

Non-negative integral matrices with given spectral radius and controlled dimension

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Abstract. A celebrated theorem of Douglas Lind states that a positive real number is equal to the spectral radius of some integral primitive matrix, if and only if, it is a Perron algebraic integer. Given a Perron number p , we prove that there is an integral irreducible matrix with spectral radius p , and with dimension bounded above in terms of the algebraic degree, the ratio of the first two largest Galois conjugates, and arithmetic information about the ring of integers of its number field. This arithmetic information can be taken to be either the discriminant or the minimal Hermite-like thickness. Equivalently, given a Perron number p , there is an irreducible shift of finite type with entropy $\log(p)$ defined as an edge shift on a graph whose number of vertices is bounded above in terms of the aforementioned data.

Key words: non-negative matrices, Perron–Frobenius theory, Perron numbers, shifts of finite type, entropy

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1. Introduction

A real algebraic integer $\lambda \geq 1$ is *Perron* if λ is strictly larger than all its other Galois conjugates in absolute value. In what follows, all matrices are considered to be square size. For a real matrix A , by $A > 0$ we mean that all entries of A are positive. A non-negative real matrix A is *primitive* (or *aperiodic*) if there is some natural number k such that $A^k > 0$. A non-negative real matrix is *irreducible* if for any two indices i and j , there is some natural number $k = k(i, j)$ such that $(A^k)_{ij} > 0$. The Perron–Frobenius theorem implies that:

- (1) the spectral radius of any *integral* primitive matrix is a Perron number; and
- (2) the spectral radius of any *integral* irreducible matrix is equal to the n th root of a Perron number where n is the *period* of the irreducible matrix. Moreover, the spectral radius of an integral non-negative non-nilpotent matrix is equal to the spectral radius of an integral irreducible one.

Lind [Lin84, Theorem 1] proved a converse to the ‘integer’ Perron–Frobenius theorem, namely for every Perron number λ there is an integral primitive matrix with spectral radius equal to λ . It readily follows that for any Perron number λ and any natural number n , there is an integral irreducible matrix with spectral radius equal to $\sqrt[n]{\lambda}$; see [Lin84, Theorem 3].

Associated to a non-negative and non-degenerate (that is, with no zero rows or columns) integral matrix $A = [a_{ij}]$ with spectral radius λ is a *shift of finite type* with entropy equal to $\log(\lambda)$, which is defined as the *edge shift* on a directed finite graph G_A as follows. The graph G_A has one vertex for each row of the matrix A , and there are exactly a_{ij} oriented edges from the vertex v_i to the vertex v_j . In particular, the dimension of the matrix A (that is, its number of rows or columns) is equal to the number of vertices of the graph G_A . The matrix is primitive if and only if the associated shift of finite type is topologically mixing. If the matrix is irreducible, then the corresponding shift of finite type is called *irreducible*. Irreducible shifts of finite type are those which are topologically transitive. See e.g. [LM21].

Given a Perron number λ , the *Perron–Frobenius degree* of λ , $d_{\text{PF}}(\lambda)$, is the least dimension of an integral *primitive* matrix with spectral radius equal to λ . Clearly we have

$$d_{\text{PF}}(\lambda) \geq d,$$

where d denotes the algebraic degree of λ as an algebraic integer. It is easy to see that the equality happens if λ is quadratic (see [Yaz21, Remark 3.1]), so we consider the case of $d \geq 3$. Lind observed that if the *trace* (that is, the sum of Galois conjugates) of λ is negative, then $d_{\text{PF}}(\lambda)$ is strictly larger than d ; see [Lin84, p. 289]. In [Yaz21], using an idea of Lind, we gave a lower bound for $d_{\text{PF}}(\lambda)$ in terms of the layout of the two largest (in absolute value) Galois conjugates of λ in the complex plane. As a corollary, it was shown that there are examples of cubic Perron numbers with arbitrarily large Perron–Frobenius degrees, a result previously known to Lind, McMullen, and Thurston, although unpublished.

Definition 1.1. Given a Perron number λ , define the *spectral ratio* of λ as $\max_i(|\lambda_i|/\lambda)$, where $\lambda_i \neq \lambda$ are the remaining Galois conjugates of λ .

Notation 1.2. For a Perron number λ , let $d_{\text{PF}}^{\text{irr}}(\lambda)$ be the smallest dimension of an integral *irreducible* matrix with spectral radius λ .

In this paper, we give an explicit upper bound for $d_{\text{PF}}^{\text{irr}}(\lambda)$ in terms of the algebraic degree of λ , the spectral ratio of λ , and arithmetic information about the ring of integers $\mathcal{O}_{\mathbb{K}}$ of the number field $\mathbb{K} := \mathbb{Q}(\lambda)$. This arithmetic quantity, which we call the *minimal Hermite-like thickness* and denote it by $\tau_{\min}(\mathcal{O}_{\mathbb{K}})$, was previously defined by Bayer Fluckiger [Bay06] in relation to *Minkowski’s conjecture*; see Definition 3.1. Intuitively, once an inner product is chosen on \mathbb{R}^d , the Hermite-like thickness is defined as the square of the covering radius, normalized properly, for the inclusion of the lattice $\mathcal{O}_{\mathbb{K}}$ in \mathbb{R}^d . The minimal Hermite-like thickness is then defined by taking the infimum of Hermite-like thickness over an appropriate space of inner products on \mathbb{R}^d . As a corollary of our main result, and using an inequality due to Banaszczyk and Bayer Fluckiger (see inequality (3.25)), we

obtain a similar bound in terms of the *discriminant* $D_{\mathbb{K}}$ of $\mathcal{O}_{\mathbb{K}}$ instead of $\tau_{\min}(\mathcal{O}_{\mathbb{K}})$. See Definition 3.3.

THEOREM 1.3. *Let λ be a Perron number of algebraic degree $d \geq 3$ and spectral ratio ρ . Set $\mathbb{K} := \mathbb{Q}(\lambda)$. Let $\mathcal{O}_{\mathbb{K}}$ be the ring of integers of \mathbb{K} , and denote the discriminant and the minimal Hermite-like thickness of \mathbb{K} by, respectively, $D_{\mathbb{K}}$ and $\tau_{\min}(\mathcal{O}_{\mathbb{K}})$. Then $d_{\text{PF}}^{\text{irr}}(\lambda)$ is bounded above by each of*

$$\left(\frac{8d}{1-\rho}\right)^{d^2} \tau_{\min}(\mathcal{O}_{\mathbb{K}})^{d/2}$$

and

$$\left(\frac{8d}{1-\rho}\right)^{d^2} \sqrt{D_{\mathbb{K}}}.$$

Remark 1.4. Given a natural number n and an integral irreducible matrix A with spectral radius λ , one can readily construct an integral irreducible matrix B with spectral radius $\sqrt[n]{\lambda}$ such that $\dim(B) = n \dim(A)$. Hence, Theorem 1.3 can be used to give an upper bound for $d_{\text{PF}}^{\text{irr}}(\sqrt[n]{\lambda})$. See e.g. the proof of [Lin84, Theorem 3].

Theorem 1.3 immediately translates into the context of irreducible shifts of finite type with a given entropy. We can derive an upper bound for the Perron–Frobenius degree using Theorem 1.3. See also Remark 3.9 and Question 5.1. First we need to introduce a notation.

Notation 1.5. Let \mathbb{K} be a real number field (that is, with at least one real place), and $\rho \in (0, 1)$. Set

$$M = 1 + \frac{4}{1-\rho},$$

and denote

$$\kappa(\mathbb{K}, \rho) := \max\{d_{\text{PF}}(\alpha) \mid \alpha \in [1, M] \cap \mathbb{K} \text{ is a Perron number}\}.$$

Note that for any $M > 1$, there are only finitely many Perron numbers in the interval $[1, M]$ with algebraic degree at most d . Hence, $\kappa(\mathbb{K}, \rho)$ can be computed in theory.

THEOREM 1.6. *Let λ be a Perron number of algebraic degree $d \geq 3$ and spectral ratio ρ . Set $\mathbb{K} := \mathbb{Q}(\lambda)$. Denote the bound from Theorem 1.3 by $B(\mathbb{K}, \rho)$; note that $d = [\mathbb{K} : \mathbb{Q}]$ is uniquely determined by \mathbb{K} . The Perron–Frobenius degree of λ is bounded above by*

$$\max\{2^{d^2} B(\mathbb{K}, \rho), \kappa(\mathbb{K}, \rho)\}.$$

1.1. Previous work. In [Lin84], Lind gave a method for producing *all* integral primitive matrices with a given Perron number λ as their spectral radius. Let $B : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the *companion matrix* associated with λ . In what follows, unless otherwise specified, all eigenvectors are column (right) eigenvectors; similarly for eigenspaces. Pick an eigenvector v for the eigenvalue λ , and denote the one-dimensional eigenspace of λ by E_{λ} . Let E be the positive half-space associated with v , that is, if $\pi_1 : \mathbb{R}^d \rightarrow E_{\lambda}$ is the projection along

the complementary invariant subspace, then

$$E = \{x \in \mathbb{R}^d \mid \pi_1(x) = rv \text{ for some } r > 0\}.$$

THEOREM 1.7. (Lind) *Let λ be a Perron number of algebraic degree d , and B and E be as above. If $A = [a_{ij}]$ is an n -dimensional primitive non-negative integral matrix with spectral radius λ , then there are $z_i \in \mathbb{Z}^d \cap E$ for $1 \leq i \leq n$ such that $Bz_j = \sum_{i=1}^n a_{ij}z_i$.*

Conversely, if λ , B , and E are as above, and the points $z_i \in \mathbb{Z}^d \cap E$ and a non-negative integral matrix $A = [a_{ij}]$ satisfy $Bz_j = \sum_{i=1}^n a_{ij}z_i$, then every irreducible component of A has its spectral radius equal to λ .

The above theorem of Lind gives a practical way to produce an integral primitive matrix with a given spectral radius; see [Lin84, p. 289]. However, it does not tell us how to find such a matrix with smallest (or close to smallest) dimension, since we are not given control over the size of the coordinates of z_i . Nevertheless, the referee has kindly mentioned to me that there exists a simple, but not necessarily practical, algorithm that computes the Perron–Frobenius degree of a Perron number: Assume that λ is given by its minimal polynomial, and denote the algebraic degree of λ by d . For every positive integer n , there are only finitely many primitive $n \times n$ integral matrices with spectral radius less than or equal to λ . One can algorithmically enumerate these. For each of them, one can algorithmically determine whether the spectral radius is equal to λ . So for $n = d$, we can check whether there is an integral primitive matrix of dimension n which has λ as the spectral radius. Recursively, if we fail at n , then we try at $n + 1$. By Lind’s theorem, we eventually find an n where we succeed; that n is equal to $d_{\text{PF}}(\lambda)$.

For matrices with non-negative integral *polynomial* entries, the situation is different. See the work of Boyle and Lind, which gives a uniform upper bound (in fact, a 2 by 2 matrix) in this context [BL02]. For the related topic of the inverse spectral problem for non-negative integral matrices, see the works of Boyle and Handelmann [BH91] and Kim, Ormes and Roush [KOR00] and the references therein.

1.2. *Idea of the proof.* In [Thu14], Thurston gave a simpler proof of Lind’s converse to the integer Perron–Frobenius theorem. Our proof of Theorem 1.3 follows Thurston’s approach, while controlling the dimension of a constructed matrix.

The tensor product $\mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{R}$ can be identified with \mathbb{R}^d . Let M_λ be the linear endomorphism of $\mathbb{R}^d \cong \mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{R}$ induced by multiplication by λ in $\mathbb{Q}(\lambda)$. The eigenvalues of M_λ are the Galois conjugates of λ . Then \mathbb{R}^d decomposes into invariant subspaces of M_λ corresponding to real places and pairs of conjugate complex places of λ ; see the opening paragraphs to §3. Fix an eigenvector for M_λ with eigenvalue λ , and consider the positive half-space corresponding to λ .

We start with a polygonal cone with apex at the origin that lies in the positive half-space and is invariant under M_λ . We then perturb the vertices of the cone to obtain an invariant polygonal cone \mathcal{C} with *integral* vertices; see Steps 2–4 of the proof of Theorem 1.3. It is during this perturbation that the minimal Hermite-like thickness appears in the picture. Since the polygonal cone has integral vertices, the semigroup S generated by the set of

integral points in the cone \mathcal{C} under addition of vectors is finitely generated; see Proposition 2.1. The cardinality of a generating set for the semigroup S gives an upper bound for the dimension of an integral non-negative matrix A with spectral radius λ . Moreover, after possibly passing to an irreducible component of A , an integral irreducible matrix with spectral radius λ is obtained; see Step 5. Finally we give an upper bound for the dimension of A ; see Step 6.

1.3. *Plan of the paper.* In §2, we present a few preliminary lemmas. The proof of Theorem 1.3 is given in §3. Theorem 1.6 is proved in §4. In §5, a few questions are posed.

2. Preliminaries

2.1. *Lattice points.* Throughout this article, by a *lattice* $\Lambda \subset \mathbb{R}^d$ we mean a lattice of full rank; that is, a discrete subgroup of \mathbb{R}^d isomorphic to \mathbb{Z}^d .

PROPOSITION 2.1. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice, and $x_1, \dots, x_k \in \Lambda$. Denote by \mathcal{C} the convex cone over the points x_i inside \mathbb{R}^d , that is,*

$$\mathcal{C} := \{\alpha_1 x_1 + \dots + \alpha_k x_k \mid \alpha_i \geq 0 \text{ for every } i\}.$$

Let S be the semigroup generated by elements of $\mathcal{C} \cap \Lambda$ under vector addition. Define the compact set C , and the finite set $C_\Lambda \subset C$ as

$$C := \{\alpha_1 x_1 + \dots + \alpha_k x_k \mid 0 \leq \alpha_i \leq 1 \text{ for every } i\}, \quad C_\Lambda := C \cap \Lambda.$$

Then C_Λ is a finite generating set for the semigroup S .

Proof. For $\alpha \in \mathbb{R}$, denote the fractional part of α by $\{\alpha\}$, and let $[\alpha] = \alpha - \{\alpha\}$. Any point $y \in S$ can be written as

$$y = \alpha_1 x_1 + \dots + \alpha_k x_k = ([\alpha_1] x_1 + \dots + [\alpha_k] x_k) + (\{\alpha_1\} x_1 + \dots + \{\alpha_k\} x_k),$$

where $\alpha_i, [\alpha_i] \geq 0$ for each i . Note that we have $x_i \in C_\Lambda$ for each i , so the first parenthesis is a sum of elements of C_Λ . As y and the first parenthesis are both in Λ , the second parenthesis should represent a point in Λ as well. On the other hand, the coefficients of the second parenthesis are in the interval $[0, 1)$, and so the second parenthesis lies in $C_\Lambda = C \cap \Lambda$. We have written y as a sum of elements of C_Λ , so C_Λ is a generating set for the semigroup S . Since C is compact and Λ is a lattice, $C_\Lambda = C \cap \Lambda$ is a finite set. \square

By a *Euclidean space* of dimension d , we mean a d -dimensional vector space \mathbb{R}^d equipped with an inner product. A *polytope* is the convex hull of finitely many points in \mathbb{R}^d . Let $\Lambda \subset \mathbb{R}^d$ be a lattice. If $\{v_1, \dots, v_d\}$ is a basis for $\Lambda \cong \mathbb{Z}^d$, then a *fundamental domain* for Λ is

$$\{\alpha_1 v_1 + \dots + \alpha_d v_d \mid 0 \leq \alpha_i < 1 \text{ for every } i\}.$$

If \mathbb{R}^d is a Euclidean space, then the *covolume* of Λ is defined as the volume of any fundamental domain for Λ . A *lattice polytope* is a polytope whose vertices are lattice points.

PROPOSITION 2.2. Let \mathbb{R}^d be a Euclidean space, $\Lambda \subset \mathbb{R}^d$ be a lattice with covolume $\text{Covol}(\Lambda)$, and $P \subset \mathbb{R}^d$ be a d -dimensional lattice polytope with volume $\text{Vol}(P)$. Denote the number of lattice points in P by $|P \cap \Lambda|$. Then

$$|P \cap \Lambda| \leq \frac{\text{Vol}(P)}{\text{Covol}(\Lambda)} \cdot (d + 1)!.$$

The equality happens exactly when P is a d -simplex with $|P \cap \Lambda| = d + 1$.

Proof. First consider the special case that P has exactly $d + 1$ vertices, and every lattice point in P is a vertex of P . If $v_0, v_1, \dots, v_d \in \mathbb{R}^d$ are the vertices of P , then the volume of the parallelepiped formed by the vectors $v_1 - v_0, \dots, v_d - v_0$ is equal to $d! \times \text{Vol}(P)$. Since the volume of this parallelepiped is at least as large as the volume of a fundamental domain for Λ , we have

$$d! \times \text{Vol}(P) \geq \text{Covol}(\Lambda),$$

implying that

$$|P \cap \Lambda| = d + 1 \leq \frac{\text{Vol}(P)}{\text{Covol}(\Lambda)} \cdot (d + 1)!.$$

In general, decompose P into d -simplices $\Delta_1, \dots, \Delta_n$ with disjoint interiors such that each simplex Δ_i contains no lattice point except for its vertices. This can be done for example as follows. Decompose P into smaller polyhedra by coning off from one of the vertices of P . Here, by coning off from a point $v \in P$, we mean that for every facet F of P , the polyhedron which is the convex hull of $F \cup v$ is added unless its dimension is strictly smaller than that of P ; for example, if the starting polyhedron is a polygon, then coning off from a vertex v is just decomposing the polygon into triangles via adding all the diagonals emanating from v . For any of the resulting polyhedra, successively take a lattice point inside or on the boundary, and cone off from that lattice point. After finitely many repetitions, we arrive at the decomposition into Δ_i .

The desired inequality follows from adding up the corresponding inequalities for simplices $\Delta_1, \dots, \Delta_n$. Note that if P is not a d -simplex or $|P \cap \Lambda| > d + 1$, then at least one lattice point in $P \cap \Lambda$ is counted for more than one simplex Δ_i , and so the inequality is strict. □

2.2. *Minkowski sum and difference.* For sets $A, B \subset \mathbb{R}^d$, define their *Minkowski sum* as

$$A + B := \{a + b \mid a \in A, \text{ and } b \in B\} \subset \mathbb{R}^d.$$

Intuitively, $A + B$ is the union of all translates of A by elements of B

$$A + B = \bigcup_{b \in B} (A + b).$$

Define the *Minkowski difference* of A and B by

$$A \div B := \{c \in \mathbb{R}^d \mid B + c \subseteq A\}.$$

If B is empty, $A \div B$ is, by convention, equal to \mathbb{R}^d . Intuitively, $A \div B$ is the intersection of all translates of A by the antipodes of elements of B

$$A \div B = \bigcap_{b \in B} (A - b).$$

We have used the rather odd notation \div for the Minkowski difference, in order to distinguish it from the set

$$\{a - b \mid a \in A, \text{ and } b \in B\} \subset \mathbb{R}^d.$$

The Minkowski sum and difference are *not* in general the inverse of each other.

LEMMA 2.3. *The following properties hold for sets $A, B, C \subset \mathbb{R}^d$:*

$$\begin{aligned} A &\subset (A + B) \div B, \\ A \subset B &\implies A \div C \subset B \div C, \\ A \subset B &\implies A + C \subset B + C. \end{aligned}$$

Moreover, if A and B are non-empty compact, convex sets, then

$$(A + B) \div B = A.$$

Proof. The first three properties directly follow from the definition. We sketch the proof of the last implication, and refer the reader to e.g. [Sch13, Lemma 3.1.11 and Section 1.7] for details. Pick an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d , and denote the *support functions* for A and B by, respectively, h_A and h_B . By definition, for any $x \in \mathbb{R}^d$,

$$h_A(x) := \sup\{\langle x, y \rangle \mid y \in A\},$$

and h_B is defined similarly. By the *separation theorem* for convex sets, for a non-empty compact convex set A , we have

$$a \in A \iff \langle a, x \rangle \leq h_A(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Assuming $x \in (A + B) \div B$, we would like to show that $x \in A$. By hypothesis, $x + B \subset A + B$. Equivalently, the support function for $x + B$ does not exceed that of $A + B$ pointwise. This implies

$$h_{\{x\}} + h_B \leq h_A + h_B,$$

using the fact that $h_{A+B} = h_A + h_B$ for non-empty compact convex sets A and B ; see [Sch13, Theorem 1.7.5]. Cancelling h_B from both sides gives us the inequality $h_{\{x\}} \leq h_A$, which implies that $x \in A$. □

3. Proof of Theorem 1.3

We follow Bayer Fluckiger [Bay06] and Jarvis [Jar14] for the definitions below. Let λ be an algebraic integer, and $\mathbb{K} = \mathbb{Q}(\lambda)$ be the number field obtained by adjoining λ to \mathbb{Q} . We may interpret the points of \mathbb{K} as lying in a d -dimensional real linear space as follows. Let $\sigma_1, \dots, \sigma_r$ be the real embeddings of \mathbb{K} , and $\sigma_{r+1}, \bar{\sigma}_{r+1}, \dots, \sigma_{r+s}, \bar{\sigma}_{r+s}$ be the pairwise

conjugate complex embeddings of \mathbb{K} , where

$$r + 2s = d. \tag{3.1}$$

Consider the embedding

$$\begin{aligned} \sigma : \mathbb{K} &\longrightarrow \mathbb{R}^r \times \mathbb{C}^s, \\ \sigma(x) &= (\sigma_1(x), \dots, \sigma_{r+s}(x)). \end{aligned}$$

We may identify \mathbb{C} with \mathbb{R}^2 , by identifying $a + bi$ with (a, b) , then σ becomes an embedding

$$\sigma : \mathbb{K} \longrightarrow \mathbb{R}^d.$$

The mapping $\sigma : \mathbb{K} \rightarrow \mathbb{R}^d$ identifies the vector space \mathbb{R}^d with the tensor product $\mathbb{K}_{\mathbb{R}} := \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R}$,

$$\begin{aligned} \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R} &\longleftrightarrow \mathbb{R}^d \\ x \otimes a &\mapsto (\sigma x)a. \end{aligned}$$

Note that $\mathbb{R}^r \times \mathbb{C}^s$ has a canonical involution, which is identity on \mathbb{R}^r and complex conjugation on \mathbb{C}^s . Let

$$\mathfrak{B} := \{\alpha \in \mathbb{R}^r \times \mathbb{C}^s \mid \alpha = \bar{\alpha}, \text{ and all components of } \alpha \text{ are positive}\}. \tag{3.2}$$

Given $\alpha \in \mathfrak{B}$, define the symmetric positive definite bilinear form q_{α} on $\mathbb{K}_{\mathbb{R}}$ by

$$\begin{aligned} q_{\alpha} : \mathbb{K}_{\mathbb{R}} \times \mathbb{K}_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ (x, y) &\mapsto \text{Trace}(\alpha x \bar{y}). \end{aligned}$$

Here, $\text{Trace}(x_1, \dots, x_{r+s}) := \sum_{i=1}^r x_i + \sum_{j=r+1}^{r+s} (x_j + \bar{x}_j)$ denotes the trace of $\mathbb{K}_{\mathbb{R}} \cong \mathbb{R}^r \times \mathbb{C}^s$, and $\alpha x \bar{y}$ denotes component-wise product in $\mathbb{R}^r \times \mathbb{C}^s$. Then q_{α} induces the norm $|\cdot|_{\alpha}$ on $\mathbb{K}_{\mathbb{R}}$ given by the following formula, where $\alpha = (\alpha_1, \dots, \alpha_{r+s})$ with $\alpha_i \in \mathbb{R}^{>0}$:

$$|x|_{\alpha}^2 = \sum_{i=1}^r \alpha_i \cdot |\sigma_i(x)|^2 + 2 \sum_{j=r+1}^{r+s} \alpha_j \cdot |\sigma_j(x)|^2. \tag{3.3}$$

Denote the ring of integers of \mathbb{K} by $\mathcal{O}_{\mathbb{K}}$. Let $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ denote the lattice $\mathcal{O}_{\mathbb{K}}$ equipped with the inner product q_{α} . The *maximum* of $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$, denoted by $\max(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$, is defined as

$$\inf\{u \in \mathbb{R} \mid \text{for all } x \in \mathbb{K}_{\mathbb{R}}, \text{ there exists } y \in \mathcal{O}_{\mathbb{K}} \text{ with } q_{\alpha}(x - y, x - y) \leq u\}.$$

The *covering radius* of $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ is, by definition, the square root of $\max(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$. Define the *determinant* of $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ as the determinant of the matrix of q_{α} in a \mathbb{Z} -basis of $\mathcal{O}_{\mathbb{K}}$; that is, if $\omega_1, \dots, \omega_d$ is a basis for the abelian group $\mathcal{O}_{\mathbb{K}} \cong \mathbb{Z}^d$ (under addition), then $\det(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ is the determinant of the $d \times d$ matrix $(q_{\alpha}(w_i, w_j))$. With this definition, the determinant of $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ is equal to the *square* of the volume of any fundamental domain for the lattice $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$. We remark that some texts define the determinant as the volume of a fundamental domain, but we preferred to follow Bayer Fluckiger’s convention as in [Bay06].

Definition 3.1. Define the *Hermite-like thickness* $\tau(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ of $(\mathcal{O}_{\mathbb{K}}, q_{\alpha})$ as

$$\tau(\mathcal{O}_{\mathbb{K}}, q_{\alpha}) := \frac{\max(\mathcal{O}_{\mathbb{K}}, q_{\alpha})}{\det(\mathcal{O}_{\mathbb{K}}, q_{\alpha})^{1/d}}.$$

Define the *minimal Hermite-like thickness* as

$$\tau_{\min}(\mathcal{O}_{\mathbb{K}}) = \inf\{\tau(\mathcal{O}_{\mathbb{K}}, q_{\alpha}) \mid \alpha \in \mathfrak{B}\},$$

where \mathfrak{B} is as in equation (3.2).

Remark 3.2. Although we called $\tau_{\min}(\mathcal{O}_{\mathbb{K}})$ the minimal Hermite-like thickness, it should be noted that the minimum is taken over the set of inner products coming from elements of \mathfrak{B} and not all possible inner products on \mathbb{R}^d .

It is clear that the concepts of maximum, covering radius, and Hermite-like thickness can be defined more generally for a lattice in a Euclidean space; see [Bay06].

Definition 3.3. Let \mathbb{K} be a number field. Assume that $\omega_1, \dots, \omega_d$ is any integral basis for $\mathcal{O}_{\mathbb{K}}$. Denote the complete list of places of \mathbb{K} by $\sigma_1, \dots, \sigma_d$. The *discriminant* of \mathbb{K} is defined as the square of the determinant of the $d \times d$ matrix $(\sigma_i(\omega_j))$.

See [Jar14, Chs 3 and 7] for further properties of the discriminant as well as the embedding of $\mathcal{O}_{\mathbb{K}}$ in \mathbb{R}^d .

THEOREM 1.3. *Let λ be a Perron number of algebraic degree $d \geq 3$ and spectral ratio ρ . Set $\mathbb{K} := \mathbb{Q}(\lambda)$. Let $\mathcal{O}_{\mathbb{K}}$ be the ring of integers of \mathbb{K} , and denote the discriminant and the minimal Hermite-like thickness of \mathbb{K} by, respectively, $D_{\mathbb{K}}$ and $\tau_{\min}(\mathcal{O}_{\mathbb{K}})$. Then $d_{\text{PF}}^{\text{irr}}(\lambda)$ is bounded above by each of*

$$\left(\frac{8d}{1-\rho}\right)^{d^2} \tau_{\min}(\mathcal{O}_{\mathbb{K}})^{d/2}$$

and

$$\left(\frac{8d}{1-\rho}\right)^{d^2} \sqrt{D_{\mathbb{K}}}.$$

Proof. Let $\mathbb{K}_{\mathbb{R}} := \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\sigma_1, \dots, \sigma_{r+s}$ be as before, and identify $\mathbb{K}_{\mathbb{R}}$ with $\mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^d$. A place σ_j is a field homomorphism $\sigma_j: \mathbb{Q}(\lambda) \rightarrow (\mathbb{R} \text{ or } \mathbb{C})$ and so σ_j is completely determined by $\sigma_j(\lambda)$, which is one of the Galois conjugates of λ . Assume that σ_1 is the real place corresponding to λ itself; that is, $\sigma_1(\lambda) = \lambda$. Let M_{λ} be the linear endomorphisms of $\mathbb{K}_{\mathbb{R}}$ induced by multiplication by λ in \mathbb{K} . The eigenvalues of M_{λ} are the Galois conjugates of λ . For any Galois conjugate λ_i , denote by E_i the invariant subspace for M_{λ} with eigenvalue λ_i , and let $\pi_i: \mathbb{K}_{\mathbb{R}} \rightarrow E_i$ be the projection along the complementary invariant subspace. Therefore, π_i is the projection onto the i th factor under the identification $\mathbb{K}_{\mathbb{R}} \cong \mathbb{R}^r \times \mathbb{C}^s$.

Before going into the details, we explain the main idea when λ is a cubic algebraic integer that is not totally real. In order to find a non-negative integral matrix with spectral radius λ , we follow Thurston’s proof of Lind’s theorem. See [Thu14, pp. 353–354] and

[Lin84]. Let E_1 be the one-dimensional invariant subspace for M_λ with eigenvalue λ , and E_2 be the two-dimensional invariant subspace corresponding to the pair of complex Galois conjugates $\{\delta, \bar{\delta}\}$ of λ . The endomorphism M_λ of $\mathbb{K}_\mathbb{R}$ leaves E_1 and E_2 invariant, and it acts on $E_1 \cong \mathbb{R}$ and $E_2 \cong \mathbb{C}$ by multiplication by, respectively, the numbers λ and δ . Pick a large positive integer $N = N(\delta, \lambda)$ such that if P_δ is a regular N -gon inscribed in a circle of radius R around the origin in E_2 , then $P_\delta \subset E_2 \cong \mathbb{C}$ is invariant under multiplication by the complex number δ/λ . Such an integer N exists since, by the Perron condition, the absolute value of δ/λ is strictly less than 1. Let v be an eigenvector of M_λ with eigenvalue λ , and E_2^v be the affine plane $Rv + E_2$. Then E_2^v lies in the positive half-space E corresponding to λ and v .

Denote the vertices of the shifted polygon $P_v := Rv + P_\delta \subset E_2^v$ by v_1, \dots, v_k , and choose integral points $z_1, \dots, z_k \in \mathbb{R}^3$ such that the distance between z_i and v_i is ‘small’. Since the cone over the points v_1, \dots, v_k lies in the positive half-space and is invariant under M_λ , and the distance between z_i and v_i is small, it is reasonable to expect that the cone \mathcal{C} over the points z_i also lies in the positive half-space E and is invariant under M_λ for ‘large’ R . Let S be the semigroup generated by the set of integral points in the cone \mathcal{C} under vector addition. Then M_λ preserves S and induces an action M_λ^S on S . Moreover, S has a finite generating set, and we can estimate an upper bound for the size $|G|$ of a generating set G using Proposition 2.1. If we write the action of M_λ^S on S in the generating set G , we obtain a non-negative integral matrix of size $|G|$ whose spectral radius is equal to λ . The details of the proof are as follows.

Step 1: Choosing an inner product on $\mathbb{R}^r \times \mathbb{C}^s$. Define \mathfrak{B} as in equation (3.2). Pick $\alpha \in \mathfrak{B}$ and equip $\mathbb{R}^r \times \mathbb{C}^s$ with the inner product q_α . Note that for the norm $|\cdot|_\alpha$ and for any $x \in \mathbb{K}_\mathbb{R}$,

$$|x|_\alpha^2 = \sum_{j=1}^{r+s} |\pi_j(x)|_\alpha^2, \tag{3.4}$$

and so

$$|x|_\alpha \geq |\pi_j(x)|_\alpha \quad \text{for } 1 \leq j \leq r + s. \tag{3.5}$$

Let ℓ be the covering radius of $(\mathcal{O}_\mathbb{K}, q_\alpha)$. Then

$$\ell := \max(\mathcal{O}_\mathbb{K}, q_\alpha)^{1/2} = \tau(\mathcal{O}_\mathbb{K}, q_\alpha)^{1/2} \cdot \det(\mathcal{O}_\mathbb{K}, q_\alpha)^{1/2d}. \tag{3.6}$$

Step 2: Defining the polygon P_j in the invariant subspace E_j for each $j > 1$. Define

$$\rho_j = \frac{|\sigma_j(\lambda)|}{\lambda} \in \mathbb{R} \quad \text{for } 1 < j \leq r + s. \tag{3.7}$$

Hence

$$\rho = \max_{j>1} \{\rho_j\}. \tag{3.8}$$

By the Perron condition, $\rho_j \in (0, 1)$ for each $j > 1$. Set

$$R_j = \frac{(2\sqrt{d} + 4)\ell}{1 - \rho} \quad \text{for } 1 < j \leq r. \tag{3.9}$$

Remark 3.4. Clearly R_j does not depend on $1 < j \leq r$. However, we decided that this notation would be more suitable if one would like to improve the bounds in the article by substituting ρ_j instead of ρ in the definition of R_j . Similarly, in what follows, R_j for $j > r$ will not depend on j .

For each real place σ_j with $j > 1$, define P_j as the set of points of distance at most R_j from the origin in E_j ; in particular, P_j is an interval. Note, for future use, that

$$R_j \leq \left(\frac{8\sqrt{d}}{1 - \rho}\right)\ell \quad \text{for } 1 < j \leq r. \tag{3.10}$$

For $j > r$, define the natural number $N_j \geq 3$ as the smallest positive integer satisfying

$$N_j^2 \geq \frac{2\sqrt{d} + 9}{1 - \rho}. \tag{3.11}$$

Note, for later use, that

$$N_j^2 \leq \frac{16\sqrt{d}}{1 - \rho}. \tag{3.12}$$

In the above, we used the fact that the smallest whole square exceeding $u \geq 16$ is less than or equal to $(\sqrt{u} + 1)^2 \leq 3u/2 + 1$. Set

$$R_j = N_j^2\ell \quad \text{for } r < j \leq r + s. \tag{3.13}$$

For $j > r$, define the solid polygon $P_j \subset E_j$ as a regular N_j -gon inscribed in the circle of radius R_j around the origin. Define the polytope P as the product of P_j for $j > 1$,

$$P := \left\{ z \in \prod_{j>1} E_j \mid \pi_j(z) \in P_j \text{ for every } j \right\} \subset \prod_{j>1} E_j.$$

The number of vertices of P is equal to the product of the number of vertices of P_j for $j > 1$, which is equal to $2^{r-1} \times \prod_{j>r} N_j$.

Step 3: Defining the cone \mathcal{C} in the positive half-space of λ . Let $v \in E_1$ be the positive (with respect to π_1) unit length eigenvector for the eigenvalue λ . Here, the unit length is considered with respect to the distance $|\cdot|_\alpha$. Let E be the positive half-space corresponding to v . Set

$$L = \max\{R_j \mid 1 < j \leq r + s\}. \tag{3.14}$$

Note, for future use, that

$$L > 3\ell. \tag{3.15}$$

Denote by P_v the translated polytope $Lv + P$, and let $\mathcal{C}_v \subset E$ be the cone over the polytope P_v . Equivalently, if $1/L \cdot P_j$ is the dilation of P_j by the factor $1/L$ centered at the origin

of E_j , then

$$C_v = \left\{ z \in E \mid \frac{\pi_j(z)}{|\pi_1(z)|_\alpha} \in \frac{1}{L} \cdot P_j \text{ for every } j > 1 \right\}. \tag{3.16}$$

Denote the vertices of P_v by v_1, v_2, \dots, v_k , where

$$k = 2^{r-1} \times \prod_{j>r} N_j. \tag{3.17}$$

Pick integral points z_1, z_2, \dots, z_k in $\mathcal{O}_{\mathbb{K}}$ such that the distance between v_i and z_i does not exceed the covering radius ℓ of $(\mathcal{O}_{\mathbb{K}}, q_\alpha)$.

First we show that each z_i lies in the positive half-space E . Identify the subspace E_1 with \mathbb{R} via

$$x = rv \in E_1 \longleftrightarrow r \in \mathbb{R}.$$

It is enough to show that for each i , we have $\pi_1(z_i) > 0$. By the triangle inequality

$$\pi_1(z_i) \geq \pi_1(v_i) - |\pi_1(v_i - z_i)| \geq \pi_1(v_i) - |v_i - z_i|_\alpha \geq L - \ell > 0,$$

where the second inequality holds by inequality (3.5), and the last inequality is true by inequality (3.15).

Define \mathcal{C} as the cone over the points z_1, z_2, \dots, z_k with apex at the origin,

$$\mathcal{C} = \{z \in E \mid z = \beta_1 z_1 + \dots + \beta_k z_k, \beta_i \geq 0 \text{ for every } i\}. \tag{3.18}$$

For each z_i , define w_i as the intersection point of the ray through z_i from the origin and the affine hyperplane $Lv + E_1^c$, where $E_1^c = \bigoplus_{j>1} E_j$ is the complementary invariant subspace to E_1 . Then

$$\mathcal{C} = \{z \in E \mid z = \beta_1 w_1 + \dots + \beta_k w_k, \beta_i \geq 0 \text{ for every } i\}. \tag{3.19}$$

Equivalently, if $Q_v \subset Lv + E_1^c$ is the convex hull of w_1, w_2, \dots, w_k , then

$$\mathcal{C} = \left\{ z \in E \mid \frac{z}{|\pi_1(z)|_\alpha} \in \frac{1}{L} \cdot Q_v \right\} = \left\{ z \in E \mid \frac{Lz}{|\pi_1(z)|_\alpha} \in Q_v \right\}. \tag{3.20}$$

Hence, \mathcal{C} is the cone over the points z_1, z_2, \dots, z_k , or, equivalently, the cone over the points w_1, w_2, \dots, w_k . In what follows, we will use the two descriptions of \mathcal{C} as needed.

Step 4: Showing that the cone \mathcal{C} is invariant. We need a few lemmas to prove that \mathcal{C} is invariant.

LEMMA 3.5. *For each i , we have*

$$\begin{aligned} |v_i|_\alpha &\leq \sqrt{d}L, \\ |v_i - w_i|_\alpha &\leq (2\sqrt{d} + 2)\ell. \end{aligned}$$

Proof. For the first inequality, we have

$$|v_i|_\alpha^2 = \sum_{j=1}^{r+s} |\pi_j(v_i)|_\alpha^2 \leq L^2 + \sum_{j>1} R_j^2 \leq (r+s)L^2 \leq dL^2,$$

where we used equations (3.4) and (3.14).

Assume that $w_i = r_i z_i$ for some positive real number r_i . Note that

$$\begin{aligned} |\pi_1(z_i) - \pi_1(v_i)|_\alpha &= |\pi_1(z_i - v_i)|_\alpha \leq |z_i - v_i|_\alpha \leq \ell \\ \implies \pi_1(z_i) &= \pi_1(v_i) + \pi_1(z_i - v_i) \in [L - \ell, L + \ell], \end{aligned}$$

where we used inequality (3.5) in the first line above. Therefore,

$$\begin{aligned} r_i &= \frac{\pi_1(w_i)}{\pi_1(z_i)} = \frac{L}{\pi_1(z_i)} \in \left[\frac{L}{L + \ell}, \frac{L}{L - \ell} \right] \\ \implies r_i &\leq \frac{L}{L - \ell}, \quad \text{and} \quad |r_i - 1| \leq \frac{\ell}{L - \ell}. \end{aligned}$$

Hence, by the triangle inequality,

$$\begin{aligned} |w_i - v_i|_\alpha &= |r_i z_i - v_i|_\alpha \leq |r_i z_i - r_i v_i|_\alpha + |r_i v_i - v_i|_\alpha \\ &= r_i |z_i - v_i|_\alpha + |r_i - 1| |v_i|_\alpha \leq \left(\frac{L}{L - \ell} \right) \ell + \left(\frac{\ell}{L - \ell} \right) \sqrt{d} L \\ &= (\sqrt{d} + 1) \frac{L \ell}{L - \ell} \leq (2\sqrt{d} + 2) \ell. \end{aligned}$$

The last implication above, $L \leq 2(L - \ell)$, holds by inequality (3.15).

Set

$$b = (2\sqrt{d} + 2) \ell. \tag{3.21}$$

LEMMA 3.6. *The following estimates hold for every z_i , where M_λ is the linear endomorphism of $\mathbb{K}_\mathbb{R}$ induced by multiplication by λ in \mathbb{K} .*

(a)

$$\frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} \leq \rho_j \left(\frac{R_j + \ell}{L - \ell} \right) \quad \text{for every } j > 1.$$

(b)

$$\frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} \leq \frac{R_j - b}{L} \quad \text{for every } 1 < j \leq r.$$

(c)

$$\frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} \leq \frac{1}{L} \left(R_j \left(1 - \frac{5}{N_j^2} \right) - b \right) \quad \text{for every } j > r.$$

Proof. (a) We have

$$\begin{aligned} \frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} &= \frac{|\sigma_j(\lambda)|}{|\sigma_1(\lambda)|} \cdot \frac{|\pi_j(z_i)|_\alpha}{|\pi_1(z_i)|_\alpha} = \rho_j \cdot \frac{|\pi_j(z_i)|_\alpha}{|\pi_1(z_i)|_\alpha} \\ &\leq \rho_j \cdot \frac{|\pi_j(v_i)|_\alpha + |\pi_j(z_i - v_i)|_\alpha}{|\pi_1(v_i)|_\alpha - |\pi_1(v_i - z_i)|_\alpha} \\ &\leq \rho_j \cdot \frac{|\pi_j(v_i)|_\alpha + |z_i - v_i|_\alpha}{|\pi_1(v_i)|_\alpha - |v_i - z_i|_\alpha} \leq \rho_j \left(\frac{R_j + \ell}{L - \ell} \right), \end{aligned}$$

where we have used inequality (3.5).

(b) Assume that σ_j is a real place where $1 < j \leq r$, and set $R'_j = R_j/\ell$, $L' = L/\ell$, and $b' = b/\ell$. Then

$$\begin{aligned} \rho_j \left(\frac{R_j + \ell}{L - \ell} \right) \leq \frac{R_j - b}{L} &\iff \rho_j \left(\frac{R'_j + 1}{L' - 1} \right) \leq \frac{R'_j - b'}{L'} \\ &\iff (\rho_j + b')L' + R'_j - b' \leq (1 - \rho_j)L'R'_j. \end{aligned}$$

Hence, using $\rho_j \in (0, 1)$, it is enough to have

$$\begin{aligned} (1 + b')L' + R'_j \leq (1 - \rho_j)L'R'_j &\iff \frac{1 + b'}{R'_j} + \frac{1}{L'} \leq 1 - \rho_j \\ &\iff \frac{2\sqrt{d} + 3}{R'_j} + \frac{1}{L'} \leq 1 - \rho_j. \end{aligned}$$

On the other hand, $L' \geq R'_j$ by equation (3.14). Hence, it suffices to have

$$\frac{2\sqrt{d} + 4}{R'_j} \leq 1 - \rho_j \iff R'_j \geq \frac{2\sqrt{d} + 4}{1 - \rho_j},$$

which holds by equation (3.9).

(c) Now assume that σ_j is a complex place where $j > r$. Then

$$\begin{aligned} \rho_j \left(\frac{R_j + \ell}{L - \ell} \right) &\leq \frac{1}{L} \left(R_j \left(1 - \frac{5}{N_j^2} \right) - b \right) \\ &\iff \rho_j(R_j + \ell) \leq \left(\frac{L - \ell}{L} \right) \left(R_j \left(1 - \frac{5}{N_j^2} \right) - b \right). \end{aligned}$$

By equation (3.14), we have $L \geq R_j = N_j^2\ell$, and so

$$\frac{L - \ell}{L} = 1 - \frac{\ell}{L} \geq 1 - \frac{1}{N_j^2}.$$

Set $b' = b/\ell$. After substituting $R_j = N_j^2\ell$ and dividing both sides by ℓ , it is enough to have

$$\rho_j(N_j^2 + 1) \leq \left(1 - \frac{1}{N_j^2} \right) (N_j^2 - 5 - b').$$

Multiplying both sides by N_j^2 and then expanding, the last inequality is equivalent to

$$N_j^4(1 - \rho_j) + (5 + b') \geq N_j^2(b' + 6 + \rho_j).$$

So, using $\rho_j \in (0, 1)$ and neglecting the positive term $5 + b'$, it suffices to have

$$N_j^4(1 - \rho_j) \geq N_j^2(b' + 7) \iff N_j^2 \geq \frac{b' + 7}{1 - \rho_j} = \frac{2\sqrt{d} + 9}{1 - \rho_j},$$

and the last inequality holds by inequality (3.11). □

Recall that \mathcal{C} is the cone over the points w_1, \dots, w_k with apex at the origin, that is, the cone over the polytope Q_v (that is, the convex hull of w_i). For each i , the point w_i of Q_v is of distance at most $b = (2\sqrt{d} + 2)\ell$ from the corresponding point v_i of P_v . Since b is ‘small’, we expect the polytope Q_v to contain a ‘smaller version’ of P_v . More precisely, define

$$P_v^b = \{x \in P_v \mid \text{dist}(x, \partial P_v) \geq b\},$$

where

$$\text{dist}(X, Y) := \min\{|x - y|_\alpha \mid x \in X, y \in Y\}.$$

LEMMA 3.7. *We have $P_v^b \subset Q_v$.*

Proof. Let B be the ball of radius b around the origin in $E_1^c = \bigoplus_{j>1} E_j$. We show that

$$P_v \subset Q_v + B,$$

where $Q_v + B$ denotes the Minkowski sum. To see this, pick an arbitrary point $x \in P_v$ and write it as a convex combination

$$x = \sum_{i=1}^k \alpha_i v_i,$$

with $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^k \alpha_i = 1$. Then

$$x = \sum_{i=1}^k \alpha_i w_i + \sum_{i=1}^k \alpha_i (v_i - w_i).$$

If we set $y = \sum_{i=1}^k \alpha_i w_i$ and $z = \sum_{i=1}^k \alpha_i (v_i - w_i)$, then $y \in Q_v$. Moreover, by the triangle inequality, $z \in B$, since each term $v_i - w_i$ lies in B . This shows that $P_v \subseteq Q_v + B$.

If we denote the Minkowski difference by \div , then we have

$$P_v \div B \subset (Q_v + B) \div B = Q_v,$$

where the last equality holds since Q_v and B are non-empty, compact, and convex. See Lemma 2.3. Therefore, it suffices to show that $P_v^b \subset P_v \div B$. It follows from the definition

that $P_v^b + B \subset P_v$. Hence, assuming that P_v^b is non-empty, we have

$$P_v^b = (P_v^b + B) \div B \subset P_v \div B,$$

where we used Lemma 2.3 twice. This completes the proof. □

Define

$$P_j^b := \{x \in P_j \mid \text{dist}(x, \partial P_j) \geq b\},$$

and set

$$V := Lv + \{x \in E_1^c \mid \pi_j(x) \in P_j^b \text{ for every } j > 1\},$$

$$W := Lv + \left\{ x \in E_1^c \mid \begin{aligned} &|\pi_j(x)|_\alpha \leq R_j - b \text{ for } 1 < j \leq r, \\ &|\pi_j(x)|_\alpha \leq R_j \left(1 - \frac{5}{N_j^2}\right) - b \text{ for } j > r \end{aligned} \right\}.$$

LEMMA 3.8. *We have $W \subset V \subset P_v^b \subset Q_v$.*

Proof. The inclusion $W \subset V$ follows from the following simple fact: Let T be a regular N -gon inscribed in a circle of radius R centered at the point O . Every point in ∂T is of distance at least $R(1 - 5/N^2)$ from O .

For completeness, we give a proof of the above fact. After scaling, we may assume that $R = 1$. The minimum distance $d(O, x)$ for $x \in \partial T$ is obtained by drawing the perpendicular from O to a side of T . Such a perpendicular has length $\cos(\pi/N)$. Hence

$$d(O, x) \geq \cos\left(\frac{\pi}{N}\right) > 1 - \frac{\pi^2}{2N^2} > 1 - \frac{5}{N^2},$$

where we have used the inequality $\cos(x) > 1 - (x^2/2)$ for $0 < x < 1$.

The inclusion $P_v^b \subset Q_v$ was proved in Lemma 3.7. Now we show that $V \subset P_v^b$. Pick arbitrary points $x \in V$ and $y \in \partial P_v = \partial(Lv + P)$. Since $P = \prod_{j>1} P_j$ is a product, there is $j > 1$ such that $\pi_j(y) \in \partial P_j$. Therefore

$$\text{dist}(x, y) \geq \text{dist}(\pi_j(x), \pi_j(y)) \geq \text{dist}(P_j^b, \partial P_j) \geq b.$$

In the above, we used the fact that projection onto a non-empty closed convex set in a Euclidean space is distance decreasing; see e.g. [Sch13, Theorem 1.2.1]. Hence $x \in P_v^b$, proving the inclusion $V \subset P_v^b$. □

Using equation (3.20), to show that \mathcal{C} is invariant, it is enough to prove that for every z_i ,

$$L \cdot \frac{M_\lambda(z_i)}{|\pi_1(M_\lambda(z_i))|_\alpha} \in W \subset Q_v.$$

Equivalently, we would like to show that for every $j > 1$ and every z_i , the following inequalities hold:

$$\frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} \leq \frac{R_j - b}{L} \quad \text{for } 1 < j \leq r,$$

$$\frac{|\pi_j(M_\lambda(z_i))|_\alpha}{|\pi_1(M_\lambda(z_i))|_\alpha} \leq \frac{1}{L} \left(R_j \left(1 - \frac{5}{N_j^2} \right) - b \right) \quad \text{for } j > r.$$

But the above inequalities hold by Lemma 3.6. This finishes the proof of the invariance of \mathcal{C} .

Step 5: Defining a non-negative integral matrix A , where each irreducible component of A has spectral radius equal to λ . Let S be the semigroup generated by elements of $\mathcal{C} \cap \mathcal{O}_{\mathbb{K}}$ under vector addition. By Proposition 2.1, the set of integral points (that is, elements of $\mathcal{O}_{\mathbb{K}}$) in the compact region

$$C := \{\alpha_1 z_1 + \alpha_2 z_2 + \dots + \alpha_k z_k \mid 0 \leq \alpha_i \leq 1 \text{ for every } i\} \tag{3.22}$$

generates S . Enumerate the set of points in $C \cap \mathcal{O}_{\mathbb{K}}$ by c_1, \dots, c_n . Recall that M_λ acts on S . Moreover, M_λ can be represented by the companion matrix B when written in the basis $\{1, \lambda, \dots, \lambda^{d-1}\}$ of $\mathbb{K}_{\mathbb{R}} = \mathbb{Q}(\lambda) \otimes \mathbb{R}$. Let $A = [a_{ij}]$ be a non-negative integral matrix corresponding to the action of B on S in the basis $\{c_1, \dots, c_n\}$. Hence

$$Bc_j = \sum_{i=1}^n a_{ij}c_i, \tag{3.23}$$

where $B: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the companion matrix associated with λ . There might be various choices for A , when Bc_j can be written as a non-negative linear combination of c_1, \dots, c_n in more than one way.

Lind’s argument [Lin84, pp. 288–289] shows that every irreducible component of A has spectral radius λ . We follow the proof given in [LM21, Lemma 11.1.10] briefly for the reader’s convenience. Replace A by one of its irreducible components r ; this amounts to taking a minimal subset of $\{c_1, \dots, c_n\}$ for which equation (3.23) holds. Let μ be the spectral radius of A . Let e_i be the i th unit vector in \mathbb{R}^n , and define the linear map $P: \mathbb{R}^n \rightarrow \mathbb{R}^d$ by $P(e_i) = c_i$. Then by equation (3.23), we have $PA = BP$. Since A is non-negative and irreducible, by the Perron–Frobenius theorem A has a positive eigenvector v_μ corresponding to μ . Since Pv_μ is a positive linear combination of the c_j , and each c_j satisfies $\pi_1(c_j) > 0$, we necessarily have $\pi_1(Pv_\mu) > 0$ and so $Pv_\mu \neq 0$. Moreover,

$$B(Pv_\mu) = P(Av_\mu) = \mu(Pv_\mu).$$

Hence, Pv_μ is an eigenvector for B with eigenvalue μ . However, the only eigenvectors of B with positive E_1 component lie in E_1 . Hence Pv_μ is a multiple of v (the eigenvector with eigenvalue λ) and

$$\begin{aligned} \mu(Pv_\mu) &= B(Pv_\mu) = \lambda(Pv_\mu) \\ \implies \mu &= \lambda. \end{aligned}$$

Remark 3.9. In [Thu14, p. 354], Thurston gave a new proof of Lind’s converse to the integer Perron–Frobenius theorem. Thurston assumed that the matrix constructed via his method can be taken to be primitive as well. If this is the case, then Theorem 1.3 readily upgrades to give an upper bound for the Perron–Frobenius degree of a Perron number.

Step 6: Bounding the dimension of the matrix A . By Proposition 2.2, the number of integral points in C is at most

$$\frac{\text{Vol}(C)}{\text{Covol}(\mathcal{O}_{\mathbb{K}})} \cdot (d + 1)! = \frac{\text{Vol}(C)}{\det(\mathcal{O}_{\mathbb{K}}, q_{\alpha})^{1/2}} \cdot (d + 1)! .$$

We give an upper bound for $\text{Vol}(C)$ via a series of lemmas.

LEMMA 3.10. *For every $j > 1$ and every w_i , we have*

$$|\pi_j(w_i)|_{\alpha} < 2R_j .$$

Proof. By the triangle inequality,

$$\begin{aligned} |\pi_j(w_i)|_{\alpha} &\leq |\pi_j(v_i)|_{\alpha} + |\pi_j(w_i - v_i)|_{\alpha} \\ &\leq |\pi_j(v_i)|_{\alpha} + |w_i - v_i|_{\alpha} \\ &\leq R_j + (2\sqrt{d} + 2)\ell < 2R_j \\ &\iff R_j > (2\sqrt{d} + 2)\ell . \end{aligned}$$

In the first line above, inequality (3.5) is used. The second line follows from Lemma 3.5, and the last inequality holds by equations (3.9) and (3.13), and inequality (3.11). □

LEMMA 3.11. *Let $M := 2k\sqrt{d}$, where k is the number of vertices of P_v . Then*

$$C \subset \{rx \in \mathbb{R}^d \mid x \in Q_v, \text{ and } 0 \leq r \leq M\} .$$

Proof. Since Q_v is the convex hull of w_i , the ray from the origin and passing through an arbitrary point p of C intersects Q_v at some point w . Therefore, if we define M_1 as

$$M_1 = \frac{\max_{p \in C} |p|_{\alpha}}{\min_{w \in Q_v} |w|_{\alpha}} ,$$

then

$$C \subset \{rx \in \mathbb{R}^d \mid x \in Q_v, \text{ and } 0 \leq r \leq M_1\} .$$

It is enough to show that $M_1 \leq M$. For every point w in Q_v ,

$$|w|_{\alpha} \geq |\pi_1(w)|_{\alpha} = L .$$

On the other hand, if p is a point in C , then by the triangle inequality, we have

$$|p|_{\alpha} \leq k \cdot \max_i |z_i|_{\alpha} \leq k \cdot \max_i (|v_i|_{\alpha} + \ell) \leq k(\sqrt{d}L + \ell) < 2k\sqrt{d}L ,$$

where we used Lemma 3.5 for the implication $|v_j|_\alpha \leq \sqrt{d}L$. Combining the previous inequalities gives

$$M_1 = \frac{\max_{p \in C} |p|_\alpha}{\min_{w \in Q_v} |w|_\alpha} \leq \frac{2k\sqrt{d}L}{L} = 2k\sqrt{d} = M. \quad \square$$

LEMMA 3.12. *We have*

$$\text{Vol}(Q_v) \leq 2^{2d-2} \times \prod_{j>1}^r R_j \times \prod_{j>r} R_j^2,$$

where $\text{Vol}(Q_v)$ is the $(d - 1)$ -dimensional volume of Q_v .

Proof. We have

$$\text{Vol}(Q_v) \leq \prod_{j>1}^r \text{Length}(\pi_j(Q_v)) \times \prod_{j>r} \text{Area}(\pi_j(Q_v)),$$

where Vol , Length , and Area are with respect to the inner product q_α . For $j > 1$, define \mathbf{R}_j as

$$\mathbf{R}_j = \max_{w \in Q_v} |\pi_j(w)|_\alpha.$$

Therefore

$$\text{Vol}(Q_v) \leq \prod_{j>1}^r (2\mathbf{R}_j) \times \prod_{j>r} (\pi \mathbf{R}_j^2).$$

By Lemma 3.10 and the triangle inequality, we have $\mathbf{R}_j < 2R_j$. The lemma now follows from the inequalities $\mathbf{R}_j < 2R_j$ and $\pi < 4$, and the equality $r + 2s = d$. \square

LEMMA 3.13. *We have*

$$\text{Vol}(C) < 2^{d^2+6d-1} \times d^{(ds/4)+(3d/2)-1} \times \frac{\ell^d}{(1 - \rho)^{d+(ds/2)}},$$

where C is defined as in equation (3.22).

Proof. Note that

$$M = 2k\sqrt{d} = 2^r \times \prod_{j>r} N_j \times \sqrt{d} \leq 2^r \times \left(\frac{16\sqrt{d}}{1 - \rho}\right)^{s/2} \times \sqrt{d} < 2^d \times \frac{d^{(s/4)+1}}{(1 - \rho)^{s/2}}, \tag{3.24}$$

where we used equation (3.17), inequality (3.12), and the identity $r + 2s = d$. By Lemma 3.11,

$$\text{Vol}(C) \leq \text{Vol}(\{rx \in \mathbb{R}^d \mid x \in Q_v, \text{ and } 0 \leq r \leq M\}).$$

The upper bound above is the volume of a cone over $M \cdot Q_v$, where $M \cdot Q_v$ denotes a dilation of Q_v with the factor of M from the point $0 \in \mathbb{R}^d$. The height of this cone is ML , since the height of Q_v measured perpendicularly from its apex along the E_1 -axis is L .

Hence, the volume of the cone is equal to

$$\frac{(ML)\text{Vol}(M \cdot Q_v)}{d} = \frac{M^d L}{d} \text{Vol}(Q_v).$$

Therefore, we have

$$\text{Vol}(C) \leq \frac{M^d L}{d} \times \text{Vol}(Q_v) \leq \left(2^d \times \frac{d^{(s/4)+1}}{(1-\rho)^{s/2}}\right)^d \times \frac{L}{d} \times 2^{2d-2} \times \prod_{j>1}^r R_j \times \prod_{j>r} R_j^2,$$

where we used Lemma 3.12. Moreover, using inequalities (3.10) and (3.12), and equation (3.13), we have

$$\begin{aligned} \prod_{j>1}^r R_j \times \prod_{j>r} R_j^2 &\leq \prod_{j>1}^r \left(\frac{8\sqrt{d}\ell}{1-\rho}\right) \times \prod_{j>r} \left(\frac{16\sqrt{d}\ell}{1-\rho}\right)^2 \\ &= 2^{3r+8s-3} \times d^{(d-1)/2} \times \frac{\ell^{d-1}}{(1-\rho)^{d-1}}. \end{aligned}$$

Combining with the previous upper bound for Vol(C), and using $L < (16\sqrt{d}\ell/(1-\rho))$, we have

$$\begin{aligned} \text{Vol}(C) &\leq 2^{d^2+2d+3r+8s-1} \times d^{(ds/4)+(3d/2)-1} \times \frac{\ell^d}{(1-\rho)^{d+(ds/2)}} \\ &\leq 2^{d^2+6d-1} \times d^{(ds/4)+(3d/2)-1} \times \frac{\ell^d}{(1-\rho)^{d+(ds/2)}}, \end{aligned}$$

where we used the relation $r + 2s = d$. □

Therefore, the number of integral points in C is at most

$$\begin{aligned} &\frac{\text{Vol}(C)}{\det(\mathcal{O}_{\mathbb{K}}, q_\alpha)^{1/2}} \cdot (d+1)! \\ &\leq 2^{d^2+6d} \times d^{(ds/4)+(5d/2)-1} \times \frac{\ell^d}{\det(\mathcal{O}_{\mathbb{K}}, q_\alpha)^{1/2}} \times \frac{1}{(1-\rho)^{d+(ds/2)}} \\ &= 2^{d^2+6d} \times d^{(ds/4)+(5d/2)-1} \times \tau(\mathcal{O}_{\mathbb{K}}, q_\alpha)^{d/2} \times \frac{1}{(1-\rho)^{d+(ds/2)}}. \end{aligned}$$

In the first line above, we used the inequality

$$(d+1)! = (d+1)d! < (2d)d^{d-1} = 2d^d.$$

The second line above uses the definition of ℓ and Definition 3.1. By taking infimum over all $\alpha \in \mathfrak{B}$, we obtain the upper bound

$$2^{d^2+6d} \times d^{(ds/4)+(5d/2)-1} \times \frac{\tau_{\min}(\mathcal{O}_{\mathbb{K}})^{d/2}}{(1-\rho)^{d+(ds/2)}},$$

for the number of integral points in C and hence for the dimension of the matrix A.

Bayer Fluckiger showed in [Bay06, Proposition 4.2], as a corollary of the work of Banaszczyk [Ban93, Theorem 2.2], that for any $\alpha \in \mathfrak{B}$,

$$\tau(\mathcal{O}_{\mathbb{K}}, q_\alpha) \leq \frac{d}{4} \cdot D_{\mathbb{K}}^{1/d}. \tag{3.25}$$

This gives the upper bound

$$2^{d^2+5d} \times d^{(ds/4)+3d-1} \times \frac{\sqrt{D_{\mathbb{K}}}}{(1-\rho)^{d+(ds/2)}}$$

for the dimension of A . Both parts of the theorem now follow from the crude estimates

$$2^{d^2+6d} \leq 8^{d^2}, \quad \frac{ds}{4} + 3d - 1 < d^2, \quad d + \frac{ds}{2} < d^2. \quad \square$$

The bound in Theorem 1.3 is perhaps enormous compared to the Perron–Frobenius degree, so we have not tried to make the constants optimal. The point is to have an explicit bound in terms of data that we believe are relevant to the Perron–Frobenius degree; see Question 5.2.

Remark 3.14. Denote the covering conjecture for dimension d by C_d .

CONJECTURE 3.15. (Covering conjecture) *The covering radius of any well-rounded unimodular lattice L in \mathbb{R}^d (with the standard norm $|\cdot|$) satisfies*

$$\sup_{x \in \mathbb{R}^d} \inf_{y \in L} |x - y| \leq \frac{\sqrt{d}}{2}.$$

Equality happens if and only if $L = g \cdot \mathbb{Z}^d$ for some $g \in \text{SO}_d(\mathbb{R})$.

See McMullen [McM05] for the definition of *well-rounded lattice* and the application of the covering conjecture to *Minkowski’s conjecture*. The covering conjecture is proved for $d \leq 9$; see [KR16] and the references therein. Moreover, it is known to be false for $d \geq 30$; see [RSW17].

In [Bay06, p. 313], Bayer Fluckiger pointed out that McMullen’s results [McM05] together with the covering conjecture C_d imply that for any totally real λ of degree d , we have $\tau_{\min}(\mathcal{O}_{\mathbb{K}}) \leq (d/4)$. Hence, for any $d \leq 9$ and for any totally real Perron number λ of degree d ,

$$d_{\text{PF}}^{\text{irr}}(\lambda) \leq \left(\frac{8d}{1-\rho} \right)^{d^2}.$$

It would be interesting to know which number fields satisfy an inequality similar to that of $\tau_{\min}(\mathcal{O}_{\mathbb{K}}) \leq (d/4)$ with upper bound only depending on the degree d . As pointed out kindly by McMullen to me, it is easy to see that such a bound does not exist for imaginary quadratic fields $\mathbb{Q}(\sqrt{n})$ for $n < 0$. On the other hand, Bayer Fluckiger showed that the inequality $\tau_{\min}(\mathcal{O}_{\mathbb{K}}) \leq (d/4)$ holds for fields of the form $\mathbb{K} = \mathbb{Q}(\zeta_{p^r})$ or $\mathbb{K} = \mathbb{Q}(\zeta_{p^r} + \zeta_{p^r}^{-1})$, where p is an odd prime number, r is a natural number, and ζ_{p^r} is a primitive p^r th root of unity; see respectively [Bay06, p. 319, line 4] and [Bay06, Lemma 8.5].

Example 3.16. (Pisot numbers) A real algebraic integer $\alpha > 1$ is called *Pisot* if all other Galois conjugates of α lie in the unit circle $\{z \in \mathbb{C} \mid |z| < 1\}$. Note, a Pisot number is always Perron. We collect a few facts about Pisot numbers.

- (1) A number field is called *real* if it has at least one real place. It is clear that every number field containing a Pisot number with the same degree should be real. Pisot

[Pis38] proved that every real number field \mathbb{K} of degree d contains a Pisot number of degree d . Moreover, the set of Pisot numbers of degree d in \mathbb{K} is closed under multiplication; see [Mey72, Corollary, p. 33].

- (2) The smallest Pisot number is the largest root p of $x^3 - x - 1$ and is known as the *plastic constant*. This was identified as the smallest known Pisot number by Salem [Sal44], and Siegel proved it to be the smallest possible Pisot number [Sie44]. In particular, if λ is a Pisot number with spectral ratio ρ , then $\rho < p^{-1}$ implying that

$$\frac{1}{1 - \rho} < \frac{1}{1 - p^{-1}}.$$

Let \mathbb{K} be a real number field of degree $d \geq 3$. Let λ be any Pisot number in \mathbb{K} of degree d , and note that $\mathbb{Q}(\lambda) = \mathbb{K}$. By Theorem 1.3, $d_{\text{PF}}^{\text{irr}}(\lambda)$ is bounded above by

$$\left(\frac{8d}{1 - p^{-1}}\right)^{d^2} \sqrt{D_{\mathbb{K}}}.$$

Note this upper bound only depends on \mathbb{K} and not on the Pisot number $\lambda \in \mathbb{K}$. Every number field \mathbb{K} has only finitely many subfields $\mathbb{K}' \subset \mathbb{K}$. Hence there is a similar upper bound depending only on \mathbb{K} for an arbitrary Pisot number $\lambda \in \mathbb{K}$ (with any degree).

4. Primitive matrices

In this section, we use Theorem 1.3 to derive an upper bound for the Perron–Frobenius degree of a Perron number λ . A useful observation is that if there exists an irreducible matrix A with spectral radius $\lambda - 1$, then λ is the spectral radius of the *primitive* matrix $I + A$. This is because $I + A$ is irreducible and has positive trace; hence it is primitive.

THEOREM 1.6. *Let λ be a Perron number of algebraic degree $d \geq 3$ and spectral ratio ρ . Set $\mathbb{K} := \mathbb{Q}(\lambda)$. Denote the bound from Theorem 1.3 by $B(\mathbb{K}, \rho)$, and let $\kappa(\mathbb{K}, \rho)$ be as in Notation 1.5. The Perron–Frobenius degree of λ is bounded above by*

$$\max\{2^{d^2} B(\mathbb{K}, \rho), \kappa(\mathbb{K}, \rho)\}.$$

Proof. We may assume that $\lambda \geq M = 1 + (4/(1 - \rho))$. First we show that $\lambda - 1$ is Perron. The Galois conjugates of $\lambda - 1$ are equal to $\lambda_i - 1$. We have

$$\begin{aligned} \frac{|\lambda_i - 1|}{\lambda - 1} &\leq \frac{|\lambda_i| + 1}{\lambda - 1} \leq \frac{\rho\lambda + 1}{\lambda - 1} = \rho + \frac{\rho + 1}{\lambda - 1} \\ &\leq \rho + \frac{2}{\lambda - 1} \leq \rho + \frac{1 - \rho}{2} < 1, \end{aligned}$$

where we used the assumptions $\rho < 1$ and $\lambda \geq 1 + (4/(1 - \rho))$. Denote the spectral ratio for $\lambda - 1$ by μ . Note that

$$1 - \mu \geq \frac{1 - \rho}{2}. \tag{4.1}$$

In fact, we just showed that for every i ,

$$\frac{|\lambda_i - 1|}{\lambda - 1} \leq \rho + \frac{1 - \rho}{2},$$

so

$$1 - \frac{|\lambda_i - 1|}{\lambda - 1} \geq 1 - \left(\rho + \frac{1 - \rho}{2} \right) = \frac{1 - \rho}{2}.$$

Inequality (4.1) in particular implies that

$$B(\mathbb{K}, \mu) \leq 2^{d^2} B(\mathbb{K}, \rho). \quad (4.2)$$

Now $\lambda - 1$ is a Perron number of spectral ratio μ and it generates the same number field \mathbb{K} as λ . By Theorem 1.3, there is an integral irreducible matrix A with spectral radius $\lambda - 1$ and dimension at most $B(\mathbb{K}, \mu)$. Then $I + A$ is a primitive matrix of the same dimension and with spectral radius λ . \square

5. Questions

In Theorem 1.3, for technical reasons, we constructed an integral *irreducible* matrix with spectral radius equal to a given Perron number, although it would have been more natural to construct an integral *primitive* matrix instead. This motivates the following questions.

Question 5.1.

- (1) Are there upper bounds for $d_{\text{PF}}(\lambda)$ in terms of $d_{\text{PF}}^{\text{irr}}(\lambda)$?
- (2) Does $d_{\text{PF}}(\lambda) = d_{\text{PF}}^{\text{irr}}(\lambda)$ always hold?

The lower bound for the Perron–Frobenius degree in [Yaz21] is in terms of the two largest Galois conjugates in the complex plane. Therefore, the following is a natural question.

Question 5.2. Let d and ρ denote respectively the algebraic degree and the spectral ratio. Is there an upper bound $B = B(d, \rho)$ for $d_{\text{PF}}^{\text{irr}}(\lambda)$ (respectively $d_{\text{PF}}(\lambda)$) where λ is an arbitrary Perron number?

We expect the above question to have a negative answer, meaning that in Theorem 1.3, the arithmetic information on $\mathcal{O}_{\mathbb{K}}$ cannot be overlooked. A unit algebraic integer λ is *bi-Perron* if all other Galois conjugates of λ lie in the annulus $\{c \in \mathbb{C} \mid \lambda^{-1} < |z| < \lambda\}$ except possibly for $\pm\lambda^{-1}$. See [McM14].

Question 5.3. How can the bound in Theorem 1.3 be improved for special classes of algebraic integers such as totally real Perron numbers, Pisot numbers, Salem numbers, or bi-Perron numbers?

In particular, we ask the following about Pisot numbers.

Question 5.4. Let d denote the algebraic degree. Is there an upper bound $B = B(d)$ for $d_{\text{PF}}^{\text{irr}}(\lambda)$ (respectively $d_{\text{PF}}(\lambda)$) where λ is an arbitrary Pisot number?

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REFERENCES

- [Ban93] W. Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Math. Ann.* **296**(1) (1993), 625–635.
- [Bay06] E. Bayer Fluckiger. Upper bounds for Euclidean minima of algebraic number fields. *J. Number Theory* **121**(2) (2006), 305–323.
- [BH91] M. Boyle and D. Handelman. The spectra of nonnegative matrices via symbolic dynamics. *Ann. of Math. (2)* **133**(2) (1991), 249–316.
- [BL02] M. Boyle and D. Lind. Small polynomial matrix presentations of non-negative matrices. *Linear Algebra Appl.* **355** (2002), 49–70.
- [Jar14] F. Jarvis. *Algebraic Number Theory*. Springer, Cham, 2014.
- [KOR00] K. Kim, N. Ormes and F. Roush. The spectra of nonnegative integer matrices via formal power series. *J. Amer. Math. Soc.* **13**(4) (2000), 773–806.
- [KR16] L. Kathuria and M. Raka. On conjectures of Minkowski and Woods for $n = 9$. *Proc. Indian Acad. Sci. (Math. Sci.)* **126**(4) (2016), 501–548.
- [Lin84] D. A. Lind. The entropies of topological Markov shifts and a related class of algebraic integers. *Ergod. Th. & Dynam. Sys.* **4**(2) (1984), 283–300.
- [LM21] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding (Cambridge Mathematical Library)*, 2nd edn. Cambridge University Press, Cambridge, UK, 2021.
- [McM05] C. McMullen. Minkowski’s conjecture, well-rounded lattices and topological dimension. *J. Amer. Math. Soc.* **18**(3) (2005), 711–734.
- [McM14] C. McMullen. Slides for dynamics and algebraic integers: perspectives on Thurston’s last theorem, 2014, <https://people.math.harvard.edu/~ctm/expositions/home/text/talks/cornell/2014/slides/slides.pdf>.
- [Mey72] Y. Meyer. *Algebraic Numbers and Harmonic Analysis. (North-Holland Mathematical Library, 2)*. North-Holland Publishing Co., Amsterdam; American Elsevier Publishing Co., Inc., New York, 1972.
- [Pis38] C. Pisot. La répartition modulo 1 et les nombres algébriques. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **7**(3–4) (1938), 205–248.
- [RSW17] O. Regev, U. Shapira and B. Weiss. Counterexamples to a conjecture of Woods. *Duke Math. J.* **166**(13) (2017), 2443–2446.
- [Sal44] R. Salem. A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan. *Duke Math. J.* **11**(1) (1944), 103–108.
- [Sch13] R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory (Encyclopedia of Mathematics and Its Applications)*, 2nd edn. Cambridge University Press, Cambridge, UK, 2013.
- [Sie44] C. L. Siegel. Algebraic integers whose conjugates lie in the unit circle. *Duke Math. J.* **11**(3) (1944), 597–602.
- [Thu14] W. Thurston. Entropy in dimension one. *Frontiers in Complex Dynamics: In Celebration of John Milnor’s 80th Birthday, (Princeton Mathematical Series, 51)* Eds A. Bonifant, M. Lyubich and S. Sutherland, Princeton University Press, Princeton, NJ, 2014.
- [Yaz21] M. Yazdi. Lower bound for the Perron–Frobenius degrees of Perron numbers. *Ergod. Th. & Dynam. Sys.* **41**(4) (2021), 1264–1280.