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ON THE RADON-NIKODYM DERIVATIVE WITH A CHAIN RULE IN A VON NEUMANN ALGEBRA

BY
GEORGE A. ELLIOTT

1. The purpose of this paper is to show that by a reorganization of the proofs of the main results concerning Radon-Nikodym derivatives in a von Neumann algebra of Pedersen and Takesaki in [12] and of Connes in paragraphs 1.1 and 1.2 of [5], considerable technical simplification can be achieved. Roughly speaking, the analytic vector techniques developed by these authors for the study of weights on a von Neumann algebra can be replaced, to a large extent, by the tensor product methods introduced by Connes, which are essentially algebraic in nature. In the exposition which follows, analytic vectors are not used at all (see, however, 4.4).

The present approach does not lead to Proposition 5.9 of [12] (or its consequence in 1.1.2 of [5]), or to the most general case of Proposition 5.10 of [12] (the case that the modular automorphisms of the two weights, while they are permutable, do not leave both weights invariant). These results indicate that analytic vectors will continue to play an important role in the study of weights (and states). This is confirmed by the work of Araki in [1] and [2] and of Connes in [7] and [8] on the analytic properties of the Radon-Nikodym derivative, which reflect various order relations between the weights that the derivative compares.

2. In terminology and notation we shall follow [3], [4] and [13]. We shall denote by M a von Neumann algebra which will be fixed throughout. Let us recall the definition of a weight on M, and some basic properties of weights that we shall use.

A weight on M is a function φ defined on M^+, with values in the interval $[0, \infty]$, such that

$$\varphi(x+y) = \varphi(x)+\varphi(y), \qquad x, y \in M^+,$$
$$\varphi(\lambda x) = \lambda\varphi(x), x \in M^+, \lambda \in \mathbb{R}^+ \qquad (\text{here } 0.\ \infty = 0).$$

A weight φ is said to be faithful if

$$x \in M^+, \varphi(x) = 0 \Rightarrow x = 0,$$

and to be semifinite if the linear span $\mathfrak{M}_\varphi$ of $x \in M^+$ such that $\varphi(x) < \infty$ is ultraweakly dense in M. It is equivalent to say that the set $\mathfrak{N}_\varphi$ of $y \in M$ such that

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$\varphi(y^*y)<\infty$ is ultraweakly dense. $\mathfrak{N}_\varphi$ is a left ideal of M such that

$$\mathfrak{N}_\varphi^*\mathfrak{N}_\varphi = (\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)^2 = \mathfrak{M}_\varphi.$$

A weight φ on M is said to be normal if for every bounded, upward directed set S in M^+,

$$\varphi(\sup S) = \sup \varphi(S).$$

It has recently been shown by Haagerup (1.8, 2.1 and 2.2 of [11]) that a normal weight is a supremum of finite normal weights (i.e., normal positive functionals).

Let φ be a faithful normal weight on M. Then an inner product is defined on $\mathfrak{N}_\varphi$ by

$$(x \mid y)_\varphi = \varphi(y^*x), \qquad x, y \in \mathfrak{N}_\varphi.$$

By 2.13 of [4] (coupled with the results of [11]), the antilinear map

$$\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* \ni x \mapsto x^*$$

is densely defined and preclosed in the completion of $\mathfrak{N}_\varphi$ with respect to the inner product determined by φ. Denote by S_φ the closure of this map in the completion of $\mathfrak{N}_\varphi$, and denote the domain of S_φ by $\mathscr{D}(S_\varphi)$. Then by 2.13 of [4],

$$\mathfrak{N}_\varphi \cap \mathscr{D}(S_\varphi) = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*.$$

Let φ be a faithful semifinite normal weight on M. Denote by π_0 the representation of M by left multiplication in $\mathfrak{N}_\varphi$; π_0 is injective since φ is semifinite. For each $x \in M$, $\pi_0(x)$ may be extended to a bounded operator in the Hilbert space completion of $\mathfrak{N}_\varphi$, which we shall denote by $\pi_\varphi(x)$. Set $S_\varphi^*S_\varphi=\Delta_\varphi$. Then Δ_φ is nonsingular and for all $t \in \mathbb{R}$,

$$\Delta_\varphi^{it}\pi_\varphi(M)\,\Delta_\varphi^{-it} = \pi_\varphi(M).$$

This is the main result described in [13]. (For a proof free of analytic vectors, see [14]; simultaneously, a different short proof was given in [10].) It makes possible the definition of a one-parameter group $t\mapsto\sigma_t^\varphi$ of automorphisms of M such that

$$\Delta_\varphi^{it}\pi_\varphi(x)\,\Delta_\varphi^{-it} = \pi_\varphi(\sigma_t^\varphi(x)), \qquad x \in M, \qquad t \in \mathbb{R}.$$

By 2.13 of [4], π_φ is a normal isomorphism, whence for each $x \in M, t\mapsto \sigma_t^\varphi(x)$ is strongly continuous. The group σ^φ is sometimes called the modular automorphism group associated with φ.

3. Invariant elements.

3.1. Lemma. *Let φ be a faithful semifinite normal weight on the von Neumann algebra M. Let h be an element of M such that $\sigma_t^\varphi(h)=h$ for all $t \in \mathbb{R}$. Then $h\mathfrak{M}_\varphi\subset \mathfrak{M}_\varphi$, $\mathfrak{M}_\varphi h\subset\mathfrak{M}_\varphi$, and $\varphi(hx)=\varphi(xh)$ for all $x \in \mathfrak{M}_\varphi$.*

Proof. Adding a scalar to h if necessary, we may suppose that h is invertible.

The hypothesis may then be stated as $\pi_\varphi(h)^{-1}\Delta_\varphi\pi_\varphi(h)=\Delta_\varphi$. Using the identity $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* = \mathfrak{N}_\varphi \cap \mathscr{D}(\Delta_\varphi^{1/2})$, we deduce that the maps $x \mapsto hx$ and $x \mapsto xh=(h^*x^*)^*$ are bijections of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$. From this and the relation $\mathfrak{M}_\varphi=(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)^2$ follow the inclusions $h\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$ and $\mathfrak{M}_\varphi h \subset \mathfrak{M}_\varphi$.

The first of the preceding maps is of course $\pi_\varphi(h)$; denote the second by R_h. Then for $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$,

$$R_h S_\varphi \pi_\varphi(h^*)^{-1}x = R_h(h^{*-1}x)^* = x^* = S_\varphi x,$$

whence, with $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ the polar decomposition of S_φ,

$$R_h x = S_\varphi \pi_\varphi(h^*) S_\varphi x = J_\varphi \Delta_\varphi^{1/2} \pi_\varphi(h^*) \Delta_\varphi^{-1/2} J_\varphi x = J_\varphi \pi_\varphi(h^*) J_\varphi x.$$

This shows that

$$R_h = J_\varphi \pi_\varphi(h^*) J_\varphi \mid \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*.$$

Hence, if $x, y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$:

$$\varphi(xyh) = (yh \mid x^*)_\varphi = (R_h y \mid x^*)_\varphi = (J_\varphi \pi_\varphi(h^*) J_\varphi y \mid x^*)_\varphi;$$

$$\varphi(hxy) = (y \mid x^*h^*)_\varphi = (y \mid R_{h^*}x^*)_\varphi = (y \mid J_\varphi \pi_\varphi(h) J_\varphi x^*)_\varphi;$$

$$\varphi(hxy) = \varphi(xyh).$$

3.2. LEMMA. *Let φ be a faithful semifinite normal weight on the von Neumann algebra M. Let v be a unitary element of M such that $v^*\varphi v=\varphi$, that is, such that $v\mathfrak{M}_\varphi v^*=\mathfrak{M}_\varphi$ and $\varphi(vxv^*)=\varphi(x)$ for $x \in \mathfrak{M}_\varphi$. Then $\sigma_t^\varphi(v)=v$ for all $t \in \mathbb{R}$.*

Proof. From $v^*\varphi v=\varphi$ follows $\mathfrak{N}_\varphi v=\mathfrak{N}_\varphi$ and hence that $x \mapsto vx$ and $x \mapsto xv$ are unitary bijections of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, with respect to the inner product determined by φ. The first is $\pi_\varphi(v)$; denote the second by R_v. For $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$,

$$R_v S_\varphi \pi_\varphi(v) x = R_v(vx)^* = x^* = S_\varphi x.$$

Hence, since $\pi_\varphi(v)$ and R_v are unitary,

$$R_v S_\varphi \pi_\varphi(v) = S_\varphi.$$

Then

$$\Delta_\varphi = S_\varphi^* S_\varphi = \pi_\varphi(v)^* S_\varphi^* R_v^* R_v S_\varphi \pi_\varphi(v) = \pi_\varphi(v)^* \Delta_\varphi \pi_\varphi(v);$$

this is equivalent to the conclusion of the lemma.

3.3. PROBLEM. 3.1 is (i)$\Rightarrow$(ii) of Theorem 3.6 of [12]. 3.2 is a special case of (ii)$\Rightarrow$(i) of Theorem 3.6 of [12], and is the only case that will be required below. This is fortunate, since it is not clear how to obtain the general result without using the fact that $\mathfrak{M}_\varphi$ contains analytic vectors for Δ_φ. More precisely, it does not seem to be known whether, if $\mathfrak{M}$ is a dense subset of the domain of $\Delta^{1/2}$ and h is a bounded operator such that $h\mathfrak{M} \subset \mathfrak{M}$, $h^*\mathfrak{M} \subset \mathfrak{M}$ and $(\Delta^{1/2}h\xi \mid \Delta^{1/2}\eta)=(\Delta^{1/2}\xi \mid \Delta^{1/2}h^*\eta)$ for all $\xi, \eta \in \mathfrak{M}$, then h and Δ commute.

4. The Radon-Nikodym derivative of Connes.

4.1. Definition (see 1.2.2 of [5]). Let φ_1 and φ_2 be faithful semifinite normal weights on the von Neumann algebra M. Denote by M_2 the algebra of all linear operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and by (e_{ij}) the system of matrix units of M_2 associated with the standard basis of $\mathbb{C}^2$. Define a weight θ on the von Neumann algebra $M\otimes M_2$ by

$$\theta(\Sigma x_{ij}\otimes e_{ij}) = \Sigma\varphi_i(x_{ii}), \qquad \Sigma x_{ij}\otimes e_{ij} \in (M\otimes M_2)^+.$$

It is clear that θ is faithful and normal. Semifiniteness of θ follows from the equivalence of $\sum x_{ij}\otimes e_{ij} \in \mathfrak{N}_\theta$ and $x_{ij} \in \mathfrak{N}_{\varphi_j}$. By 3.2 with $v=1\otimes(1-2e_{11})$, $\sigma_t^\theta(1\otimes e_{11})= 1\otimes e_{11}$, all $t \in \mathbb{R}$. Hence for each $t \in \mathbb{R}$ there exists a unique $u_t \in M$ such that $\sigma_t^\theta(1\otimes e_{21}+1\otimes e_{12})=u_t\otimes e_{21}+u_t^*\otimes e_{12}$. Since $1\otimes e_{21}+1\otimes e_{12}$ is unitary, so is each u_t. The map $t\mapsto u_t$ will be called the Radon-Nikodym derivative of φ_2 with respect to φ_1, and will be denoted by $(D\varphi_2 : D\varphi_1)$.

4.2. Theorem (*see* 1.2.2 *of* [5]). *Let φ_1 and φ_2 be faithful semifinite normal weights on the von Neumann algebra M. Then, with $u_t=(D\varphi_2 : D\varphi_1)_t$,*

(i) $u_{s+t}=u_s\sigma_s^{\varphi_1}(u_t), \quad s, t \in \mathbb{R}$;
(ii) $\sigma_t^{\varphi_2}(x)=u_t\sigma_t^{\varphi_1}(x)u_t^*, \quad t \in \mathbb{R}, x \in M$.

Proof. Since $(\mathfrak{N}_{\varphi_1} \cap \mathfrak{N}_{\varphi_1}^*)\otimes e_{11} \subset \mathfrak{N}_\theta \cap \mathfrak{N}_\theta^*$ and $S_\theta \mid (\mathfrak{N}_{\varphi_1} \cap \mathfrak{N}_{\varphi_1}^*)\otimes e_{11}=S_{\varphi_1}\otimes 1 \mid (\mathfrak{N}_{\varphi_1} \cap \mathfrak{N}_{\varphi_1}^*)\otimes e_{11}$, $\sigma_t^\theta \mid M\otimes e_{11}=\sigma_t^{\varphi_1}\otimes 1$ for all $t \in \mathbb{R}$. Hence, as in [5], for $s, t \in \mathbb{R}$, and $x \in M$:

$$u_{s+t}\otimes e_{21} = \sigma_{s+t}^\theta(1\otimes e_{21}) = \sigma_s^\theta(u_t\otimes e_{21}) = \sigma_s^\theta(1\otimes e_{21})\sigma_s^\theta(u_t\otimes e_{11}) = u_s\sigma_s^{\varphi_1}(u_t)\otimes e_{21};$$
$$\sigma_t^{\varphi_2}(x)\otimes e_{22} = \sigma_t^\theta(x\otimes e_{22}) = \sigma_t^\theta(1\otimes e_{21})\sigma_t^\theta(x\otimes e_{11})\sigma_t^\theta(1\otimes e_{12}) = u_t\sigma_t^{\varphi_1}(x)u_t^*\otimes e_{22}.$$

4.3. Remark. Property 4.2(i) is sometimes expressed by saying that u is a cocycle with respect to the one-parameter group of automorphisms σ^{φ_1}. Cf. 6 below.

4.4. Problem. The properties 4.2(i) and 4.2(ii) determine u only up to a one-parameter unitary group in the centre of M. In Theorem 4 of [6], Connes described a property of u in the case that φ_1 and φ_2 are finite which, coupled only with the property $u_0=1$, determines u completely. This property may be reformulated as follows for general φ_1 and φ_2: for any $x, y^* \in \mathfrak{N}_{\varphi_1}^* \cap \mathfrak{N}_{\varphi_2}$ there should exist a bounded continuous complex-valued function f on the strip $0\leq \mathrm{Im} z\leq 1$, holomorphic in the interior, such that

$$f(t) = \varphi_2(\sigma_t^{\varphi_2}(y)u_t x), \quad f(t+i) = \varphi_1(xu_t\sigma_t^{\varphi_1}(y)), \qquad t \in \mathbb{R}.$$

(Connes stated this with y equal to 1, which is not in general an admissable value of y.) Unfortunately the proof of Connes is no longer valid; the inequality in the third line of the proof of Lemma 5 of [6] is not available for an infinite weight.

It can be shown that if 4.2(i) and 4.2(ii) also are verified by u then u is unique. Indeed, the property 4.2(i) implies that the one-parameter family $t \mapsto (1 \otimes e_{11} + u_t \otimes e_{22})(\sigma_t^{\theta_1} \otimes 1)(\cdot)(1 \otimes e_{11} + u_t \otimes e_{22})^*$ of automorphisms of $M \otimes M_2$ is a group, and the property 4.2(ii) together with 4.4 of [4] and the property stated above ensures that this group verifies the so-called Kubo-Martin-Schwinger boundary conditions with respect to θ (see definitions 13.1 of [12] and 4.1 of [4]), whence by 4.8 of [4] this group must be $t \mapsto \sigma_t^{\theta}$; it follows immediately that $u_t = \sigma_t^{\theta}(1 \otimes e_{21})$.

4.5. Remark. It is an interesting question how various possible relations between ψ and φ are described by properties of $(D\psi : D\varphi)$. It is easy to see that two relations considered in [12], respectively ψ σ_t^{φ}-invariant for all $t \in R$, and σ_s^{ψ}, σ_t^{φ} permutable for all $s, t \in \mathbb{R}$, are expressed by the property that $(D\psi : D\varphi)$ is, respectively, a group, and a group modulo the centre of M.

5. The chain rule.

5.1. Theorem (1.2.3(a) *of* [5]). *Let φ_1, φ_2, and φ_3 be faithful semifinite normal weights on the von Neumann algebra M. Then*

$$(D\varphi_3 : D\varphi_1)_t = (D\varphi_3 : D\varphi_2)_t (D\varphi_2 : D\varphi_1)_t, \qquad t \in \mathbb{R}.$$

Proof. We repeat the argument of [5]. Define a weight ρ on $M \otimes M_3$ by $\rho(\Sigma\, x_{ij} \otimes e_{ij}) = \Sigma\, \varphi_i(x_{ii})$. As in 4.1, ρ is faithful, semifinite and normal. Then

$$(D\varphi_3 : D\varphi_1)_t \otimes e_{31} = \sigma_t^{\rho}(1 \otimes e_{31}) = \sigma_t^{\rho}(1 \otimes e_{32}) \sigma_t^{\rho}(1 \otimes e_{21}) = (D\varphi_3 : D\varphi_2)_t (D\varphi_2 : D\varphi_1)_t \otimes e_{31}.$$

5.2. Theorem (*see* 1.2.4 *of* [5]). *Let φ, ψ_1 and ψ_2 be faithful semifinite normal weights on the von Neumann algebra M. Suppose that $(D\psi_1 : D\varphi) = (D\psi_2 : D\varphi)$. Then $\psi_1 = \psi_2$.*

Proof. By 5.1,

$$\begin{aligned}(D\psi_2 : D\psi_1) &= (D\psi_2 : D\varphi)(D\varphi : D\psi_1) = (D\psi_1 : D\varphi)(D\varphi : D\psi_1) \\ &= (D\psi_1 : D\psi_1) = (D\psi_1 : D\psi_1)(D\psi_1 : D\psi_1) = 1.\end{aligned}$$

This means that $\sigma_t^{\theta}(v) = v$ for all $t \in \mathbb{R}$ where θ is as in 4.1 with φ_i replaced by ψ_i, and $v = 1 \otimes e_{21} + 1 \otimes e_{12}$. Hence by 3.1, for $x \in M^+$,

$$\psi_1(x) = \theta(x \otimes e_{11}) = \theta(v^*(x \otimes e_{11})v) = \theta(x \otimes e_{22}) = \psi_2(x).$$

6. Every cocycle is $(D\psi : D\varphi)$ for some ψ.

6.1. Lemma (5.12 *of* [12]). *Let φ be a faithful semifinite normal weight on the von Neumann algebra M. Let $t \mapsto u_t$ be a strongly continuous one-parameter group of unitaries in M such that*

$$\sigma_s^{\varphi}(u_t) = u_t, \qquad s, t \in \mathbb{R}.$$

Then there exists a faithful semifinite normal weight ψ on M such that

$$(D\psi : D\varphi) = u.$$

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Proof. We shall assume first that the infinitesimal generator of the group u is bounded, so that there exists a bounded invertible $h \in M^+$ such that $u_t = h^{it}$, $t \in \mathbb{R}$. Then $\sigma_t^\varphi(h) = h$, $t \in \mathbb{R}$, and by 3.1 the maps $x \mapsto hx$, $x \mapsto xh$ and $x \mapsto h^{1/2}xh^{1/2}$ are bijections of $\mathfrak{M}_\varphi$ onto itself. Hence the weight

$$M^+ \ni x \mapsto \varphi(h^{1/2}xh^{1/2})$$

on M is faithful, semifinite and normal. Since by 3.1, $\varphi(h^{1/2}xh^{1/2}) = \varphi(hx) = \varphi(xh)$ for $x \in \mathfrak{M}_\varphi$, we may denote this weight by φh.

We have $\mathfrak{M}_{\varphi h} = \mathfrak{M}_\varphi$; hence $\mathfrak{N}_{\varphi h} = \mathfrak{N}_\varphi$. It was shown in the proof of 3.1 that

$$xh = J_\varphi \pi_\varphi(h) J_\varphi x, \qquad x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$$

(recall $h = h^*$). Hence $J_\varphi \pi_\varphi(h) J_\varphi \in \pi_\varphi(M)'$. If $x, y \in \mathfrak{N}_\varphi = \mathfrak{N}_{\varphi h}$, then

$$(x \mid y)_{\varphi h} = \varphi(hy^*x) = \varphi((yh)^*x) = (x \mid J_\varphi \pi_\varphi(h) J_\varphi y)_\varphi.$$

Hence, since $\|h^{-1}\|^{-1} \leq J_\varphi \pi_\varphi(h) J_\varphi \leq \|h\|$ the inner products on $\mathfrak{N}_\varphi = \mathfrak{N}_{\varphi h}$ determined by φ and φh are equivalent, so that the completions, as topological linear spaces, may be identified.

We shall show that

$$\sigma_t^{\varphi h}(x) = h^{it} \sigma_t^\varphi(x) h^{-it}, \qquad t \in \mathbb{R}, \qquad x \in M.$$

We have $S_{\varphi h} = S_\varphi$. If T is any densely defined preclosed linear (or antilinear) operator in the completion of $\mathfrak{N}_\varphi$ with respect to the inner product determined by φ, and if T^* denotes the adjoint of T with respect to this inner product, then the adjoint of T with respect to the inner product determined by φh is

$$J_\varphi \pi_\varphi(h)^{-1} J_\varphi T^* J_\varphi \pi_\varphi(h) J_\varphi.$$

Hence, since $\Delta_\varphi^{1/2} \pi_\varphi(h) = \pi_\varphi(h) \Delta_\varphi^{1/2}$,

$$\begin{aligned}
\Delta_{\varphi h} &= J_\varphi \pi_\varphi(h)^{-1} J_\varphi S_\varphi^* J_\varphi \pi_\varphi(h) J_\varphi S_\varphi \\
&= J_\varphi \pi_\varphi(h)^{-1} J_\varphi \, \Delta_\varphi^{1/2} J_\varphi J_\varphi \pi_\varphi(h) J_\varphi S_\varphi \\
&= J_\varphi \pi_\varphi(h)^{-1} J_\varphi \pi_\varphi(h) \, \Delta_\varphi^{1/2} J_\varphi J_\varphi \, \Delta_\varphi^{1/2} \\
&= J_\varphi \pi_\varphi(h)^{-1} J_\varphi \pi_\varphi(h) \, \Delta_\varphi.
\end{aligned}$$

Here we have used (twice) $J_\varphi J_\varphi = 1$; this, and also $J_\varphi \Delta_\varphi J_\varphi = \Delta_\varphi^{-1}$ follow from $S_\varphi = S_\varphi^{-1}$ and uniqueness of the polar decomposition of S_φ. Since $\pi_\varphi(h)$ and (since $J_\varphi \Delta_\varphi J_\varphi = \Delta_\varphi^{-1}$) also $J_\varphi \pi_\varphi(h)^{-1} J_\varphi$ are permutable with Δ_φ, and since $J_\varphi \pi_\varphi(h)^{-1} J_\varphi$ is in $\pi_\varphi(M)'$ and in particular is permutable with $\pi_\varphi(h)$,

$$\Delta_{\varphi h}^{it} = J_\varphi \pi_\varphi(h^{it}) J_\varphi \pi_\varphi(h^{it}) \, \Delta_\varphi^{it}, \qquad t \in \mathbb{R}$$

(recall that J_φ, like S_φ, is antilinear). The assertion at the beginning of this paragraph follows (see 2).

It is clear that with θ defined on $M \otimes M_2$ by $\theta(\Sigma\, x_{ij} \otimes e_{ij}) = \varphi(x_{11}) + \varphi(x_{22})$ (see

4.1) the weight $\Sigma\, x_{ij}\otimes e_{ij}\mapsto\varphi(x_{11})+\varphi h(x_{22})$ is θk where $k=1\otimes e_{11}+h\otimes e_{22}$. Hence

$$\begin{aligned}(D\varphi h\colon D\varphi)_t\otimes e_{21} &= \sigma_t^{\theta k}(1\otimes e_{21}) = k^{it}\sigma_t^{\theta}(1\otimes e_{21})k^{-it}\\ &= k^{it}(1\otimes e_{21})k^{-it} = (h^{it}\otimes e_{22})(1\otimes e_{21})(1\otimes e_{11})\\ &= h^{it}\otimes e_{21} = u_t\otimes e_{21}.\end{aligned}$$

Now, if the exponential of the infinitesimal generator of the group u, say h, is not bounded and invertible, the preceding construction may still be carried out in eMe for each projection e in $\{h\}''$ (therefore by 3.1 such that $(1-2e)\varphi(1-2e)=\varphi$) such that eh is bounded and invertible in eMe. The sum of the weights φe_ih for a family of such projections with sum 1 is clearly a faithful, semifinite (cf. 4.1), normal weight with modular automorphism group $t\mapsto u_t\sigma_t^{\varphi}(\cdot)u_t^*$. (Denote the weight by ψ. If e is a finite sum of the e_i, then the projection $\pi_\psi(e)J_\psi\pi_\psi(e)J_\psi$ commutes with S_ψ, and if H_e denotes its range, $S_\psi\mid H_e=S_{\psi e}=S_{\varphi eh}$.) Although, a priori, this weight depends on the family (e_i), the calculation in the preceding paragraph shows that its Radon-Nikodym derivative with respect to φ is u, as desired, whence by 5.2 the weight is independent of (e_i) and may be denoted by φh.

6.2. THEOREM (1.2.4 *of* [5]). *Let φ be a faithful semifinite normal weight on the von Neumann algebra M. Let $t\mapsto u_t$ be a strongly continuous family of unitaries in M such that*

$$u_{s+t} = u_s\sigma_s^{\varphi}(u_t),\qquad s,t\in\mathbb{R}.$$

Then there exists a (unique) faithful semifinite normal weight ψ on M such that

$$(D\psi\colon D\varphi) = u.$$

Proof. Uniqueness follows from 5.2. Existence in the case that u is a group was proved in 6.1. In the general case we shall construct, in order, weights Φ, Φ', Ψ'', Ψ', ψ' and ψ. The first four will be weights on $M\otimes M_\infty$, where $M_\infty=B(L^2(\mathbb{R}))$, and the last two will be weights on M.

The weight Φ on $M\otimes M_\infty$ is constructed as the tensor product of φ with the trace on M_∞. Explicitly, choose a family (v_i) of partial isometries in M_∞ such that the vv^*'s are minimal projections with sum 1 and the v^*v's are equal to a fixed projection e_0. Necessarily, e_0 is minimal in M_∞, so that $(1\otimes e_0)(M\otimes M_\infty)(1\otimes e_0)$ is equal to $M\otimes e_0$ and may be identified with M. Then for x positive in $M\otimes M_\infty$, set $\Sigma\,\varphi(v_i^*xv_i)=\Phi(x)$. Φ is faithful, semifinite and normal (cf. 4.1). By 3.2 (with $v=1-2v_iv_i^*$), $\sigma_t^{\Phi}(v_iv_i^*)=v_iv_i^*$, $t\in\mathbb{R}$. Hence, since $v_i\Phi v_i^*=\Phi$, $\sigma^{\Phi}(v_i)=v_i$ and

$$\sigma_t^{\Phi}(x\otimes y) = \sigma_t^{\varphi}(x)\otimes y,\qquad x\in M,\qquad y\in M_\infty,\qquad t\in\mathbb{R}.$$

The weight Φ' on $M\otimes M_\infty$ is constructed so that

$$\sigma_t^{\Phi'}(x\otimes y) = \sigma_t^{\varphi}(x)\otimes U_tyU_t^*,\qquad x\in M,\qquad y\in M_\infty,\qquad t\in\mathbb{R}$$

where $t\mapsto U_t$ is the left regular representation of $\mathbb{R}$ (recall $M_\infty=B(L^2(\mathbb{R}))$).

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Since $\sigma_s^{\Phi}(1\otimes U_t)=1\otimes U_t$, $s, t\in\mathbb{R}$, the hypotheses of 6.1 are verified with $M=M\otimes M_\infty$, $\varphi=\Phi$ and $u=1\otimes U$; the weight yielded by 6.1 has the property desired of Φ'.

The weight Ψ'' is defined to be $u\Phi' u^*$, where u is considered as a unitary in $L^\infty(\mathbb{R}, M)\subset M\otimes L^\infty(\mathbb{R})\subset M\otimes M_\infty$. Since $\sigma_t^{\Phi'}(u)$ is the unitary $s\mapsto\sigma_t^{\varphi}(u_{s-t})$ the cocycle condition on u implies

$$u\sigma_t^{\Phi'}(u^*) = u_t\otimes 1, \qquad t\in\mathbb{R}.$$

Hence

$$\sigma_t^{\Psi''}(x\otimes y) = u_t\sigma_t^{\varphi}(x)u_t^*\otimes U_t y U_t^*, \qquad x\in M, \qquad y\in M_\infty, \qquad t\in\mathbb{R}.$$

The weight Ψ' is constructed so that

$$\sigma_t^{\Psi'}(x\otimes y) = u_t\sigma_t^{\varphi}(x)u_t^*\otimes y, \qquad x\in M, \qquad y\in M_\infty, \qquad t\in\mathbb{R}.$$

Since $\sigma_s^{\Psi''}(1\otimes U_t^*)=1\otimes U_t^*$, $s, t\in\mathbb{R}$, the hypotheses of 6.1 are verified with $M=M\otimes M_\infty$, $\varphi=\Psi''$ and $u=1\otimes U^*$; the resulting weight on $M\otimes M_\infty$ possesses by 4.2(ii) the property desired of Ψ'.

The weight ψ' on M is defined by

$$\psi'(x) = \Psi'(x\otimes e_0), \qquad x\in M^+,$$

where e_0 is a minimal projection in M_∞. Since $\sigma_t^{\Psi'}(1\otimes e_0)=1\otimes e_0$, $t\in\mathbb{R}$, by 3.1 we have $(1\otimes e_0)\mathfrak{M}_{\Psi'}(1\otimes e_0)\subset\mathfrak{M}_{\Psi'}$, so that ψ' is semifinite, as well as faithful and normal. Moreover, $\sigma_t^{\psi'}(x)\otimes e_0=\sigma_t^{\Psi'}(x\otimes e_0)$, $x\in M$, $t\in\mathbb{R}$, whence

$$\sigma_t^{\psi'}(x) = u_t\sigma_t^{\varphi}(x)u_t^*, \qquad x\in M, \qquad t\in\mathbb{R}.$$

We shall now repeat the argument on page 150 of [5] to show that for some positive h affiliated with the centre of M,

$$(D\psi' : D\varphi)_t = h^{it}u_t, \qquad t\in\mathbb{R}.$$

With $(D\psi' : D\varphi)t=v_t$ and $u_tv_t^*=a_t$, $t\in\mathbb{R}$, a_t is central in M. Hence, for $s, t\in\mathbb{R}$,

$$\begin{aligned} a_{s+t} = u_{s+t}v_{s+t}^* &= u_s\sigma_s^{\varphi}(u_t)(v_s\sigma_s^{\varphi}(v_t))^* \\ &= u_s\sigma_s^{\varphi}(u_t)\sigma_s^{\varphi}(v_t^*)v_s^* = u_s\sigma_s^{\varphi}(u_tv_t^*)v_s^* = a_sa_t \end{aligned}$$

(σ_t^{φ} fixes elements of the centre of M, as follows directly from the definition—see 2).

The preceding paragraph shows that the hypotheses of 6.1 are verified with $\varphi=\psi'$ and $u_t=h^{it}$. The resulting weight ψ verifies $(D\psi : D\psi')_t=h^{-it}$, $t\in\mathbb{R}$. Hence by 5.1

$$(D\psi : D\varphi) = (D\psi : D\psi')(D\psi' : D\varphi) = u.$$

6.3. THEOREM (7.2 *of* [12]). *Let ψ be a normal weight on the von Neumann algebra M. Then ψ is a sum of finite normal weights.*

Proof. As shown in the first paragraph of the proof of 7.2 of [12], it is enough to suppose that ψ is faithful and semifinite.

Let φ be a faithful semifinite normal weight on M which is a sum of finite normal weights. For example, φ could be taken to be the sum of a family of finite normal weights on M the supports of which are orthogonal and have sum 1. Then by 4.2 the cocycle $(D\psi : D\varphi)$ verifies the hypothesis for u in 6.2. Hence by uniqueness of ψ with given $(D\psi : D\varphi)$, ψ must be obtained from φ by the construction in the proof of 6.2. It is easily verified that the property of being a sum of finite normal weights is preserved at each step of this construction. For example, consider the last step. If for some projection e, $(1-2e)\varphi(1-2e)=\varphi$, and if $\varphi=\sum_i \varphi_i$ with each φ_i finite, then $\varphi e=\sum_i e\varphi_i e$ where $(e\varphi_i e)(x)=\varphi_i(exe)$, $x \in M$.

REFERENCES

1. H. Araki, *Relative Hamiltonian for faithful states of von Neumann algebras*, Publ. Res. Inst. Math. Sci. **9** (1973), 165–209.
2. H. Araki, *One-parameter family of Radon-Nikodym theorems for a von Neumann algebra*, Publ. Res. Inst. Math. Sci., **10** (1974), 1–10.
3. F. Combes, *Poids sur une C*-algèbre*, J. Math. Pures Appl. **47** (1968), 57–100.
4. F. Combes, *Poids associé à une algèbre hilbertienne à gauche*, Compositio Math. **23** (1971), 49–77.
5. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. **6** (1973), 133–252.
6. A. Connes, *Applications of Tomita-Takesaki theory to classification of factors of type III*, Proceedings of the International School of Physics "Enrico Fermi", Varenna, July 23 to August 4, 1973, to appear in Nuovo Cimento,
7. A. Connes, *Sur le théorème de Radon-Nikodym pour les poids normaux fidèles semi-finis*, Bull. Sci. Math. **97** (1973), 253–258.
8. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier (Grenoble) **24** (1974), 121–155.
9. J. Dixmier, *Algèbres quasi-unitaires*, Comment, Math. Helv. **26** (1952), 275–322.
10. U. Haagerup, *Tomita's theory for von Neumann algebras with a cyclic separating vector*, preprint, University of Copenhagen.
11. U. Haagerup, *Normal weights on W*-algebras*, J. Functional Analysis **19** (1975), 302–317.
12. G. K. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math. **130** (1973), 53–87.
13. M. Takesaki, *Tomita's theory of modular Hilbert algebras*, Lecture Notes in Mathematics 128, Springer, New York, 1970.
14. A. Van Daele, *A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras*, J. Functional Analysis **15** (1974), 378–392.

MATHEMATICS INSTITUTE
UNIVERSITY OF COPENHAGEN, DENMARK