

Shock waves and compactons for fifth-order non-linear dispersion equations

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The following *first problem* is posed: to justify that the *standing shock wave*

$$S_-(x) = -\text{sign } x = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0 \end{cases}$$

is a correct ‘entropy solution’ of the Cauchy problem for the fifth-order degenerate non-linear dispersion equations (NDEs), same as for the classic Euler one $u_t + uu_x = 0$,

$$u_t = -(uu_x)_{xxxx} \quad \text{and} \quad u_t = -(u_{xxxx})_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+.$$

These two quasi-linear degenerate partial differential equations (PDEs) are chosen as typical representatives; so other $(2m + 1)$ th-order NDEs of non-divergent form admit such shocks waves. As a related *second problem*, the opposite initial shock $S_+(x) = -S_-(x) = \text{sign } x$ is shown to be a non-entropy solution creating a *rarefaction wave*, which becomes C^∞ for any $t > 0$. Formation of shocks leads to non-uniqueness of any ‘entropy solutions’. Similar phenomena are studied for a *fifth-order in time* NDE $u_{tttt} = (uu_x)_{xxxx}$ in *normal form*.

On the other hand, related NDEs, such as

$$u_t = -(|u|u_x)_{xxxx} + |u|u_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

are shown to admit smooth *compactons*, as oscillatory *travelling wave* solutions with compact support. The well-known non-negative compactons, which appeared in various applications (first examples by Dey, 1998, *Phys. Rev. E*, vol. 57, pp. 4733–4738, and Rosenau and Levy, 1999, *Phys. Lett. A*, vol. 252, pp. 297–306), are non-existent in general and are not robust relative to small perturbations of parameters of the PDE.

1 Introduction: Non-linear dispersion partial differential equations, Riemann problems and the main directions of the study

1.1 Five main problems and layout: Shocks, rarefaction waves and compactons for fifth-order non-linear dispersion equations

Let us introduce our basic models, which are five fifth-order non-linear dispersion equations (NDEs). These are ordered by numbers of derivatives inside and outside the

quadratic differential operators involved on the right-hand sides:

$$u_t = -uu_{xxxxx} \quad (\text{NDE}-(5,0)), \tag{1.1}$$

$$u_t = -(uu_{xxxx})_x \quad (\text{NDE}-(4,1)), \tag{1.2}$$

$$u_t = -(uu_{xxx})_{xx} \quad (\text{NDE}-(3,2)), \tag{1.3}$$

$$u_t = -(uu_{xx})_{xxx} \quad (\text{NDE}-(2,3)), \tag{1.4}$$

$$u_t = -(uu_x)_{xxxx} \quad (\text{NDE}-(1,4)). \tag{1.5}$$

The only fully divergent operator is in the last NDE, namely NDE-(1,4), which, being written as

$$u_t = -(uu_x)_{xxxx} \equiv -\frac{1}{2}(u^2)_{xxxxx} \quad (\text{NDE}-(1,4) = \text{NDE}-(0,5)), \tag{1.6}$$

furthermore becomes NDE-(0,5), or simply NDE-5. This completes the list of such quasi-linear degenerate partial differential equations (PDEs) under consideration.

The main feature of these degenerate odd-order PDEs is that they admit *shock* and *rarefaction waves*, similar to the first-order conservation laws such as *Euler’s equation*

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}. \tag{1.7}$$

Before explaining the physical significance of the NDEs and their role in general PDE theory, we pose four main problems for the above NDEs (the same as for (1.7)):

(I) Problem ‘blow-up to S_- ’ (Section 2): *to show that the shock of the shape $-\text{sign } x$ can be obtained by the blow-up limit from a smooth self-similar solution $u_-(x,t)$ of (1.1)–(1.5) in $\mathbb{R} \times (0, T)$; i.e. the following holds:*

$$u_-(x,t) \rightarrow S_-(x) = -\text{sign } x = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0, \end{cases} \quad \text{as } t \rightarrow T^- \quad \text{in } L^1_{\text{loc}}(\mathbb{R}). \tag{1.8}$$

(II) The Riemann problem S_+ (RP+) (Section 3): *to show that the initial shock*

$$S_+(x) = \text{sign } x = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0 \end{cases} \tag{1.9}$$

for NDEs (1.1)–(1.5) generates a ‘rarefaction wave’, which is C^∞ -smooth for $t > 0$.

(III) The Riemann problem S_- (RP-) (Section 4): *introducing a ‘ δ -entropy test’ (smoothing of discontinuous solutions at shocks via a ‘ δ -deformation’), to show that*

$$S_-(x) \text{ is an ‘}\delta\text{-entropy’ shock wave and } S_+(x) \text{ is not.} \tag{1.10}$$

(IV) Problem – non-uniqueness/entropy (Section 5): *to show that a single-point ‘gradient catastrophe’ for NDE (1.5) leads to the principal non-uniqueness of a shock wave extension after singularity. This also suggests non-existence of any proper entropy mechanism for choosing any ‘right’ solution after single-point blow-up.*

In Section 6, we discuss these problems in application to other NDEs including the rather unusual

$$u_{tttt} = (uu_x)_{xxxx}, \quad (1.11)$$

which indeed can be reduced to a first-order system that, nevertheless, is not hyperbolic, so that modern advanced theory of 1D hyperbolic systems (see e.g. Bressan [1] or Dafermos [10]) does not apply. The main convenient mathematical feature of (1.11) is that it is in the *normal form*; so it obeys the Cauchy–Kovalevskaya theorem that guarantees local existence of a unique analytic solution and makes easier the application of our δ -entropy (smoothing) test. Regardless of this, (1.11) is shown to create in finite-time shocks of the type $S_-(x)$ in (1.8) and rarefaction waves for other discontinuous data $\sim S_+(x)$ in (1.9).

Finally, we consider the last problem.

(V) Problem – ‘oscillatory smooth compactons’ (Section 7): to show that the perturbed version of NDE (1.5), as a typical example,

$$u_t = -(|u|u_x)_{xxxx} + |u|u_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (1.12)$$

admits compactly supported travelling wave (TW) solutions of changing sign near finite interfaces. Equation (1.12) is written for solutions with infinitely many sign changes, by replacing u^2 by the monotone function $|u|u$.

Non-negative compact structures have been known since the beginning of the 1990s as *compactons* [46]. We show that *non-negative compactons* of fifth-order NDEs such as (1.12) that are more standard in literature are non-existent in general. Moreover, these are *not robust* (not ‘structurally stable’); i.e. they do not exhibit continuous dependence upon the parameters of PDEs (say arbitrarily small perturbations of non-linearities).

1.2 A link to classic entropy shocks for conservation laws

Indeed, problems **(I)–(III)** given above are classic for the entropy theory of 1D conservation laws from the 1950s. It is well recognised that shock waves first appeared in gas dynamics that led to the mathematical theory of entropy solutions of the first-order conservation laws and Euler’s equation (1.7) as a key representative. The entropy theory for PDEs such as (1.7), with arbitrary measurable initial data u_0 , was created by Oleinik [35, 36] and Kruzhkov [30] (analogous scalar equations in \mathbb{R}^N) in the 1950s and 1960s; see the details on the history, main results and modern developments in the well-known monographs [1, 10, 49]. Note that the first analysis of the formation of shocks for (1.7) was performed by Riemann in 1858 [41]; see further details and the history in [3]. It is worth mentioning that the implicitly given solution $u = u(x, t)$ of the Cauchy problem (1.7), via the characteristic formula

$$u = u_0(x - ut),$$

containing the key wave ‘overturning’ effect, was obtained earlier by Poisson in 1808 [38] (see [39]).

According to the entropy theory for conservation laws such as (1.7), it is well known that (1.10) holds. This means that

$$u_-(x, t) \equiv S_-(x) = -\text{sign } x \quad (1.13)$$

is the unique entropy solution of PDE (1.7) with the same initial data $S_-(x)$. On the contrary, taking S_+ -type initial data (1.9) in the Cauchy problem (1.7) yields the continuous rarefaction wave with a simple similarity piece-wise linear structure,

$$u_0(x) = S_+(x) = \text{sign } x \quad \implies \quad u_+(x, t) = g\left(\frac{x}{t}\right) = \begin{cases} -1 & \text{for } x < -t, \\ \frac{x}{t} & \text{for } |x| < t, \\ 1 & \text{for } x > t. \end{cases} \quad (1.14)$$

Our first goal is to justify the same conclusions for the fifth-order NDEs, where, of course, the rarefaction wave in the RP_+ is supposed to be different from that in (1.14).

We now return to main applications of the NDEs.

1.3 NDEs from theory of integrable PDEs and water waves

Talking about odd-order PDEs under consideration, these naturally appear in the classic theory of integrable PDEs from shallow-water applications, beginning with the *Korteweg–de Vries (KdV) equation*

$$u_t + uu_x = u_{xxx} \quad (1.15)$$

and the *fifth-order KdV equation*

$$u_t + u_{xxxxx} + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} = 0,$$

among others. These are *semilinear* dispersion equations, which, being endowed with smooth semigroups (groups), generate smooth flows; so discontinuous weak solutions are unlikely, though strong oscillatory behaviour of solutions is typical (see the references in [23, Chapter 4]).

The situation is changed for the quasi-linear case. In particular, consider the quasi-linear *Harry Dym equation*

$$u_t = u^3u_{xxx}, \quad (1.16)$$

which is one of the most exotic integrable soliton equations (for a survey, see [23, Section 4.7] and the references therein). Here, (1.16) indeed belongs to the NDE family, though it seems that proper semigroups of its discontinuous solutions (if any) have never been examined. On the other hand, moving blow-up singularities and other types of complex singularities of the *modified Harry Dym equation*

$$u_t = u^3u_{xxx} - u_x - \frac{1}{2}u^3$$

have been described in [6] by delicate asymptotic expansion techniques.

In addition, integrable equation theory produced various hierarchies of quasi-linear higher-order NDEs, such as the fifth-order *Kawamoto equation* [29], as a typical example,

$$u_t = u^5 u_{xxxxx} + 5u^4 u_x u_{xxxx} + 10u^5 u_{xx} u_{xxx}. \quad (1.17)$$

We can enlarge this list talking about possible quasi-linear extensions of the integrable *Lax's seventh-order KdV equation*

$$u_t + [35u^4 + 70(u^2 u_{xx} + u(u_x)^2) + 7(2uu_{xxxx} + 3(u_{xx})^2 + 4u_x u_{xxx}) + u_{xxxxx}]_x = 0$$

and the *seventh-order Sawada–Kotara equation*

$$u_t + [63u^4 + 63(2u^2 u_{xx} + u(u_x)^2) + 21(uu_{xxxx} + (u_{xx})^2 + u_x u_{xxx}) + u_{xxxxx}]_x = 0$$

(see the references in [23, p. 234]).

The modern mathematical theory of odd-order quasi-linear PDEs is partially originated and continues to be strongly connected with the class of integrable equations. Special advantages of integrability by using the inverse scattering transform method, Lax pairs, Liouville transformations and other explicit algebraic manipulations have made it possible to create a rather complete theory for some of these difficult quasi-linear PDEs. Nowadays, well-developed theory and most of the rigorous results on existence, uniqueness and various singularity and non-differentiability properties are associated with NDE-type integrable models such as the *Fuchssteiner–Fokas–Camassa–Holm* (FFCH) equation

$$(I - D_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx} \equiv -(I - D_x^2)(uu_x) - \left[u^2 + \frac{1}{2}(u_x)^2 \right]. \quad (1.18)$$

Equation (1.18) is an asymptotic model describing the wave dynamics at the free surface of fluids under gravity. It is derived from Euler equations for inviscid fluids under the long-wave asymptotics of shallow-water behaviour (where the function u is the height of the water above a flat bottom). Applying to (1.18) the integral operator $(I - D_x^2)^{-1}$ with the L^2 -kernel $\omega(s) = \frac{1}{2}e^{-|s|} > 0$ reduces it, for a class of solutions, to the conservation law (1.7) with a compact *first-order* perturbation,

$$u_t + uu_x = - \left[\omega * \left(u^2 + \frac{1}{2}(u_x)^2 \right) \right]_x. \quad (1.19)$$

Almost all mathematical results (including entropy inequalities and Oleinik's condition (E)) have been obtained by using this integral representation of the FFCH equation (see the references in [23, p. 232]).

There is another integrable PDE from the family with third-order quadratic operators,

$$u_t - u_{xxt} = \alpha uu_x + \beta u_x u_{xx} + uu_{xxx} \quad (\alpha, \beta \in \mathbb{R}), \quad (1.20)$$

where $\alpha = -3$ and $\beta = 2$ yields the FFCH equation (1.18). This is the *Degasperis–Procesi* (DP) equation for another choice, $\alpha = -4$ and $\beta = 3$:

$$u_t - u_{xxt} = -4uu_x + 3u_x u_{xx} + uu_{xxx}, \quad \text{or} \quad u_t + uu_x = - \left[\omega * \left(\frac{3}{2}u^2 \right) \right]_x. \quad (1.21)$$

On the existence, uniqueness (of entropy solutions in $L^1 \cap BV$), parabolic ε -regularisation, Oleinik's entropy estimate and generalised PDEs, see [5].

Note that since the non-local term in the DP equation (1.21) does not contain u_x , the differential properties of its solutions are distinct from those for the FFCH one (1.19). Namely, the solutions are less regular, and (1.21) admits *shock waves*, e.g., of the form

$$u_{\text{shock}}(x, t) = -\frac{1}{t} \text{sign } x e^{-|x|},$$

with the rather standard (induced by (1.7)) but more involved entropy theory (see [14,31]).

Besides (1.18) and (1.21), family (1.20) does not contain other integrable entries. A list of more applied papers related to various NDEs is also available in [23, Chapter 4].

1.4 NDEs from compacton theory

Other important applications of odd-order PDEs are associated with *compacton phenomena* for more general non-integrable models. For instance, the *Rosenau–Hyman* (RH) equation

$$u_t = (u^2)_{xxx} + (u^2)_x \quad (1.22)$$

has special important applications as a widely used model of the effects of non-linear dispersion in the pattern formation in liquid drops [46]. It is the $K(2,2)$ equation from the general $K(m,n)$ family of the NDEs:

$$u_t = (u^n)_{xxx} + (u^m)_x \quad (u \geq 0) \quad (1.23)$$

that describe the phenomena of compact pattern formation [42,43]. Such PDEs also appear in curve motion and shortening flows [45]. Similar to well-known parabolic models of the porous medium type, the $K(m,n)$ equation (1.23) with $n > 1$ is degenerate at $u = 0$ and may therefore exhibit finite speed of propagation and admit solutions with finite interfaces. The crucial advantage of the RH equation (1.22) is that it possesses *explicit* moving compactly supported soliton-type solutions, called *compactons* [42,46], which are TW solutions to be discussed for the PDEs under consideration.

Various families of quasi-linear third-order KdV-type equations can be found in [4], where further references concerning such PDEs and their exact solutions are given. Higher-order generalised KdV equations are of increasing interest; see e.g. the quintic KdV equation in [27]; see also [54], where the seventh-order PDEs are studied.

The more general $B(m,k)$ equations,

$$u_t + a(u^m)_x = \mu(u^k)_{xxx},$$

which coincide with the $K(m,k)$ after scaling, also admit simple semi-compacton solutions [47]. The same is true for the $Kq(m,\omega)$ NDE (another non-linear extension of the KdV) [42]

$$u_t + (u^m)_x + [u^{1-\omega}(u^\omega u_x)_x]_x = 0.$$

Setting $m = 2$ and $\omega = \frac{1}{2}$ yields a typical quadratic PDE,

$$\mathbf{B}(u) \equiv u_t + (u^2)_x + uu_{xxx} + 2u_x u_{xx} = 0. \quad (1.24)$$

It is curious that (1.24) admits an extended compacton-like dynamics on a standard trigonometric-exponential subspaces, on which

$$u(x, t) = C_0(t) + C_1(t) \cos \lambda x + C_2(t) \sin \lambda x \in W_3 = \text{Span}\{1, \cos \lambda x, \sin \lambda x\}, \quad (1.25)$$

where $\lambda = \sqrt{\frac{2}{3}}$. This subspace is invariant under the quadratic operator \mathbf{B} in the usual sense that $\mathbf{B}(W_3) \subseteq W_3$. Therefore substituting (1.25) into PDE (1.24) yields for the expansion coefficients on W_3 $\{C_0, C_1, C_2\}$ a 3D non-linear dynamical system; see further such examples of exact solutions of NDEs on invariant subspaces in [23, Chapter 4].

Combining the $K(m, n)$ and $B(m, k)$ equations gives the dispersive-dissipativity entity $DD(k, m, n)$ [44],

$$u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx},$$

which can also admit solutions on invariant subspaces for some values of parameters.

For the fifth-order NDEs, such as

$$u_t = \alpha(u^2)_{xxxxx} + \beta(u^2)_{xxx} + \gamma(u^2)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (1.26)$$

compacton solutions were first constructed in [12], where the more general $K(m, n, p)$ family of PDEs,

$$u_t + \beta_1(u^m)_x + \beta_2(u^n)_{xxx} + \beta_3 D_x^5(u^p) = 0,$$

with $m, n, p > 1$, was introduced. Some of these equations will be treated later on. Equation (1.26) is also associated with the family $Q(l, m, n)$ of more general quintic evolution PDEs with non-linear dispersion,

$$u_t + a(u^{m+1})_x + \omega [u(u^n)_{xx}]_x + \delta [u(u^l)_{xxxx}]_x = 0, \quad (1.27)$$

possessing multi-hump, compact solitary solutions [48].

Concerning quasi-linear PDEs that are higher-order in time, let us mention a generalisation of the *combined dissipative double-dispersive* (CDDD) equation (see, e.g., [40]),

$$u_{tt} = \alpha u_{xxxx} + \beta u_{xxtt} + \gamma (u^2)_{xxxxt} + \delta (u^2)_{xxt} + \varepsilon (u^2)_t, \quad (1.28)$$

and also the *non-linear modified dispersive Klein–Gordon equation* ($mKG(1, n, k)$),

$$u_{tt} + a(u^n)_{xx} + b(u^k)_{xxxx} = 0, \quad n, k > 1 \quad (u \geq 0) \quad (1.29)$$

(see some exact TW solutions in [28]). For $b > 0$, (1.29) is of hyperbolic (or Boussinesq) type in the class of non-negative solutions. We also mention related 2D *dispersive Boussinesq equations* denoted by $B(m, n, k, p)$ [53],

$$(u^m)_{tt} + \alpha (u^n)_{xx} + \beta (u^k)_{xxxx} + \gamma (u^p)_{yyyy} = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}.$$

See [23, Chapters 4–6] for more references and examples of exact solutions on invariant subspaces of NDEs of various types and orders.

1.5 On canonical third-order NDEs

Until recently, quite little was known about proper mathematics concerning discontinuous solutions, rarefaction waves and ‘entropy-like’ approaches, even for the simplest third-order NDEs such as (1.22) or (see [18, 22])

$$u_t = (uu_x)_{xx}. \quad (1.30)$$

However, the smoothing results for sufficiently regular solutions of linear and non-linear third-order PDEs are well known from the 1980s and the 1990s. For instance, infinite smoothing results were proved in [7] (see also [26]) for the general linear equation

$$u_t + a(x, t)u_{xxx} = 0 \quad (a(x, t) \geq c > 0) \quad (1.31)$$

and in [8] for the corresponding fully non-linear PDE

$$u_t + f(u_{xxx}, u_{xx}, u_x, u, x, t) = 0 \quad (f_{u_{xxx}} \geq c > 0) \quad (1.32)$$

(see also [2] for semilinear equations). Namely, for a class of such equations, it is shown that for data with minimal regularity and sufficient (say exponential) decay at infinity, there exists a unique solution $u(x, t) \in C_x^\infty$ for small $t > 0$. Similar smoothing local-in-time results for unique solutions are available for the equations in \mathbb{R}^2 ,

$$u_t + f(D^3u, D^2u, Du, u, x, y, t) = 0 \quad (1.33)$$

(see [32] and the references therein).

These smoothing results have been used in [18] for developing a kind of a δ -entropy test for discontinuous solutions by using techniques of smooth deformations. We will follow these ideas applied now to shock and compacton solutions of higher-order NDEs and others.

2 (I) Problem ‘blow-up’: existence of shock S_- similarity solutions

We now show that Problem (I) on blowing up to the shock $S_-(x)$ can be solved in a unified manner by constructing self-similar solutions. As often happens in non-linear evolution PDEs, the refined structure of such bounded and generic shocks is described in a scaling-invariant manner.

2.1 Finite-time blow-up formation of the shock wave $S_-(x)$

One can see that all five NDEs (1.1)–(1.5) admit the following similarity substitution:

$$u_-(x, t) = g(z), \quad z = x/(-t)^{\frac{1}{5}} \quad (t < 0), \quad (2.1)$$

where, by translation, the blow-up time reduces to $T = 0$. Substituting (2.1) into the NDEs yields for g the following ODEs in \mathbb{R} , respectively:

$$gg^{(5)} = -\frac{1}{5}g'z, \quad (2.2)$$

$$(gg^{(4)})' = -\frac{1}{5}g'z, \quad (2.3)$$

$$(gg''')'' = -\frac{1}{5}g'z, \quad (2.4)$$

$$(gg'')''' = -\frac{1}{5}g'z, \quad (2.5)$$

$$(gg')^{(4)} = -\frac{1}{5}g'z, \quad (2.6)$$

with the conditions

$$g(\mp\infty) = \pm 1. \quad (2.7)$$

at infinity for the shocks S_- . In view of the symmetry of the ODEs,

$$\left. \begin{array}{l} g \mapsto -g, \\ z \mapsto -z, \end{array} \right\} \quad (2.8)$$

it suffices to get odd solutions for $z < 0$ posing anti-symmetry conditions at the origin,

$$g(0) = g''(0) = g^{(4)}(0) = 0. \quad (2.9)$$

2.2 Shock similarity profiles exist and are unique: numerical results

Before performing a rigorous approach to Problem **(I)**, it is convenient and inspiring to check whether the shock similarity profiles $g(z)$ announced in (2.1) actually exist and are unique for each of the ODEs (2.2)–(2.6). This is done by numerical methods that supply us with positive and convincing conclusions. Moreover, these numerics clarify some crucial properties of profiles, which will determine the actual strategy of rigorous study.

A typical structure of this shock similarity profile $g(z)$ satisfying the problem (2.2), (2.9) is shown in Figure 1. As a key feature, we observe a highly oscillatory behaviour of $g(z)$ about ± 1 as $z \rightarrow \mp\infty$, which can essentially affect the metric of the announced convergence in (1.8). Therefore, we will need to describe this oscillatory behaviour in detail. In Figure 2, we show the same profile $g(z)$ for smaller z . It is crucial that in all numerical experiments, we obtained the same profile that indicates that it is the unique solution of (2.2), (2.9).

Figures 3(a)–(d) show the shock similarity profiles for the rest of the NDEs (1.2)–(1.5). They differ from each other rather slightly.

Remark: on regularisation in numerical methods. For the fifth-order NDEs, this and further numerical constructions are performed by MatLab by using the `bvp4c` solver. Typically, we take the relative and absolute tolerances

$$\text{Tols} = 10^{-4}. \quad (2.10)$$

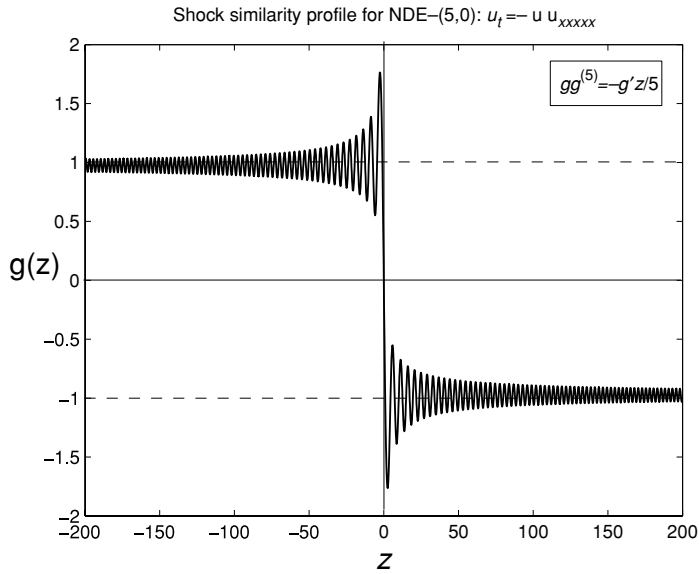


FIGURE 1. The shock similarity profile $g(z)$ as the unique solution of the problem (2.2), (2.9); $z \in [-200, 200]$.

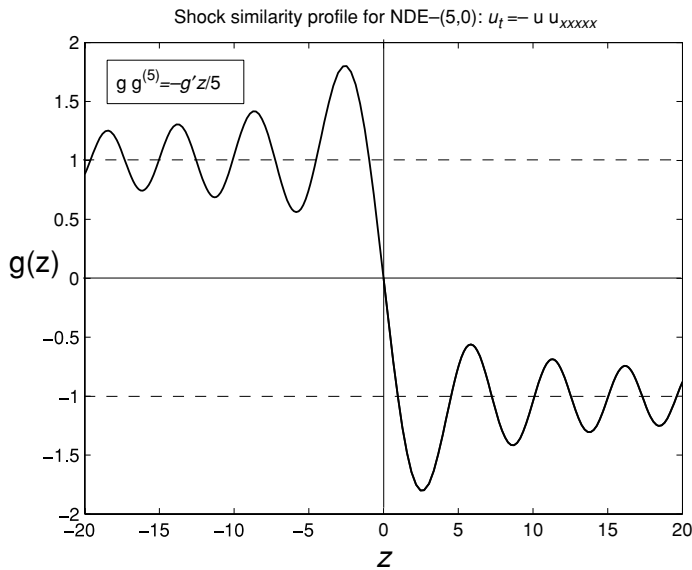


FIGURE 2. The shock similarity profile $g(z)$ as the unique solution of the problem (2.2), (2.9); $z \in [-20, 20]$.

Instead of the degenerate ODE (2.2) (or others), we solve the regularised equation

$$g^{(5)} = -\frac{\text{sign } g}{\sqrt{\nu^2 + g^2}} \left(\frac{1}{5} g' z \right), \quad \text{with the regularisation parameter } \nu = 10^{-4}, \quad (2.11)$$

where the choice of small ν is coherent with the tolerances in (2.10). Sometimes, we will need to use the enhanced parameters $\text{Tols} = \nu = 10^{-7}$ or even $\sim 10^{-9}$.

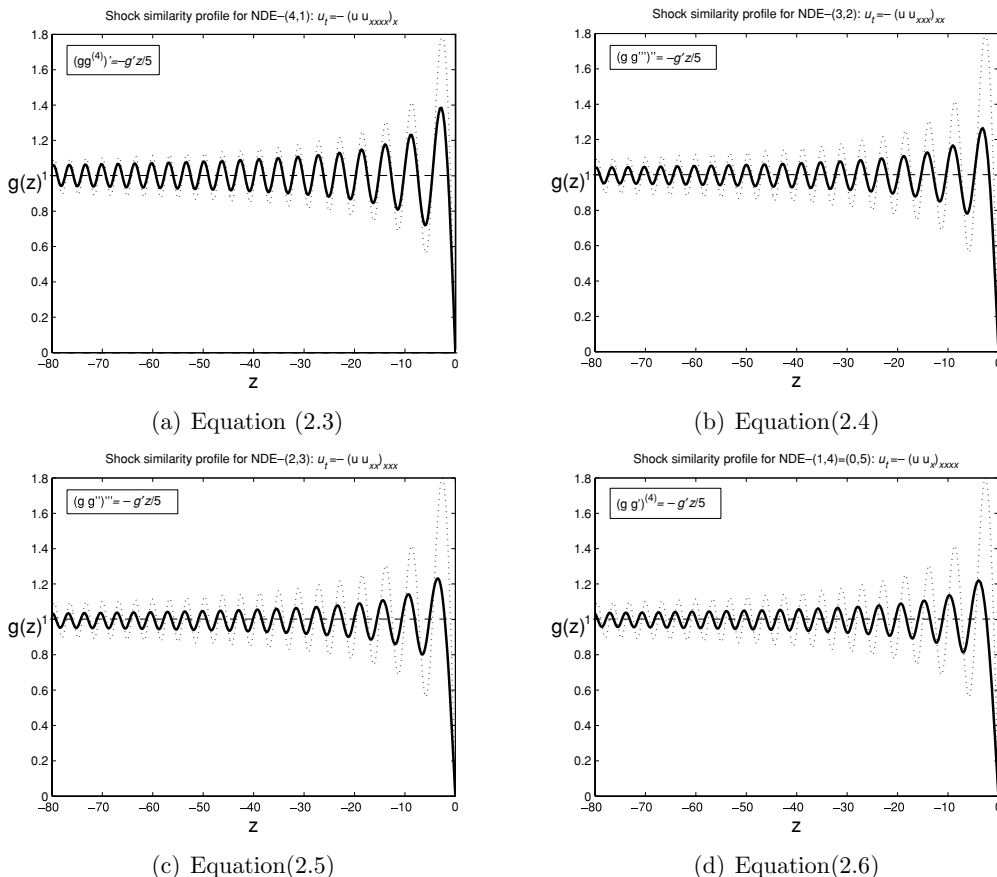


FIGURE 3. The shock similarity profiles as solutions of (2.3)–(2.6) and (2.9) respectively. For comparison, the dotted lines denote the profile from Figures 1 and 2.

2.3 Justification of oscillatory behaviour about equilibria ± 1 and other asymptotics

Thus, the shock profiles $g(z)$ are oscillatory about $+1$ as $z \rightarrow -\infty$. In order to describe these oscillations in detail, we linearise all the ODEs (2.2)–(2.6) about the regular equilibrium $g(z) \equiv 1$ by setting $g = 1 + \hat{g}$ to get the linear ODE

$$\mathbf{B}_5^* \hat{g} \equiv -\hat{g}^{(5)} - \frac{1}{5} \hat{g}'z = 0. \tag{2.12}$$

Note that this equation reminds us of that for the rescaled kernel $F(z)$ of the fundamental solution of the corresponding linear dispersion equation,

$$u_t = -u_{xxxxx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{2.13}$$

The fundamental solution of the corresponding linear operator $\frac{\partial}{\partial t} + D_x^5$ in (2.13) has the standard similarity form

$$b(x, t) = t^{-\frac{1}{5}} F(y), \quad \text{with } y = x/t^{1/5}, \tag{2.14}$$

where $F(y)$ is a unique solution of the ODE problem

$$\mathbf{B}_5 F \equiv -F^{(5)} + \frac{1}{5} (Fy)' = 0 \text{ in } \mathbb{R}, \quad \int F = 1, \quad \text{or } F^{(4)} = \frac{1}{5} Fy \text{ on integration.} \quad (2.15)$$

However, the operator \mathbf{B}_5 in (2.15) is not identical to that in (2.12). Moreover, this \mathbf{B}_5^* is adjoint to \mathbf{B}_5 in some indefinite metric, and both the operators possess countable families of eigenfunctions, which particularly are generalised Hermite polynomials for \mathbf{B}_5^* . We will not use this Hermitian spectral theory later on; please therefore refer to [17, Section 9] and [19, Section 8.2] for further results and applications.

Let us return to the linearised ODE (2.12). Looking for possible asymptotics as $z \rightarrow -\infty$ yields the following exponential ones with the characteristic equation:

$$\hat{g}(z) \sim e^{a|z|^{5/4}} \implies a^4 = \frac{4^4}{5^5}. \quad (2.16)$$

Finally, choosing the purely imaginary root of the algebraic equation in (2.16) with $\text{Re } a = 0$ gives a refined WKBJ-type asymptotics of solutions of (2.2):

$$g(z) = 1 + |z|^{-5/8} [A \sin(a_0|z|^{5/4}) + B \cos(a_0|z|^{5/4})] + \dots \text{ as } z \rightarrow -\infty, \quad (2.17)$$

where A and B are some real constants satisfying $A^2 + B^2 \neq 0$.

The asymptotic behaviour (2.17) implies two important conclusions, which are as follows.

Proposition 2.1 *The shock wave profiles $g(z)$ solving (2.2)–(2.6) and (2.7) satisfy the following: (i)*

$$g(z) - 1 \notin L^1(\mathbb{R}_-) \text{ and} \quad (2.18)$$

(ii) *the total variation of $g(z)$ (and hence of $u_-(x, t)$ for any $t < 0$) is infinite.*

Proof. Setting $|z|^{5/4} = v$ in the integrals below yields by (2.17)

$$\begin{aligned} \text{(i)} \quad & \int_{-\infty}^{\infty} |g(z) - 1| \, dz \sim \int_{-\infty}^{\infty} \frac{|\cos z^{5/4}|}{z^{5/8}} \, dz \sim \int_{-\infty}^{\infty} \frac{|\cos v|}{v^{7/10}} \, dv = \infty \quad \text{and} \\ \text{(ii)} \quad & |g(\cdot)|_{\text{Tot.Var.}} = \int_{-\infty}^{+\infty} |g'(z)| \, dz \sim \int_{-\infty}^{\infty} \frac{|\cos z^{5/4}|}{z^{3/8}} \, dz \sim \int_{-\infty}^{\infty} \frac{|\cos v|}{\sqrt{v}} \, dv = \infty. \quad \square \end{aligned} \quad (2.19)$$

This is in striking contrast with the case of conservation laws (1.7), where finite total variation approaches and Helly’s second theorem (compact embedding of sets of bounded functions of bounded total variations into L^∞) used to be key (see Oleinik’s pioneering approach [35]). In view of the presented properties of the similarity shock profile $g(z)$, the convergence in (1.8) takes place for any $x \in \mathbb{R}$, uniformly in $\mathbb{R} \setminus (-\mu, \mu)$, $\mu > 0$ small, and in $L^p_{\text{loc}}(\mathbb{R})$ for $p \in [1, \infty)$, which, for convenience, we fix in the following.

Proposition 2.2 *For the shock similarity profile $g(z)$ the convergence (1.8) with $T = 0$*

(i) *does not hold in $L^1(\mathbb{R})$ and*

(ii) does hold in $L^1_{\text{loc}}(\mathbb{R})$, and moreover, for any fixed finite $l > 0$,

$$\|u_-(\cdot, t) - S_-(\cdot)\|_{L^1(-l, l)} = O((-t)^{\frac{1}{5}}) \rightarrow 0 \quad \text{as } t \rightarrow 0^-. \quad (2.20)$$

Proof of (2.20) is the same as in (2.19) with a finite interval of integration: for $l = 1$,

$$\|\cdot\|_{L^1(-1, 1)} \sim (-t)^{\frac{1}{5}} \int^{(-t)^{-1/5}} z^{-\frac{5}{4}} |\cos z^{\frac{5}{4}}| dz \sim (-t)^{\frac{1}{5}} \int^{(-t)^{-1/4}} v^{-\frac{7}{10}} |\cos v| dv \sim (-t)^{\frac{1}{5}}. \quad \square$$

Finally, note that each $g(z)$ has a regular asymptotic expansion near the origin. For instance, for the first ODE (2.2), there exist solutions such that

$$g(z) = Cz + Dz^3 - \frac{1}{600} z^5 + \frac{D}{6300C} z^7 + \dots, \quad (2.21)$$

where $C < 0$ and $D \in \mathbb{R}$ are some constants. The local uniqueness of such asymptotics is traced out by using Banach's contraction principle applied to the equivalent integral equation in the metric of $C(-\mu, \mu)$, with $\mu > 0$ small. Moreover, it can be shown that (2.21) is the expansion of an analytic function. Other ODEs admit similar local representations of solutions.

We now need the following scaling invariance of the ODEs (2.2)–(2.6): if $g_1(z)$ is a solution, then

$$g_a(z) = a^5 g_1\left(\frac{z}{a}\right) \quad \text{is a solution for any } a \neq 0. \quad (2.22)$$

2.4 Existence of a shock similarity profile

Using the asymptotics derived above, we are now in a position to prove the following.

Proposition 2.3 *The problem (2.7), (2.9) for ODEs (2.2)–(2.6) admits a solution $g(z)$, which is an odd analytic function.*

Uniqueness for such higher-order ODEs is a more difficult problem, which is not studied here, though it has been seen numerically. Moreover, there are some analogous results. We refer to [24] (to be used later on), where uniqueness of a fourth-order semilinear ODE was established by an improved shooting argument.

Notice another difficult aspect of the problem. Figures 1–3, which were obtained by careful numerics, clearly convince that the positivity holds,

$$g(z) > 0 \quad \text{for } z < 0, \quad (2.23)$$

which is also difficult to prove rigorously (see further comments below). Actually, (2.23) is not that important for the key convergence (1.8), since possible sign changes (if any) disappear in the limit as $t \rightarrow T^-$. It seems that nothing prevents the existence of some ODEs from family (2.2)–(2.6), with different non-linearities, for which the shock profiles can change sign for $z < 0$.

Proof. As above, we consider the first ODE (2.2) only. We use a shooting argument using the 2D bundle of asymptotics (2.21). By scaling (2.22), we put $C = -1$; so, actually, we deal with the *one-parameter shooting problem* with the 1D family of orbits satisfying

$$g(z; D) = -z + D z^3 - \frac{1}{600} z^5 - \frac{D}{6300} z^7 + \dots, \quad D \in \mathbb{R}. \tag{2.24}$$

It is not hard to check that besides stabilisation to unstable constant equilibria,

$$g(z) \rightarrow C_- > 0 \quad \text{as } z \rightarrow -\infty, \tag{2.25}$$

ODE (2.3) admits an unbounded stable behaviour given by

$$g(z) \sim g_*(z) = -\frac{1}{120} z^5 \rightarrow +\infty \quad \text{as } z \rightarrow -\infty. \tag{2.26}$$

The overall asymptotic bundle about the exact solution $g_*(z)$ is obtained by linearisation: as $z \rightarrow -\infty$,

$$g(z) = g_*(z) + Y(z) \implies g_* Y^{(5)} = -\frac{1}{5} Y' z + \dots \quad \text{or} \quad z^5 Y^{(5)} = 24 Y' z + \dots \tag{2.27}$$

This is the Euler-type homogeneous equation with the characteristic equation

$$Y(z) = z^m \implies m_1 = 0 \quad (Y_2(z) \equiv 1) \quad \text{or} \quad (m-1)(m-2)(m-3)(m-4) = 24. \tag{2.28}$$

This yields another $m_2 = 0$ (hence there exists $Y_2(z) = \ln|z|$), $m_3 = 5$ (not suitable) and a proper single complex root with $\text{Re } m = \frac{5}{2} < 5$ yielding oscillatory $Y_{3,4}(z)$. Thus

$$\text{as } z \rightarrow -\infty, \quad \text{there exists a 4D asymptotic bundle about } g_*(z) = -\frac{1}{120} z^5. \tag{2.29}$$

Therefore, at $z = -\infty$, we are given a 2D bundle of proper solutions (2.17), as well as 4D fast-growing profiles from (2.29). This determines the strategy of the 1D shooting via the D -family (2.24):

(i) obviously, for all $D \ll -1$, we have that $g(z; D) > 0$ is monotone decreasing and approaches the stable behaviour (2.26), (2.29), and

(ii) on the contrary, for all $D \gg 1$, $g(z; D)$ becomes non-monotone and has a zero value at some finite $z_0 = z_0(D) < 0$, satisfying $z_0(D) \rightarrow 0^-$ as $D \rightarrow +\infty$, and eventually approaches the bundle in (2.29) but in an essentially *non-monotone* way.

It follows from different and opposite ‘topologies’ of the behaviour announced in (i) and (ii) that there exists a constant D_0 such that $g(z; D_0)$ does not belong to those two sets of orbits (both are open) and hence does not approach $g_*(z)$ as $z \rightarrow -\infty$ at all. This is precisely the necessary shock similarity profile. \square

This 1D shooting approach is explained in Figure 4 obtained numerically, where

$$D_0 = 0.069192424 \dots \tag{2.30}$$

It seems that as $D \rightarrow D_0^+$, the zero of $g(z; D)$ must disappear at infinity, i.e.

$$z_0(D) \rightarrow -\infty \quad \text{as } D \rightarrow D_0^+, \tag{2.31}$$

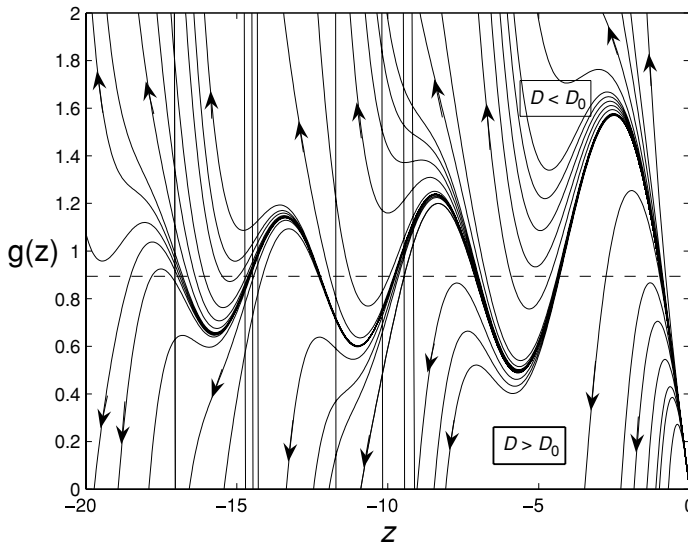


FIGURE 4. Shooting the shock similarity profile $g(z)$ via family (2.24); $D_0 = 0.069192424 \dots$

and this actually happens as Figure 4 shows. Then this would justify the positivity (2.23). Unfortunately, in general (i.e. for similar ODEs with different sufficiently arbitrary nonlinearities), this is not true; i.e. it cannot be guaranteed by a topological argument. The actual operator structure of the ODEs should be involved in the study; so, theoretically, the positivity is difficult to guarantee in general. Note again that if the shock similarity profile $g(z)$ had a few zeros for $z < 0$, this would not affect the crucial convergence property such as (1.8).

2.5 Self-similar formation of other shocks

NDE-(1,4). Let us first briefly consider the last ODE (2.6) for the fully divergent NDE (1.5). Similarly, by the same arguments, we show that according to (2.1), there exist other non-symmetric shocks as non-symmetric step-like functions, so that as $t \rightarrow 0^-$,

$$u_-(x, t) \rightarrow \begin{cases} C_- > 0 & \text{for } x < 0, \\ C_0 & \text{for } x = 0, \\ C_+ < 0 & \text{for } x > 0, \end{cases} \quad (2.32)$$

where $C_- \neq -C_+$ and $C_0 \neq 0$. Figure 5 shows a few of such similarity profiles $g(z)$, where three of these are strictly positive. The most interesting is the boldface one with

$$C_- = 1.4 \quad \text{and} \quad C_+ = 0,$$

which has the finite right-hand interface at $z = z_0 \approx 5$, with the expansion

$$g(z) = -\frac{z_0}{4200} (z_0 - z)_+^4 (1 + o(1)) \rightarrow 0^- \quad \text{as } z \rightarrow z_0. \quad (2.33)$$

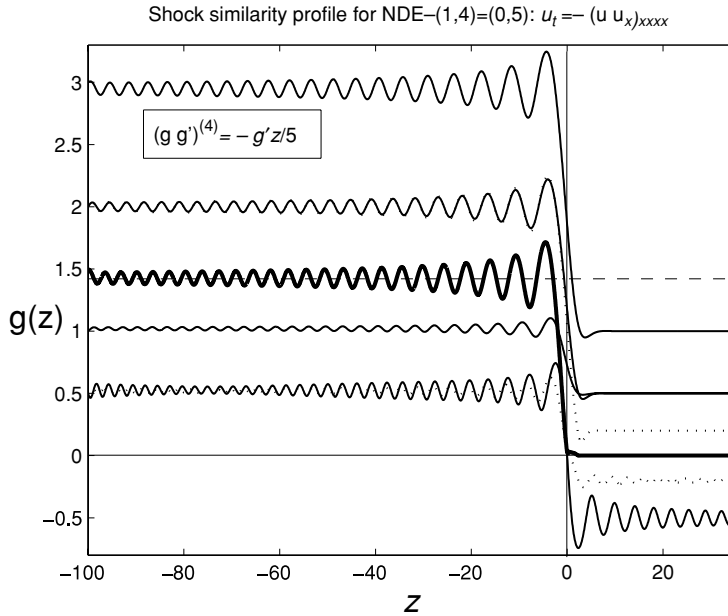


FIGURE 5. Various shock similarity profiles $g(z)$ as solutions of the problem (2.6), (2.9).

It follows that this $g(z) < 0$ near the interface; so the function changes sign there, which is also seen in Figure 5 by carefully checking the shape of profiles above the boldface one with the finite interface, bearing in mind a natural continuous dependence on parameters.

NDE-(5, 0). Consider next the first ODE (2.2) for the fully non-divergent *NDE*-(5, 0) (1.1). We can again describe formation of shocks (2.32) (see Figure 6). The boldface profile with $C_- = 1.4$ and $C_+ = 0$ has finite right-hand interface at $z = z_0 \approx 5$, with a different expansion,

$$g(z) = \frac{6z_0}{5} (z_0 - z)^4 |\ln(z_0 - z)| (1 + o(1)) \rightarrow 0^+ \text{ as } z \rightarrow z_0^-. \tag{2.34}$$

2.6 Shock formation for a uniformly dispersive NDE: an example

Here, as a key example to be continued, we show shocks for *uniform* (non-degenerate) NDEs, such as the fully divergent one,

$$u_t = -((1 + u^2)u_x)_{xxxx}, \tag{2.35}$$

where the dispersion coefficient $-(1 + u^2)$ of the principal operator is an *even* function. Recall that for all the previous ones (1.1)–(1.5), the dispersion coefficient is equal to $-u$ and is an odd function of u . Equation (2.35) is non-degenerate and represents a ‘uniformly dispersive’ NDE. The ODE for self-similar solutions (2.1) then takes the form

$$((1 + g^2)g')^{(4)} = -\frac{1}{5} g'z. \tag{2.36}$$

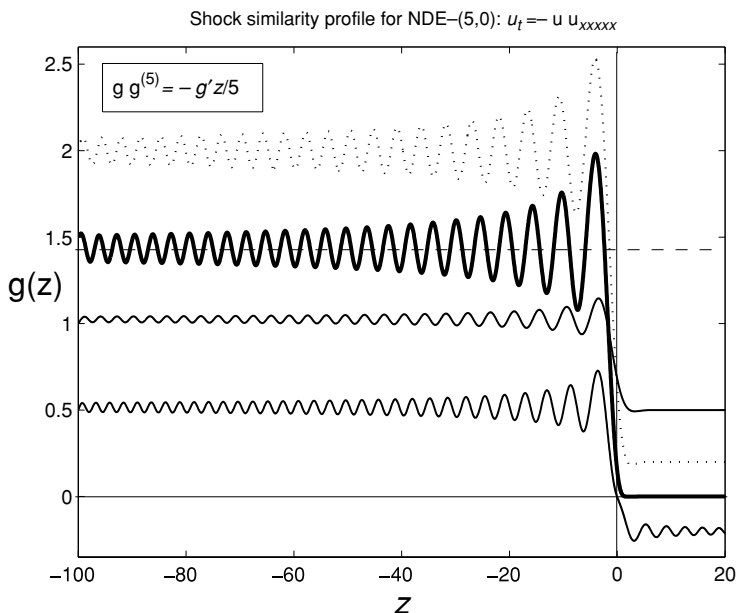


FIGURE 6. Various shock similarity profiles $g(z)$ as solutions of the problem (2.2), (2.9).

The mathematics of such equations is similar to that in Section 2.4. In Figure 7, we present a few shock similarity profiles for (2.36). Note that both shocks $S_{\pm}(x)$ are admissible, since for ODE (2.36) (and for NDE (2.35)), we have, instead of symmetry (2.8), that

$$-g(z) \text{ is also a solution.} \tag{2.37}$$

3 (II) Riemann Problem S_+ : similarity rarefaction waves

Using the reflection symmetry of all the NDEs (1.1)–(1.5),

$$\left. \begin{aligned} u &\mapsto -u, \\ t &\mapsto -t, \end{aligned} \right\} \tag{3.1}$$

we conclude that these admit *global* similarity solutions defined for all $t > 0$,

$$u_+(x, t) = g(z), \quad \text{with } z = x/t^{\frac{1}{5}}. \tag{3.2}$$

Then $g(z)$ solves ODEs (2.2)–(2.6) with the opposite terms

$$\dots = \frac{1}{5} g' z \tag{3.3}$$

on the right-hand side. Conditions (2.7) also take the opposite form

$$f(\pm\infty) = \pm 1. \tag{3.4}$$

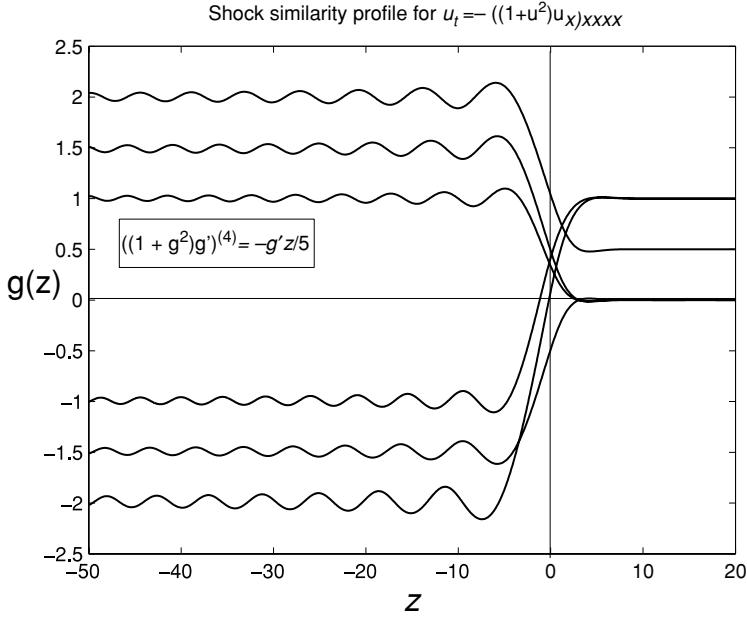


FIGURE 7. Various shock similarity profiles $g(z)$ satisfying ODE (2.36).

Thus, these profiles are obtained from the blow-up ones in (2.1) by reflection; i.e.

$$\text{if } g(z) \text{ is a shock profile in (2.1), then } g(-z) \text{ is a rarefaction one in (3.2).} \tag{3.5}$$

These are sufficiently regular similarity solutions of NDEs that have the necessary initial data: by Proposition 2.2(ii), in L^1_{loc} ,

$$u_+(x, t) \rightarrow S_+(x) \text{ as } t \rightarrow 0^+. \tag{3.6}$$

Other profiles $g(-z)$ from shock wave similarity patterns generate further rarefaction solutions including those with finite left-hand interfaces.

4 (III) Riemann problem S_- : towards δ -entropy test

4.1 Uniform NDEs

In this section, for definiteness, we consider the fully non-divergent NDE (1.1),

$$u_t = \mathbf{A}(u) \equiv -uu_{xxxx} \text{ in } \mathbb{R} \times (0, T), \quad u(x, 0) = u_0(x) \in C^\infty_0(\mathbb{R}). \tag{4.1}$$

In order to concentrate on shocks and to avoid difficulties with finite interfaces or transversal zeros at which $u = 0$ (these are weak discontinuities via non-uniformity of the PDE), we deal with strictly positive solutions satisfying

$$\frac{1}{C} \leq u \leq C, \text{ where } C > 1 \text{ is a constant.} \tag{4.2}$$

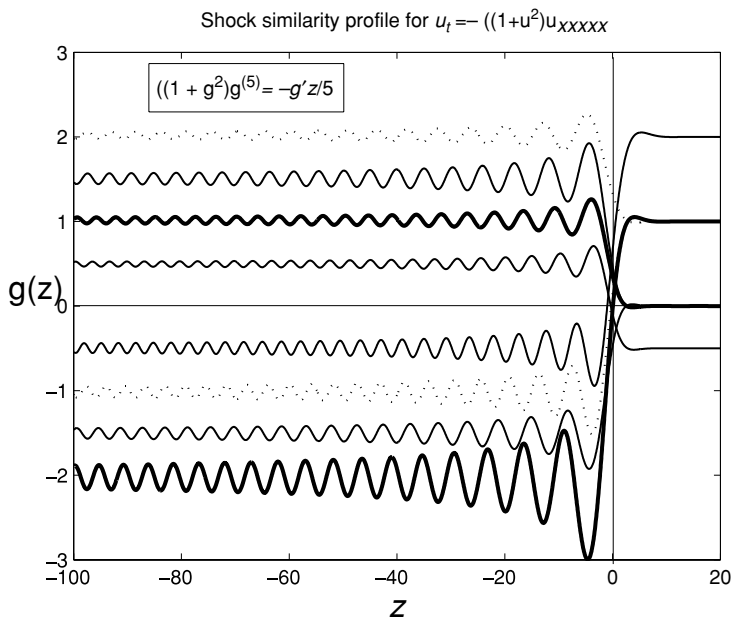


FIGURE 8. Various shock similarity profiles $g(z)$ satisfying ODE (4.4).

Remark: uniformly non-degenerate NDEs. Alternatively, in order to avoid assumptions like (4.2), we can consider the uniform equations such as (cf. (2.35))

$$u_t = -(1 + u^2)u_{xxxxx}, \quad (4.3)$$

for which no finite interfaces are available. Of course, (4.3) admits analogous blow-up similarity formation of shocks by (2.1). In Figure 8, we show a few profiles satisfying

$$(1 + g^2)g^{(5)} = -\frac{1}{5}g'z, \quad z \in \mathbb{R}. \quad (4.4)$$

Recall that for (4.4), (2.37) holds; so both $S_{\pm}(x)$ are admissible and entropy solutions (see below).

4.2 On uniqueness, continuous dependence and *a priori* bounds for smooth solutions

Actually, in our δ -entropy construction, we will need just a local semigroup of smooth solutions that is continuous as L^1_{loc} . The fact that such results are true for fifth-order (or other odd-order NDEs) is easily illustrated as follows: one can see that since (4.1) is a dispersive equation, which contains no dissipative terms, the uniqueness follows as for parabolic equations such as

$$u_t = -uu_{xxxx} \quad \text{or} \quad u_t = uu_{xxxxx} \quad \left(\text{in the class } \left\{ \frac{1}{C} \leq u \leq C \right\} \right).$$

Thus, we assume that $u(x, t)$ solves (4.1) with initial data $u_0(x) \in H^{10}(\mathbb{R})$, satisfies (4.2) and is sufficiently smooth, $u \in L^\infty([0, T], H^{10}(\mathbb{R}))$, $u_t \in L^\infty([0, T], H^5(\mathbb{R}))$ and so on.

Assuming that $v(x, t)$ is the second smooth solution, we subtract equations to obtain for the difference $w = u - v$ the PDE

$$w_t = -uw_{xxxxx} - v_{xxxxx}w. \tag{4.5}$$

We next divide by $u \geq \frac{1}{C} > 0$ and multiply by w in L^2 ; so integrating by parts that vanish the dispersive term w_{xxxxx} yields

$$\int \frac{ww_t}{u} \equiv \frac{1}{2} \frac{d}{dt} \int \frac{w^2}{u} + \frac{1}{2} \int \frac{u_t}{u^2} w^2 = - \int \frac{v_{xxxxx}}{u} w^2. \tag{4.6}$$

Therefore, using (4.2) and the assumed regularity yields

$$\frac{1}{2} \frac{d}{dt} \int \frac{w^2}{u} = \int \left(-\frac{1}{2} \frac{u_t}{u^2} - \frac{v_{xxxxx}}{u} \right) w^2 \leq C_1 \int \frac{w^2}{u}, \tag{4.7}$$

where the derivatives $u_t(\cdot, t)$ and $v_{xxxxx}(\cdot, t)$ are from $L^\infty([0, T])$. By Gronwall’s inequality, (4.7) yields $w(t) \equiv 0$. Obviously, these estimates can be translated to the continuous dependence result in L^2 and hence in L^1_{loc} .

Other *a priori* bounds on solutions can also be derived along the lines of the computations in [8, Section 2 and 3] that lead to rather technical manipulations. The principal fact is the same as seen from (4.7): differentiating (4.1) α times in x equation and setting $v = D_x^\alpha u$ yields the equations with the same principal part as in (4.5):

$$v_t = -uv_{xxxxx} + \dots \tag{4.8}$$

Multiplying this by $\zeta \frac{v}{u}$, with ζ being a cut-off function, and using various interpolation inequalities makes it possible to derive necessary *a priori* bounds and hence to observe the corresponding smoothing phenomenon for exponentially decaying initial data.

4.3 On local semigroup of smooth solutions of uniform NDEs and linear operator theory

We recall that local C^∞ -smoothing phenomena are known for third-order linear and fully non-linear dispersive PDEs (see [7, 8, 26, 32] and the references therein). We claim that having obtained *a priori* bounds, a smooth local solution can be constructed by the iteration techniques as in [8, Section 3] by using a standard scheme of iteration of the equivalent integral equation for spatial derivatives. We present further comments concerning other approaches to local existence, where we return to integral equations.

We then need a detailed spectral theory of fifth-order operators such as

$$\mathbf{P}_5 = a(x)D_x^5 + b(x)D_x^4 + \dots, \quad x \in (-L, L) \quad \left(a(x) \geq \frac{1}{C} > 0 \right), \tag{4.9}$$

with bounded coefficients. This theory can be found in, e.g., Naimark’s book [34, Chapter 2]. For *regular boundary conditions* (e.g. for periodic ones that are regular for any order, which suits us well), operators (4.9) admit a discrete spectrum $\{\lambda_k\}$, where the eigenvalues λ_k are all simple for all large k .

It is crucial for further use of eigenfunction expansion techniques that the subset of eigenfunctions $\{\psi_k\}$ that is complete in L^2 creates a *Riesz basis*; i.e. for any $f \in L^2$,

$$\sum |\langle f, \psi_k \rangle|^2 < \infty, \quad \text{where} \quad \langle f, \psi_k \rangle = \int f \bar{\psi}_k, \quad (4.10)$$

and for any $\{c_k\} \in l^2$ (i.e. $\sum |c_k|^2 < \infty$), there exists a function $f \in L^2$ such that

$$\langle f, \psi_k \rangle = c_k. \quad (4.11)$$

Then there exists a unique set of ‘adjoint’ generalised eigenfunctions $\{\psi_k^*\}$ (attributed to the ‘adjoint’ operator \mathbf{P}_5^*), which is also a Riesz basis that is bi-orthonormal to $\{\psi_k\}$:

$$\langle \psi_k, \psi_l^* \rangle = \delta_{kl} \quad (\text{Kronecker's delta}). \quad (4.12)$$

Hence, for any $f \in L^2$, in the sense of the mean convergence,

$$f = \sum c_k \psi_k, \quad \text{with} \quad c_k = \langle f, \psi_k^* \rangle \quad (4.13)$$

(see further details in [34, Section 5]).

The eigenvalues of (4.9) have the asymptotics

$$\lambda_k \sim (\pm 2ki)^5 \quad \text{for all} \quad k \gg 1. \quad (4.14)$$

In particular, it is known that \mathbf{P}_5 has compact resolvent, which makes it possible to use it in the integral representation of the NDEs (cf. [8, Section 3], where integral equations are used to construct a unique smooth solution of third-order NDEs).

On the other hand, this means that $\mathbf{P}_5 - aI$ for any $a \gg 1$ is not a sectorial operator, which makes suspicious the use of the advanced theory of analytic semigroups [9,15,33], as is natural for even-order parabolic flows (see further discussion below). Analytic smoothing effects for higher-order dispersive equations were studied in [50]. Concerning unique continuation and continuous dependence properties for dispersive equations, see [11] and the references therein and also [51] for various estimates.

4.4 Hermitian spectral theory and analytic semigroups

Let us continue to discuss related spectral issues for odd-order operators. For the linear dispersion equation with constant coefficients (2.13), the Cauchy problem with integrable data $u_0(x)$ admits the unique solution

$$u(x, t) = b(x - \cdot, t) * u_0(\cdot), \quad (4.15)$$

where $b(x, t)$ is the fundamental solution (2.14). Analyticity of solutions in t (and x) can be associated with the rescaled operator

$$\mathbf{B}_5 = -D_z^5 + \frac{1}{5} z D_z + \frac{1}{5} I \quad \text{in} \quad L_\rho^2(\mathbb{R}), \quad \text{where} \quad \rho(z) = \begin{cases} e^{a|z|^{5/4}}, & z < 0, \\ e^{-az^{5/4}}, & z > 0, \end{cases} \quad (4.16)$$

and $a > 0$ is a sufficiently small constant. Here, \mathbf{B}_5 in (4.16) is the operator in (2.15) that generates the rescaled kernel F of the fundamental solution in (2.14).

Next, using in (2.13) the same rescaling as in (2.14), we set

$$u(x, t) = t^{-\frac{1}{5}}v(y, \tau), \quad y = x/t^{\frac{1}{5}}, \quad \tau = \ln t, \tag{4.17}$$

to get the rescaled PDE with operator (4.16),

$$v_\tau = \mathbf{B}_5 v. \tag{4.18}$$

Next, the Taylor expansion of the kernel in (4.15) yields

$$v(y, \tau) = \int F(y - ze^{-\tau/5})u_0(z) dz = \sum_{(k)} \frac{(-1)^k}{\sqrt{k!}} F^{(k)}(y)e^{-\frac{k}{5}\tau} \frac{1}{\sqrt{k!}} \int z^k u_0(z) dz, \tag{4.19}$$

where the series converges uniformly on compact subsets, defining an analytic solution, and also in the mean in L^2_ρ . According to the eigenfunctions expansion (4.19) of the semigroup, there is a proper definition of operator (4.16) with a real spectrum and eigenfunctions (see the details in [17, Section 9] and [19, Section 8.2]),

$$\sigma(\mathbf{B}_5) = \left\{ -\frac{k}{5}, k = 0, 1, 2, \dots \right\} \quad \text{and} \quad \psi_k(y) = \frac{(-1)^k}{\sqrt{k!}} F^{(k)}(y), \quad k \geq 0.$$

The basis of the ‘adjoint’ operator (cf. (2.12)), in a space with an indefinite metric,

$$\mathbf{B}_5^* = -D_y^5 - \frac{1}{5}yD_y \quad \text{in} \quad L^2_{\rho^*}(\mathbb{R}), \quad \rho^*(z) = e^{-a|z|^{5/4}} \quad \text{in} \quad \mathbb{R},$$

has the same point spectrum and eigenfunctions $\{\psi_k^*\}$, which are generalised Hermite polynomials (cf. a full ‘parabolic’ version of such a Hermitian spectral theory in [13, 17]). This implies that $\mathbf{B}_5 - aI$ is sectorial for $a \geq 0$ ($\lambda_0 = 0$ is simple), and this justifies the fact that (4.15) is an analytic (in t) flow. Let us mention again that analytic smoothing effects are well known for higher-order dispersive equations with operators of the principal type [50].

Actually, this also suggests the treatment of (4.1) and (4.2) by a classic approach as in Da Prato and Grisvard [9] by linearising about a sufficiently smooth $u_0 = u(t_0)$, $t_0 \geq 0$, by setting $u(t) = u_0 + v(t)$ giving the linearised equation

$$v_t = \mathbf{A}'(u_0)v + \mathbf{A}(u_0) + g(v), \quad t > t_0, \quad v(t_0) = 0, \tag{4.20}$$

where $g(v)$ is a quadratic perturbation. Using the good semigroup $e^{\mathbf{A}'(u_0)t}$, this makes it possible to study local regularity properties of the corresponding integral equation

$$v(t) = \int_{t_0}^t e^{\mathbf{A}'(u_0)(t-s)}(\mathbf{A}(u_0) + g(v(s))) ds. \tag{4.21}$$

Note that this smoothing approach demands a fast exponential decay of solutions $v(x, t)$ as $x \rightarrow \infty$, since one needs that $v(\cdot, t) \in L^2_\rho$ (cf. [32], where C^∞ -smoothing for third-order

NDEs was also established under the exponential decay). Equation (4.21) can be used to guarantee local existence of smooth solutions of a wide class of odd-order NDEs.

Thus, we state the following conclusion to be used later on:

$$\begin{aligned} &\text{any sufficiently smooth solution } u(x, t) \text{ of (4.1), (4.2) at } t = t_0 \\ &\text{can be uniquely extended to some interval } t \in (t_0, t_0 + \nu), \nu > 0. \end{aligned} \quad (4.22)$$

4.5 Smooth deformations and δ -entropy test for solutions with shocks

The situation dramatically changes if we want to treat solutions with shocks. Namely, it is known that even for NDE-3 (1.30), the similarity formation mechanism of shocks immediately shows non-unique extensions of solutions after a typical ‘gradient’ catastrophe [20]. Therefore, we do not have a chance to get, in such an easy (or any) manner, a uniqueness/entropy result for more complicated NDEs such as (1.5) by using the δ -deformation (evolutionary smoothing) approach. However, we will continue using these ideas, which have turned out to be fruitful, in order to develop a much weaker ‘ δ -entropy test’ for distinguishing some simple shock and rarefaction waves.

Thus, given a small $\delta > 0$ and a sufficiently small bounded continuous (and, possibly, compactly supported) solution $u(x, t)$ of the Cauchy problem (4.1), satisfying (4.2), we construct its smooth δ -deformation, aiming for smoothing in a small neighbourhood of bounded shocks, as follows. Note that we deal here with simple shock configurations (mainly with one-shock structures) and do not aim to cover more general shock geometry, which can be very complicated, especially since we do not know all types of simple single-point moving shocks.

(i) We perform a smooth δ -deformation of the initial data $u_0(x)$ by introducing a suitable C^1 function $u_{0\delta}(x)$ such that

$$\int |u_0 - u_{0\delta}| < \delta. \quad (4.23)$$

If u_0 is already sufficiently smooth, this step *must be abandoned* (now and always later on). By $u_{1\delta}(x, t)$, we denote the unique local smooth solution of the Cauchy problem with the data $u_{0\delta}$, so that by (4.22), the continuous function $u_{1\delta}(x, t)$ is defined on the maximal interval $t \in [t_0, t_1(\delta))$, where we denote $t_0 = 0$ and $t_1(\delta) = \Delta_{1\delta}$. At this step, we are able to eliminate non-evolutionary (evolutionary unstable) initially posed shocks, which then create corresponding smooth rarefaction waves.

(ii) At $t = \Delta_{1\delta}$, a shock-type discontinuity (or possibly infinitely many shocks) is supposed to occur, since otherwise we extend the continuous solution by (4.22); so we perform another suitable δ -deformation of the ‘data’ $u_{1\delta}(x, \Delta_{1\delta})$ to get a unique continuous solution $u_{2\delta}(x, t)$ on the maximal interval $t \in [t_1(\delta), t_2(\delta))$, with $t_2(\delta) = \Delta_{1\delta} + \Delta_{2\delta}$ and the like. Here and in what follows, we always mean a ‘ δ -smoothing’ performed in a small neighbourhood of occurring singularities *ONLY* as discontinuous shocks.

...

We continue in this manner with suitable choices of each δ -deformations of ‘data’ at the moments $t = t_j(\delta)$; when $u_{j\delta}(x, t)$ has a shock, there exists a $t_k(\delta) > 1$ for some finite

$k = k(\delta)$, where $k(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$. It is easy to see that for bounded solutions, $k(\delta)$ is always finite. A contradiction is obtained by assuming that $t_j(\delta) \rightarrow \bar{t} < 1$ as $j \rightarrow \infty$ for arbitrarily small $\delta > 0$, meaning a kind of ‘complete blow-up’ that was excluded by the assumption of smallness of the data.

This gives a *global δ -deformation* in $\mathbb{R} \times [0, 1]$ of the solution $u(x, t)$, which is the discontinuous orbit denoted by

$$u^\delta(x, t) = \{u_{j\delta}(x, t) \text{ for } t \in [t_{j-1}(\delta), t_j(\delta)), \quad j = 1, 2, \dots, k(\delta)\}. \tag{4.24}$$

One can see that this δ -deformation construction aims at checking a kind of *evolution stability* of possible shock wave singularities and therefore to exclude those that are not entropy and evolutionarily generate smooth rarefaction waves.

Finally, by an arbitrary *smooth δ -deformation*, we will mean function (4.24) constructed by any sufficiently refined finite partition $\{t_j(\delta)\}$ of $[0, 1]$, without reaching a shock of S_- -type at some or all intermediate points $t = t_j^-(\delta)$.

We next say that given a solution $u(x, t)$, it is *stable relative smooth deformations*, or simply δ -stable (*δ -deformation-stable*), if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any finite δ -deformation of u given by (4.24),

$$\iint |u - u^\delta| < \varepsilon. \tag{4.25}$$

Recall that (4.24) is an δ -orbit and, in general, is not and cannot be aimed to represent a fixed solution in the limit $\delta \rightarrow 0$ (see below).

4.6 On δ -entropy solutions

Having checked that the local smooth solvability problem above is well posed, we now present the corresponding definition that will be applied to particular weak solutions. Recall that the metric of convergence, L^1_{loc} under present consideration for (1.30) was justified by a similarity analysis presented in Proposition 2.2. For other types of shocks and/or NDEs, the metric may be different.

Thus, under the given hypotheses, a function $u(x, t)$ is called a δ -entropy solution of the Cauchy problem (4.1) if there exists a sequence of its smooth δ -deformations $\{u^{\delta_k}, k = 1, 2, \dots\}$, where $\delta_k \rightarrow 0$, which converges in L^1_{loc} to u as $k \rightarrow \infty$.

This is slightly weaker than (but equivalent to) the condition of δ -stability.

Remark: δ -entropy solution is unique for 1D conservation law. Consider, as a typical example, (1.7) for general measurable L^1 -data. The classical Oleinik–Kruzhkov entropy theory for (1.7) defines the unique semigroup of contractions in L^1 (see [49]), i.e. for an arbitrary pair of entropy solutions $u(\cdot, t)$ and $v(\cdot, t)$, in the sense of distributions,

$$\frac{d}{dt} \|u(t) - v(t)\|_{L^1} \leq 0 \quad \text{for a.a. } t \geq 0. \tag{4.26}$$

Consider now the above δ -deformation construction of an orbit $\{u_\delta\}$ in the case in which the entropy solution $u(x, t)$ is continuous a.e. for all $t \geq 0$, i.e. shocks have zero measure. It means that $u_\delta(x, t)$ for $t \geq 0$ is smooth and essentially differs from $u(x, t)$ on a set of arbitrarily small measure $\sim \delta \rightarrow 0$. Therefore, under these (possibly non-constructive)

assumptions, (4.26) implies that any smooth δ -deformation in L^1 inevitably leads to the unique entropy solution of (1.7) as $\delta \rightarrow 0$. In other words,

$$\boxed{\text{for Euler's equation (1.7), classic entropy solutions} = \delta\text{-entropy ones.}} \quad (4.27)$$

Of course, this is just the trivial consequence of the L^1 -contractivity (4.26), which, in its turn, is induced by the ‘maximum principle’. It is also worth mentioning that, somehow, (4.26) reflects the fact that the conservation laws such as (1.7) admit the direct algebraic solution via characteristics. Indeed, the characteristic method guarantees the unique solvability in the regularity domain, while the ‘shocks cut-off’ can be performed at the necessary points by the corresponding Rankine–Hugoniot relations. Thus, the entropy conditions just describe the correct evolution from initially posed singularities (evolutionary, such ‘rarefaction waves’ cannot appear by characteristics).

Therefore, the absence of the maximum principle and absence of any characteristic-based approach for higher-order NDEs recall that a result such as (4.27) cannot be expected in principle here. The situation is even more terrible: we will show that any uniqueness/entropy result for such NDEs fails always and anyway.

4.7 δ -entropy test and non-existent uniqueness

Since, for obvious reasons, the δ -deformation construction gets rid of non-evolutionary shocks (leading to non-singular rarefaction waves), a first consequence of the construction is that it defines the δ -entropy test for solutions, which allows one, at least, to distinguish the true simple isolated shocks from smooth rarefaction waves.

In Section 5, we show that it is completely unrealistic to expect from this construction something essentially stronger in the direction of uniqueness and/or entropy-like selection of proper solutions. Though these expectations correspond well to previous classical PDE entropy-like theories, these are excessive for higher-order models, where such a universal property is not achievable at all any more. Even proving convergence for a fixed special δ -deformation is not easy at all. Thus, for particular cases, we will use the above notions with convergence along a subsequence of δ 's to classify and distinguish shocks and rarefaction waves of simple geometric configurations:

4.8 First easy conclusions of δ -entropy test

As a first application, we have the following.

Proposition 4.1 *Shocks of the type $S_-(x)$ are δ -entropy for (4.1).*

The result follows from the properties of similarity solutions (2.1), with $-t \mapsto T_t$, which, by varying the blow-up time $T \mapsto T + \delta$, can be used as their local smooth δ -deformations at any point $t \geq 0$.

Proposition 4.2 *Shocks of the type $S_+(x)$ are not δ -entropy for (4.1).*

Indeed, taking initial data $S_+(x)$ and constructing its smooth δ -deformation via the self-similar solution (3.2) with shifting $t \mapsto t + \delta$, we obtain the global δ -deformation $\{u^\delta = u_+(x, t + \delta)\}$, which goes away from S_+ .

Thus, the idea of smooth δ -deformations allows us to distinguish basic δ -entropy and non-entropy shocks *without* any use of mathematical manipulations associated with standard entropy inequalities, which, indeed, are illusive for higher-order NDEs (cf. [20]). We believe that successful applications of the δ -entropy test can be extended to any configuration with a *finite* number of *isolated* shocks. However, it is completely illusive to think that such a simple procedure could be applied to general solutions, especially since the uniqueness after singularity formation cannot be achieved in principle, as we show next.

In other words, the δ -entropy test allows us to *prohibit* formation of non δ -deformation stable shocks of type S_+ and proposes a smooth rarefaction wave instead. However, this approach cannot detect a unique shock of the opposite geometry S_- , since such a formation is principally non-unique.

5 (IV) Non-uniqueness after shock formation

Here we mainly follow the ideas from [20] applied there to NDE-3 (1.30); so we will omit some technical data and present more convincing analytic and numerical results concerning the non-uniqueness. For the hard 5D dynamical systems under consideration, numerics becomes more and more essential and unavoidable for understanding the nature of such non-unique extensions of solutions. Without loss of generality, we always deal with NDE-5 (1.5) of the fully divergent form.

5.1 Main strategy towards non-unique continuation: pessimistic conclusions

We begin with the study of new shock patterns, which are induced by other (cf. (2.1)) similarity solutions of (1.5):

$$u_-(x, t) = (-t)^\alpha f(y), \quad y = \frac{x}{(-t)^\beta}, \quad \beta = \frac{1+\alpha}{5}, \quad \text{where } \alpha \in (0, \frac{1}{4}) \text{ and} \tag{5.1}$$

$$\left. \begin{aligned} -(ff')^{(4)} - \beta f'y + \alpha f &= 0 \quad \text{in } \mathbb{R}_-, \quad f(0) = f''(0) = f^{(4)}(0) = 0, \\ f(y) &= C_0|y|^{\frac{\alpha}{\beta}}(1 + o(1)) \quad \text{as } y \rightarrow -\infty, \quad C_0 > 0. \end{aligned} \right\} \tag{5.2}$$

In this section, in order to match the key results in [20], in (2.1) and later on, we change the variables $\{g, z\} \mapsto \{f, y\}$. In Section 6, we return to the original notation. The anti-symmetry conditions in (5.2) allow us to extend the solution to the positive semi-axis $\{y > 0\}$ by the reflection $-f(-y)$ to get a global pattern.

Obviously, solutions (2.1), which are suitable for Riemann problems, correspond to the simple case $\alpha = 0$ in (5.1). It is easy to see that for positive α , the asymptotics in (5.2) ensures getting first *gradient blow-up* at $x = 0$ as $t \rightarrow 0^-$, as a weak discontinuity, where the final time profile remains locally bounded and continuous:

$$u_-(x, 0^-) = \begin{cases} C_0|x|^{\frac{\alpha}{\beta}} & \text{for } x < 0, \\ -C_0|x|^{\frac{\alpha}{\beta}} & \text{for } x > 0, \end{cases} \tag{5.3}$$

where $C_0 > 0$ is an arbitrary constant. Note that the standard ‘gradient catastrophe’, $u_x(0, 0^-) = -\infty$, then occurs in the range, which we will deal within,

$$\frac{\alpha}{\beta} < 1 \quad \text{provided that} \quad \alpha < \frac{1}{4}. \quad (5.4)$$

Thus, the wave braking (or ‘overturning’) begins at $t = 0$, and next we show that it is performed again in a self-similar manner and is described by similarity solutions

$$u_+(x, t) = t^\alpha F(y), \quad y = \frac{x}{t^\beta}, \quad \beta = \frac{1+\alpha}{5}, \quad \text{where} \quad (5.5)$$

$$\left. \begin{aligned} -(FF')^{(4)} + \beta F'y - \alpha F &= 0 \quad \text{in } \mathbb{R}_-, \\ F(0) = F_0 > 0, \quad F(y) &= C_0 |y|^{\frac{5}{\beta}} (1 + o(1)) \quad \text{as } y \rightarrow -\infty, \end{aligned} \right\} \quad (5.6)$$

where the constant $C_0 > 0$ is fixed by blow-up data (5.3). The asymptotic behaviour as $y \rightarrow -\infty$ in (5.6) guarantees the continuity of the global discontinuous pattern (with $F(-y) \equiv -F(y)$) at the singularity blow-up instant $t = 0$, so that

$$u_-(x, 0^-) = u_+(x, 0^+) \quad \text{in } \mathbb{R}. \quad (5.7)$$

Then any suitable couple $\{f, F\}$ defines a global solution $u_\pm(x, t)$, which is continuous at $t = 0$, and then it is called an *extension pair*. It was shown in [20] that for the typical NDE-3s, the pair is not uniquely determined, and there exist infinitely many shock-type extensions of the solution after blow-up at $t = 0$. We are going to describe a similar non-uniqueness phenomenon for the NDE-5s such as (1.5).

It is worth mentioning that for conservation laws such as (1.7), such an extension pair $\{f, F\}$ is *always unique* (see the similarity analysis in [20, Section 4]). Of course, this is not surprising because of the existing Oleinik–Kruzhkov classic uniqueness–entropy theory [30, 36]. Note again that any sufficient multiplicity of extension pairs $\{f, F\}$, obtained via small micro-scale blow-up analysis of the PDEs, would always lead to a principle non-uniqueness; so this approach could be referred to as a ‘uniqueness test’.

A first immediate consequence of our similarity blow-up/extension analysis is as follows:

$$\text{in the CP, formation of shocks for NDE (1.5) can lead to non-uniqueness.} \quad (5.8)$$

The second conclusion is subtler and is based on the fact that for some initial data at $t = 0$ (i.e. created by single-point *gradient blow-up* as $t \rightarrow 0^-$), the whole admitted solution set for $t > 0$ does not contain any ‘minimal’, ‘maximal’, ‘extremal’ in any reasonable sense, or any isolated points, which might play a role of a unique ‘entropy’ one chosen by introducing a hypothetical entropy inequalities, conditions or otherwise. If this is true for the whole set of such weak solutions of (1.5) with initial data (5.3), then for the Cauchy problem,

$$\text{there exists no general ‘entropy mechanisms’ to choose a unique solution.} \quad (5.9)$$

Actually, overall, (5.8) and (5.9) show that the problem of uniqueness of weak solutions for NDEs such as (1.5) *cannot be solved in principal*.

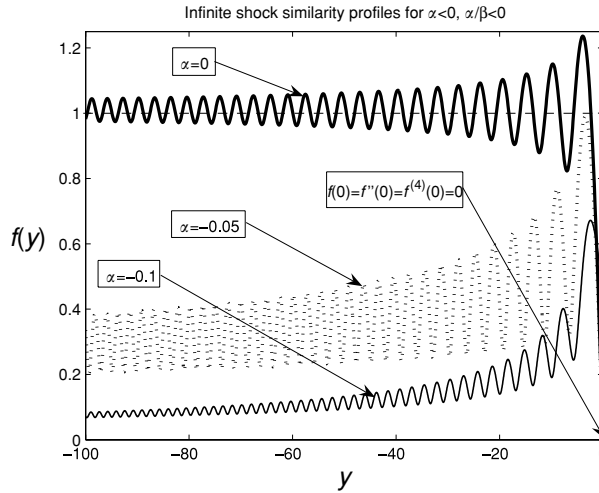


FIGURE 9. Infinite shock similarity profiles as solutions of (5.2) for $\alpha < 0$: $\alpha = -0.05$ and $\alpha = -0.1$.

On the other hand, in a free-boundary problem (FBP) setting by adding an extra suitable condition on shock lines, the problem might be well posed with a unique solution, though proofs can be very difficult. We refer again to a more detailed discussion of these issues for NDE-3 (1.30) in [20]. Though we must admit that for NDE-5 (1.5), which induces 5D dynamical systems for the similarity profiles (and hence 5D phase spaces), those non-uniqueness and non-entropy conclusions are more difficult and not that clear as for NDE-3s; so some of their aspects do unavoidably remain questionable and even open.

Hence, the *non-uniqueness* in the CP is a non-removable issue of PDE theory for higher-order degenerate non-linear odd-order equations (and possibly not only for those). The non-uniqueness of solutions of (1.5) has some pure dimensional natural features and, more precisely, is associated with the dimensions of ‘good’ and ‘bad’ asymptotic bundles of orbits in the 5D phase space of ODE (5.6).

5.2 Infinite shock similarity solutions for $\alpha < 0$

Let us first note that the blow-up solutions (5.1) represent an effective way to describe other types of singularities with *infinite shocks*. Namely, assuming that

$$\alpha < 0 \quad \text{and} \quad \frac{\alpha}{\beta} < 0, \tag{5.10}$$

we again obtain the same ‘data’ (5.3) but now $u_-(0, 0^-) = \infty$. We do not study in any detail such interesting new singularity phenomena and present Figure 9, which shows that such infinite shock similarity profiles do exist. For comparison, we indicate the standard S_- -type profile for $\alpha = 0$, which coincides with that in Figure 3(d).

For NDE-3 such as (1.30), the infinite shock similarity solutions in the range (5.10) were studied in [22, Section 4] in sufficient detail.

5.3 Gradient blow-up similarity solutions

Consider the blow-up ODE problem (5.2), which is a difficult one, with a 5D phase space. Note that by invariant scaling (2.22), it can be reduced to a fourth-order ODE with a also even more complicated non-linear operator composed from too many polynomial terms; so we do not rely on that and work in the original phase space. Therefore, some more delicate issues on, say, uniqueness of certain orbits become very difficult or even remain open, though some more robust properties can be detected rigorously. We will also use numerical methods for illustrating and even justifying some of our conclusions. As before, for the fifth-order equations such as (5.2), this and further numerical constructions are performed by MatLab with the standard `ode45` solver therein.

Let us describe the necessary properties of orbits $\{f(y)\}$ we are interested in. Firstly, it follows from the conditions in (5.2) that for $y \approx 0^-$,

$$\text{the set of proper orbits is 2D parameterised by } f_1 = f'(0) < 0 \text{ and } f_3 = f'''(0). \quad (5.11)$$

Secondly, and on the other hand, the necessary behaviour at infinity is as follows:

$$f(y) = C_0 |y|^{\frac{5\alpha}{1+\alpha}} (1 + o(1)) \quad \text{as } y \rightarrow -\infty \quad \left(\frac{5\alpha}{1+\alpha} = \frac{\alpha}{\beta} \right), \quad (5.12)$$

where $C_0 > 0$ is an arbitrary constant by scaling (2.22). It is key to derive the whole 4D bundle of solutions satisfying (5.12). This is done by the linearisation as $y \rightarrow -\infty$:

$$\begin{aligned} f(y) &= f_0(y) + Y(y), \quad \text{where } f_0(y) = C_0 (-y)^{\frac{\alpha}{\beta}} \\ \implies -C_0 ((-y)^{\frac{\alpha}{\beta}} Y)^{(5)} + \beta Y'(-y) + \alpha Y + \frac{1}{2} (f_0^2(y))^{(5)} + \dots &= 0. \end{aligned} \quad (5.13)$$

By WKBJ-type asymptotic techniques in the ODE theory, solutions of (5.13) have a standard exponential form with the characteristic equation

$$Y(y) \sim e^{a(-y)^\gamma}, \quad \gamma = 1 + \frac{1}{4} \left(1 - \frac{\alpha}{\beta} \right) > 1 \quad \implies \quad C_0 (\gamma a)^4 = \beta, \quad (5.14)$$

which has three roots with non-positive real parts, $\text{Re } a_k \leq 0$, where $a_1 < 0$ is real and conjugate $a_{2,3} \in i\mathbb{R}$. Hence, we conclude that

$$\text{as } y \rightarrow -\infty, \text{ bundle (5.12) is 4D (including } C_0). \quad (5.15)$$

The behaviour corresponding to bundle (5.15) gives the desired asymptotics. Indeed, by (5.12), we have the gradient blow-up behaviour at a single point: for any fixed $x < 0$, as $t \rightarrow 0^-$, where $y = x/(-t)^\beta \rightarrow -\infty$, uniformly on compact subsets,

$$u_-(x, t) = (-t)^\alpha f(y) = (-t)^\alpha C_0 \left| \frac{x}{(-t)^\beta} \right|^{\frac{\alpha}{\beta}} (1 + o(1)) \rightarrow C_0 |x|^{\frac{5\alpha}{1+\alpha}}. \quad (5.16)$$

Let us explain some other crucial properties of the phase space, now meaning ‘bad bundles’ of orbits. First, these are the fast-growing solutions according to the explicit solution

$$f_*(y) = -\frac{y^5}{15120} > 0 \quad \text{for } y \ll -1. \quad (5.17)$$

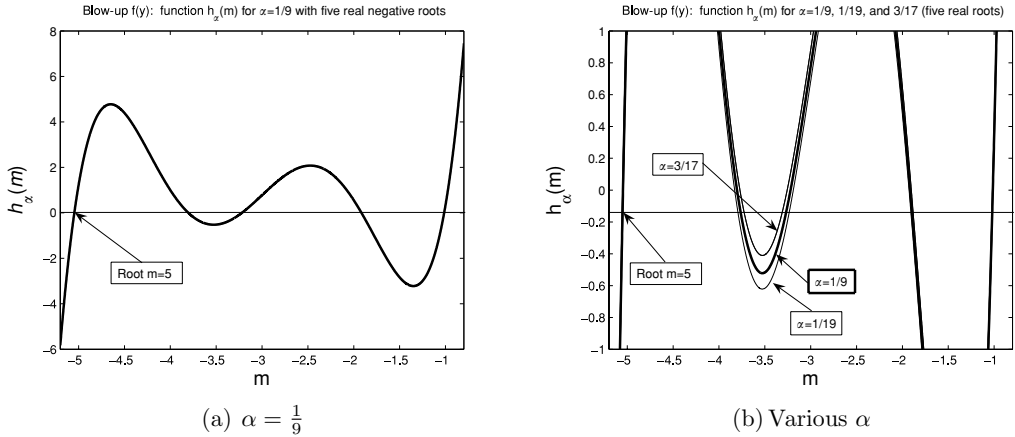


FIGURE 10. The polynomial $h_{\alpha}(m)$ in (5.19) for various $\alpha \in (0, \frac{1}{4})$: five negative roots.

Analogous to (5.13), we compute the whole bundle about (5.17):

$$f(y) = f_*(y) + Y(y) \implies \frac{1}{15120} (y^5 Y)^{(5)} - \beta Y' y + \alpha Y + \dots = 0. \quad (5.18)$$

This Euler equation has the following solutions with the characteristic polynomial:

$$Y(y) = y^m \implies h_{\alpha}(m) \equiv \frac{(m+1)(m+2)(m+3)(m+4)(m+5)}{15120} - \beta m + \alpha = 0. \quad (5.19)$$

One root $m = -5$ is obvious, which gives the solution (5.17). It turns out that this algebraic equation has precisely five negative real roots for α from range (5.4), as Figure 10 shows. Actually, Figure 10(b) explains that the graphs are rather slightly dependent on α . Thus

$$\text{the bundle about (5.17) is 5D.} \quad (5.20)$$

Second, there exists a bundle of positive solutions vanishing at some finite $y \rightarrow y_0^+ < 0$ with the behaviour (this bundle occurs from both sides, as $y \rightarrow y_0^{\pm}$ to be also used)

$$f_1(y) = A\sqrt{|y - y_0|} (1 + o(1)), \quad A > 0, \quad (5.21)$$

which is 4D, which also can be shown by linearisation about (5.21). Indeed, the linearised operator contains the leading term

$$-A^2(\sqrt{|y - y_0|} Y)^{(5)} + \dots = 0 \implies Y(y) \sim |y - y_0|^{\frac{3}{5}}, |y - y_0|^{\frac{5}{5}}, |y - y_0|^{\frac{7}{5}}, \quad (5.22)$$

which together with the parameter $y_0 < 0$ yields that

$$\text{the bundle about (5.21) is 4D.} \quad (5.23)$$

Thus, (5.11), (5.15), (5.20) and (5.23) prescribe key aspects of the 5D phase space we are dealing with. To get a global orbit $\{f(y), y \in \mathbb{R}_-\}$ as a connection of the proper bundles (5.11) and (5.15), it is natural to follow the strategy of ‘shooting from below’ by avoiding

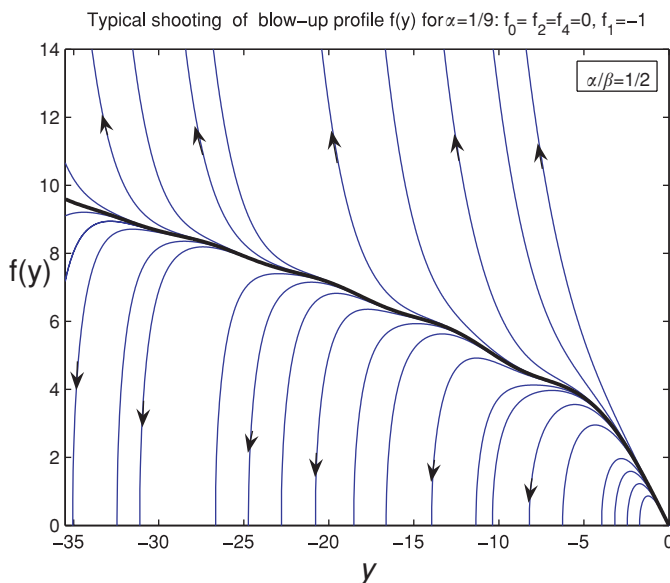


FIGURE 11. (Color online) The shooting strategy of a blow-up similarity profile $f(y)$ for $\alpha = \frac{1}{9}$, with data $f(0) = f''(0) = f^{(4)}(0) = 0$ and $f'(0) = 1$; the shooting parameter is $f'''(0) = 0.0718040128557 \dots$

the bundle (5.21), (5.23), i.e. using the parameters $f_{1,3}$ in (5.11), to obtain

$$y_0 = -\infty. \quad (5.24)$$

It is not difficult to see that this profile $f(y)$ will belong to bundle (5.15). The proof of such a 2D shooting strategy can be done by standard arguments. By scaling (2.22), we can always reduce the problem to a 1D shooting (recall that $f_0 = f_2 = f_4 = 0$ already):

$$f_1 \equiv f'(0) = -1 \quad \text{and} \quad f_3 \equiv f'''(0) \text{ is a parameter.} \quad (5.25)$$

By the above asymptotic analysis of the 5D phase space, it follows that

- (I) for $f_3 \ll -1$ the orbit belongs to the bundle about (5.17), and
- (II) for $f_3 \gg 1$ the orbit vanishes at finite y_0 along (5.21).

Hence, by continuous dependence, we obtain a solution $f(y)$ by the min-max principle (plus some usual technical details that can be omitted). Before stating the result, for convenience, in Figure 11, obtained by the ode45 solver, we explain how we are going to justify existence of a proper blow-up shock profile $f(y)$ (cf. Figure 4).

Thus, we fix the above speculations as follows.

Proposition 5.1 (i) In range (5.4), problem (5.2) admits a shock profile $f(y)$.

We have the following expectation: (ii) $f(y)$ is unique up to scaling (2.22) and is positive for $y < 0$. This remains an open problem that was confirmed numerically. In [20], for the NDE-3 (1.30), the phase space is 3D, and a full proof is available.

In fact, this is a rather typical result for higher-order dynamical systems. For example, we refer to a similar and not less complicated study of a fourth-order ODE [24], where

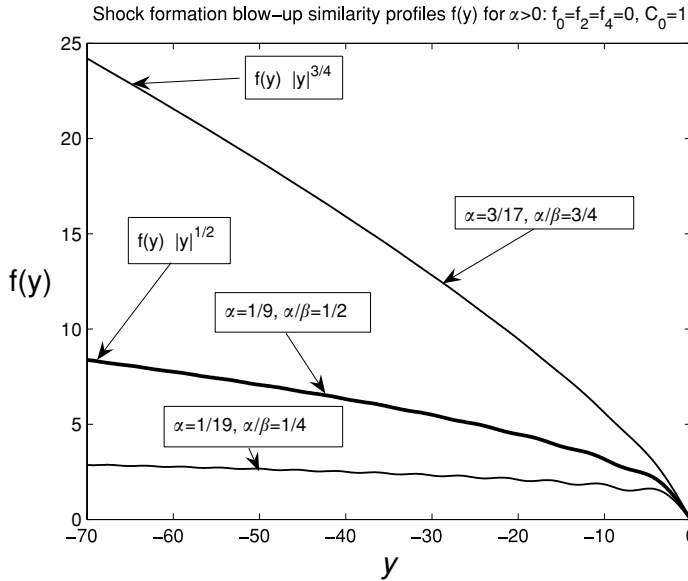


FIGURE 12. The odd blow-up similarity profiles $f(y)$ in \mathbb{R}_- with $\alpha = \frac{1}{9}, \frac{1}{19}$ and $\frac{3}{17}$.

existence and *uniqueness* of a positive solution of the radial bi-harmonic equation with source

$$\Delta_r^2 u = u^p \quad \text{for } r = |x| > 0, \quad u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u(\infty) = 0 \quad (5.26)$$

was proved in the supercritical Sobolev range $p > p_{\text{Sob}} = \frac{N+4}{N-4}$, $N > 4$. Here, analogously, there exists a single shooting parameter, which is the second derivative at the origin $u_2 = u''(0)$; the value $u_0 = u(0) = 1$ is fixed by a scaling symmetry. Proving the uniqueness of such a solution in [24] is not easy and leads to essential technicalities, which the attentive reader can consult in case of necessity. Fortunately, we are not interested in any uniqueness of such kind. Instead of the global behaviour such as (5.17), (5.26) admits the blow-up one governed by the principal operator $u^{(4)} + \dots = u^p$ ($u \rightarrow +\infty$). The solutions vanishing at finite point otherwise can be treated as in family **(I)**.

MORE NUMERICS BY `bvp4c`. We next use more advanced and enhanced numerical methods towards the existence (and uniqueness-positivity; see (ii)) of $f(y)$. Figure 12 shows blow-up profiles, with $f(0) = 0$, constructed by a different method (via the solver `bvp4c`) for convenient values $\alpha = \frac{1}{9}, \frac{1}{19}$ and $\frac{3}{17}$. Note the clear oscillatory behaviour of such patterns that is induced by the complex roots of the characteristic equation (5.14).

COLLAPSE OF SHOCKS: ‘BACKWARD NON-UNIQUENESS’. This new phenomenon is presented in Figure 13, which shows the shooting from $y = 0^-$ for

$$\alpha = \frac{1}{9} \implies \frac{\alpha}{\beta} = \frac{1}{2}. \quad (5.27)$$

This again illustrates the actual strategy in proving Proposition 5.1. However, though the phase space looks similar, note that here, as an illustration of another important evolution

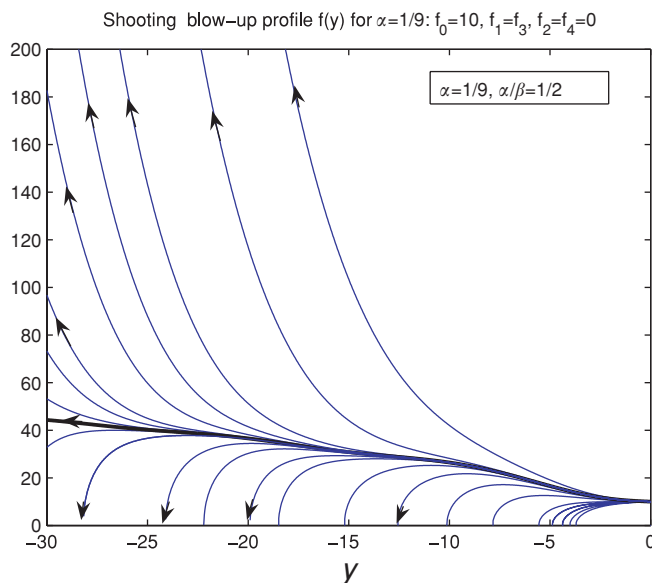


FIGURE 13. (Color online) Shooting the blow-up profile $f(y)$ for $\alpha = \frac{1}{9}$: $f(0) = 10$, $f''(0) = f^{(4)}(0) = 0$ and the shooting parameter is $f'(0) = f'''(0)$.

phenomenon, we solve the problem with $f(0) \neq 0$, so that there exists a non-zero jump of $u_-(x, t)$ at $x = 0$ denoted by $[\cdot]$:

$$f(0) = f_0 = 10 \implies [u_-(0, t)] = 2f_0(-t)^\alpha \rightarrow 0 \quad \text{as } t \rightarrow 0^-. \quad (5.28)$$

Therefore, this similarity solution describes *collapse of a shock wave* as $t \rightarrow 0^-$.

More numerical results of such types are presented in Figures 14 and 15, where we use other boundary conditions at $y = 0$. Note that being extended for $y > 0$ in the anti-symmetric way, by $-f(-y)$, this will give a proper shock wave solution with the nil speed of propagation (see the Rankine–Hugoniot condition (5.42) below).

As a whole, since all these blow-up profiles satisfy the necessary behaviour as $y \rightarrow -\infty$ as indicated in (5.2), these create as $t \rightarrow 0^-$ the same initial data (5.3). This confirms the following phenomenon of **backward non-uniqueness**: *initial data (5.3) with gradient blow-up at $x = 0$ can be created by an infinite number (in fact by a 2D subset parameterised, say, by $\{f_0, f_1\}$) of various self-similar solutions (5.1).*

Indeed, such a non-uniqueness is directly associated with the fact that because of (5.15), the proper asymptotic bundle as $y \rightarrow -\infty$ is 3D (for a fixed $C_0 > 0$, we have to subtract the dimension via the scaling invariance (2.22)). Therefore, roughly speaking, shooting from $y = 0^-$ with five parameters $f_0 = f(0), \dots, f_4 = f^{(4)}(0)$ allows a 2D ($2 = 5 - 3$) subset of solutions $f(y)$ with shocks at $y = 0$. A full justification of such a conclusion requires a more careful analysis of the phase space including geometry of two ‘bad’ bundles, which we do not perform here and concentrate on other more important solutions and true non-uniqueness phenomena.

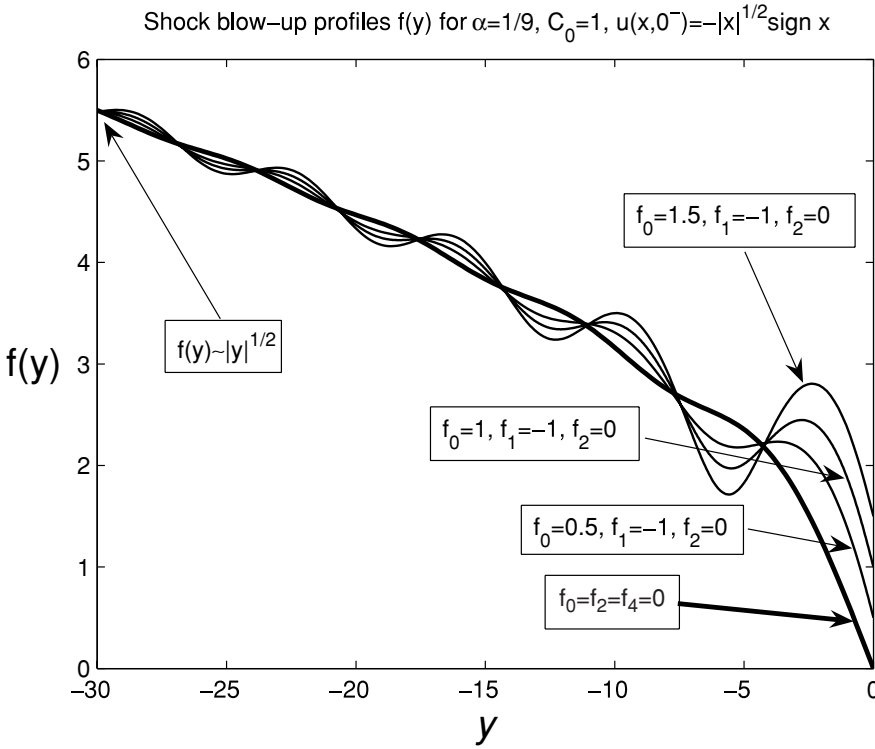
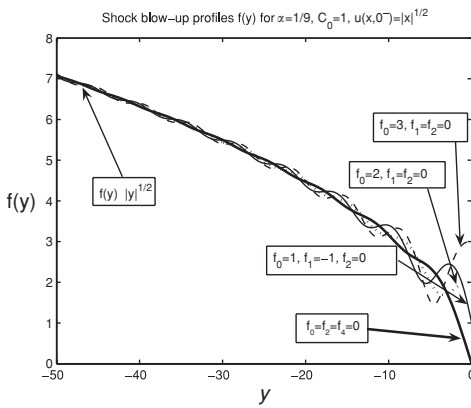
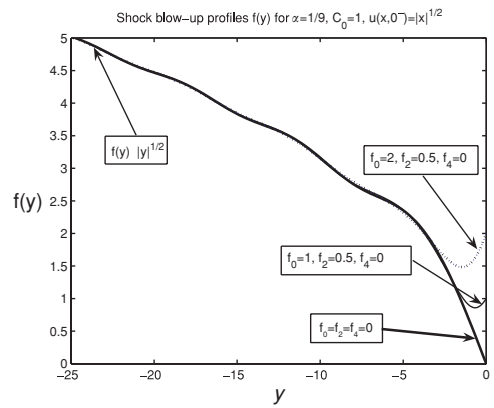


FIGURE 14. Blow-up profiles $f(y)$ for $\alpha = \frac{1}{9}$, with $f'(0) = -1, f''(0) = 0$; $f(0) \in [0, 1.5]$ is a parameter.



(a) $f'(0) = f''(0) = 0$



(b) $f''(0) = \frac{1}{2}, f^{(4)}(0) = 0$

FIGURE 15. (Color online) Blow-up profiles $f(y)$ for $\alpha = \frac{1}{9}$, for $f'(0) = f''(0) = 0$ (a) and $f''(0) = \frac{1}{2}, f^{(4)}(0) = 0$ (b); $f(0)$ is a parameter.

STATIONARY SOLUTIONS WITH A ‘WEAK SHOCK’. The ODE in (5.2) and hence PDE (1.5) admit a number of simple continuous ‘stationary’ solutions. For example, consider

$$\alpha = \frac{1}{9}, \frac{\alpha}{\beta} = \frac{1}{2} : \hat{f}(y) = \sqrt{|y|} \operatorname{sign} y \quad \text{and} \quad \hat{u}(x, t) \equiv \pm \sqrt{|x|} \operatorname{sign} x. \quad (5.29)$$

Note that these *are not* weak solutions of the stationary equation

$$\frac{1}{2} (u^2)_{xxxxx} = 0 \quad \text{in} \quad \mathcal{D}'. \quad (5.30)$$

The classic stationary solution of (5.30) $\hat{u}(x, t) = \pm x^2$ is smoother at $x = 0$.

We will show that such ‘weak stationary shocks’ as in (5.29) also lead to non-uniqueness.

Remark: an exact solution for a critical α . One can see that the quadratic operator $\mathbf{B}(f) = (ff')^{(4)}$ in (5.2) admits the following polynomial invariant subspace:

$$W_6 = \operatorname{Span}\{1, y, y^2, y^3, y^4, y^5\} \implies \mathbf{B}(W_6) \subseteq W_6.$$

Restricting ODE (5.2) to W_6 yields an algebraic system, which admits an exact solution for the following value of the *critical* α_c :

$$\alpha = \alpha_c = \frac{17}{84} = 0.202381 \dots \implies \exists f(y) = Cy - \frac{4!}{9!} y^5, \quad C \in \mathbb{R}. \quad (5.31)$$

Since $\alpha_c > 0$, it does not deliver a ‘saw-type’ blow-up profile (having infinite number of positive humps) as it used to be for NDE-3 (1.30) for $\alpha_c = -\frac{1}{10}$ (see [22, Section 4]).

5.4 Non-uniqueness of similarity extensions beyond blow-up

As in [20] for NDEs-3, a discontinuous shock wave extension of blow-up similarity solutions (5.1) and (5.2) is assumed to be done by using the global ones (5.5) and (5.6). Actually, this leads to watching a whole 5D family of solutions parameterised by their Cauchy values at the origin:

$$F(0) = F_0 > 0, \quad F'(0) = F_1 < 0, \quad F''(0) = F_2, \quad F'''(0) = F_3, \quad F^{(4)}(0) = F_4. \quad (5.32)$$

Thus, unlike (5.11), the proper bundle in (5.32) is 5D. Note that at $y = -\infty$, the solution must have the form

$$F(y) = C_0 |y|^{\frac{5x}{1+x}} (1 + o(1)) \quad \text{as} \quad y \rightarrow -\infty \quad (C_0 > 0). \quad (5.33)$$

As above, the 5D phase space for the ODE in (5.6) has two stable ‘bad’ bundles:

- (I) Positive solutions with ‘singular extinction’ in finite y , where $F(y) \rightarrow 0$ as $y \rightarrow y_0^+ < 0$. This is an unavoidable singularity following from the degeneracy of the equations with the principal term $FF^{(5)}$ leading to the singular potential $\sim \frac{1}{F}$. As in (5.22), this bundle is 4D.
- (II) Negative solutions with the fast growth (cf. (5.17)),

$$F_*(y) = \frac{y^5}{15120} (1 + o(1)) \rightarrow -\infty \quad \text{as} \quad y \rightarrow -\infty. \quad (5.34)$$

The characteristic polynomial is the same as in (5.19), so that the bundle is 5D (cf. (5.20)).

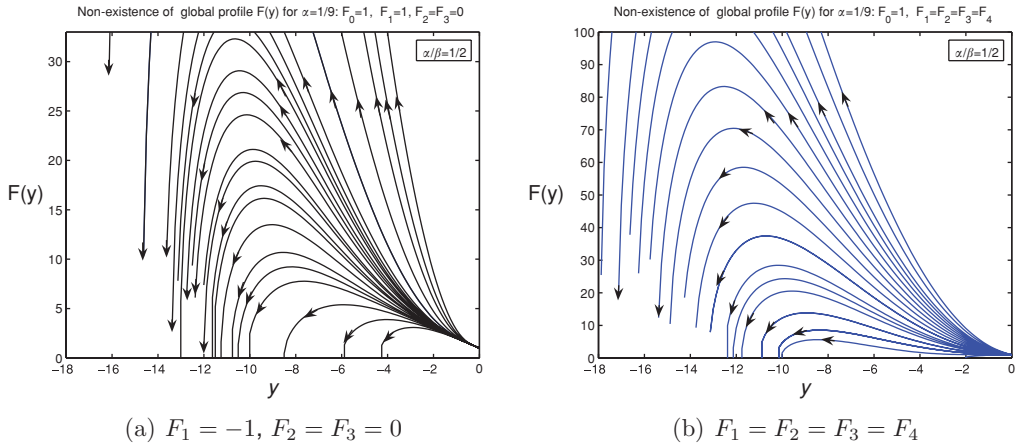


FIGURE 16. (Color online) Unsuccessful examples of 1D shooting of $F(y)$ of (5.6) from $y = 0^-$.

Both sets of such solutions are *open* by the standard continuous dependence of solutions of ODEs on parameters. The whole bundle of solutions satisfying (5.12) is obtained by linearisation as $y \rightarrow -\infty$ in (5.6):

$$\begin{aligned}
 f(y) &= F_0(y) + Y(y), \quad \text{where } F_0(y) = C_0(-y)^{\frac{\alpha}{\beta}} \\
 \implies & -C_0((-y)^{\frac{\alpha}{\beta}} Y)^{(5)} - \beta Y'(-y) - \alpha Y + \frac{1}{2} (F_0^2(y))^{(5)} + \dots = 0.
 \end{aligned}
 \tag{5.35}$$

The WKBJ method now leads to a different characteristic equation:

$$Y(y) \sim e^{\alpha(-y)^\gamma}, \quad \gamma = 1 + \frac{1}{4} \left(1 - \frac{\alpha}{\beta}\right) > 1 \implies C_0(\gamma a)^4 = -\beta,
 \tag{5.36}$$

so that there exist just two complex conjugate roots with $\text{Re} \leq 0$, and hence, unlike (5.15),

$$\text{the bundle (5.12) of global orbits } \{F(y)\} \text{ is 3D.}
 \tag{5.37}$$

However, the geometry of the whole phase space and the structure of key asymptotic bundles change dramatically in comparison with the blow-up cases, so that the standard shooting of *positive* global profiles $F(y)$ by the ode45 solver yields no encouraging results. We refer to Figure 16, which illustrates typical negative results of a standard shooting. Figure 17 looks better and presents shooting a kind of ‘separatrix’, which however does not belong to the necessary family as in (5.6). Actually, this means that a 1D shooting is not possible, and as we will see, there occurs a more complicated 2D one, i.e. using two parameters.

Therefore, we now use the bvp4c solver, and this gives the following results for case (5.27), with $C_0 = 1$, as usual. Namely, we show that there are two parameters, say

$$F_0 = F(0) \quad \text{and} \quad F_1 = F'(0),
 \tag{5.38}$$

such that for their arbitrary values from some connected subset in \mathbb{R}^2 , including all points with $F_0 > 0$ and $F_1 \leq 0$, problem (5.6) admits a solution. This is confirmed in Figure 18

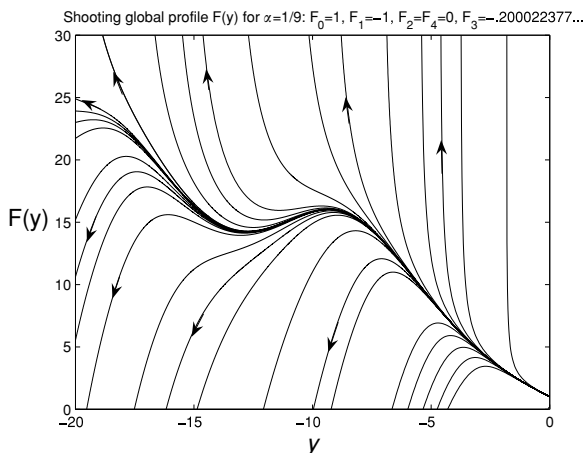


FIGURE 17. Unsuccessful 1D shooting of $F(y)$ satisfying (5.6) from $y = 0^-$, with conditions $F(0) = 1, F'(0) = -1, F''(0) = F^{(4)}(0) = 0$ and with $F'''(0) = -0.200022377\dots$ being a parameter.

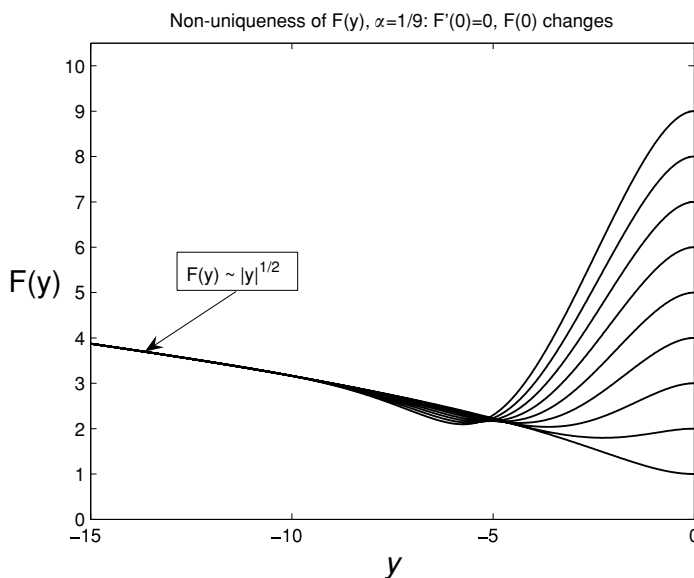


FIGURE 18. Global profiles $F(y)$ of (5.6) for $\alpha = \frac{1}{9}, C_0 = 1, F'(0) = 0$, with $F(0) \in [1, 9]$ being a parameter.

for the case $F'(0) = 0$ and in Figure 19 for the cases (a) $F'(0) = +1$ and (b) $F'(0) = -1$. Obviously, all these profiles are different and exhibit fast and ‘non-oscillatory’ convergence as $y \rightarrow -\infty$ to the ‘good’ bundle as in (5.6) with $C_0 = 1$.

Finally, carefully analysing the dimensions of all the ‘bad’ and ‘good’ asymptotic bundles indicated in (i) and (ii) above, plus (5.37), unlike the result for blow-up profiles in Proposition 5.1, we arrive at an even stronger non-uniqueness.

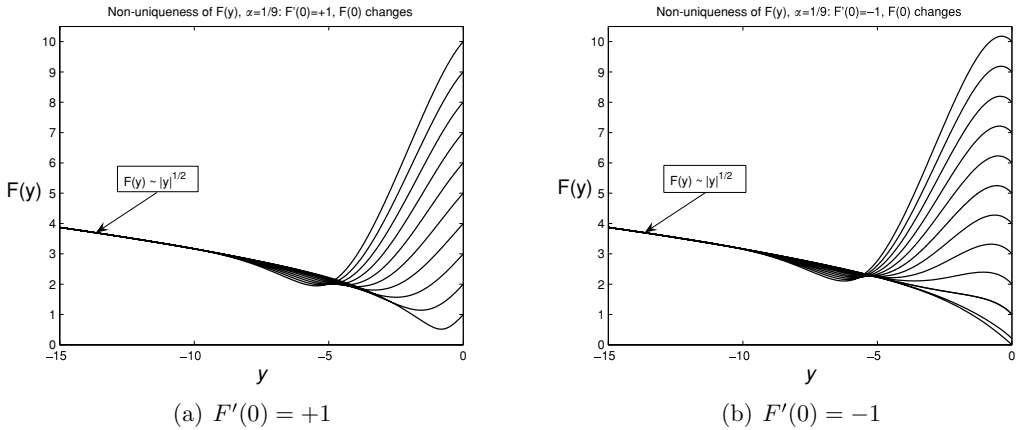


FIGURE 19. Global profiles $F(y)$ of (5.6) for $\alpha = \frac{1}{9}$, $C_0 = 1$ and $F'(0) = +1$ (a), $F'(0) = -1$ (b), with $F(0) \in [0, 10]$ being a parameter.

Proposition 5.2 *In range (5.4) and any fixed $C_0 > 0$, problem (5.6) admits a 2D family of solutions, which can be parameterised by F_0 and F_1 .*

Recall again that for any hope of uniqueness, the extension pair $\{f, F\}$ must be unique (or at least its subset should contain some ‘minimal’ and/or isolated points as proper candidates for unique entropy solutions) for any fixed constant $C_0 > 0$, which defines the ‘initial data’ (5.3) at the blow-up time $t = 0^-$. This actually happens for the Euler equation (1.7) (see [20, Section 4], where the similarity analysis is indeed easier and is reduced to algebraic manipulations but is not that straightforward anyway even for such a ‘first-order NDE’).

5.5 ‘Initial non-uniqueness’

A new ‘non-uniqueness’ phenomenon is achieved for the values of parameters

$$F(0) = F_0 < 0 \quad \text{and} \quad F'(0) = F_1 \geq 0. \tag{5.39}$$

Figures 20(a) and (b) shows such shock profiles leading to the non-uniqueness, obtained by a standard 1D shooting via the `ode45` solver. Here, two similarity profiles $F(y)$ are obtained via distinct types of shooting: relative to the parameter $F'(0) = F'''(0)$ in (a) and relative to $F''(0)$ in (b).

The proof of existence of such profiles F is based on the same geometric arguments as that of Proposition 2.3 (with the evident change of the geometry of the phase space). These two different profiles posed into the similarity solutions (5.5) show a non-unique way to get solutions with initial data ($C_0 = 1$ by scaling) at $t = 0^+$,

$$u_0(x) = |x|^{\frac{\alpha}{\beta}} \text{sign } x \quad \text{in } \mathbb{R}, \tag{5.40}$$

which already have a gradient blow-up singularity at $x = 0$. This is another potential type of non-uniqueness in the Cauchy problem for (1.5), showing the non-unique way of formation of shocks from weak discontinuities, including the stationary ones as in (5.29).

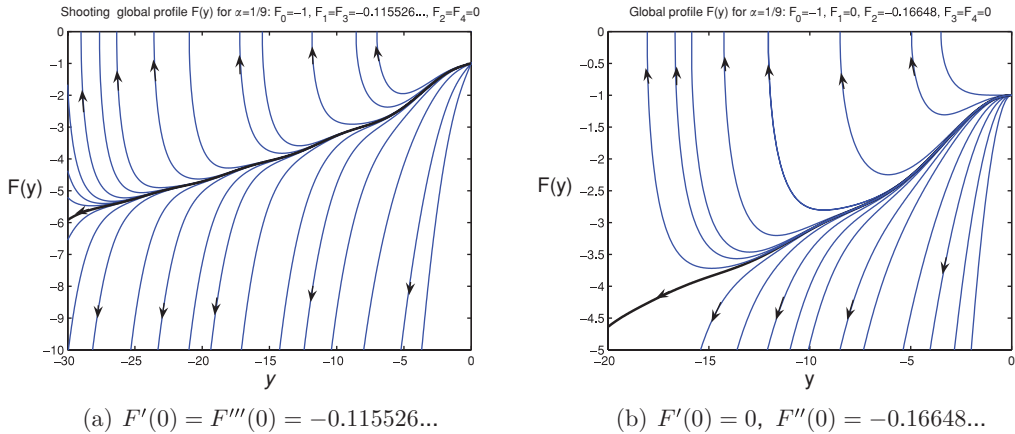


FIGURE 20. (Color online) Shooting a proper solution $F(y)$ of (5.6) for $\alpha = \frac{1}{9}$ with data $F(0) = -1, F_4 = 0$ and $F'(0) = F'''(0) = -0.115526 \dots$ (shooting parameter), $F_2 = 0$ (a), and $F(0) = -1, F'(0) = 0, F_2 = -0.16648 \dots$ (shooting parameter), $F_3 = F_4 = 0$ (b).

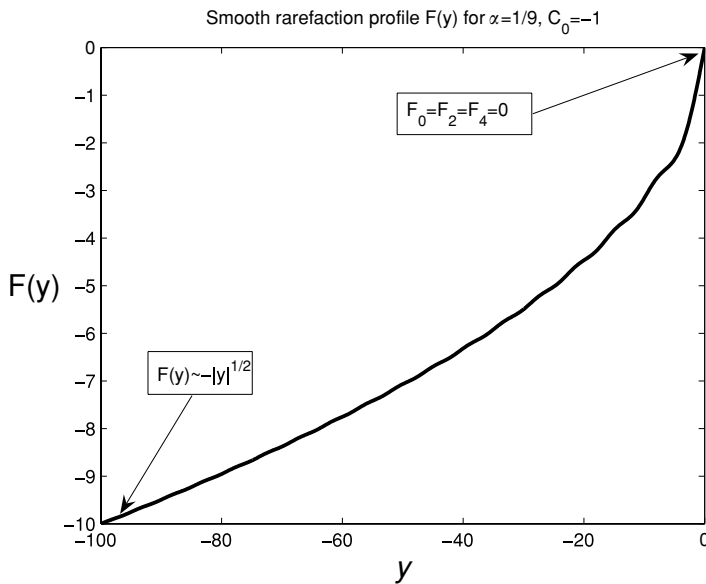


FIGURE 21. Global rarefaction profile $F(y)$ of (5.6) for $\alpha = \frac{1}{9}, C_0 = -1; F(0) = F''(0) = F^{(4)}(0) = 0$.

However, bearing in mind that Proposition 4.2 says that the shocks of S_+ -type are not δ -entropy (i.e. not stable relative small smooth deformations), one can expect that the shocks as in (5.39) are also unstable. Indeed, smooth extensions of weak pointwise shocks (5.40) via rarefaction self-similar waves given by (5.6) are δ -entropy. In Figure 21, we show such a global rarefaction profile $F(y)$ for $\alpha = \frac{1}{9}$, which describes smooth collapse of the ‘weak equilibrium’ (5.29). One can see that such rarefaction profiles satisfy $F(y) \equiv -f(y)$, where f are the corresponding blow-up ones, as shown in Figure 12 for various α .

Overall, it seems that the δ -entropy test rules out such an ‘initial non-uniqueness’ with the data of type S_+ as in (5.40), where a unique smooth rarefaction extension is available. On the other hand, for other classes of data of S_- -shape (according to Proposition 4.1), such a non-uniqueness can take place (see Section 5.4).

5.6 More on the non-uniqueness and well-posedness of FBPs

The non-uniqueness (5.8) in the Cauchy problem (1.5), (5.3) is as follows: any $F(y)$ yields the self-similar continuation (5.5), with the behaviour of the jump at $x = 0$ (profiles $F(y)$ as in Figure 20)

$$-[u_+(x, t)]|_{x=0} \equiv -(u_+(0^+, t) - u_+(0^-, t)) = 2F_0t^\alpha < 0 \quad \text{for } t > 0. \tag{5.41}$$

In the similarity ODE representation, this non-uniqueness has a pure geometric-dimensional origin associated with the dimension and mutual geometry of the good and bad asymptotic bundles of the 5D phase spaces of both blow-up and global equations. Since these shocks are stationary, the corresponding Rankine–Hugoniot condition on the speed λ of the shock propagation

$$\lambda = \frac{[(uu_x)_{xxx}]|_{x=0}}{[u]} \equiv \frac{[(u^2)_{xxxx}]|_{x=0}}{2[u]} = \frac{[(f^2)^{(4)}]|_{y=0}}{2[f]} = 0, \tag{5.42}$$

is valid by anti-symmetry. As usual, (5.42) is obtained by integration of (1.30) in a small neighbourhood of the shock. The Rankine–Hugoniot condition does not assume any novelty and is a corollary of integrating the PDE about the line of discontinuity.

Moreover, the Rankine–Hugoniot condition (5.42) also indicates another origin of non-uniqueness: a *symmetry breaking*. Indeed, the solution for $t > 0$ is not obliged to be an odd function of x ; so the self-similar solution (5.5) for $x < 0$ and $x > 0$ can be defined using 10 different parameters $\{F_0^\pm, \dots, F_4^\pm\}$, and the only extra condition one needs is the Rankine–Hugoniot one:

$$[(FF')'''](0) = 0, \quad \text{i.e.} \quad F_0^-F_4^- + 4F_1^-F_3^- + 3(F_2^-)^2 = F_0^+F_4^+ + 4F_1^+F_3^+ + 3(F_2^+)^2. \tag{5.43}$$

This algebraic equation with 10 unknowns admits many other solutions rather than the obvious anti-symmetric one:

$$F_0^- = -F_0^+, \quad F_1^- = F_1^+, \quad F_2^- = -F_2^+, \quad F_3^- = F_3^+ \quad \text{and} \quad F_4^- = -F_4^+.$$

Finally, we note that the uniqueness can be restored by posing specially designed conditions on moving shocks, which provide the overall guarantee of the unique solvability of the algebraic equation in (5.43) and hence the unique continuation of the solution beyond blow-up. This construction is analytically similar to that for NDEs–3 (1.30) in [20].

6 Shocks for an NDE obeying the Cauchy–Kovalevskaya theorem

In this short section, we touch on the problem of formation of shocks for NDEs that are higher order in time. Instead of studying the PDEs such as (cf. [18,22])

$$u_{tt} = -(uu_x)_{xxxx}, \quad u_{ttt} = -(uu_x)_{xxxx}, \quad (6.1)$$

we consider NDE (1.11) that is fifth order in time; it exhibits certain simple and, at the same time, exceptional properties. Writing it for $W = (u, v, w, g, h)^T$ as

$$\left. \begin{aligned} u_t &= v_x, \\ v_t &= w_x, \\ w_t &= g_x, \\ g_t &= h_x, \\ h_t &= uu_x, \end{aligned} \right\} \text{ or } W_t = AW_x, \quad \text{with the matrix } A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ u & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (6.2)$$

(1.11) becomes a first-order system with the characteristic equation for eigenvalues

$$-\lambda^5 + u = 0.$$

Hence, for any $u \neq 0$, there exist complex roots, so that advanced results on hyperbolic systems [1, 10] cannot be applied.

6.1 Evolution formation of shocks

For (1.11), the blow-up similarity solution is

$$u_-(x, t) = g(z), \quad z = x/(-t), \quad \text{where} \quad (6.3)$$

$$\begin{aligned} (gg')^{(4)} &= (z^5 g')^{(4)} \equiv 120g'z + 240g''z^2 + 120g'''z^3 \\ &+ 20g^{(4)}z^4 + g^{(5)}z^5 \quad \text{in } \mathbb{R}, \quad f(\mp\infty) = \pm 1. \end{aligned} \quad (6.4)$$

Integrating (6.4) four times yields

$$gg' = z^5 g' + Az + Bz^3, \quad \text{with constants } A = (g'(0))^2 > 0, \quad B = \frac{2}{3} g'(0)g'''(0); \quad (6.5)$$

so that the necessary similarity profile $g(z)$ solves the first-order ODE

$$\frac{dg}{dz} = \frac{Az + Bz^3}{g - z^5}. \quad (6.6)$$

By the phase-plane analysis of (6.6) with $A > 0$ and $B = 0$, we easily get the following.

Proposition 6.1 *Problem (6.4) admits a solution $g(z)$ satisfying the anti-symmetry conditions (2.9) that is positive for $z < 0$, is monotone decreasing and is real analytic.*

Actually, involving the second parameter $B > 0$ yields that there exist infinitely many shock similarity profiles. The boldface profile $g(z)$ in Figure 22 (by (6.3), it gives $S_-(x)$ as

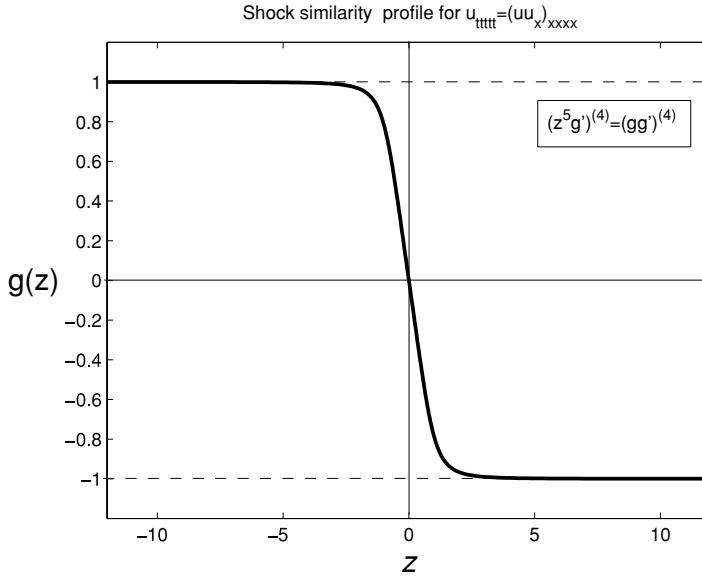


FIGURE 22. The shock similarity profile satisfying (6.4).

$t \rightarrow 0^-$) is non-oscillatory about ± 1 , with the following algebraic rate of convergence to the equilibrium as $z \rightarrow -\infty$:

$$g(z) = \begin{cases} 1 + \frac{A}{3z^5} + \dots & \text{for } B = 0, \\ 1 + \frac{B}{z} + \dots & \text{for } B > 0. \end{cases}$$

Note that the fundamental solutions of the corresponding linear PDE

$$u_{tttt} = u_{xxxx} \tag{6.7}$$

is also not oscillatory as $x \rightarrow \pm\infty$. This has the form

$$b(x, t) = t^3 F(y), \quad y = x/t, \quad \text{so that } b(x, 0) = \dots = b_{ttt}(x, 0) = 0, \quad b_{tttt}(x, 0) = \delta(x).$$

The linear equation (6.7) exhibits some features of finite propagation via TWs, since

$$u(x, t) = f(x - \lambda t) \implies -\lambda^5 f^{(5)} = f^{(5)}, \quad \text{i.e. } \lambda = -1,$$

since the profile $f(y)$ disappears from the ODE. This is similar to some canonical equations of mathematical physics, such as

$$u_t = u_x \text{ (dispersion, } \lambda = -1) \quad \text{and} \quad u_{tt} = u_{xx} \text{ (wave equation, } \lambda = \pm 1).$$

The blow-up solution (6.3) gives in the limit $t \rightarrow 0^-$ the shock $S_-(x)$, and (1.8) holds.

Since (1.11) has the same symmetry (3.1) as (1.30), similarity solutions (6.3), with $-t \mapsto t$ and $g(z) \mapsto g(-z)$ according to (3.5), also give the rarefaction waves for $S_+(x)$ as well as other types of collapse of initial non-entropy discontinuities.

6.2 Analytic δ -deformations by Cauchy–Kovalevskaya theorem

The great advantage of (1.11) is that it is in the *normal form*; so it obeys the Cauchy–Kovalevskaya theorem [52, p. 387]. Hence, for any analytic initial data $u(x, 0)$, $u_t(x, 0)$, $u_{tt}(x, 0)$, $u_{ttt}(x, 0)$ and $u_{tttt}(x, 0)$, there exists a unique local-in-time analytic solution $u(x, t)$. Thus, (1.11) generates a local semigroup of analytic solutions, and this makes it easier to deal with smooth δ -deformations that are chosen to be analytic. This defines a special analytic δ -entropy test for shock/rarefaction waves. On the other hand, such non-linear PDEs can admit other (say, weak) solutions that are not analytic. Actually, Proposition 6.1 shows that the shock $S_-(x)$ is a δ -entropy solution of (1.11), which is obtained by finite-time blow-up as $t \rightarrow 0^-$ from the analytic similarity solution (6.3).

6.3 On the formation of single-point shocks and extension non-uniqueness

Similar to the analysis in Section 5, for model (1.11) (and (6.1)), these assume studying extension similarity pairs $\{f, F\}$ induced by the easily derived analogies of the blow-up (5.2) and global (5.6) 5D dynamical systems, with

$$\beta = \frac{5 + \alpha}{5},$$

These are very difficult, so that checking three types (standard, backward and initial) of possible non-uniqueness and non-entropy of such flows with strong and weak shocks becomes a hard open problem, though some auxiliary analytic steps towards non-uniqueness are doable. Overall, in view of complicated multi-dimensional phase spaces involved, we do not have any reason for having a unique continuation after singularity. In other words, for such higher-order NDEs, uniqueness can occur *accidentally* only for very special phase spaces and hence, at least, is not robust (in a natural ODE–PDE sense) anyway.

7 (V) Problem – ‘oscillatory smooth sompactons’ of fifth-order NDEs

We begin with an easier explicit example of *non-negative* compactons for a third-order NDE.

7.1 Third-order NDEs: δ -entropy compactons

Compactons as compactly supported TW solutions of the $K(2, 2)$ equation (1.22) were introduced in 1993 [46] as

$$u_c(x, t) = f_c(y), \quad y = x + t \quad \Longrightarrow \quad f_c : \quad f = (f^2)'' + f^2. \quad (7.1)$$

Integrating yields the following explicit compacton profile:

$$f_c(y) = \begin{cases} \frac{4}{3} \cos^2\left(\frac{y}{4}\right) & \text{for } |y| \leq 2\pi, \\ 0 & \text{for } |y| \geq 2\pi. \end{cases} \quad (7.2)$$

The corresponding compacton (7.1), (7.2) is G-admissible in the sense of Gel'fand¹ (1959) [25, Sections 2 and 8] and is a δ -entropy solution [18, Section 4], i.e. can be constructed by smooth (and moreover analytic) approximation via strictly positive solutions of the full third-order ODE for $f_c(y)$,

$$f' = (f^2)''' + (f^2)'$$

Since the PDE is not involved unlike in Section 4.5, the δ -entropy notion coincides with the G-admissibility.

It is curious that the same compactly supported blow-up patterns occur in the combustion problem for the related reaction–diffusion parabolic equation

$$u_t = (u^2)_{xx} + u^2. \tag{7.3}$$

Then the standing-wave blow-up (as $t \rightarrow T^-$) solution of S-regime leads to the same ODE:

$$u_S(x, t) = (T - t)^{-1} f(x) \implies f = (f^2)'' + f^2. \tag{7.4}$$

This yields the *Zmitrenko–Kurdyumov blow-up localised solution*, which has been known since 1975 (see more historical details in [23, Section 4.2]).

7.2 Examples of C^3 -smooth non-negative compacton for higher-order NDEs

Such an example was given in [12, p. 4734]. Following [23, p. 189], we construct this explicit solution as follows. The operator $F_5(u)$ of the *quintic NDE*

$$u_t = F_5(u) \equiv (u^2)_{xxxxx} + 25(u^2)_{xxx} + 144(u^2)_x \tag{7.5}$$

is shown to preserve the 5D invariant subspace

$$W_5 = \text{Span}\{1, \cos x, \sin x, \cos 2x, \sin 2x\}, \tag{7.6}$$

i.e. $F_5(W_5) \subseteq W_5$. Therefore, (7.5) restricted to the invariant subspace W_5 is a 5D dynamical system for the expansion coefficients of the solution

$$u(x, t) = C_1(t) + C_2(t) \cos x + C_3(t) \sin x + C_4(t) \cos 2x + C_5(t) \sin 2x \in W_5.$$

Solving this yields the explicit compacton TW,

$$u_c(x, t) = f_c(x + t), \quad \text{where } f_c(y) = \begin{cases} \frac{1}{105} \cos^4\left(\frac{y}{2}\right) & \text{for } |y| \leq \pi, \\ 0 & \text{for } |y| \geq \pi. \end{cases} \tag{7.7}$$

This C_x^3 solution can be attributed to the Cauchy problem for (7.5), since smooth solutions are not oscillatory near interfaces (see the discussion in [23, p. 184]).

The above invariant subspace analysis applies also to the seventh-order PDE

$$u_t = F_7(u) \equiv D_x^7(u^2) + \beta D_x^5(u^2) + \gamma(u^2)_{xxx} + \nu(u^2)_x. \tag{7.8}$$

¹ I. M. Gel'fand, 2.09.1913–5.10.2009.

Here \mathbf{F}_7 admits W_5 if

$$\beta = 25, \quad \gamma = 144 \quad \text{and} \quad \nu = 0.$$

Moreover [23, p. 190], the only operator \mathbf{F}_7 in (7.8) preserving the 7D subspace

$$W_7 = \mathcal{L}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x\} \quad (7.9)$$

is in the following NDE, NDE-7:

$$u_t = \mathbf{F}_7(u) \equiv D_x^7(u^2) + 77D_x^5(u^2) + 1876(u^2)_{xxx} + 14400(u^2)_x. \quad (7.10)$$

This makes it possible to reduce (7.10) on W_7 to a complicated dynamical system.

7.3 Why non-negative compactons for fifth-order NDEs are not robust: a saddle–saddle homoclinic

Recall that as usual in dynamical system theory, by robustness of trajectories we mean that these are stable with respect to small perturbations of the parameters entering the NDE or the corresponding ODEs. In other words, the dynamical systems (ODEs) admitting such non-negative ‘heteroclinic’ *saddle-like* orbits $0 \rightarrow 0$ are not *structurally stable* in a natural sense. This reminds us of the classic Andronov–Pontriagin–Peixoto theorem, where one of the four conditions for the structural stability of dynamical systems in \mathbb{R}^2 reads as follows [37, p. 301]:

$$\text{‘(ii) there are no trajectories connecting saddle points’} \quad (7.11)$$

Actually, non-negative compactons, such as (7.7), are special homoclinics of the origin, and we will show that the nature of their non-robustness is in the fact that these represent a stable–unstable manifold of the origin consisting of a *single orbit*. Therefore, in consistency with (7.11), the origin is indeed a saddle in \mathbb{R}^4 in the plane $\{f, f', f'', f'''\}$, obtained after integration once (see below).

In order to illustrate the lack of such a robustness in view of a sole heteroclinic involved, consider NDE (7.5), for which, substituting the TW solution, on integration, we obtain the ODE

$$u_c(x, t) = f_c(x + t) \implies f_c : \quad 2f = (f^2)^{(4)} + \dots, \quad (7.12)$$

where we omit the lower-order terms as $f \rightarrow 0$. Looking for the compacton profile $f \geq 0$, we set $f^2 = F$ to get

$$F^{(4)} = 2\sqrt{F} + \dots \quad \text{for} \quad y > 0, \quad F'(0) = F'''(0) = 0. \quad (7.13)$$

As usual, we look for a symmetric $F(y)$ by putting two symmetry conditions at the origin.

Let $y = y_0 > 0$ be the interface point of $F(y)$. Then, looking for the expansion as $y \rightarrow y_0^-$ in the form

$$F(y) = \frac{1}{840^2} (y_0 - y)^8 + \varepsilon(y), \quad \text{with} \quad \varepsilon(y) = o((y_0 - y)^8), \quad (7.14)$$

we obtain Euler’s equation for the perturbation $\varepsilon(y)$,

$$\frac{1}{840}(y_0 - y)^4 \varepsilon^{(4)} - \varepsilon = 0. \tag{7.15}$$

Hence, $\varepsilon(y) = (y_0 - y)^m$, with the characteristic equation

$$m(m - 1)(m - 2)(m - 3) - 840 = 0 \implies m_1 = -4, \quad m_{2,3} = \frac{3 \pm i\sqrt{111}}{2}, \quad m_4 = 7. \tag{7.16}$$

Hence, $\text{Re } m_i < 8$, and in other words, (7.15) does not admit any non-trivial solution satisfying the condition in (7.14) (see further comments in [23, p. 142]). In fact, it is easy to see that (7.15) with $\varepsilon = 0$ is the unique positive smooth solution of $F^{(4)} = 2\sqrt{F}$. Thus,

$$\text{the asymptotic bundle of solutions (7.14) is 1D,} \tag{7.17}$$

where the only parameter is the position of the interface $y_0 > 0$.

Obviously, as a typical property, this 1D bundle is not sufficient to satisfy (by shooting) two conditions at the origin in (7.13); so such TW profiles $F(y) \geq 0$ are non-existent for almost all NDEs like that. In other words, the condition of positivity of the solution,

$$\text{to look for a non-trivial solution } F \geq 0 \text{ for the ODE in (7.13),} \tag{7.18}$$

creates a free-boundary ‘obstacle’ problem that, in general, is inconsistent. Skipping the obstacle condition (7.18) will return such ODEs (or elliptic equations), with a special extension, into the consistent variety, as we will illustrate below.

Thus, non-negative TW compactons are not generic (robust) solutions of the $(2m + 1)$ th-order quadratic NDEs with $m = 2$ and also for larger m ’s, where some kind of (7.17), as a ‘dimensional defect’ (the bundle dimension is smaller than the number of conditions at $y = 0$ to shoot), remains valid.

7.4 Non-negative compactons are robust for third-order NDEs only

The third-order case $m = 1$, i.e. NDEs such as (1.22), is the only one in which propagation of perturbations via non-negative TW compactons is structurally stable, i.e. with respect to small perturbation of the parameters (and non-linearities) of equations. Mathematically speaking, then the 1D bundle in (7.17) perfectly matches with the SINGLE symmetry condition at the origin,

$$F'' = 2\sqrt{F} + \dots \quad \text{and} \quad F'(0) = 0.$$

7.5 Robustness of the compactons of changing sign, which are δ -entropy for NDE–5s

As a typical example, we consider the perturbed version (1.12) of NDE–(1,4) (1.5). As we have mentioned, this is written for solutions of changing sign, since non-negative compactons do not exist in general. Looking for the TW compacton (7.12) yields the ODE

$$f = -\frac{1}{2}(|f|f)^{(4)} + \frac{1}{2}|f|f \implies F^{(4)} = F - 2|F|^{-\frac{1}{2}}F \text{ for } F = |f|f. \tag{7.19}$$

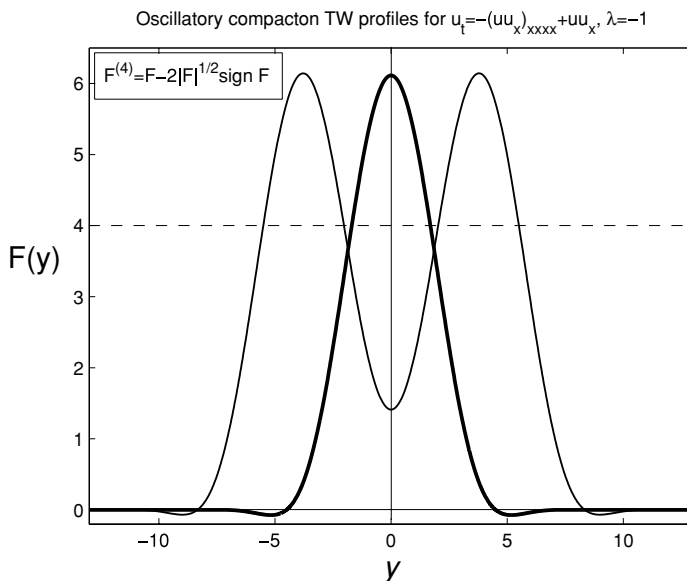


FIGURE 23. First two compacton TW profiles $F(y)$ satisfying the ODE in (7.19).

Such ODEs with non-Lipschitz non-linearities are known to admit countable sets of compactly supported solutions, which are studied by a combination of Lusternik–Schnirel’man and Pohozaev’s fibring theory (see [21]).

In Figure 23, we present the first TW compacton patterns (the boldface line) and the second one that is essentially non-monotone. These look like standard compacton profiles, but careful analysis of the behaviour near the finite interface at $y = y_0$ shows that $F(y)$ changes sign infinitely many times according to the asymptotics

$$F(y) = (y_0 - y)^8 [\varphi(s + s_0) + o(1)], \quad s = \ln(y_0 - y) \quad \text{as } y \rightarrow y_0^-. \tag{7.20}$$

Here, the oscillatory component $\varphi(s)$ is a periodic solution of a certain non-linear ODE and s_0 is an arbitrary phase shift (see [23, Section 4.3] and [21, Section 4] for further details). Thus, unlike (7.17),

$$\text{the asymptotic bundle of solutions (7.20) is 2D (the parameters are } y_0 \text{ and } s_0) \tag{7.21}$$

and exhibits some features of a ‘non-linear focus’ (not a saddle as above) on some manifold. Hence, this is enough to match also two symmetry boundary conditions given in (7.13). Such a robust solvability is confirmed by variational techniques that apply to rather arbitrary equations such as in (7.19) with similar singular non-Lipschitz non-linearities.

Let us also note that such oscillatory compactons are also δ -entropy in the sense that can be approximated by analytic TW solutions of the same ODE but having finite number of zeros (i.e. admit smooth analytic δ -deformations; see [18, Section 8.3] for a related NDE–5).

Regardless of the existence of such sufficiently smooth compacton solutions, it is worth recalling again that for NDE (1.12), as well as (2.35) and (4.3), both containing monotone

non-linearities, the generic behaviour, for other initial data, can include formation of shocks in finite time, with the local similarity mechanism as in Section 2 and in Section 5.1, representing more generic single-point shock pattern formations.

8 Final conclusions

The fifth-order NDEs (1.1)–(1.5) (NDE–5s), which are associated with a number of important applications, are considered. The main achieved properties of such 1D degenerate non-linear PDEs are as follows:

- (i) SECTION 2. These NDEs admit blow-up self-similar formation from smooth solutions of *shock waves* of a specific oscillatory structure, which correspond to initial data $S_-(x) = -\text{sign } x$, i.e. the same as for the first-order conservation law (1.30).
- (ii) SECTION 3. As customary, *self-similar rarefaction waves*, which get smooth for any $t > 0$, are created by the reversed data $S_+(x) = \text{sign } x$.
- (iii) Unlike the classical theory of first-order conservation laws developed in the 1950s and the 1960s, the entropy-type techniques are no longer applied for distinguishing general proper (unique) solutions. A δ -*deformation test* via smoothing the solutions is developed in SECTION 4, which is able to separate shocks and rarefaction waves for particular classes of initial data $\sim S_{\pm}(x)$ with a simple geometry of initial shocks;
- (iv) SECTION 5. By studying more general self-similar solutions of NDEs, it was shown that uniqueness of the solutions after formation of a shock is principally impossible. Namely, there exist single-point gradient blow-up similarity solutions, which admit an infinite number of self-similar extensions beyond. This 2D set of shock wave extensions after singularity does not have any distinguished solution (say, maximal, minimal and isolated). This also suggests that any entropy-like mechanism for a unique continuation does not exist either. However, using a proper *free-boundary setting*, i.e. posing special conditions on shocks, can restore uniqueness.
- (v) SECTION 7. *Non-negative compacton solutions* for some NDE–5s are shown to be non-robust (not ‘structurally stable’); i.e. these disappear after a.a. arbitrarily small perturbations of the parameters (non-linearities) of the equations. However, *oscillatory compactons* of changing sign near finite interface are shown to exist and to be robust.

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