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Perfect Packings in Quasirandom Hypergraphs II

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For each of the notions of hypergraph quasirandomness that have been studied, we identify a large class of hypergraphs F so that every quasirandom hypergraph H admits a perfect F-packing. An informal statement of a special case of our general result for 3-uniform hypergraphs is as follows. Fix an integer $r \geqslant 4$ and $0 < p < 1$. Suppose that H is an n-vertex triple system with $r|n$ and the following two properties:

- for every graph G with $V(G) = V(H)$, at least p proportion of the triangles in G are also edges of H,
- for every vertex x of H, the link graph of x is a quasirandom graph with density at least p.

Then H has a perfect $K_r^{(3)}$-packing. Moreover, we show that neither of the hypotheses above can be weakened, so in this sense our result is tight. A similar conclusion for this special case can be proved by Keevash's Hypergraph Blow-up Lemma, with a slightly stronger hypothesis on H.

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1. Introduction

A k-uniform hypergraph H (k-graph for short) is a collection of k-element subsets (edges) of a vertex set $V(H)$. For a k-graph H and a subset S of vertices of size at most $k-1$, define the $(k-|S|)$-graph $N_H(S) := \{T \subseteq V(H) - S : T \cup S \in H\}$. Also, let $d_H(S) = |N_H(S)|$. When $S = \{x\}$, we write $N_H(x)$ and $d_H(x)$. The *minimum* ℓ*-degree* of H, written $\delta_\ell(H)$, is the minimum of $d_H(S)$ taken over all ℓ-sets $S \in \binom{V(H)}{\ell}$. The *minimum codegree* of H is $\delta_{k-1}(H)$ and the *minimum degree* is $\delta(H) = \delta_1(H)$. The complete k-graph on r vertices, denoted $K_r^{(k)}$ (or sometimes just K_r) is the k-graph with vertex set $[r]$ and all

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$\binom{r}{k}$ edges. If H is a k-graph and $x \in V(H)$, the *link of* x, written $L_H(x)$, is the $(k-1)$-graph whose vertex set is $V(H) - \{x\}$ and whose edge set is $N_H(x)$. We write $v(H)$ for $|V(H)|$.

Let G and F be k-graphs. We say that G has a *perfect F-packing* if the vertex set of G can be partitioned into copies of F. Minimum degree conditions that force perfect F-packings in graphs have a long history and have been well studied [1, 11, 21, 23]. In the past decade there has been substantial interest in extending these result to k-graphs [9, 12, 15, 16, 17, 22, 24, 25, 30, 31, 32, 33, 34, 39, 40]. Despite this activity many basic questions in this area remain open. For example, for $k \geqslant 5$ the minimum degree threshold which forces a perfect matching in k-graphs is not known.

A key ingredient in the proofs of most of the previously cited results is specially designed random-like or quasirandom properties of k-graphs implying the existence of perfect F-packings. There is a rather well-defined notion of quasirandomness for graphs that originated in early work by Thomason [36, 37] and Chung, Graham and Wilson [7]. These graph quasirandom properties, when generalized to k-graphs, provide a rich structure of inequivalent hypergraph quasirandom properties (see [28, 38]). In Lenz and Mubayi [27], the authors studied in detail the packing problem for the simplest of these quasirandom properties, the so-called weak hypergraph quasirandomness. A hypergraph is *linear* if every two edges share at most one vertex. The results of [27] showed that weak hypergraph quasirandomness and an obvious minimum degree condition suffice to obtain perfect F-packings for all linear F, but the result does not hold for certain F that are very close to being linear.

In this paper, we address the packing problem for the other quasirandom properties. A special case of our result identifies what hypergraph quasirandom property and what condition on the link of each vertex is required in order to be able to guarantee a perfect $K_r^{(k)}$-packing for all r (which implies a perfect F-packing for all F). The quasirandom property naturally has a great resemblance to those used in the various (strong) hypergraph regularity lemmas. Keevash's Hypergraph Blow-up Lemma [14] has as a corollary that the super-regularity of complexes implies the existence of perfect packings, but our main result below (Theorem 1.1) shows that a weaker notion of quasirandomness is enough to obtain perfect packings of complete hypergraphs. In fact, we are able to do more: for many of the hypergraph quasirandom properties that have been studied previously in the literature, we give a class of hypergraphs F for which we can find a perfect packing. Before stating Theorem 1.1, we need to define these notions of hypergraph quasirandomness.

1.1. Notions of hypergraph quasirandomness

Our definitions are closely related to the definitions by Towsner [38], which gives the most general treatment of hypergraph quasirandomness.

Definitions. **(1)** Let X be a finite set and let $2^X = \{A : A \subseteq X\}$. An *antichain* is an $\mathcal{I} \subseteq 2^X$ such that $A \subsetneq B$ for all $A, B \in \mathcal{I}$. A *full antichain* is an antichain $\mathcal{I} \subseteq 2^X$ such that $|\mathcal{I}| \geqslant 2$, and for all $x \in X$, there exists $I \in \mathcal{I}$ with $x \in I$.

(2) Let $k \geqslant 1$, let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain, and let H be a k-graph. An *$\mathcal{I}$-layout in H* is a tuple of uniform hypergraphs $\Lambda = (\lambda_I)_{I \in \mathcal{I}}$ where λ_I is an $|I|$-uniform hypergraph

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on vertex set $V(H)$. If Λ is an $\mathcal{I}$-layout, then the *k-cliques of* Λ, denoted $K_k(\Lambda)$, is the set of all vertex tuples $(x_1,\dots,x_k)$ such that $x_1,\dots,x_k$ are distinct vertices, and for each $I \in \mathcal{I}$, $\{x_i : i \in I\} \in \lambda_I$. In an abuse of notation, we will let $H \cap K_k(\Lambda)$ denote the k-tuples $(x_1,\dots,x_k)$ such that $(x_1,\dots,x_k) \in K_k(\Lambda)$ and $\{x_1,\dots,x_k\} \in H$.

We are now ready to define hypergraph quasirandomness.

Definition. Let $0 < \mu, p < 1$. A k-graph H satisfies $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ if, for every $\mathcal{I}$-layout Λ,

$$|H \cap K_k(\Lambda)| \geqslant p|K_k(\Lambda)| - \mu n^k.$$

The stronger property $\texttt{Disc}^{(k)}(\mathcal{I}, p, \mu)$ stipulates that for every $\mathcal{I}$-layout Λ,

$$||H \cap K_k(\Lambda)| - p|K_k(\Lambda)|| \leqslant \mu n^k.$$

Example. Let $k = 3$ and $\mathcal{I} = \{\{1,2\},\{2,3\}\}$. A 3-graph H satisfies $\texttt{Disc}^{(3)}(\mathcal{I}, \geqslant p, \mu)$ if, for every two graphs λ_{12} and λ_{23} with vertex set $V(H)$, the number of tuples (x,y,z) with $\{x,y,z\} \in H$, $xy \in \lambda_{12}$, and $yz \in \lambda_{23}$ is at least $p|K_3(\lambda_{12},\lambda_{23})| - \mu n^3$, where $K_3(\lambda_{12},\lambda_{23})$ is the set of tuples (x,y,z) with $xy \in \lambda_{12}$ and $yz \in \lambda_{23}$.

Several special cases of this definition deserve mention, since essentially all previously studied hypergraph quasirandomness properties are related to $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ for some $\mathcal{I}$.

- When $\mathcal{I} = \{\{1\},\dots,\{k\}\}$, then $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ is exactly the property $(p, \mu/k!)$-dense from [27], and is closely related to weak quasirandomness studied in [8, 10, 18, 35].
- More generally, when $\mathcal{I}$ is a partition, the property $\texttt{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is essentially the property $\texttt{Expand}[\pi]$ studied in [29, 26, 28]. In particular, when $\mathcal{I} = \{\{1,\dots,k-1\},\{k\}\}$, then $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ is essentially equivalent to the property considered recently by Keevash (the property called 'typical' in [13]) in his recent proof of the existence of designs.
- When

$$\mathcal{I} = \binom{[k]}{\ell},$$

then $\texttt{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is closely related to the property $\texttt{CliqueDisc}[\ell]$ studied in [2, 3, 4, 5, 6, 19, 28].
- When $\mathcal{I} = \{I \in \binom{[k]}{k-1} : \{1,\dots,\ell\} \subseteq I\}$, then $\texttt{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is essentially the same as the property $\texttt{Deviation}[\ell]$ studied in [4, 5, 3, 19, 28].
- Finally, note that $\texttt{Disc}^{(k)}(\{\emptyset\}, \geqslant p, \mu)$ is equivalent to

$$|H| \geqslant p\binom{v(H)}{k} - \frac{\mu}{k!}n^k,$$

since $K_k(\{\emptyset\})$ is the set of all ordered k-tuples of distinct vertices.

Definition. Let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain. A k-graph F is *$\mathcal{I}$-adapted* if there exists an ordering $E_1,\dots,E_m$ of the edges of F and bijections $\phi_i : E_i \to [k]$ such that for each

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$1 \leqslant j < i \leqslant m$ the following holds: there exists an $I \in \mathcal{I}$ with $\{\phi_i(x) : x \in E_j \cap E_i\} \subseteq I \in \mathcal{I}$. In words, F is $\mathcal{I}$-adapted if the set of labels assigned to E_i which appear on $E_j \cap E_i$ is a subset of a set in $\mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains. A k-graph F is $(\mathcal{I}, \mathcal{J})$-*adapted* if F is $\mathcal{I}$-adapted and there exists $x \in V(F)$, an ordering $E_1, \ldots, E_m$ of the edges of F, and bijections $\psi_i : E_i \to [k]$ such that for all $1 \leqslant j < i \leqslant m$, the following holds.

- If $x \notin E_i$ then there exists $I \in \mathcal{I}$ with $\{\psi_i(y) : y \in E_j \cap E_i\} \subseteq I$.
- If $x \in E_i$ then $\psi_i(x) = k$ and there exists $J \in \mathcal{J}$ with $\{\psi_i(y) : y \in E_j \cap E_i, y \neq x\} \subseteq J$.

1.2. Our results

The following is our main result.

Theorem 1.1. *Let $k \geqslant 2$, $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain, $\mathcal{J} \subseteq 2^{[k-1]}$ an antichain, and $0 < \alpha, p < 1$. For every $(\mathcal{I}, \mathcal{J})$-adapted k-graph F, there exists $\mu > 0$ and n_0 so that the following holds. Let H be an n-vertex k-graph where $n \geqslant n_0$ and $v(F)|n$. Suppose that H satisfies $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ and that $L_H(x)$ satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant \alpha, \mu)$ for all $x \in V(H)$. Then H has a perfect F-packing.*

It is straightforward to see that if $\mathcal{I}$ and $\mathcal{I}'$ are such that for every $I' \in \mathcal{I}'$, there exists $I \in \mathcal{I}$ with $I' \subseteq I$, then $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu) \Rightarrow \texttt{Disc}^{(k)}(\mathcal{I}', \geqslant p, \mu)$. Also, if $\mathcal{I} = \binom{[k]}{k-1}$ and $\mathcal{J} = \binom{[k-1]}{k-2}$, then every F is $(\mathcal{I}, \mathcal{J})$-adapted. Thus, to find the weakest quasirandom condition to apply Theorem 1.1 to a given k-graph F, one should find the minimal $\mathcal{I}$ and $\mathcal{J}$ for which F is $(\mathcal{I}, \mathcal{J})$-adapted. For example, if $C = \{abc, bcd, def, aef\}$, then C is $(\mathcal{I}, \mathcal{J})$-adapted, where $\mathcal{I} = \{\{1,2\},\{3\}\}$ and $\mathcal{J} = \{\emptyset\}$ (let $x = a$ and order the edges which contain a first).

As mentioned above, special cases of $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ correspond to previously studied quasirandom properties, so that Theorem 1.1 generalizes several previous results.

(i) Let $k = 2$. The only full antichain is $\mathcal{I} = \{\{1\},\{2\}\}$. For this $\mathcal{I}$, all graphs F are $(\mathcal{I}, \mathcal{J})$-adapted if $\mathcal{J} = \{\emptyset\}$. To see this, pick $x \in V(F)$ and place all edges incident to x first in the ordering for the definition of $(\mathcal{I}, \mathcal{J})$-adapted. Now the property $\texttt{Disc}^{(2)}(\mathcal{I}, \geqslant p, \mu)$ just states that G is quasirandom (in fact only 'one-sided' quasirandom). Also, the condition '$L_H(x)$ satisfies $\texttt{Disc}^{(1)}(\{\emptyset\}, \geqslant \alpha, \mu)$ for every $x \in V(H)$' is equivalent to the condition that $\delta(H) \geqslant (\alpha - \mu)(n-1)$. To see this, recall from earlier that if H' is an r-graph, then the property 'H' satisfies $\texttt{Disc}^{(1)}(\{\emptyset\}, \geqslant \alpha, \mu)$' is equivalent to the property that

$$|H'| \geqslant \alpha \binom{v(H')}{r} - \frac{\mu}{r!} v(H')^r.$$

Thus Theorem 1.1 for $k = 2$ states that if G is an n-vertex quasirandom graph, $v(F)|n$, and $\delta(G) \geqslant (\alpha - \mu)(n-1)$, then G has a perfect F-packing. This fact is a simple consequence of the Blow-up Lemma of Komlós, Sárközy and Szemerédi [20].

(ii) For $k \geqslant 2$ with $\mathcal{I}$ a partition into singletons, we obtain exactly [27, Theorem 3]. In this case, $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ is equivalent to $(p, \mu/k!)$-dense from [27], an $\mathcal{I}$-adapted k-graph is a linear k-graph, and one can take $\mathcal{J} = \{\emptyset\}$. Similar to the previous paragraph, the condition '$L_H(x)$ satisfies $\mathtt{Disc}^{(k-1)}(\{\emptyset\}, \geqslant\alpha, \mu)$ for every $x \in V(H)$' is equivalent to the condition that

$$\delta(H) \geqslant \alpha\binom{v(H)-1}{k-1} - \frac{\mu}{(k-1)!}v(H)^{k-1}.$$

(iii) If

$$\mathcal{I} = \binom{[k]}{k-1} \quad \text{and} \quad \mathcal{J} = \binom{[k-1]}{k-2},$$

then every k-graph F is $(\mathcal{I}, \mathcal{J})$-adapted. Thus Theorem 1.1 implies the following corollary.

Corollary 1.2. *Fix $2 \leqslant k \leqslant r$. For every $0 < \alpha, p < 1$, there exists $\mu > 0$ and n_0 such that the following holds. Let H be an n-vertex k-graph with $n \geqslant n_0$ and $r|n$. If H satisfies*

$$\mathtt{Disc}^{(k)}\left(\binom{[k]}{k-1}, \geqslant p, \mu\right)$$

and $L_H(x)$ satisfies

$$\mathtt{Disc}^{(k-1)}\left(\binom{[k-1]}{k-2}, \geqslant\alpha, \mu\right)$$

for every $x \in V(H)$, then H has a perfect $K_r^{(k)}$-packing.

Keevash's Hypergraph Blow-up Lemma [14] also guarantees perfect $K_r^{(k)}$-packings under certain regularity conditions, but the hypotheses of Corollary 1.2 are slightly weaker. Indeed, the main extra requirement that [14] places on H is [14, Definition 3.16 part (iii)]; translated into our language, for 3-graphs this property says roughly that for every $x \in V(H)$, if W is a set of triples where each triple contains some pair from $L_H(x)$, then $|H \cap W| \approx p|W|$.

Next, we investigate if either of the conditions $\mathtt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ or $\mathtt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ in the links from Theorem 1.1 can be weakened. This question was studied by the authors [27] in detail when $\mathcal{I}$ is a partition, and it turns out that for certain non-linear F it is possible to weaken the conditions (see [27] for details). Most likely, the constructions and results from [27] can be generalized to all $\mathcal{I}$. In this paper, we focus only on the case $\mathcal{I} = \binom{[k]}{k-1}$ and $\mathcal{J} = \binom{[k-1]}{k-2}$, which corresponds to the condition required for perfect $K_r^{(k)}$-packings. In this case, neither condition can be weakened, so that Theorem 1.1 cannot be improved in general.

Proposition 1.3. *For every $k \geqslant 3$ there exists an r (depending only on k) such that the following holds. Let $\alpha = p = (k-1)/k$ and let $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain where $\mathcal{I} \neq \binom{[k]}{k-1}$. For every $\mu > 0$, there exists n_0 such that for all $n \geqslant n_0$ there exists an n-vertex k-graph H which*

- *satisfies* $\mathtt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$,
- *fails* $\mathtt{Disc}^{(k)}(\binom{[k]}{k-1}, \geqslant p, \mu)$,
- *for every $x \in V(H)$ the link $L_H(x)$ satisfies* $\mathtt{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geqslant\alpha, \mu)$,
- *has no copy of K_r (so no perfect K_r-packing).*

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Proposition 1.4. *For every $k \geqslant 3$ there exists an r (depending only on k) such that the following holds. Let $\alpha = p = (k-1)/k$ and let $\mathcal{J} \subseteq 2^{[k-1]}$ be a full antichain where $\mathcal{J} \neq \binom{[k-1]}{k-2}$. For every $0 < \mu, p < 1$, there exists n_0 such that for all $n \geqslant n_0$ with $r|n$, there exists an n-vertex k-graph H which*

- *satisfies* $\texttt{Disc}^{(k)}(\binom{[k]}{k-1}, \geqslant p, \mu)$,
- *for every $x \in V(H)$ the link $L_H(x)$ satisfies* $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant \alpha, \mu)$,
- *there exists $x \in V(H)$ such that the link $L_H(x)$ fails* $\texttt{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geqslant \alpha, \mu)$,
- *has no perfect K_r-packing.*

The remainder of this paper is organized as follows. In Sections 2 and 3 we discuss the two main tools needed for the proof of Theorem 1.1, in Section 4 we prove Theorem 1.1, and finally in Section 5 we explain the constructions which prove Propositions 1.3 and 1.4.

2. Absorbing sets

One of the main tools for our proof of Theorem 1.1 is the absorbing technique of Rödl, Ruciński and Szemerédi [34]. We will use the following absorbing lemma from [27] without modification.

Definitions. (1) Let F and H be k-graphs and let $A, B \subseteq V(H)$. We say that A *F-absorbs* B or that A is an *F-absorbing set* for B if both $H[A]$ and $H[A \cup B]$ have perfect F-packings. When F is a single edge, we say that A *edge-absorbs* B.

(2) Let F and H be k-graphs, $\epsilon > 0$, and let a and b be multiples of $v(F)$. We say that H is *(a, b, ϵ, F)-rich* if for all $B \in \binom{V(H)}{b}$ there are at least ϵn^a sets in $\binom{V(H)}{a}$ which F-absorb B.

Lemma 2.1 (Absorbing Lemma, specialized version of Lemma 10 of [27]). *Let F be a k-graph, $\epsilon > 0$, and let a and b be multiples of $v(F)$. There exists an n_0 and $\omega > 0$ such that for all n-vertex k-graphs H with $n \geqslant n_0$, the following holds. If H is (a, b, ϵ, F)-rich, then there exists an $A \subseteq V(H)$ such that $a||A|$ and A F-absorbs all sets C satisfying the following conditions: $C \subseteq V(H) - A$, $|C| \leqslant \omega n$, and $b||C|$.*

3. Embedding lemma

Definition. Let $k \geqslant 2$ and $0 \leqslant m \leqslant f$. Let F and H be k-graphs with $V(F) = \{w_1, \dots, w_f\}$. A *labelled copy of F in H* is an edge-preserving injection from $V(F)$ to $V(H)$. A *degenerate labelled copy of F in H* is an edge-preserving map from $V(F)$ to $V(H)$ that is not an injection. Let $1 \leqslant m \leqslant f$ and let $Z_1, \dots, Z_m \subseteq V(H)$. Set $\text{inj}[F \to H; w_1 \to Z_1, \dots, w_m \to Z_m]$ to be the number of edge-preserving injections $\psi : V(F) \to V(H)$ such that $\psi(w_i) \in Z_i$ for all $1 \leqslant i \leqslant m$. If $Z_i = \{z_i\}$, we abbreviate $w_i \to \{z_i\}$ as $w_i \to z_i$.

The embedding lemma (Lemma 3.1) proved in this section shows that if H satisfies

$$\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu) \quad \text{and} \quad \texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant \alpha, \mu)$$

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in the links, then H contains many copies of F if F is $(\mathcal{I},\mathcal{J})$-adapted. In fact, it says more: if m of the vertices of F are pre-specified and F satisfies the following more technical condition, then there are many copies of F using the m pre-specified vertices.

Definition. Let $k \geqslant 2, \mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains, F a k-graph, and $s_1,\dots,s_m \in V(F)$. We say that F is $(\mathcal{I},\mathcal{J})$-*adapted at* $s_1,\dots,s_m$ if there exists an ordering $E_1,\dots,E_t$ of the edges of F such that

- for every i, $|E_i \cap \{s_1,\dots,s_m\}| \leqslant 1$,
- for every E_i with $E_i \cap \{s_1,\dots,s_m\} = \emptyset$, there exists a bijection $\phi_i : E_i \to [k]$ such that for all $j < i$, there exists $I \in \mathcal{I}$ with $\{\phi_i(x) : x \in E_j \cap E_i\} \subseteq I$,
- for every E_i with $s_\ell \in E_i$, there exists a bijection $\psi_i : E_i \setminus \{s_\ell\} \to [k-1]$ such that for all $j < i$, there exists $J \in \mathcal{J}$ with $\{\psi_i(x) : x \in E_j \cap E_i, x \neq s_\ell\} \subseteq J$.

Remarks.

- If $m = 1$, then F is $(\mathcal{I},\mathcal{J})$-*adapted at* s_1 is the same as saying that F is $(\mathcal{I},\mathcal{J})$-*adapted*.
- $m = 0$ is possible, in which case the definition is equivalent to $\mathcal{I}$-adapted.

Lemma 3.1. *Let $k \geqslant 2$, $0 < \alpha,\gamma,p < 1$, and $\mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains. Let F be an f-vertex k-graph with $V(F) = \{s_1,\dots,s_m,t_{m+1},\dots,t_f\}$. Suppose that F is $(\mathcal{I},\mathcal{J})$-adapted at $s_1,\dots,s_m$. Then there exists an n_0 and $\mu > 0$ such that the following is true.*

Let H be an n-vertex k-graph with $n \geqslant n_0$, where H satisfies $\mathtt{Disc}^{(k)}(\mathcal{I},{\geqslant}p,\mu)$. If $m > 0$, then also assume that $L_H(x)$ satisfies $\mathtt{Disc}^{(k-1)}(\mathcal{J},{\geqslant}\alpha,\mu)$ for every vertex $x \in V(H)$. Let $y_1,\dots,y_m \in V(H)$ be distinct and let $V_{m+1},\dots,V_f \subseteq V(H)$. Then

$$\begin{aligned}&\mathrm{inj}[F \to H; s_1 \to y_1,\dots,s_m \to y_m, t_{m+1} \to V_{m+1},\dots,t_f \to V_f]\\&\quad\geqslant \alpha^{d_F(s_1)}\cdots\alpha^{d_F(s_m)} p^{|F|-\sum d_F(s_i)}|V_{m+1}|\cdots|V_f| - \gamma n^{f-m}.\end{aligned}$$

Proof. We first prove the lemma under the additional assumption that the sets $V_{m+1},\dots,V_f$ are pairwise disjoint. This is proved by induction on $|F|$. If $|F| = 0$, then

$$\begin{aligned}&\mathrm{inj}[F \to H; s_1 \to y_1,\dots,s_m \to y_m, t_{m+1} \to V_{m+1},\dots,t_f \to V_f]\\&\quad\geqslant \prod_{i=m+1}^{f}(|V_i| - f)\\&\quad\geqslant \alpha^0 p^0 \prod_{i=m+1}^{f}|V_i| - \gamma n^{f-m}\end{aligned}$$

for large n. So assume F has at least one edge and let E be the last edge in an ordering of the edges of F which witness that F is $(\mathcal{I},\mathcal{J})$-adapted at $s_1,\dots,s_m$. (Recall that if $m = 0$ then $(\mathcal{I},\mathcal{J})$-adapted at $s_1,\dots,s_m$ is equivalent to $\mathcal{I}$-adapted.)

Let F_* be the hypergraph formed by deleting all vertices of E from F. Let F_- be the hypergraph formed by removing the edge E from F but keeping the same vertex set. Let Q_* be an injective edge-preserving map $Q_* : V(F_*) \to V(H)$ where $Q_*(s_i) = y_i$ for $1 \leqslant i \leqslant m$ and $Q_*(t_j) \in V_j$ for $t_j \notin E$. There are two cases.

Case 1: $E \cap \{s_1, \ldots, s_m\} = \emptyset$. Let $\phi : E \to [k]$ be the bijection from the definition of $(\mathcal{I}, \mathcal{J})$-adapted at $s_1, \ldots, s_m$ and assume the vertices of F are labelled so that $E = \{t_{m+1}, \ldots, t_{m+k}\}$, where $\phi(t_{m+i}) = i$. For each $I \in \mathcal{I}$, define an $|I|$-uniform hypergraph λ_{I,Q_*} with vertex set $V(H)$ as follows. Let $I = \{i_1, \ldots, i_{|I|}\}$. Make $\{z_{i_1}, \ldots, z_{i_{|I|}}\} \in \binom{V(H)}{|I|}$ a hyperedge of λ_{I,Q_*} if $z_{i_j} \in V_{m+i_j}$ for all j and when the map Q_* is extended to map t_{i_j} to z_{i_j} for all j, this extended map is an edge-preserving map from $F_-[V(F_*) \cup \{t_{i_1}, \ldots, t_{i_{|I|}}\}]$ to H. More informally, λ_{I,Q_*} consists of all $|I|$-sets to which Q_* can be extended to produce a copy of F_* together with the vertices of E indexed by I. Let $\Lambda_{Q_*} = (\lambda_{I,Q_*})_{I \in \mathcal{I}}$.

Now, if $(z_{m+1}, \ldots, z_{m+k})$ is a k-tuple in $K_k(\Lambda_{Q_*})$, then the map Q_* can be extended to map t_j to z_j for $m+1 \leqslant j \leqslant m+k$ to produce an edge-preserving map from F_- to H. To see this, let E' be an edge of F_-. Since E is the last edge in the ordering, if $E' \cap E = \{t_{j_1}, \ldots, t_{j_r}\}$ then there exists some $I \in \mathcal{I}$ with $\{j_1, \ldots, j_r\} \subseteq I$ since F is $\mathcal{I}$-adapted. Since $(z_{m+1}, \ldots, z_{m+k})$ is a k-clique, $\{z_{m+i} : i \in I\} \in \lambda_{I,Q_*}$. This implies that there is some permutation η of I such that extending Q_* to map t_{m+i} to $z_{m+\eta(i)}$ produces an edge-preserving map. Since the V_{m+i} are pairwise disjoint and $z_{m+i} \in V_{m+i}$ for all $i \in I$, η must be the identity permutation, *i.e.*, extending the map Q_* to map t_{m+i} to z_{m+i} for all $i \in I$ produces an edge-preserving map. Thus extending the map Q_* to map t_{j_p} to z_{j_p} for all p is an edge-preserving map and E' is one of the preserved edges. Finally, since the V_j are disjoint, each k-tuple in $K_k(\Lambda_{Q_*})$ corresponds to exactly one labelled copy of F_- in H which extend Q_* with t_j mapped into V_j for all j. Similarly, $|H \cap K_k(\Lambda_{Q_*})|$ is exactly the number of labelled copies of F in H which extend Q_* with t_j mapped into V_j for all j. Thus,

$$\begin{aligned} \mathrm{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] &= \sum_{Q_*} |H \cap K_k(\Lambda_{Q_*})|, \\ \mathrm{inj}[F_- \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] &= \sum_{Q_*} |K_k(\Lambda_{Q_*})|. \end{aligned} \tag{3.1}$$

Since H satisfies $\mathtt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$,

$$\begin{aligned} &\mathrm{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] \\ &\quad \geqslant \sum_{Q_*} (p|K_k(\Lambda_{Q_*})| - \mu n^k) \\ &\quad \geqslant p \sum_{Q_*} |K_k(\Lambda_{Q_*})| - \mu n^{f-m}, \end{aligned} \tag{3.2}$$

where the last inequality is because there are at most n^{f-m-k} maps Q_*, since F_* has $f-k$ vertices and $s_i \in V(F_*)$ must map to y_i. Combining (3.1) and (3.2) and then applying induction,

$$\begin{aligned} &\mathrm{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] \\ &\quad \geqslant p\, \mathrm{inj}[F_- \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] - \mu n^{f-m} \\ &\quad \geqslant p\big(\alpha^{\sum d(s_i)} p^{|F|-1-\sum d(s_i)} |V_{m+1}| \cdots |V_f| - \gamma n^{f-m}\big) - \mu n^{f-m}. \end{aligned}$$

Let $\mu = (1-p)\gamma$ so that the proof of this case is complete.

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Case 2: $s_\ell \in E$. (Since F is $(\mathcal{I}, \mathcal{J})$-adapted at $s_1, \ldots, s_m$, at most one vertex s_ℓ can be in E.) Let $\psi : E \setminus \{s_\ell\} \to [k-1]$ be the bijection from the definition of $(\mathcal{I}, \mathcal{J})$-adapted at $s_1, \ldots, s_m$ and assume the vertices of E are labelled such that $E = \{s_\ell, t_{m+1}, \ldots, t_{m+k-1}\}$, where $\psi(t_{m+j}) = j$. This case is very similar to the previous case, except that we will use $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ in the link of y_ℓ. For each $J \in \mathcal{J}$, define a $|J|$-uniform hypergraph λ_{J,Q_*} with vertex set $V(H)$ as follows. Let $J = \{j_1, \ldots, j_{|J|}\}$. Make $\{z_{j_1}, \ldots, z_{j_{|J|}}\}$ a hyperedge of λ_{J,Q_*} if $z_{j_r} \in V_{j_r}$ for all r and extending the map Q_* to map s_ℓ to y_ℓ and mapping t_{j_r} to z_{j_r} for all r produces an edge-preserving map. Let $\Lambda_{Q_*} = (\lambda_{J,Q_*})_{J \in \mathcal{J}}$. Similar to before, if $(z_{m+1}, \ldots, z_{m+k-1})$ is a $(k-1)$-tuple in $K_{k-1}(\Lambda_{Q_*})$, then the map Q_* can be extended to map s_ℓ to y_ℓ and map t_i to z_i for $m+1 \leqslant i \leqslant m+k-1$ to produce an edge-preserving map from F_- to H. Thus $|K_{k-1}(\Lambda_{Q_*})|$ is exactly the number of labelled copies of F_- in H which extend Q_*. Similarly, $|L_H(y_\ell) \cap K_{k-1}(\Lambda_{Q_*})|$ is exactly the number of labelled copies of F in H which extend Q_*.

Now formulas similar to (3.1) and (3.2) and the fact that $L_H(y_\ell)$ satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ completes this case. This concludes the proof of the lemma if the sets $V_{m+1}, \ldots, V_f$ are pairwise disjoint.

Now assume that the sets $V_{m+1}, \ldots, V_f$ are not necessarily pairwise disjoint. Let

$$\mathcal{P} = \{(P_{m+1}, \ldots, P_f) : P_{m+1}, \ldots, P_f \text{ is a partition of } V(H)\},$$

so that $|\mathcal{P}| = (f-m)^n$. Now

$$\begin{aligned}
&\operatorname{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] \\
&\quad = \frac{1}{(f-m)^{n-f+m}} \sum_{(P_{m+1},\ldots,P_f) \in \mathcal{P}} \operatorname{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, \\
&\qquad\qquad t_{m+1} \to V_{m+1} \cap P_{m+1}, \ldots, t_f \to V_f \cap P_f].
\end{aligned}$$

Indeed, each labelled copy of F of the right form will be counted exactly $(f-m)^{n-f+m}$ times by the sum over all partitions, since the images of $t_{m+1}, \ldots, t_f$ must map into the corresponding part of the partition and all other vertices of H can be distributed to any of the parts of the partition. Let $\delta = \alpha^{d_F(s_1)} \cdots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)}$. Since $V_{m+1} \cap P_{m+1}, \ldots, V_f \cap P_f$ are pairwise disjoint,

$$\begin{aligned}
&\operatorname{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to V_{m+1}, \ldots, t_f \to V_f] \\
&\quad \geqslant \frac{1}{(f-m)^{n-f+m}} \sum_{(P_{m+1},\ldots,P_f) \in \mathcal{P}} (\delta |V_{m+1} \cap P_{m+1}| \cdots |V_f \cap P_f| - \gamma n^{f-m}) \\
&\quad = \delta |V_{m+1}| \cdots |V_f| - \frac{\gamma n^{f-m} |\mathcal{P}|}{(f-m)^{n-f+m}} \geqslant \delta |V_{m+1}| \cdots |V_f| - \gamma n^{f-m}.
\end{aligned}$$

□

4. Packing $(\mathcal{I}, \mathcal{J})$-adapted hypergraphs

In this section we prove Theorem 1.1. The proof has several stages: we first prove that the quasirandom conditions on H imply that H is rich; then we use Lemma 2.1 to set aside a vertex set A which can absorb all reasonably sized sets; next we use the embedding

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lemma (Lemma 3.1) to produce an almost perfect packing in $H - A$; and finally we use the properties of A to absorb the remaining vertices.

4.1. Richness

In this subsection we prove that the conditions on H in Theorem 1.1 imply that H is $(f^2 - f, f, \epsilon, F)$-rich, where $f = v(F)$.

Lemma 4.1. *Let $k \geqslant 2$, $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain, and $\mathcal{J} \subseteq 2^{[k-1]}$ an antichain. Let F be an $(\mathcal{I}, \mathcal{J})$-adapted k-graph with f vertices. For every $0 < \alpha, p < 1$, there exists $\mu, \epsilon > 0$ and n_0 such that the following holds. Let H be an n-vertex k-graph where $n \geqslant n_0$. Also, assume that H satisfies $\mathtt{Disc}^{(k)}(\mathcal{I}, {\geqslant}p, \mu)$ and that $L_H(z)$ satisfies $\mathtt{Disc}^{(k-1)}(\mathcal{J}, {\geqslant}\alpha, \mu)$ for every vertex $z \in V(H)$. Then H is $(f^2 - f, f, \epsilon, F)$-rich.*

Proof. Let $a = f(f-1)$ and $b = f$. Our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \binom{V(H)}{b}$, there are at least ϵn^a vertex sets of size a which F-absorb B; we will define ϵ and μ later. Let $V(F) = \{w_0, \dots, w_{f-1}\}$, where w_0 is the special vertex in the definition that F is $(\mathcal{I}, \mathcal{J})$-adapted.

Next, form the following k-graph F'. Let

$$V(F') = \{x_{i,j} : 0 \leqslant i, j \leqslant f - 1\}.$$

(We think of the vertices of F' as arranged in a grid with i as the row and j as the column.) Form the edges of F' as follows: for each fixed $1 \leqslant i \leqslant f-1$, let $\{x_{i,0}, \dots, x_{i,f-1}\}$ induce a copy of F where $x_{i,j}$ is mapped to w_j. Similarly, for each fixed $0 \leqslant j \leqslant f-1$, let $\{x_{0,j}, \dots, x_{f-1,j}\}$ induce a copy of F where $x_{i,j}$ is mapped to w_i. Note that we therefore have a copy of F in each column and a copy of F in each row besides the zeroth row.

Now fix $B = \{b_0, \dots, b_{f-1}\} \subseteq V(H)$; we want to show that B is F-absorbed by many a-sets. Note that any labelled copy of F' in H which maps $x_{0,0} \to b_0, \dots, x_{0,f-1} \to b_{f-1}$ produces an F-absorbing set for B as follows. Let $Q : V(F') \to V(H)$ be an edge-preserving injection where $Q(b_j) = x_{0,j}$ (so Q is a labelled copy of F' in H where the set B is the zeroth row of F'). Let $A = \{Q(x_{i,j}) : 1 \leqslant i \leqslant f-1, 0 \leqslant j \leqslant f-1\}$ consist of all vertices in rows 1 to $f-1$. Then A has a perfect F-packing consisting of the copies of F on the rows, and $A \cup B$ has a perfect F-packing consisting of the copies of F on the columns. Therefore, A F-absorbs B.

To complete the proof, we therefore just need to use Lemma 3.1, where $m = f$ and $s_1 = x_{0,0}, \dots, s_f = x_{0,f-1}$, to show that there are many copies of F' with B as the zeroth row. To do so, we need to show that F' is $(\mathcal{I}, \mathcal{J})$-adapted at $s_1, \dots, s_m$. Indeed, consider the following ordering of edges of F'. First, list the edges of F' in the first column, then the edges of F' in the second column, and so on until the kth column. Next, list the edges of F' in the first row, then the second row, and so on until the $(k-1)$st row. Within each row or column, list the edges in the ordering given in the definition of F being $(\mathcal{I}, \mathcal{J})$-adapted. For the bijections ϕ or ψ, use the same bijection as in the definition of F being $(\mathcal{I}, \mathcal{J})$-adapted. Now consider $E_i, E_j \in F'$ in this ordering with $j < i$. If E_i and E_j are from the same row or the same column, then since F is $(\mathcal{I}, \mathcal{J})$-adapted, the condition on $E_i \cap E_j$ is satisfied. If E_i and E_j are in different rows or columns, the size

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of their intersection is at most one. If $E_i \cap E_j = \emptyset$ then the condition is trivially satisfied. If $E_i \cap E_j = \{u\}$, then E_i must be from a row since $i > j$. Then E_i does not contain any $s_1, \ldots, s_m$, so we must show that there is some $I \in \mathcal{I}$ so that $\phi_i(u) \in I$. This is true because $\mathcal{I}$ is full. Thus F' is $(\mathcal{I}, \mathcal{J})$-adapted at $s_1, \ldots, s_m$.

Now we apply Lemma 3.1 to F' with

$$m = f, \quad s_1 = x_{0,0}, \ldots, s_f = x_{0,f-1}, \quad V_{m+1} = \cdots = V_{f^2} = V(H) - B,$$
$$\text{and} \quad \gamma = \tfrac{1}{2}\alpha^{\sum d(x_{0,j})} p^{|F| - \sum d(x_{0,j})}.$$

Ensure that n_0 is large enough and μ is small enough to apply Lemma 3.1, to show that

$$\operatorname{inj}[F' \to H; x_{0,0} \to b_0, \ldots, x_{0,f-1} \to b_{f-1}] \geqslant \gamma \left(\frac{n}{2}\right)^{f^2 - f} = \frac{\gamma}{2^{f^2-f}} n^a.$$

Each labelled copy of F' produces a labelled F-absorbing set for B, so there are at least

$$\frac{\gamma}{a!2^{f^2-f}} n^a$$

F-absorbing sets for B. The proof is complete by letting

$$\epsilon = \frac{\gamma}{a!2^{f^2-f}}. \qquad \square$$

4.2. Almost perfect packings

In this section we prove that the conditions in Theorem 1.1 imply that there exists a perfect F-packing covering almost all the vertices of H.

Lemma 4.2. *Let $k \geqslant 2$ and $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain. Fix $0 < p < 1$ and an $\mathcal{I}$-adapted k-graph F with f vertices. Fix an integer b with $f | b$. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an n-vertex k-graph satisfying* $\mathtt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ *with $n \geqslant n_0$ and $f | n$. Then there exists $C \subseteq V(H)$ such that $|C| \leqslant \omega n$, $b || C|$, and $H[\bar{C}]$ has a perfect F-packing.*

Proof. First, select n_0 large enough and μ small enough so that any vertex set C of size $\lceil \omega n/2 \rceil$ contains a copy of F. To see this, let

$$\gamma = \frac{1}{2} p^{|F|} \left(\frac{\omega}{2}\right)^f$$

and select n_0 and $\mu > 0$ according to Lemma 3.1 with $m = 0$. (Recall that if $m = 0$ then the condition $(\mathcal{I}, \mathcal{J})$-adapted on F at $\emptyset$ just reduces to the statement that F is $\mathcal{I}$-adapted.) Now if $C \subseteq V(H)$ with $|C| \geqslant \omega n/2$, then let $V_1 = \cdots = V_f = C$ so that $|V_i| \geqslant \omega n/2$ for all i. Then Lemma 3.1 implies that there are at least

$$p^{|F|} \prod |V_i| - \gamma n^f \geqslant p^{|F|} \left(\frac{\omega}{2}\right)^f n^f - \gamma n^f = \gamma n^f > 0$$

copies of F inside C.

Now let $F_1, \ldots, F_t$ be a greedily constructed F-packing. That is, $F_1, \ldots, F_t$ are disjoint copies of F and $C := V(H) - V(F_1) - \cdots - V(F_t)$ has no copy of F. By the previous paragraph, $|C| \leqslant \omega n/2$. Since $f|n$ and $H[\bar{C}]$ has a perfect F-packing, $f||C|$. Thus we can let

$$y \equiv -\frac{|C|}{f} \pmod{b}$$

with $0 \leqslant y < b$ and take y of the copies of F in the F-packing of $H[\bar{C}]$ and add their vertices into C so that $b||C|$. □

4.3. Proof of Theorem 1.1

Proof of Theorem 1.1. First, apply Lemma 4.1 to produce $\epsilon > 0$ and μ_1. Next, select $\omega > 0$ according to Lemma 2.1 and $\mu_2 > 0$ according to Lemma 4.2. Also, make n_0 large enough so that both Lemma 2.1 and 4.2 can be applied. Let $\mu = \min\{\mu_1, \mu_2\omega^k\}$. All the parameters have now been chosen.

By Lemmas 2.1 and 4.1, there exists a set $A \subseteq V(H)$ such that A F-absorbs C for all $C \subseteq V(H) \setminus A$ with $|C| \leqslant \omega n$ and $b \mid |C|$. If $|A| \geqslant (1-\omega)n$, then A F-absorbs $V(H) \setminus A$ so that H has a perfect F-packing. Thus $|A| \leqslant (1-\omega)n$. Next, let $H' := H[\bar{A}]$ and notice that H' satisfies $\texttt{Disc}^{(k)}(\mathcal{I}, {\geqslant}p, \mu_2)$ since $v(H') \geqslant \omega n$ and

$$\mu n^k \leqslant \frac{\mu}{\omega^k} v(H')^k \leqslant \mu_2 v(H')^k.$$

Therefore, by Lemma 4.2, there exists a vertex set $C \subseteq V(H') = V(H) \setminus A$ such that $|C| \leqslant \omega n$, $|C|$ is a multiple of b, and $H'[\bar{C}]$ has a perfect F-packing. Now Lemma 2.1 implies that A F-absorbs C. The perfect F-packing of $A \cup C$ and the perfect F-packing of $H'[\bar{C}]$ produces a perfect F-packing of H. □

5. Constructions

In this section we prove Propositions 1.3 and 1.4 using the following construction.

Construction. Let $k \geqslant 2$. Let $A_n^{(k)}$ be the following probability distribution over n-vertex k-graphs. Let

$$f : \binom{V(A_n^{(k)})}{k-1} \to \{0, \dots, k-1\}$$

be a random k-colouring of the $(k-1)$-sets. Make $E \in \binom{V(A_n^{(k)})}{k}$ an edge of $A_n^{(k)}$ if

$$\sum_{\substack{T \subseteq E \\ |T| = k-1}} f(T) \neq 0 \pmod{k}.$$

Lemma 5.1. *Let $p = (k-1)/k$ and $\epsilon > 0$. Then with probability going to one as n goes to infinity,*

$$\left| |A_n^{(k)}| - p\binom{n}{k} \right| < \epsilon n^k.$$

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The proof of Lemma 5.1 is a standard second moment argument so the proof is omitted.

Lemma 5.2. *There exists a μ_0 such that for all $0 < \mu < \mu_0$, with probability going to one as n goes to infinity, $A_n^{(k)}$ fails* $\texttt{Disc}^{(k)}\big(\binom{[k]}{k-1}, \geqslant p, \mu\big)$.

Proof. Let Z be the $(k-1)$-graph whose edges are all the $(k-1)$-sets coloured zero. Let $\Lambda = (Z, \dots, Z)$ be the $\binom{[k]}{k-1}$-layout consisting of Z in every coordinate. Now any k-clique $(z_1, \dots, z_k)$ of Λ is not a hyperedge of $A_n^{(k)}$, since every $(k-1)$-subset of $\{z_1, \dots, z_k\}$ has colour zero. This Λ will show that $A_n^{(k)}$ fails $\texttt{Disc}^{(k)}\big(\binom{[k]}{k-1}, \geqslant p, \mu\big)$ if $|K_k(\Lambda)|$ is large enough. Each k-tuple of vertices is a k-clique with probability $(1/k)^k$, so $\mathbb{E}[|K_k(\Lambda)|] = k^{-k}(n)_k$. A simple second moment computation shows that $|K_k(\Lambda)|$ is concentrated around its expectation, so with high probability for large n we have that $|K_k(\Lambda)| \geqslant \frac{1}{10}k^{-k}n^k$. Thus, if

$$\mu_0 = \frac{1}{20}\frac{k-1}{k^{k+1}},$$

we have that

$$0 = |H \cap K_k(\Lambda)| < \frac{k-1}{k}|K_k(\Lambda)| - \mu n^k. \qquad \square$$

Lemma 5.3. *Let $r = r_{k-1}(K_k^{(k-1)}, \dots, K_k^{(k-1)})$ be the k-colour Ramsey number, where the $(k-1)$-sets are coloured and a monochromatic k-clique is forced. Then $A_n^{(k)}$ has no copy of $K_r^{(k)}$.*

Proof. Let $X \subseteq V(A_n^{(k)})$ be such that $|X| = r$ and $A_n^{(k)}[X]$ is a clique. Then by the property of r, there exists a $Y \subseteq X$ such that $|Y| = k$ and all $(k-1)$-subsets of Y have the same colour c. But now

$$\sum_{\substack{T \subseteq Y \\ |T| = k-1}} f(T) = ck = 0 \pmod{k}.$$

Thus $Y \notin A_n^{(k)}$, which contradicts that $A_n^{(k)}[X]$ is a clique. $\square$

To show that $A_n^{(k)}$ satisfies $\texttt{Disc}^{(k)}(\mathcal{I}, \geqslant p, \mu)$ when $\mathcal{I} \neq \binom{[k]}{k-1}$, we will use a theorem of Towsner [38] that equates $\mathcal{I}$-discrepancy with counting $\mathcal{I}$-adapted hypergraphs. Therefore, we prove that the count of any $\mathcal{I}$-adapted hypergraph F in $A_n^{(k)}$ is correct with high probability.

Lemma 5.4. *Let $p = (k-1)/k$ and let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain such that $\mathcal{I} \neq \binom{[k]}{k-1}$. Let F be an $\mathcal{I}$-adapted k-graph. For every $\mu > 0$, with probability going to one as n goes to infinity, the number of labelled copies of F in $A_n^{(k)}$ satisfies*

$$|\text{inj}[F \to A_n^{(k)}] - p^{|F|}n^{v(F)}| < \mu n^{v(F)}.$$

Proof. Let $E_1, \dots, E_m$ be the ordering of edges in the definition of F being $\mathcal{I}$-adapted. First we show that if $Q : V(F) \to V(A_n^{(k)})$ is any injection, then the probability that $Q(E_i) \in A_n^{(k)}$

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is exactly p independently of whether the edges E_j with $j < i$ map to hyperedges. Indeed, since $\mathcal{I} \neq \binom{[k]}{k-1}$, let $I \in \binom{[k]}{k-1} - \mathcal{I}$. Now consider some E_i and let $\phi_i : E_i \to [k]$ be the bijection from the definition of F being $\mathcal{I}$-adapted. Now since $I \notin \mathcal{I}$, there is no $j < i$ such that $\phi_i(E_i \cap E_j) = I$. Thus conditioning on whether the edges E_j with $j < i$ map to edges of $A_n^{(k)}$ potentially fixes the colours on $(k-1)$-subsets of $Q(E_i)$ besides the $(k-1)$-subset indexed by I. Since the colour of $\{Q(x) : x \in E_i, \phi_i(x) \in I\}$ (which has size $k-1$) has probability exactly p of making the colour sum of $Q(E_i)$ once all other colours are fixed, with probability p we have that $Q(E_i)$ is an edge. Therefore, the probability that Q is an edge-preserving map is $p^{|F|}$. This implies that the expected number of labelled copies of F in $A_n^{(k)}$ is $p^{|F|}n(n-1)\cdots(n-v(F)+1)$.

A simple second moment calculation shows that with high probability the number of labelled copies of F in $A_n^{(k)}$ is $p^{|F|}n^{v(F)} \pm \mu n^{v(F)}$ for large n. Indeed, for each injective map $T : V(F) \to V(A_n^{(k)})$, define an indicator random variable X_T where $X_T = 1$ if the map is edge-preserving. Let $X = \sum X_T$ so that $\mathbb{E}[X] = p^{|F|}n(n-1)\cdots(n-V(F)+1)$. The event '$X_T = 1$' will depend on '$X_{T'} = 1$' only if the images of T and T' intersect. Therefore, given a fixed injective map $T : V(F) \to V(A_n^{(k)})$, there are at most $v(F)n^{v(F)-1}$ possible maps T' that intersect the image of T. Thus $\mathrm{Var}(X) = o(\mathbb{E}[X]^2)$, so by the second moment method, with probability going to one as n goes to infinity, the number of labelled copies of F in $A_n^{(k)}$ satisfies $|\,\mathrm{inj}[F \to A_n^{(k)}] - p^{|F|}n^{v(F)}| < \mu n^{v(F)}$. □

Lastly, we need to show that $A_n^{(k)}$ satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ in every link for every $\mathcal{J}$. We could do that similarly to the previous lemma by showing that the count of $\mathcal{J}$-adapted k-graphs is correct, but instead are able to directly show that $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ holds.

Lemma 5.5. *Let $\mathcal{J} \subseteq 2^{[k-1]}$ be an antichain and $\alpha = (k-1)/k$. Then for every $\mu > 0$, with probability going to one as n goes to infinity, $L(x)$ satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J}, \geqslant\alpha, \mu)$ for each $x \in V(A_n^{(k)})$.*

Proof. Fix $x \in V(A_n^{(k)})$ and view $L_{A_n^{(k)}}(x)$ as a probability distribution over $(k-1)$-graphs with vertex set $V(A_n^{(k)}) - x$. That is, an element from this probability distribution is generated by first generating $A_n^{(k)}$ and then outputting the link of x. We claim that the probability distribution $L(x)$ is isomorphic to the probability distribution $G^{(k-1)}(n-1, \alpha)$. To see this, consider $S \in \binom{V(A_n^{(k)})-x}{k-1}$. Then $S \in L(x)$ if

$$\sum_{\substack{T \subseteq S \cup \{x\} \\ |T|=k-1}} f(T) \neq 0 \pmod{k}.$$

We could rewrite this as

$$f(S) \neq \sum_{\substack{T \subseteq S \\ |T|=k-2}} f(T \cup x) \pmod{k}.$$

The sum on the left-hand side is some integer w_S between 0 and $k-1$, so that S is a hyperedge of $L(x)$ if and only if the colour of S is not w_S. Since this is for every S and the colours assigned to S are mutually independent, $L(x)$ is isomorphic to $G^{(k-1)}(n-1, \alpha)$.

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The proof is now complete, since for large n, $G^{(k-1)}(n-1,\alpha)$ satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J},\geqslant\alpha,\mu)$ with very high probability as follows. Fix any $\mathcal{J}$-layout Λ. Each $(k-1)$-clique in Λ is a hyperedge with probability α, and two $(k-1)$-cliques are independent unless one is a permutation of the other. So divide $K_{k-1}(\Lambda)$ up into at most $(k-1)!$ sets $R_1,\ldots,R_{(k-1)!}$ such that within a single R_i there are no $(k-1)$-tuples which are permutations of each other. Then the expected size of $H\cap R_i$ is $\alpha|R_i|$, and by Chernoff's inequality,

$$\mathbb{P}\big[||H\cap R_i|-\alpha|R_i||>\epsilon n^{k-1}\big]<2e^{-\epsilon^2 n^{2k-2}/2|R_i|}.$$

Since $|R_i|\leqslant n^{k-1}$, the probability is at most $e^{-cn^{k-1}}$ for some constant c. There are $(k-1)!$ sets R_i and there are at most $2^{k-2}2^{n^{k-2}}$ $\mathcal{J}$-layouts Λ, so with probability at most $e^{-c'n^{k-1}}$, the link of x fails $\texttt{Disc}^{(k-1)}(\mathcal{J},\geqslant\alpha,\mu)$. There are n vertices of $A_n^{(k)}$, so with probability at most $ne^{-c'n^{k-1}}\to 0$, there is some vertex x of $A_n^{(k)}$ whose link fails $\texttt{Disc}^{(k-1)}(\mathcal{J},\geqslant\alpha,\mu)$. □

Proof of Proposition 1.3. As mentioned above, to show that $A_n^{(k)}$ satisfies $\texttt{Disc}^{(k)}(\mathcal{I},\geqslant p,\mu)$, we combine Lemma 5.4 with a theorem of Towsner [38] which is stated in the language of k-graph sequences. Converting from the probability distribution $A_n^{(k)}$ to a k-graph sequence is very similar to the proofs of [28, Lemmas 30 and 31], so we only briefly sketch the technique here. By the previous lemmas and the probabilistic method, for every $\mu>0$ there exists an n_0 such that for every $n\geqslant n_0$ there exists some k-graph satisfying the properties in the previous lemmas (it has the right edge density, it fails $\texttt{Disc}^{(k)}(\binom{[k]}{k-1},\geqslant p,\mu)$, no copy of K_r, it has the right count of all $\mathcal{I}$-adapted hypergraphs, and it satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J},\geqslant\alpha,\mu)$ in the links). Construct a k-graph sequence $\mathcal{H}=\{H_n\}_{n\in\mathbb{N}}$ by diagonalization by setting $\mu=1/n$.

By Lemma 5.4, $\mathcal{H}$ satisfies the property that for every $\mathcal{I}$-adapted F, $\lim_{n\to\infty}t_F(H_n)=p^{|F|}$, so by [38, Theorem 1.1] $\mathcal{H}$ is $\texttt{Disc}_p[\mathcal{I}]$ (where $t_F(H_n)$ and $\texttt{Disc}_p[\mathcal{I}]$ are defined in [38]). Thus for large n, the k-graphs in the sequence $\mathcal{H}$ are the k-graphs which prove Proposition 1.3. □

Proof of Proposition 1.4. Let $G=G^{(k)}(n,p)$ be the random k-graph with density p. Modify G by picking a single vertex $x\in V(G)$, removing all edges which contain x, and adding edges so that $L(x)=A_n^{(k-1)}$. Now the link of x has no copy of $K_r^{(k-1)}$ so that G has no perfect $K_{r+1}^{(k)}$-packing. Also, G satisfies $\texttt{Disc}^{(k)}(\binom{[k]}{k-1},\geqslant p,\mu)$ since the random k-graph satisfies $\texttt{Disc}^{(k)}(\binom{[k]}{k-1},\geqslant p,\mu)$ (see the proof of Lemma 5.5) and we only modified at most n^{k-1} hyperedges. By the previous lemmas, the link of x fails $\texttt{Disc}^{(k-1)}(\binom{[k-1]}{k-2},\geqslant\alpha,\mu)$ and satisfies $\texttt{Disc}^{(k-1)}(\mathcal{J},\geqslant\alpha,\mu)$ for all $\mathcal{J}\neq\binom{[k-1]}{k-2}$. □

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