 + C_\delta\|u_{xxt}\|_{L^2}^{10/3}. \tag{3.35}$$

Thus,

$$\int_0^t \int_0^1 |u_{xxt}|^3\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \delta\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau + C_\delta. \tag{3.36}$$

Similarly, we have

$$-\int_0^1 (3\alpha_2 u_x^2 u_{xxt})_x u_{xtt}\,\mathrm{d}x = \int_0^1 (3\alpha_2 u_x^2 u_{xxt}) u_{xxtt}\,\mathrm{d}x$$
$$= \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 3\alpha_2 u_x^2 u_{xxt}^2\,\mathrm{d}x - \int_0^1 6\alpha_2 u_x u_{xxt}^3\,\mathrm{d}x\right) \tag{3.37}$$

and

$$\left|\int_0^t \int_0^1 6\alpha_2 u_x u_{xxt}^3\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant \delta\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau + C_\delta. \tag{3.38}$$

Finally, we deduce

$$\int_0^t \int_0^1 f(u_x)_{xxt} u_{xtt}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \delta\int_0^t \|u_{xtt}\|^2\,\mathrm{d}\tau + \delta\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau + \tilde{C}_\delta \tag{3.39}$$

with $\tilde{C}_\delta = \tilde{C}(\|u_0\|_{H^4}, \|u_1\|_{H^2}, \delta)$.

Now we integrate (3.21) with respect to t to obtain

$$\tfrac{1}{2}t\|u_{xtt}\|^2 + \tfrac{1}{2}Rt\|u_{xxxt}\|^2 + \tfrac{1}{2}\mu\int_0^t \tau\|u_{xtt}\|^2\,\mathrm{d}\tau$$
$$\leqslant \frac{1}{2}\int_0^t \|u_{xtt}\|^2\,\mathrm{d}\tau + \tfrac{1}{2}R\int_0^t \|u_{xxxt}\|^2\,\mathrm{d}\tau$$
$$+ \delta t\int_0^t \|u_{xtt}\|^2 + \|u_{xxxt}\|^2\,\mathrm{d}\tau + \tilde{C}_\delta t. \tag{3.40}$$

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Combining (3.40) with (3.27), (3.28) for any $t \geqslant 1$ and choosing sufficiently small $\delta \ll \mu$ yields

$$\tfrac{1}{2}\|u_{xtt}\|^2 + \tfrac{1}{2}R\|u_{xxxt}\|^2 \leqslant \frac{1}{2}\int_0^t \|u_{xtt}\|^2 \,\mathrm{d}\tau + \tfrac{1}{2}R\int_0^t \|u_{xxxt}\|^2 \,\mathrm{d}\tau + Ct. \tag{3.41}$$

Using Gronwall's inequality yields

$$\tfrac{1}{2}\|u_{xtt}\|^2 + \tfrac{1}{2}R\|u_{xxxt}\|^2 \leqslant Ct\mathrm{e}^t. \tag{3.42}$$

Let $t = 1$ in (3.42) to obtain

$$\tfrac{1}{2}\|u_{xtt}|_{t=1}\|^2 + \tfrac{1}{2}R\|u_{xxxt}|_{t=1}\|^2 \leqslant C\mathrm{e} \tag{3.43}$$

with $C = C(\|u_0\|_{H^4}, \|u_1\|_{H^2})$.

Integrating (3.21) again with respect to t in $[1, +\infty)$ and combining the result with (3.27), (3.28) and (3.39), we derive that there exists sufficiently large $t_1 > 1$ in (3.21) such that, when $t > t_1$, the terms on the right-hand side of (3.21), i.e.

$$\frac{1}{2}\int_1^t \|u_{xtt}\|^2 \,\mathrm{d}\tau + \tfrac{1}{2}R\int_1^t \|u_{xxxt}\|^2 \,\mathrm{d}\tau + \delta t\int_1^t \|u_{xxxt}\|^2 + \|u_{xtt}\|^2 \,\mathrm{d}\tau$$

can be absorbed by

$$\tfrac{1}{4}\mu\int_1^t \tau\|u_{xtt}\|^2 \,\mathrm{d}\tau + C.$$

Then we obtain

$$\tfrac{1}{2}t\|u_{xtt}\|^2 + \tfrac{1}{2}Rt\|u_{xxxt}\|^2 + \tfrac{1}{4}\mu\int_1^t \tau\|u_{xtt}\|^2 \,\mathrm{d}\tau \leqslant C + Ce + Ct \quad \text{for any } t \geqslant t_1 \tag{3.44}$$

with $C = C(\|u_0\|_{H^4}, \|u_1\|_{H^2})$.

Finally, we have

$$\tfrac{1}{2}\|u_{xtt}\|^2 + \tfrac{1}{2}R\|u_{xxxt}\|^2 \leqslant \frac{C}{t} + C \quad \text{for any } t \geqslant t_1. \tag{3.45}$$

Combining (3.45) with equation (1.4), we conclude our argument. The proof is complete. □

The compactness of the orbit in $H^4 \times H^2$ follows from the last lemma.

In a similar manner to § 2, applying the Temam theorem again, which can be rephrased as follows, we deduce the results of Theorem 1.3(ii).

Theorem 3.6. *Suppose that the following hold.*

(a) *The mapping $S(t)$, $t \geqslant 0$, defined by the solution to problems (1.2)–(1.4) is a nonlinear continuous semigroup from H into itself.*

(b) *The operators $S(t)$ are uniformly compact for t large, i.e. for every bounded set B contained in H_β, there exists t_1 which may depend on B such that $\bigcup_{t \geqslant t_1} S(t)B$ is relatively compact in H.*

(c) *The orbit starting from any bounded set of H_β will reenter in H_β after a finite time, which depends only on this bounded set, and stay there forever. There exists a bounded set B_β in H_β such that B_β is absorbing in H_β.*

Then the ω-limit set of B_β, A_β is a global attractor which is compact and attracts the bounded sets of H_β.

Therefore, the proof of theorem 1.3(ii) is complete.

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