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Construction of foliations with prescribed separatrix

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Abstract. A germ of a singular foliation in $\mathbb{C}^2$ is built, with its analytical class of separatrix and holonomy representations prescribed. Thanks to this construction, we study the link between the moduli space of a foliation and the moduli space of its separatrix.

0. *Introduction*

Considering the problem of moduli for a germ of singular holomorphic foliation $\mathcal{F}$ in $\mathbb{C}^2$ highlights many kinds of topological and analytical invariants. Invariants of the first kind are derived from the reduction of singularities $E : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of:

(1) the topological class of the manifold $\mathcal{M}$ (a combinatorial invariant);
(2) the analytical class of the pointed divisor $\mathcal{D}$; and
(3) the analytical class of the manifold $\mathcal{M}$.

Invariants of the second kind are more related to the foliation itself: the collection of projective holonomy representations defined over each component of the divisor $\mathcal{D}$, the so-called holonomy pseudo-group. A natural problem is to know if *coherent* data of the above invariants correspond to a concrete foliation. The first step towards answering this question is a theorem of Lins Neto [**7**], which establishes the possibility of constructing a foliation with invariant (1) and projective holonomy prescribed. In his thesis [**12**], Seguy shows that it is possible to fix invariant (2). The aim of this article is to prove that one can even prescribe invariant (3) and the holonomy invariants in the construction of a foliation.

The first three sections of this paper are devoted to proving the above result. The main tool of our construction is the equisingular unfolding of foliations. Basically, it is a deformation of a foliation obtained as the family of planar sections of a codimension one foliation on a manifold of higher dimension. Such a deformation is topologically trivial and preserves the holonomy. It was introduced in [**8**] to study the local moduli space of a foliation. It is not easy to find in a constructive way a non-analytically trivial deformation of this type; except in the very special case where the foliation admits a first integral, no example is known. One is lead to the study of the cohomological interpretation in order to build non-trivial equisingular unfoldings. In the first section, we fix a manifold $\mathcal{M}$ which is built over the reduction tree of a foliation and we define a special class of manifold denoted by $\mathrm{Glu}_0(\mathcal{M}, \mathcal{U}, Z)$ related to $\mathcal{M}$. In the second section, $\mathcal{M}$ is supposed to be foliated by $\mathcal{F}$. A property of cobordism type is highlighted and allows us to detect the existence of a foliation on any element of $\mathrm{Glu}_0(\mathcal{M}, \mathcal{U}, Z)$. This foliation will automatically be linked to $\mathcal{F}$ by an equisingular unfolding. Theorem 3.1 states that, under the general hypothesis of being of *second kind* (§2.2), the cobordism property holds for any element of $\mathrm{Glu}_0(\mathcal{M}, \mathcal{U}, Z)$. In the third section, we deduce the following result from the cobordism property.

THEOREM 0.1. *Let $\mathcal{F}$ be of second kind. If $\mathcal{M}$ and $\mathcal{M}'$ are topologically equivalent then there exists a holomorphic foliation on $\mathcal{M}'$ linked to $\mathcal{F}$ by equisingular unfolding.*

A trivial but perhaps more explicit corollary of the above theorem is the following.

THEOREM 0.2. *Let ω_0 be a germ of singular holomorphic 1-form of the second kind at $0 \in \mathbb{C}^2$ and $E : \mathcal{M} \to \mathbb{C}^2$ the reduction of its singularities. For any blowing-up process E' with the same dual tree as E, there exists a 1-form ω' at $0 \in \mathbb{C}^2$ linked to ω by an equisingular unfolding such that the reduction of the singularities of ω' is exactly E'.*

The last section is devoted to the study of the relations between the moduli of a foliation and its invariant analytical curves, the so-called separatrix. After establishing a property of finite determination, we use the existence theorem to prove the following.

THEOREM 0.3. *Let $\mathcal{F}_0$ be a non-dicritical generalized curve at $0 \in \mathbb{C}^2$. For any curve $S \subset \mathbb{C}^2, 0$ topologically equivalent to $\mathrm{Sep}(\mathcal{F}_0)$, there exists a germ of a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ topologically equivalent to $\mathcal{F}$ with $\mathrm{Sep}(\mathcal{F}) = S$.*

The previous theorem can be expressed in terms of moduli spaces: the natural map $\mathtt{M}(\mathcal{F}) \to \mathtt{M}(\mathrm{Sep}(\mathcal{F}))$ is onto, where $\mathtt{M}(\cdot)$ refers to the moduli space.

1. *The categories* $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$

The aim of this section is to describe a family of sub-categories in *the marked trees category* (§1.1). These categories are built due to a *gluing process* (§1.3) over a fixed tree and present some good computational properties (§1.5).

1.1. *The marked trees category.*

A blowing-up process over $\mathbb{C}^2$ is a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{M}^h & \xrightarrow{E^h} \cdots \longrightarrow & \mathcal{M}^j & \xrightarrow{E^j} & \mathcal{M}^{j-1} & \longrightarrow \cdots \xrightarrow{E^1} & \mathcal{M}^0 & = & \mathbb{C}^2 \\
\cup & & \cup & & \cup & & \cup & & \\
\Sigma^h & \longrightarrow \cdots \longrightarrow & \Sigma^j & \longrightarrow & \Sigma^{j-1} & \longrightarrow \cdots \longrightarrow & \Sigma^0 & = & \{0\} \\
\cup & & \cup & & \cup & & \cup & & \\
S^h & \longrightarrow \cdots \longrightarrow & S^j & \longrightarrow & S^{j-1} & \longrightarrow \cdots \longrightarrow & S^0 & = & \{0\}
\end{array} \tag{1}$$

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where $\mathcal{M}^j$ is an analytical two-dimensional manifold; Σ^j is a finite subset of the exceptional divisor $\mathcal{D}^j := (E^1 \circ \cdots \circ E^j)^{-1}(S^0)$; E^{j+1} is the standard blowing-up process centered at S_j. The set Σ_j is called the set of *singular points*. The set of irreducible components of $\mathcal{D}^j$ is denoted by $\mathrm{Comp}(\mathcal{D}^j)$. The integer h is called the *height* of the blowing-up process and $(\mathcal{M}^h, \mathcal{D}^h, \Sigma^h)$ the *top* of the process. The composed map $E_h := E^1 \circ \cdots \circ E^h$ is called the *total morphism* of the process.

The blowing-up process appears naturally in the desingularization theory. This article focuses on isolated singularities of holomorphic foliations in $\mathbb{C}^2$. In this context, the Seidenberg's theorem [**10, 13**] states that there exists a blowing-up process $(E^j, \mathcal{M}^j, \Sigma^j, S^j)_{j=1\ldots h}$ reducing the singularities of $\mathcal{F}$. A singularity of a foliation is *reduced* when given by a holomorphic 1-form with linear part

$$\lambda x\, dy + \beta y\, dx, \quad \beta \neq 0, \quad \frac{\lambda}{\beta} \notin \mathbb{Q}_{<0}.$$

In the commutative diagram (1), Σ^j refers to the singularities of $E^{j*}\mathcal{F}$ and S^j to the non-reduced ones. The foliation $E^{h*}\mathcal{F}$ is required to have only reduced singularities.

More generally, we refer to a triplet $(\mathcal{M}, \mathcal{D}, \Sigma)$ as a *tree* where $\mathcal{M}$ is a two-dimensional holomorphic manifold germ with a closed normal crossing curve $\mathcal{D}$. Each irreducible component of $\mathcal{D}$ is biholomorphic to $\mathbb{P}^1$; Σ is a finite subset of $\mathcal{D}$ which contains the singular locus of $\mathcal{D}$. Let us denote by $\langle \mathcal{M}, \mathcal{D} \rangle$ the matrix

$$[\langle D, D' \rangle]_{D, D' \in \mathrm{Comp}(\mathcal{D})}$$

where $\langle D, D' \rangle$ is the intersection number of the components D and D'. It is referred to as the *tree intersection matrix*.

The *marked tree notion* was introduced in [**12**] in order to compare the foliation semi-local invariants. Let $(\mathcal{M}_0^h, \mathcal{D}_0^h, \Sigma_0^h)$ be the top of a blowing-up process. We refer to the two bijections

$$\sigma : \Sigma_0^h \xrightarrow{\sim} \Sigma, \quad \kappa : \mathrm{Comp}(\mathcal{D}_0^h) \xrightarrow{\sim} \mathrm{Comp}(\mathcal{D})$$

as the *indexation* of $(\mathcal{M}, \mathcal{D}, \Sigma)$ related to $(\mathcal{M}_0^h, \mathcal{D}_0^h, \Sigma_0^h)$ such that for any irreducible component $D \subset \mathcal{D}_0^h$, $\sigma(D \cap \Sigma_0^h) = \kappa(D) \cap \Sigma$. Moreover, let us denote by $\Sigma \check{\times} \mathcal{D}$ the set

$$\{(s, D) \mid s \in D\} \subset \Sigma \times \mathrm{Comp}(\mathcal{D}).$$

Definition 1.1. A marking of $(\mathcal{M}, \mathcal{D}, \Sigma)$ related to $(\mathcal{M}_0^h, \mathcal{D}_0^h, \Sigma_0^h)$ is an indexation (σ, κ) such that κ conjugates the intersection matrix:

$$[\langle \kappa(D), \kappa(D') \rangle]_{D, D' \in \mathrm{Comp}(\mathcal{D}_0^h)} = \langle \mathcal{M}_0^h, \mathcal{D}_0^h \rangle.$$

The *marked weighted dual tree*, denoted by $\mathbb{A}^*[\mathcal{M}, \mathcal{D}, \Sigma]$, is the weighted dual graph whose incidence matrix is the intersection matrix and the tree indexation. Two marked weighted dual trees are *topologically equivalent* when there exists a bijection between the two graphs which respect the intersection matrix and the indexations. Two trees with topologically equivalent marked weighted dual trees are topologically equivalent as topological spaces. Two trees $(\mathcal{M}, \mathcal{D}, \Sigma)$ and $(\mathcal{N}, \mathcal{E}, \Delta)$ with markings (σ, κ) and (ρ, θ) are *conjuguated* when there is a biholomorphism germ H defined on a neighborhood of $\mathcal{D}$ such that:

(1) $H(\mathcal{D}) = \mathcal{E}$, $H(\Sigma) = \Delta$;
(2) $H^*\sigma = \rho$, $H^*\kappa = \theta$.

The assumption (2) is referred to as *marked conjugation compatibility*. Let us denote by $\mathfrak{A}(\mathcal{M}, \mathcal{D}, \Sigma)$ the category of marked trees by $(\mathcal{M}, \mathcal{D}, \Sigma)$ with marked compatible conjugation as arrows. It is not difficult to prove the following proposition by induction on the height.

PROPOSITION 1.1. *Let* $(\mathcal{N}, \mathcal{E}, \Delta) \in \mathfrak{A}(\mathcal{M}, \mathcal{D}, \Sigma)$. *Then there exists a blowing-up process with a marked top conjugated to* $(\mathcal{N}, \mathcal{E}, \Delta)$.

Proposition 1.1 allows us to extend in an easy way all natural invariants of the blowing-up process to the category of marked trees. We define the *component multiplicity* below.

Let us denote by $\mathcal{O}(\mathcal{M}_0^h)$ the sheaf of germs of the holomorphic function over $\mathcal{M}_0^h$. Let $i_{\mathcal{D}_0^h}$ be the divisor inclusion $\mathcal{D}_0^h \subset \mathcal{M}_0^h$. We define

$$\mathcal{O}_{\mathcal{M}_0^h} := i_{\mathcal{D}_0^h}^{-1}(\mathcal{O}(\mathcal{M}_0^h)).$$

The sheaf $\mathcal{O}_{\mathcal{M}_0^h}$ is the restriction of $\mathcal{O}$ over $\mathcal{D}_0^h$. Let D be an irreducible component of the divisor. We denote

$$I_D \subset \mathcal{O}_{\mathcal{M}_0^h},$$

the ideal subsheaf of germs of a function vanishing along D. Let us consider the subsheaf $\mathfrak{M} \subset \mathcal{O}_{\mathcal{M}_0^h}$ pull-back of the maximal ideal at $0 \in \mathbb{C}^2$. The sheaf $\mathfrak{M}$ is locally free and generated by two global sections. Furthermore, we have the decomposition

$$\mathfrak{M} = \mathcal{O}\left(- \sum_{D \in \mathrm{Comp}(\mathcal{D})} \nu(D) D\right)\Big|_{\mathcal{D}}.$$

Here, $\nu(D)$ is called the *multiplicity of the component* D. One can see that it is well determined by the intersection matrix. When $(\mathcal{M}, \mathcal{D}, \Sigma)$ is a general tree marked by $(\mathcal{M}_0^h, \mathcal{D}_0^h, \Sigma_0^h)$, the multiplicity of D is naturally the multiplicity of the component associated by the indexation in $\mathcal{D}_0^h$.

1.2. *The sheaves* $\mathcal{G}_Z^n$, $n \geq 0$. From this point onwards, we fix an element $(\mathcal{M}, \mathcal{D}, \Sigma)$ in $\mathfrak{A}(\mathcal{M}_0, \mathcal{D}_0^h, \Sigma_0^h)$.

In order to define some sub-categories of $\mathfrak{A}(\mathcal{M}_0, \mathcal{D}_0^h, \Sigma_0^h)$ we are interested in, we introduce a family of sheaves of groups over $\mathcal{D}$. This construction leads to a key property, Property 1.1.

In order to overcome a technical difficulty which appears in the final induction (§2.1.3), the tree is enhanced with a *cross*. Let $\mathcal{M}^h$ be the top of a blowing-up process with $\mathcal{M}^h \overset{\phi}{\simeq} \mathcal{M}$ given by Proposition 1.1; let E_h be the total morphism of the process and $E := \phi \circ E_h$.

Definition 1.2. (Cross) A cross on $\mathcal{M}$ is the strict transform $Z = E^* Z_0$ of a single $Z_0 = \{Z_1\}$ or of a couple $Z_0 = \{Z_1, Z_2\}$ of germs of smooth transversal curves at the origin of $\mathbb{C}^2$.

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Throughout this article, we will often have to describe objects in coordinates. *Adapted coordinates* will always refer to (x, y) local coordinates such that:

- in the neighborhood of a regular point of $\mathcal{D}$, $\{x = 0\}$ is a local equation of $\mathcal{D}$;
- in the neighborhood of a singular point of $\mathcal{D}$, $\{xy = 0\}$ is a local equation of $\mathcal{D}$; and
- in the neighborhood of the cross, $\{x = 0\}$ is a local equation of $\mathcal{D}$ and $\{y = 0\}$ an equation of Z.

We consider $\mathrm{Aut}(\mathcal{M}, Z)$ to be the group sheaf over $\mathcal{D}$ of germs of automorphisms defined in a neighborhood of points of $\mathcal{D}$ such that

$$\Phi_{|\mathcal{D}} = \mathrm{Id}, \quad \Phi_{|Z} = \mathrm{Id}.$$

Let us have a closer look at the form of the $\mathrm{Aut}(\mathcal{M}, Z)$ sections in an adapted coordinate system. At a regular point c of $\mathcal{D} \cup Z$, the stack $\mathrm{Aut}(\mathcal{M}, Z)_c$ is the set of germs $(x, y) \mapsto (x(\alpha + A), y + xB)$ where A and B belong to $\mathbb{C}\{x, y\}$, $A(0, 0) = 0$ and $\alpha \in \mathbb{C}^*$. At a singular point s of $\mathcal{D} \cup Z$, the stack $\mathrm{Aut}(\mathcal{M}, Z)_s$ is the set of germs $(x, y) \mapsto (x(1 + yA), y(1 + xB))$ where A and B belong to $\mathbb{C}\{x, y\}$.

We will be naturally led to look at the infinitesimal neighborhood of the divisor. To take care of the cross, we consider a filtration of $\mathcal{O}_{\mathcal{M}}$ defined by $\mathfrak{M}^n_Z = I_Z \cdot \mathfrak{M}^n$, $n \geq 1$. In the same way, we denote by $\mathfrak{I}_Z \subset \mathcal{O}_{\mathcal{M}}$ the sheaf

$$\mathfrak{I}_Z = \mathcal{O}\Big(-Z - \sum_{D \in \mathrm{Comp}(\mathcal{D})} D\Big)\Big|_{\mathcal{D}}. \tag{2}$$

Definition 1.3. (Infinitesimal crossed tree) We refer to the analytical space

$$\mathcal{M}^{[n],Z} = (\mathcal{D}, \mathcal{O}_{\mathcal{M}}/\mathfrak{M}^n_Z)$$

as the nth infinitesimal crossed tree. The neighborhood of order 0 is $\mathcal{M}^{[0],Z} = (\mathcal{D}, \mathcal{O}_{\mathcal{M}}/\mathfrak{I}_Z)$. We also consider the ringed spaces

$$\mathcal{M}^{\underline{n},Z} = (\mathcal{D}, \mathfrak{I}_Z/\mathfrak{I}_Z\mathfrak{M}^n_Z), \quad \mathcal{M}^{\underline{0},Z} = (\mathcal{D}, \mathfrak{I}_Z/\mathfrak{I}^2_Z).$$

The sequence of canonical imbeddings

$$\ldots \mathcal{M}^{[p],Z} \hookrightarrow \mathcal{M}^{[p-1],Z} \hookrightarrow \ldots \hookrightarrow \mathcal{M}^{[1],Z} \hookrightarrow \mathcal{M}^{[0],Z} \hookrightarrow \mathcal{M}$$

induces a natural filtration of the sheaf $\mathrm{Aut}(\mathcal{M}, Z)$.

Definition 1.4. We denote by $\mathrm{Aut}_n(\mathcal{M}, Z)$ the subsheaf of $\mathrm{Aut}(\mathcal{M}, Z)$ of germs that coincide with Id when restricted to the infinitesimal neighborhood of order n.

We now study the form of $\mathrm{Aut}_n(\mathcal{M}, Z)$ sections in coordinates in order to construct a special morphism of sheaves.

At a regular point c of $\mathcal{D} \cup Z$: let p be the multiplicity of the component containing c. In an adapted coordinate system, the elements of $\mathrm{Aut}_n(\mathcal{M}, Z)_c$ can be written as $\phi(x, y) = (x + x^{pn}A, y + x^{pn}B)$, where A, B belong to $\mathbb{C}\{x, y\}$. Let $\mathcal{J}_n$ be the function defined by

$$\phi(x, y) = (x + x^{pn}A, y + x^{pn}B) \in \mathrm{Aut}_n(\mathcal{M}, Z)_c \stackrel{\mathcal{J}_n}{\longmapsto} x^{pn-1}A \in (\mathcal{O}_{\mathcal{M}^{[n],Z}})_c.$$

One can see that $\mathcal{J}_n$ is a morphism of groups that does not depend on the adapted coordinates.

At a singular point s of $\mathcal{D}$: let p and q be the multiplicities of the local components. The elements of $\mathrm{Aut}_n(\mathcal{M}, Z)_s$ are those of the form $\phi(x, y) = (x + x^{pn}y^{qn}A, y + x^{pn}y^{qn}B)$. In the same way, we define an intrisic group morphism by

$$\begin{aligned}\phi = (x + x^{pn}y^{qn}A, y + x^{pn}y^{qn}B) &\in \mathrm{Aut}_n(\mathcal{M}, Z)_s\\ \stackrel{\mathcal{J}_n}{\longmapsto} x^{pn-1}y^{qn-1}(yA + xB) &\in (\mathcal{O}_{\mathcal{M}^{[n],Z}})_s.\end{aligned}$$

At an attachment point z of Z: the multiplicity of the local component is one since the components of Z are smooth curves blown down at the origin. The elements of $\mathrm{Aut}_n(\mathcal{M}, Z)_z$ are of the form $\phi(x, y) = (x + x^n yA, y + x^n yB)$ and the morphism is defined by

$$\phi(x, y) = (x + x^n yA, y + x^n yB) \in \mathrm{Aut}_n(\mathcal{M}, Z)_z \stackrel{\mathcal{J}_n}{\longmapsto} x^{n-1}(yA + xB) \in (\mathcal{O}_{\mathcal{M}^{[n],Z}})_z.$$

Finally, we obtain a morphism of sheaves defined by its previous local description $\mathrm{Aut}_n(\mathcal{M}, Z) \stackrel{\mathcal{J}_n}{\longmapsto} \mathcal{O}_{\mathcal{M}^{[n],Z}}$. Likewise, we have a morphism of sheaves $\mathcal{J}_0$ defined, for example, near a regular point by

$$\phi = (x + xA, y + xB) \in \mathrm{Aut}_0(\mathcal{M}, Z)_c \stackrel{\mathcal{J}_0}{\longmapsto} A \in (\mathcal{O}_{\mathcal{M}^{[0],Z}})_c. \tag{3}$$

Definition 1.5. We denote by $\mathcal{G}_Z^n$ the subsheaf of the $\mathrm{Aut}_n(\mathcal{M}, Z)$ kernel of the morphism $\mathcal{J}_n$.

Remark 1.1. At any point c of the divisor, which is not a point of attachment of Z, the two decreasing filtrations $\{\mathcal{G}_Z^n\}_{n\geq 0}$ and $\{\mathrm{Aut}_n(\mathcal{M}, Z)\}_{n\geq 0}$ are *equivalent* in the following sense. One has the sequence of inclusions

$$\cdots \subset (\mathcal{G}_Z^p)_c \subset (\mathrm{Aut}_p(\mathcal{M}, Z))_c \subset (\mathcal{G}_Z^{p-1})_c \subset \cdots .$$

Hence, the main difference between the two sheaves $\mathcal{G}_Z^n$ and $\mathrm{Aut}_n(\mathcal{M}, Z)$ is located near the cross. A section of $\mathcal{G}_Z^{p-1}$ near the cross is written in adapted coordinates

$$(x, y) \mapsto (x + x^p yA, y + x^p y^2 B)$$

whereas a section of $\mathrm{Aut}_p(\mathcal{M}, Z)$ is given by

$$(x, y) \mapsto (x + x^p yA, y + x^p yB).$$

Hence, the order of tangency along the curve $\{y = 0\}$ of the second factor is bigger for sections of $\mathcal{G}_Z^{p-1}$ than for sections of $\mathrm{Aut}_p(\mathcal{M}, Z)$. It will be seen that this difference is very important.

Lemma 1.1 gives an intrisic criterion for a germ to be a section of $\mathcal{G}_Z^n$.

LEMMA 1.1. *Let ϕ be a germ of a section of* $\mathrm{Aut}(\mathcal{M}, Z)$. *The following properties are equivalent:*

(1) *ϕ is a section of $\mathcal{G}_Z^n$;*

(2) *ϕ is the identity restricted to $\mathcal{M}^{[n],Z}$ and to $\mathcal{M}^{\underline{n},Z}$.*

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The following property shows the interest of the introduced sheaves $\mathcal{G}_Z^n$: basically, these are sheaves of Lie groups associated to some sheaves of Lie algebras which are natural in our context.

PROPERTY 1.1. (Key property) *Let X be a germ of the vector field tangent to $\mathcal{D}$ and to Z. Let f be a germ of the $\mathfrak{M}_Z^n$ section. Then the flow of $f \cdot X$ is a germ of the $\mathcal{G}_Z^n$ section.*

Proof. The property can be read in the form of the flow in coordinates. Let (x, y) be the local adapted coordinates. If Y is a germ of a vector field, we denote by $Y^{(k)}$ the kth power of Y as a differential operator on $\mathcal{O}_0^2$; for any $t \in [0, 1]$, the flow of Y at time t can be expanded in a neighborhood of the origin as

$$e^{tY} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Y^{(k)}.$$

In the case of a divisor singular point, putting $Y = f \cdot X$, an induction on k shows that there exists A_k, B_k, sections of $\mathfrak{M}_Z^{n+k}$, such that $Y^{(k)}(x, y) = (xA_k(x, y), yB_k(x, y))$. Hence, at time one, the flow has the form $(x, y) \longmapsto (x, y) + x^{pn}y^{qn}(xA, yB)$ where p and q are the multiplicities of the local components of $\mathcal{D}$. The latter automorphism is a $\mathcal{G}_Z^n$ section. □

1.3. *The tree gluing.* As a consequence of the sheaf $\mathrm{Aut}(\mathcal{M}, Z)$, we can introduce a process called *gluing* on $\mathcal{M}$. This construction will allow us to define a large class of trees with the same divisor analytical type. These trees will inherit a canonical marking and a cross.

1.3.1. *Distinguished covering.* Let us define a particular type of open covering of the divisor. Open sets of that covering will play the role of gluing 'bricks'.

Let $\mathcal{U} = \{U_i\}_{i \in \mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1}$ be the covering of $\mathcal{D}$ comprising two kinds of open sets:

- if i belongs to $\mathbb{I}_0$, U_i is the trace on $\mathcal{D}$ of a neighborhood of a unique singular; and
- if i belongs to $\mathbb{I}_1$, U_i is an irreducible component of $\mathcal{D}$ deprived of the singular points of $\mathcal{D}$.

Definition 1.6. Every covering is said to be *distinguished* when there is no 3-intersection.

Distinguished coverings contain Stein open sets having fundamental systems of Stein neighborhoods. From here onwards, a covering denoted by $\mathcal{U}$ will always be assumed to be distinguished. The spaces $\mathcal{Z}^0(\mathcal{U}, \mathcal{G})$ and $\mathcal{Z}^1(\mathcal{U}, \mathcal{G})$ are the sets of 0-cocycles and 1-cocycles in the sense of Cech for the sheaf $\mathcal{G}$ and the covering $\mathcal{U}$. Let us consider

$$\mathbb{I}_0 \check{\times} \mathbb{I}_1 = \{(i, j) \in \mathbb{I}_0 \times \mathbb{I}_1 \mid U_i \cap U_j \neq \emptyset\}.$$

We define $\tilde{\mathcal{Z}}^1 := \prod_{(i,j) \in \mathbb{I}_0 \check{\times} \mathbb{I}_1} \mathcal{Z}^0(U_i \cap U_j, \mathrm{Aut}(\mathcal{M}, Z))$. Since a distinguished covering does not have any 3-intersection, $\tilde{\mathcal{Z}}^1$ and $\mathcal{Z}^1(\mathcal{U}, \mathrm{Aut}(\mathcal{M}, Z))$ are isomorphic. Hence, if no confusion is possible, we will denote by $\mathcal{Z}^1$ the space $\tilde{\mathcal{Z}}^1$.

1.3.2. *Gluing.* As a result of covering, we are able to glue the open sets of that covering by identifying points according to a 1-cocycle in $\text{Aut}(\mathcal{M}, Z)$. Let (ϕ_{ij}) be a 1-cocycle in $\mathcal{Z}^1(\mathcal{U}, \text{Aut}(\mathcal{M}, Z))$. We define

$$\mathcal{M}[\phi_{ij}] = \bigcup_i \mathcal{U}_i \times \{i\}/_{(\{x\}\times\{i\}\sim\{\phi_{ij}(x)\}\times\{j\})},$$

where $\mathcal{U}_i$ is a neigborhood of U_i in $\mathcal{M}$ such that ϕ_{ij} is represented as an automorphism of $\mathcal{M}$ along $U_i \cap U_j$. The obtained manifold comes with an embedding

$$\mathcal{D} \hookrightarrow \mathcal{M}[\phi_{ij}] \tag{4}$$

whose image is denoted by $\mathcal{D}[\phi_{ij}]$ and $\mathcal{M}[\phi_{ij}]$ is considered as a germ of the neighborhood of $\mathcal{D}[\phi_{ij}]$.

Definition 1.7. The germ of the manifold $\mathcal{M}[\phi_{ij}]$ is called the gluing of $\mathcal{M}$ along $\mathcal{U}$ by the cocycle (ϕ_{ij}).

The gluing of a marked crossed tree comes naturally with a marking and a cross: the marking is the direct image by the embedding (4) of the marking σ; the cross is the direct image of Z by the quotient map for the gluing relation. Such a tree, marking and cross are respectively denoted by

$$(\mathcal{M}[\phi_{ij}], \mathcal{D}[\phi_{ij}], \Sigma[\phi_{ij}]), \quad \sigma[\phi_{ij}] \quad \text{and} \quad Z[\phi_{ij}].$$

In the same way, the direct image of the covering $\mathcal{U}$ by the quotient map is a distinguished covering of the new tree and is denoted by $\mathcal{U}[\phi_{ij}]$.

We associate the data of morphisms on an infinitesimal neigborhood with any gluing generalizing the embedding (4). The intrisic description of the $\text{Aut}_n(\mathcal{M}, Z)$ sections (Lemma 1.1) reveals the following property.

PROPERTY 1.2. *Let n be an integer and $\mathcal{N} = \mathcal{M}[\phi_{ij}]$ be a gluing of $\mathcal{M}$ by a cocycle in $\mathcal{Z}^1(\mathcal{U}, \text{Aut}_n(\mathcal{M}, Z))$. Then there are canonical isomorphisms of analytical and ringed spaces*

$$\rho_{\mathcal{N}}^{[n]} : \mathcal{M}^{[n],Z} \xrightarrow{\sim} \mathcal{N}^{[n],Z[\phi_{ij}]},$$
$$\rho_{\mathcal{N}}^{\underline{n}} : \mathcal{M}^{\underline{n},Z} \xrightarrow{\sim} \mathcal{N}^{\underline{n},Z[\phi_{ij}]}.$$

1.4. *The* $\text{Glu}_n(\mathcal{M}, Z, \mathcal{U})$ *categories.* Let p be an integer. Let us consider the marked crossed tree built by a succession of gluings

$$\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p] \tag{5}$$

where

- (ϕ_{ij}^1) is a 1-cocyle of $\mathcal{G}_Z^n$; and
- for $k = 2, \ldots, p$, $(\phi_{ij}^k) \in \mathcal{Z}^1(\mathcal{U}[\phi_{ij}^1]\cdots[\phi_{ij}^{k-1}], \mathcal{G}^n_{Z[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}]})$.

Following Property 1.2, we have canonical isomorphisms

$$\rho^{[n]}_{\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]} : \mathcal{M}^{[n],Z} \xrightarrow{\sim} \mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]^{[n],Z[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]}, \tag{6}$$

$$\rho^{\underline{n}}_{\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]} : \mathcal{M}^{\underline{n},Z} \xrightarrow{\sim} \mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]^{\underline{n},Z[\phi_{ij}^1][\phi_{ij}^2][\cdots][\phi_{ij}^p]}. \tag{7}$$

Definition 1.8. The $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$ category is the category whose objects are marked crossed trees built as equation (5) with the data of isomorphisms (6) and (7). Arrows are biholomorphic germs Φ between two trees $\mathcal{N}$ and $\mathcal{P}$ that respect marking and commute with these isomorphisms:

$$\Phi \circ \rho_{\mathcal{N}}^{[n]} = \rho_{\mathcal{P}}^{[n]} \quad \text{and} \quad \Phi \circ \rho_{\mathcal{N}}^{n} = \rho_{\mathcal{P}}^{n}.$$

If $\mathcal{N}$ and $\mathcal{P}$ are isomorphic in $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$, we write

$$\mathcal{N} \overset{\mathrm{G}_n}{\simeq} \mathcal{P}.$$

1.5. *Computations in* $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$. To compute in $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$, we state three properties to manipulate gluings towards their defining cocycles. In view of the intrisic description of $\mathcal{G}_n^Z$ sections, it is easy to check the following property.

PROPERTY 1.3. *Let $\mathcal{N}$ and $\mathcal{P}$ be in $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$ defined by the 1-cocyles (ρ_{ij}) and (γ_{ij}), respectively. The following properties are equivalent:*

(1) $\mathcal{N} \overset{\mathrm{G}_n}{\simeq} \mathcal{P}$;

(2) *there exists a 0-cocyle (ϕ_i) in $\mathcal{G}_Z^n$ such that $\rho_{ij} = \phi_j \gamma_{ij} \phi_i^{-1}$.*

Proof. Suppose that $\mathcal{N}$ and $\mathcal{P}$ are isomorphic in $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$ and denote the isomorphism by ϕ. Taking the restriction of ϕ on each open set of the distinguished covering of $\mathcal{M}$, one finds a 0-cocycle (ϕ_i) in $\mathrm{Aut}(\mathcal{M}, Z)$, which provides the cohomological relation

$$\rho_{ij} = \phi_j \gamma_{ij} \phi_i^{-1}. \tag{8}$$

Since the isomorphism ϕ commutes with the canonical isomorphisms associated to $\mathcal{N}$ and $\mathcal{P}$, the 0-cocycle (ϕ_i) has its values in $\mathcal{G}_Z^n$. □

Remark 1.2. Note that if $\mathcal{N}$ and $\mathcal{P}$ are simply conjugated as trees, there still exists a 0-cocycle (ϕ_i) with values in $\mathrm{Aut}(\mathcal{M}, Z)$ verifying the cohomological relation (8). However, to ensure that (ϕ_i) belongs to the $\mathcal{G}_Z^n$ class, one needs the assumption of commutativity with both isomorphisms (6) and (7) in the definition of the $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$ categories.

The family of maps $I_i : U_i \subset \mathcal{M} \to \mathcal{M}[\phi_{ij}]$ induces the following canonical isomorphism:

$$\zeta^0 : \begin{cases} \mathcal{Z}^0(\mathcal{U}, \mathrm{Aut}(\mathcal{M}, Z)) & \longrightarrow \quad \mathcal{Z}^0(\mathcal{U}[\phi_{ij}], \mathrm{Aut}(\mathcal{M}[\phi_{ij}], Z[\phi_{ij}])) \\ \qquad (\phi_i) & \longrightarrow \qquad\qquad (I_i \phi_i I_i^{-1}). \end{cases}$$

We are able to define such an isomorphism for 1-cocycles as a result of the $\tilde{\mathcal{Z}}^1$ spaces (1.3.1). We define ζ^1 as the isomorphism

$$\begin{array}{rcl} \tilde{\mathcal{Z}}^1 & \to & \tilde{\mathcal{Z}}^1[\phi_{ij}] := \displaystyle\prod_{(i,j) \in \mathbb{I}_0 \check{\times} \mathbb{I}_1} \mathcal{Z}^0(U_i \cap U_j[\phi_{ij}], \mathrm{Aut}(\mathcal{M}[\phi_{ij}], Z[\phi_{ij}])) \\ (\phi_{ij}) & \to & (I_j \phi_{ij} I_j^{-1}). \end{array}$$

The following useful property can now be stated.

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PROPERTY 1.4. *Let* (ϕ_{ij}) *and* (ψ_{ij}) *be in* $\mathcal{Z}^1(\mathcal{U}, \mathcal{G}_Z^n)$*. Then*

$$\mathcal{M}[\phi_{ij}][\zeta^1\psi_{ij}] \overset{\mathbb{G}_n}{\simeq} \mathcal{M}[\phi_{ij}\psi_{ij}].$$

1.5.1. *Stability property.* In this section, we identify a class of isomorphic trees in the gluing category. Basically, if the gluing cocycle is sufficiently tangent to the identity along the divisor, then the glued tree is isomorphic to the initial one.

PROPERTY 1.5. (Stability) *For n big enough, the image of*

$$\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U}) \longrightarrow \mathrm{Glu}_1(\mathcal{M}, Z, \mathcal{U})$$

is a set of trees conjugated to $\mathcal{M}$ *in the category* $\mathrm{Glu}_1(\mathcal{M}, Z, \mathcal{U})$.

This property can be stated in the following way: the map

$$H^1(\mathcal{D}, \mathcal{G}_Z^n) \longrightarrow H^1(\mathcal{D}, \mathcal{G}_Z^1) \tag{9}$$

is constant, equal to $[\mathrm{Id}]_{\mathcal{G}_Z^1}$. First, let us establish the equivalent statement for the initial sheaf $\mathrm{Aut}_n(\mathcal{D}, Z)$.

LEMMA 1.2. *For any integer n, there is an integer* $\delta(n) > n$ *such that the image of the map*

$$H^1(\mathcal{D}, \mathrm{Aut}_{\delta(n)}(\mathcal{M}, Z)) \to H^1(\mathcal{D}, \mathrm{Aut}_n(\mathcal{M}, Z))$$

is trivial.

Proof. The proof of a similar result can be found in [**11**]. Here, we reproduce the main arguments. One can suppose $\mathcal{M}$ to be the top of a blowing-up process. Let $p \geq n$ and (ϕ_{ij}) be an element of $Z^1(D, \mathrm{Aut}_p(\mathcal{M}, Z))$. There exists a germ of a biholomorphism θ between $\mathcal{M}[\phi_{ij}]$ and the top of a blowing-up process $\mathcal{M}'$. Let us denote by Z' the induced cross on $\mathcal{M}'$. The map θ induces an isomorphism $\theta^{[p]}$ between the infinitesimal crossed neighborhoods $\mathcal{M}[\phi_{ij}]^{[p],Z[\phi_{ij}]}$ and $\mathcal{M}'^{[p],Z'}$. Hence, $\theta^{[p]} \circ \rho^{[p]}_{\mathcal{M}[\phi_{ij}]}$ identifies infinitesimal neighborhoods of both top blowing-up processes. One can show that for $p = \delta(n)$ big enough, $\theta^{[p]} \circ \rho^{[p]}_{\mathcal{M}[\phi_{ij}]}$ can be extended as a biholomorphism T of trees such that

$$T^{[n]} = \theta^{[n]} \circ \rho^{[n]}_{\mathcal{M}[\phi_{ij}]}.$$

Hence, $H = T^{-1} \circ \theta$ is a germ of a biholomorphism between $\mathcal{M}[\phi_{ij}]$ and $\mathcal{M}$ with $H^{[n]} \circ \rho^{[n]}_{\mathcal{M}[\phi_{ij}]} = \mathrm{Id}^{[n]} = \rho^{[n]}_{\mathcal{M}}$. This is equivalent to the triviality of (ϕ_{ij}) in $H^1(\mathcal{D}, \mathrm{Aut}_n(\mathcal{M}, Z))$. □

Proof of Property 1.5. Let (ϕ_{ij}) be a 1-cocycle in $\mathcal{G}_Z^{\delta(n)}$. Its trivialization in $\mathrm{Aut}_n(D, Z)$ can be written as

$$\phi_{ij} = \phi_i \circ \phi_j^{-1}. \tag{10}$$

We are going to *correct* this trivialization in order to obtain one with values in $\mathcal{G}_Z^1$. Let us denote by p_1 and p_2 the points of attachment of the cross. The cross is the strict pull-back of a couple of transversal curves at the origin in $\mathbb{C}^2$ which define a local

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system of coordinates. Near p_1 and p_2, there are two local adapted coordinate systems (x_1, y_1) and (x_2, y_2) such that E takes the following forms: $E(x_1, y_1) = (x_1, y_1 x_1^{N_1})$ and $E(x_2, y_2) = (x_2 y_2^{N_2}, y_2)$. The components of the cocycle ϕ_1 and ϕ_2 defined near p_1 and p_2 can be expanded as

$$\phi_1(x_1, y_1) = (x_1 + x_1^n y_1 U_1(x_1, y_1), y_1 + x_1^n y_1 V_1(x_1, y_1)),$$
$$\phi_2(x_2, y_2) = (x_2 + y_2^n x_2 U_2(x_2, y_2), y_2 + y_2^n x_2 V_2(x_2, y_2)).$$

Let ϕ_0 be the germ of a biholomorphism near 0 in $\mathbb{C}^2$ defined by

$$\phi_0(x, y) = (x + xy^n U_2(0, y), y + yx^n V_1(x, 0)).$$

For n big enough, ϕ_0 can be raised to an automorphism ϕ that fixes each point of the divisor and the cross. Moreover, for any point c different from p_1 and p_2, the evaluation through $\mathcal{J}_1$ provides the equality $\mathcal{J}_1(\phi^{-1})_c \equiv 0$. Now, the choice of ϕ ensures that $\mathcal{J}_1(\phi_1)_{p_1} \equiv \mathcal{J}_1(\phi)_{p_1}$ and that $\mathcal{J}_1(\phi_2)_{p_2} \equiv \mathcal{J}_1(\phi)_{p_2}$. Finally, $\mathcal{J}_1(\phi_i \circ \phi^{-1}) \equiv 0$. Hence, the 0-cocycle $(\phi_i \circ \phi^{-1})$ is a trivialization of (ϕ_{ij}) in $\mathcal{G}_Z^1$.

Remark 1.3. Since the points of attachment of the cross can be chosen outside of any 2-intersection of the distinguished covering, in view of Remark 1.1 there is a canonical embedding

$$\mathcal{Z}^1(\mathcal{U}, \mathrm{Aut}_{\delta(n)+1}(\mathcal{M}, Z)) \hookrightarrow \mathcal{Z}^1(\mathcal{U}, \mathcal{G}_Z^{\delta(n)}). \tag{11}$$

Hence, the proof of Property 1.5 would work if we began with $(\phi_{ij}) \in \mathcal{Z}^1(\mathcal{U}, \mathrm{Aut}_{\delta(n)+1}(\mathcal{M}, Z))$. Nevertheless, the embedding (11) cannot induce a map in cohomology.

2. *Cobordism in* $\mathrm{Glu}_n(\mathcal{M}, Z, \mathcal{U})$

From here onwards, we assume the marked tree $\mathcal{M}$ to be foliated by $\mathcal{F}$. We are going to define a cobordism notion in order to detect in any element of $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$ the existence of a foliation linked to $\mathcal{F}$ in the sense of equisingular unfolding. For a precise definition of equisingular unfolding, we refer to [**8**]. We assume the singularities of $\mathcal{F}$ to be reduced.

Definition 2.1. (Cross adapted to $\mathcal{F}$) Let Z be a cross on $\mathcal{M}$. Z is said to be adapted to $\mathcal{F}$ when each Z_i verifies at least one of the following properties:
(1) Z_i is a separatrix of $\mathcal{F}$; and
(2) Z_i is attached at a regular point of $\mathcal{F}$.
In the latter case, Z_i will be transversal to the foliation.

From now on, the crosses will always be adapted. Let us denote by S the union of analytical invariant curves of $\mathcal{F}$, which are transversal to the divisor. By extension, the curve S is called the separatrix of $\mathcal{F}$. Now, we consider two natural sheaves.

- The sheaf $\mathfrak{X}_{S,Z}$ is a sheaf over $\mathcal{D}$. At each point of $\mathcal{D}$, its fiber is the space of germs of holomorphic vector fields of $\mathcal{M}$ that are tangent to $\mathcal{D}$, to the separatrix and to the cross.
- The sheaf $\mathfrak{X}_{\mathcal{F},Z} \subset \mathfrak{X}_{S,Z}$ is the subsheaf of germs of vector fields tangent to the foliation.

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From this point onwards, e^{tX} refers to the flow of the vector field X at time t. One notices that if X is a section of $\mathfrak{I}_Z\mathfrak{X}_{S,Z}$ then, for all $t \in \mathbb{C}$, e^{tX} exists as a germ and defines a section of $\mathrm{Aut}(\mathcal{M}, Z)$.

Definition 2.2. (Elementary cobordism) Let $\mathcal{N} \in \mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$.

$\mathcal{N}$ is elementary $\mathcal{F}$-cobordant to $\mathcal{M}$ when there exists a 1-cocycle (T_{ij}) of $Z^1(\mathcal{U}, \mathfrak{I}_Z\mathfrak{X}_{\mathcal{F},Z})$ such that $\mathcal{N}$ and $\mathcal{M}[e^{T_{ij}}]$ are isomorphic in $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$.

Here, $\mathcal{N}$ inherits a canonical foliation defined by

$$\mathcal{F}[e^{T_{ij}}] := \coprod_i \mathcal{F}|_{\mathcal{U}_i} \Big/_{x \sim e^{T_{ij}} x}.$$

Moreover, a rigidity result of Grauert [**8**] implies that the deformation $t \to \mathcal{M}_t = \mathcal{M}[e^{(t)T_{ij}}]$, $t \in \overline{\mathbb{D}}$ carries an equisingular unfolding between $\mathcal{F}$ and $\mathcal{F}[e^{T_{ij}}]$ in the sense of [**8**]. To be more specific, the foliation of codimension one in a manifold of dimension three defined by

$$\mathcal{F}_{\overline{\mathbb{D}}} = \coprod_i \mathcal{F}|_{\mathcal{U}_i} \times \overline{\mathbb{D}} \Big/_{(x,t) \sim (e^{(t)T_{ij}} x, t)} \quad \text{on} \quad \coprod_i \mathcal{U}_i \times \overline{\mathbb{D}} \Big/_{(x,t) \sim (e^{(t)T_{ij}} x, t)}$$

comes with a transversal fibration π, which is the projection on the second factor. The family of foliated fibers of π is an equisingular unfolding. The fiber at 0 is the tree $\mathcal{M}$ foliated by $\mathcal{F}$ and the fiber at 1 is the tree $\mathcal{M}[e^{T_{ij}}]$ foliated by $\mathcal{F}[e^{T_{ij}}]$.

Definition 2.3. (General cobordism) Let $\mathcal{N}$ be in $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$. $\mathcal{N}$ is said to be $\mathcal{F}$-cobordant to $\mathcal{M}$ if there exists a finite sequence of 1-cocyles $(T^k_{ij})_{k=1,\ldots,N}$ such that the following two conditions are verified.

(1) For any $p = 0, \ldots, N-1$, let $\mathfrak{X}_{\mathcal{F}_p, Z_p}$ be the sheaf over $\mathcal{D}[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}]$ of germs of the vector field tangent to the foliation and cross

$$\mathcal{F}_p = \mathcal{F}[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}], \quad Z_p = Z[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}].$$

We assume (T^{p+1}_{ij}) is a 1-cocycle with values in $\mathfrak{X}_{\mathcal{F}_p, Z_p}$.

(2) $\mathcal{N} \overset{\mathrm{G}_0}{\simeq} \mathcal{M}[e^{T^1_{ij}}][\cdots][e^{T^N_{ij}}]$.

We summarize this definition with the following notation:

$$\mathcal{M} \overset{\mathcal{F}_1, Z_1}{\dashrightarrow} \mathcal{M}_2 \overset{\mathcal{F}_2, Z_2}{\dashrightarrow} \cdots \overset{\mathcal{F}_{N-1}, Z_{N-1}}{\dashrightarrow} \mathcal{M}_N \overset{\mathrm{G}_0}{\simeq} \mathcal{N}.$$

2.1. *Construction of cobordism.* A foliation $\mathcal{F}$ is a non-dicritical generalized curve when all its singularities are reduced with two non-vanishing eigenvalues and when the divisor is invariant.

PROPOSITION 2.1. *Let $\mathcal{F}$ be a non-dicritical generalized curve on a marked tree $\mathcal{M}$ crossed by Z. Any element in* $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$ *is $\mathcal{F}$-cobordant to $\mathcal{M}$.*

The proof highlights three different steps: first, we establish the result at an infinitesimal level, then for the sub-category $\mathrm{Glu}_1(\mathcal{M}, Z, \mathcal{U})$ and finally, due to an induction on the height of the tree, for the $\mathrm{Glu}_0(\mathcal{M}, \mathcal{Z}, \mathcal{U})$ category.

2.1.1. *Step 1: the infinitesimal level.* Let H be a biholomorphism between $\mathcal{M}$ and the top of a blowing-up process $\mathcal{M}^h$. The induced foliation $\mathcal{F}^h$ on $\mathcal{M}^h$ is defined by a germ of the holomorphic 1-form ω at the origin of $\mathbb{C}^2$. We define $E := E_h \circ H$ where E_h is the total morphism of the process (§1.1). The global 1-form $E^*\omega$ defines a morphism of sheaves

$$X \in \mathfrak{X}_{S,Z} \to E^*(\omega)(X) \in \mathcal{O}_{\mathcal{M}}.$$

Let a reduced equation of the separatrix of ω be denoted by f. For any component D of $\mathcal{D}$, we denote by $\nu_D(f)$ and $\nu_D(\omega)$ the respective vanishing orders of $E^*(f)$ and $E^*(\omega)$ along the component D.

LEMMA 2.1. *There exists an exact sequence of sheaves*

$$0 \longrightarrow \mathfrak{M}_Z^n \mathfrak{X}_{\mathcal{F},Z} \longrightarrow \mathfrak{M}_Z^n \mathfrak{X}_{S,Z} \xrightarrow{E^*\omega(\cdot)} \mathfrak{M}_Z^n (f \circ E) \longrightarrow 0.$$

Here, $(f \circ E)$ is the sheaf of ideals generated by $f \circ E$ in $\mathcal{O}_{\mathcal{M}}$.

Proof. Let us show that there exists an exact sequence of sheaves

$$0 \longrightarrow \mathfrak{X}_{\mathcal{F},Z} \longrightarrow \mathfrak{X}_{S,Z} \xrightarrow{E^*\omega(\cdot)} (f \circ E) \longrightarrow 0. \tag{12}$$

The exactness of the first part in equation (12) is obvious since $\mathcal{X}_{\mathcal{F},Z} = \ker(E^*\omega)$. Let us compute the image of $E^*\omega(\cdot)$. In view of [**4**] and as $\mathcal{F}$ is a non-dicritical generalized curve, for any component D of $\mathcal{D}$ one has the relation

$$\nu_D(f) = \nu_D(\omega) + 1.$$

Hence, near a divisor singular point s there exist local coordinates such that $f \circ E = Ux^{p+1}y^{q+1}$ and $E^*\omega = Vx^p y^q(\lambda x(1 + A)\, dy + y(1 + B)\, dx)$, where U and V are unities. Let g be any element of $(\mathcal{O}_{\mathcal{M}})_s$. The vector field $X = yg(U/V)\partial y$ belongs to $(\mathfrak{X}_{S,Z})_s$ and $E^*\omega(X) = gf \circ E$. At a regular point c of the foliation, there exists coordinates such that $f \circ E = Ux^{p+1}$ and $E^*\omega = Vx^p\, dx$, where U and V are unities. The vector field $X = xg(U/V)\partial x$ belongs to $(\mathfrak{X}_{S,Z})_c$ and verifies $E^*\omega(X) = gf \circ E$ for any germ of a function g. The other cases can be studied in the same way. Now, the sheaf $\mathfrak{M}_Z^n$ is locally principal. Hence, the sequence (12) multiplied by $\mathfrak{M}_Z^n$ remains exact. □

LEMMA 2.2. *For any $n \geq 1$, $H^1(\mathcal{D}, \mathfrak{M}_Z^n) = 0$.*

Proof. The long exact sequence associated to the short exact sequence $0 \to \mathfrak{M}_Z^n \to \mathfrak{M}^n \to \mathfrak{M}^n/\mathfrak{M}_Z^n \to 0$ is written as [**6**]

$$\cdots \to H^0(\mathcal{D}, \mathfrak{M}^n) \xrightarrow{\delta} H^0(\mathcal{D}, \mathfrak{M}^n/\mathfrak{M}_Z^n) \to H^1(\mathcal{D}, \mathfrak{M}_Z^n) \to H^1(\mathcal{D}, \mathfrak{M}^n) \cdots .$$

The sheaf $\mathfrak{M}^n$ is generated by its global sections. Hence $H^1(\mathcal{D}, \mathfrak{M}^n) = 0$ since $H^1(\mathcal{D}, \mathcal{O}_{\mathcal{M}}) = 0$ [**8**]. In order to conclude, it remains to prove that δ is onto. Outside the points p_1 and p_2 of attachment of Z, the sheaf I_Z coincides with $\mathcal{O}_{\mathcal{M}}$, therefore the fiber of $\mathfrak{M}^n/\mathfrak{M}_Z^n$ is trivial. Let us consider the local coordinate systems (x_1, y_1) and (x_2, y_2) near p_1 and p_2 introduced in the proof of Property 1.5. We have

$$(\mathfrak{M}^n/\mathfrak{M}_Z^n)_{p_1} \simeq x_1^n\mathbb{C}\{x_1, y_1\}/x_1^n y_1\mathbb{C}\{x_1, y_1\} \simeq x_1^n\mathbb{C}\{x_1\}.$$

In the same way, $(\mathfrak{M}^n/\mathfrak{M}^n_Z)_{p_2}$ is isomorphic to $y_2^n\mathbb{C}\{y_2\}$. Hence, the space of $\mathfrak{M}^n/\mathfrak{M}^n_Z$ global sections is identified to $x_1^n\mathbb{C}\{x_1\} \oplus y_2^n\mathbb{C}\{y_2\}$. Let $S = x_1^n a_1(x_1) \oplus y_2^n a_2(y_2)$ be in the previous set. The germ of a function defined by $s(x, y) = x^n a_1(x) + y^n a_2(y)$ induces a global section $s \circ E$ of $\mathfrak{M}^n$. Now, the map E takes the following form in coordinates: $E(x_1, y_1) = (x_1, y_1 x_1^{N_1})$ and $E(x_2, y_2) = (x_2 y_2^{N_2}, y_2)$. Hence $s \circ E$ verifies the equalities

$$(s \circ E)_{p_1} \equiv x_1^n a_1(x_1) + y_1^n x_1^{nN_1} a_2(y_1 x_1^{n_1}) \equiv x_1^n a_1(x_1) \in (\mathfrak{M}^n/\mathfrak{M}^n_Z)_{p_1},$$
$$(s \circ E)_{p_2} \equiv x_2^n y_2^{nN_2} a_1(x_2 y_2^{N_2}) + y_2^n a_2(y_2) \equiv y_2^n a_2(x_2) \in (\mathfrak{M}^n/\mathfrak{M}^n_Z)_{p_2},$$

which means that $\delta(s \circ E) = S$ and ends the proof. □

Remark 2.1. In view of the previous proof of Lemma 2.1, the exactness of the sequence (12) holds when replacing the sheaf $\mathfrak{M}^n_Z$ by any sub-sheaf of

$$\mathcal{O}\bigg(-Z - \sum_{D \in \mathrm{Comp}(\mathcal{D})} D\bigg)\bigg|_{\mathcal{D}}.$$

However, very few have vanishing cohomology groups. In particular, the image by $E^*\omega(\cdot)$ of the Lie algebra associated to the sheaf $\mathrm{Aut}_n(\mathcal{M}, Z)$ fails to have vanishing cohomology groups. This justifies the introduction of $\mathcal{G}^n_Z$, which in Lie algebra is precisely $\mathfrak{M}^n_Z\mathfrak{X}_{S,Z}$.

Now, we can state the infinitesimal equivalent of Proposition 2.1.

PROPOSITION 2.2. (Infinitesimal cobordism) *The map*

$$H^1(\mathcal{D}, \mathfrak{M}^n_Z\mathfrak{X}_{\mathcal{F},Z}) \longrightarrow H^1(\mathcal{D}, \mathfrak{M}^n_Z\mathfrak{X}_{S,Z})$$

is onto.

Proof. The long exact sequence associated with the short one in Lemma 2.1 is

$$\cdots \to H^1(\mathcal{D}, \mathfrak{M}^n_Z\mathfrak{X}_{\mathcal{F},Z}) \to H^1(\mathcal{D}, \mathfrak{M}^n_Z\mathfrak{X}_{S,Z}) \to H^1(\mathcal{D}, \mathfrak{M}^n_Z(f \circ E)) \to \cdots. \quad (13)$$

Now, Lemma 2.2 ensures that $H^1(\mathfrak{M}^n_Z(f \circ E))$ is trivial, hence the last term of sequence (13) vanishes. □

2.1.2. *Step 2: cobordism in* $\mathrm{Glu}_1(\mathcal{M}, Z, \mathcal{U})$. This section is devoted to proving the following proposition.

PROPOSITION 2.3. *Let $\mathcal{F}$ be a non-dicritical generalized curve on a marked tree $\mathcal{M}$ crossed by Z. Any element of* $\mathrm{Glu}_1(\mathcal{M}, Z, \mathcal{U})$ *is elementary $\mathcal{F}$-cobordant to $\mathcal{M}$.*

The proof consists of getting the cobordism on an infinitesimal neighborhood of order big enough to apply the stability property.

LEMMA 2.3. (Infinitesimal cobordism of order n) *Let (ϕ_{ij}) in $\mathcal{Z}^1(\mathcal{U}, \mathcal{G}^1_Z)$. For any $n \geq 1$, there exist (T_{ij}) in $\mathcal{Z}^1(\mathcal{U}, \mathfrak{M}_Z\mathfrak{X}_{\mathcal{F},Z})$, (ϕ_i) in $\mathcal{Z}^0(\mathcal{U}, \mathcal{G}^1_Z)$ and $(\tilde{\phi}_{ij})$ in $\mathcal{Z}^1(\mathcal{U}, \mathcal{G}^n_Z)$ such that*

$$\phi_j^{-1} \circ \phi_{ij} \circ \phi_i = e^{T_{ij}} \circ \tilde{\phi}_{ij}. \quad (14)$$

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Proof. It is an induction on the integer n. Let us assume the property to be true at rank n. Let $(\tilde{\phi}_{ij}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{G}_Z^n)$ be given by the induction hypothesis. In local adapted coordinates, one can write $\tilde{\phi}_{ij}(x_{ij}, y_{ij}) = \mathrm{Id} + (A_{ij}, B_{ij})$. Let $\tilde{X}_{ij}$ be the germ of a vector field defined by $\tilde{X}_{ij} = A_{ij}\partial_{x_{ij}} + B_{ij}\partial_{y_{ij}}$. The family $(\tilde{X}_{ij})$ is a 1-cocycle with values in $\mathfrak{M}_Z^n\mathfrak{X}_{S,Z}$. Proposition 2.2 ensures the existence of a 0-cocycle $(\tilde{X}_i)$ in $Z^0(\mathcal{U}, \mathfrak{M}_Z^n\mathfrak{X}_{S,Z})$ and a 1-cocycle $(\tilde{T}_{ij})$ in $Z^1(\mathcal{U}, \mathfrak{M}_Z^n\mathfrak{X}_{\mathcal{F},Z})$ such that $\tilde{X}_{ij} = \tilde{X}_j - \tilde{X}_i + \tilde{T}_{ij}$. By expanding the flow $e^{\tilde{X}_{ij}}$, one can see that

$$\phi_{ij}^1 := e^{-\tilde{X}_{ij}} \circ \tilde{\phi}_{ij} = (\mathrm{Id} - \tilde{X}_{ij} + \cdots) \circ (\mathrm{Id} + \tilde{X}_{ij} + \cdots) \in \mathcal{G}_Z^{n+1}.$$

In view of the previous relations, we find

$$\begin{aligned} e^{-\tilde{X}_j} \circ \phi_j^{-1} \circ \phi_{ij} \circ \phi_i \circ e^{\tilde{X}_i} &= e^{-\tilde{X}_j} \circ e^{T_{ij}} \circ \tilde{\phi}_{ij} \circ e^{\tilde{X}_i} \\ &= e^{-\tilde{X}_j} \circ e^{T_{ij}} \circ e^{\tilde{X}_{ij}} \circ e^{\tilde{X}_i} \circ \phi_{ij}^1 \circ [\phi_{ij}^1, e^{\tilde{X}_i}] \end{aligned}$$

where $[a, b] = a^{-1}b^{-1}ab$. One can find in [**14**] the following result.

SUB-LEMMA 2.1. (Campbell–Hausdorff formula) *Let X and Y be two germs of a vector field vanishing along the divisor. Then there exists a formal vector field $\rho(X, Y) = \sum_{k=1}^{\infty} \rho_k(X, Y)$ such that*

$$e^{\rho(X,Y)} = e^X \circ e^Y.$$

The sum is convergent for the Krull topology in the space of formal series. Moreover, the first terms of the series are

$$\rho(X, Y) = X + Y + \tfrac{1}{2}[X, Y] + \cdots . \tag{15}$$

As a result of the above lemma, we obtain the following relation:

$$e^{-\tilde{X}_j} \circ e^{T_{ij}} \circ e^{\tilde{X}_{ij}} \circ e^{\tilde{X}_i} = e^{\rho(\rho(\rho(-\tilde{X}_j, T_{ij}), \tilde{X}_{ij}), \tilde{X}_i)}.$$

Equation (15) provides the expansion

$$\begin{aligned} &\rho(\rho(\rho(-\tilde{X}_j, T_{ij}), \tilde{X}_{ij}), \tilde{X}_i) \\ &\quad = T_{ij} + \tilde{T}_{ij} + \underbrace{\tfrac{1}{2}[T_{ij}, \tilde{X}_j] + \tfrac{1}{2}[T_{ij}, \tilde{X}_{ij}] + \tfrac{1}{2}[T_{ij}, \tilde{X}_i] + \cdots}_{Y_{ij}}. \end{aligned} \tag{16}$$

Now, for any integers n and m, $[\mathfrak{M}_Z^m\mathfrak{X}_{S,Z}, \mathfrak{M}_Z^n\mathfrak{X}_{S,Z}] \subset \mathfrak{M}_Z^{m+n}\mathfrak{X}_{S,Z}$. Hence, (Y_{ij}) is a 1-cocycle of $\mathcal{Z}^1(\mathcal{U}, \mathfrak{M}_Z^{n+1}\mathfrak{X}_{S,Z})$. We therefore obtain the expression

$$\begin{aligned} e^{-\tilde{X}_j} \circ \phi_j^{-1} \circ \phi_{ij} \circ \phi_i \circ e^{\tilde{X}_i} &= e^{T_{ij}+\tilde{T}_{ij}+Y_{ij}} \circ \phi_{ij}^1 \circ [\phi_{ij}^1, e^{-\tilde{X}_i}] \\ &= e^{T_{ij}+\tilde{T}_{ij}} \circ \phi_{ij}^2 \circ \phi_{ij}^1 \circ [\phi_{ij}^1, e^{-\tilde{X}_i}], \end{aligned} \tag{17}$$

where ϕ_{ij}^2 is equal to $e^{-T_{ij}-\tilde{T}_{ij}} \circ e^{T_{ij}+\tilde{T}_{ij}+Y_{ij}}$. Again, the Campbell–Hausdorff formula shows that ϕ_{ij}^2 is the flow of the vector field admitting the expansion $Y_{ij} - (1/2)[Y_{ij}, T_{ij} + \tilde{T}_{ij}] + \cdots$. As a consequence, in view of Property 1.1, ϕ_{ij}^2 takes its values in $\mathcal{G}_Z^{n+1}$. Finally, similar arguments ensure that the commutator in equation (17) is in the $\mathcal{G}_Z^{n+1}$ class. Hence, equation (17) is the induction hypothesis at rank $n + 1$. □

Remark 2.2. The previous lemma would be wrong if one replaces $\mathcal{G}^1_Z$ by $\mathcal{G}^0_Z$, as the infinitesimal argument does not work with the latter sheaf. In fact, as in Remark 2.1, the image by $E^*\omega(\cdot)$ of the Lie algebra associated with $\mathcal{G}^0_Z$ is

$$\mathcal{O}\Big(-Z-\sum_{D\in\mathrm{Comp}(\mathcal{D})}D\Big)\Big|_{\mathcal{D}}(f\circ E),$$

which does not have vanishing cohomology.

Let us now prove Proposition 2.3. One can write $\mathcal{N}\simeq\mathcal{M}[\phi_{ij}]$ with (ϕ_{ij}) in $\mathcal{Z}^1(\mathcal{U},\mathcal{G}^1_Z)$. The infinitesimal cobordism lemma gives us three cocycles (T_{ij}) in $\mathcal{Z}^1(\mathcal{U},\mathfrak{M}_Z\mathfrak{X}_{\mathcal{F},Z})$, (ϕ_i) in $\mathcal{Z}^0(\mathcal{U},\mathcal{G}^1_Z)$ and $(\tilde\phi_{ij})$ in $\mathcal{Z}^1(\mathcal{U},\mathcal{G}^n_Z)$ such that $\phi_j^{-1}\phi_{ij}\phi_i=e^{T_{ij}}\tilde\phi_{ij}$. Now, Properties 1.3 and 1.4 imply that

$$\mathcal{M}[\phi_{ij}]\overset{\mathrm{G}_1}{\simeq}\mathcal{M}[\phi_j^{-1}\phi_{ij}\phi_i]=\mathcal{M}[e^{T_{ij}}\tilde\phi_{ij}]\overset{\mathrm{G}_1}{\simeq}\mathcal{M}[e^{T_{ij}}][\zeta^1\tilde\phi_{ij}].$$

Moreover, the stability property applied in $\mathrm{Glu}_n(\mathcal{M}[e^{T_{ij}}],Z[e^{T_{ij}}],\mathcal{U}[e^{T_{ij}}])$ to the cocycle $(\zeta^1\tilde\phi_{ij})$ shows that $\mathcal{M}[e^{T_{ij}}][\zeta^1\tilde\phi_{ij}]\overset{\mathrm{G}_1}{\simeq}\mathcal{M}[e^{T_{ij}}]$. Hence, $\mathcal{M}[\phi_{ij}]$ is isomorphic to $\mathcal{M}[e^{T_{ij}}]$, which is the property to be verified.

2.1.3. *Step 3: cobordism in* $\mathrm{Glu}_0(\mathcal{M},Z,\mathcal{U})$. The proof of Proposition 2.1 is an induction on the height of the tree, which allows us to use the previous result established for the $\mathrm{Glu}_1(\mathcal{M},Z,\mathcal{U})$ category.

Let (ϕ_{ij}) be such that $\mathcal{N}=\mathcal{M}[\phi_{ij}]$. Let us denote by D_0 the irreducible component of $\mathcal{D}$ appearing after the first blowing-up process. Let $\{c_1,\ldots,c_N\}$ be the set of singular points of $\mathcal{D}$ on D_0. We denote by $\mathcal{D}_l$ the branch of $\mathcal{D}$ attached to c_l and $\mathcal{M}_l$ the neighborhood of $\mathcal{D}_l$ in $\mathcal{M}$. The notation $\mathcal{U}_l$ refers to the distinguished covering of $\mathcal{D}_l$ induced by restriction of $\mathcal{U}$. Moreover, we denote by $U_0\in\mathcal{U}$ the open set $D_0\backslash\{c_1,\ldots,c_N\}$ and $U_l\in\mathcal{U}$ the neighborhood of c_l. The foliation $\mathcal{F}_l$ refers to the restriction of the foliation $\mathcal{F}$ to $\mathcal{M}_l$. We consider a cross Z_l defined by the trace in $\mathcal{M}_l$ of D_0 and of the strict transform of Z (see Figure 1). One can see that $\mathcal{F}_l$ is a non-dicritical generalized curve on a marked crossed tree and that Z_l is adapted to $\mathcal{F}_l$.

Let us consider $(\phi^l_{ij})\in\mathcal{Z}^1(\mathcal{U}_l,\mathcal{G}^0_{Z_l})$, the 1-cocycle restriction of $(\phi_{ij})\in\mathcal{Z}^1(\mathcal{U},\mathcal{G}^0_Z)$ to $\mathcal{U}_l$,

$$\phi^l_{ij}=\phi_{ij},\qquad U_i,U_j\in\mathcal{U}_l.$$

Since each tree $\mathcal{D}_l$ has a height smaller than $h-1$, the induction hypothesis ensures that each tree $\mathcal{M}_l[\phi^l_{ij}]$ is $\mathcal{F}_l$-cobordant to $\mathcal{M}_l$ in the category $\mathrm{Glu}_0(\mathcal{M}_l,Z_l,\mathcal{U}_l)$.

Let us assume first that the above $\mathcal{F}_l$-cobordisms are elementary. By definition, there exists a family of 1-cocycles (T^l_{ij}), $l=1,\ldots,N$ of $\mathcal{Z}^1(\mathcal{U}_l,\mathfrak{I}_{Z_l}\mathfrak{X}_{\mathcal{F}_l,Z_l})$ such that

$$\mathcal{M}_l[\phi^l_{ij}]\overset{\mathrm{G}_0}{\simeq}\mathcal{M}_l[e^{T^l_{ij}}].$$

In view of Property 1.3, there exists a family of 0-cocycles in $\mathcal{G}^0_{Z_l}$ verifying

$$\phi^l_{ij}=\psi^l_je^{T^l_{ij}}\psi_i^{l\,-1}. \tag{18}$$

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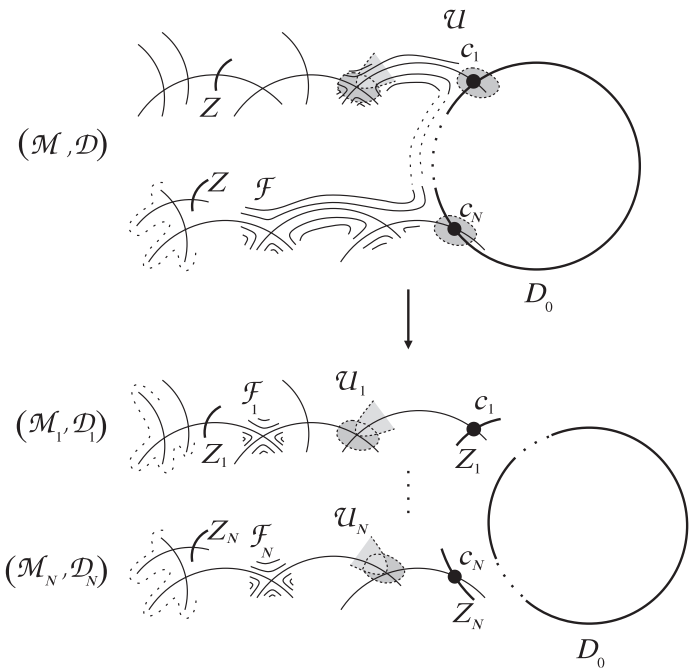

FIGURE 1. Induction construction.

Let us suppose that ψ_1^l refers to the component of the 0-cocycle (ψ_i^l) defined on the open set U_l containing the singularity c_l. Let $(e^{T_{ij}})_{\text{ext}}$ be the 1-cocycle

$$(e^{T_{ij}})_{\text{ext}} = \begin{cases} \text{Id} & \text{on } U_l \cap U_0, \\ e^{T_{ij}^l} & \text{elsewhere.} \end{cases}$$

We denote by ϕ_{l0} the component of the 1-cocycle (ϕ_{ij}) defined on $U_l \cap U_0$. Finally, we define a 1-cocycle by

$$\kappa_{ij} = \begin{cases} \psi_1^l \phi_{l0} & \text{on } U_l \cap U_0, \\ e^{T_{ij}^l} & \text{elsewhere.} \end{cases}$$

Equation (18) induces an isomorphism in $\text{Glu}_0(\mathcal{M}, Z, \mathcal{U})$

$$\mathcal{M}[\kappa_{ij}] \overset{\mathbb{G}_0}{\simeq} \mathcal{M}[\phi_{ij}]. \tag{19}$$

We define the 1-cocycle

$$\tilde{\kappa}_{ij} = \begin{cases} \psi_1^l \phi_{l0} & \text{on } U_l \cap U_0, \\ \text{Id} & \text{elsewhere.} \end{cases} \tag{20}$$

Let $\tilde{\mathcal{M}} \in \text{Glu}_0(\mathcal{M}, Z, \mathcal{U})$ be defined by $\tilde{\mathcal{M}} = \mathcal{M}[e_{\text{ext}}^{T_{ij}}]$. We have

$$\tilde{\mathcal{M}}[\zeta^1 \tilde{\kappa}_{ij}] = \mathcal{M}[e_{\text{ext}}^{T_{ij}}][\zeta^1 \tilde{\kappa}_{ij}] \tag{21}$$

$$\overset{\mathbb{G}_0}{\simeq} \mathcal{M}[e_{\text{ext}}^{T_{ij}} \tilde{\kappa}_{ij}] \tag{22}$$

$$= \mathcal{M}[\kappa_{ij}]. \tag{23}$$

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Outside $U_l \cap U_0$, the components of the cocycle $(\zeta^1 \tilde{\kappa}_{ij})$ are equal to Id. Since ψ_l is in the $\mathcal{G}^0_{Z_l}$ class, $\zeta^1 \tilde{\kappa}_{l0}$ is a germ of an automorphism in $\mathcal{G}^0_{\tilde{Z}}$ where $\tilde{Z} = Z[e^{T_{ij}}_{\text{ext}}]$.

Remark 2.3. The latter argument is the key of the induction, which explains why all our constructions come with the use of a cross: here Z_l plays the role of the *forgotten* part of the component D_0 in the induction step. The cross implies we can control the cocycles along Z_l when the hypothesis of induction is applied. It ensures that, once the cocycles are put together as in equation (20), we keep cocycles in the $\mathcal{G}^0_{\tilde{Z}}$ class.

Now, the sheaves $\mathfrak{I}_{\tilde{Z}}$ and $\mathfrak{M}^1_{\tilde{Z}}$ coincide along the regular part of $\tilde{D}_0$. Hence, the sheaves $\mathcal{G}^1_{\tilde{Z}}$ and $\mathcal{G}^0_{\tilde{Z}}$ are equal. As a consequence, the 1-cocycle $(\zeta^1 \tilde{\kappa}_{ij})$ is in $\mathcal{G}^1_{\tilde{Z}}$ and $\tilde{\mathcal{M}}[\zeta^1 \tilde{\kappa}_{ij}]$ belongs to $\text{Glu}_1(\tilde{\mathcal{M}}, \tilde{Z}, \tilde{\mathcal{U}})$. Proposition 2.3 ensures that there exists an elementary $\mathcal{F}[e^{T_{ij}}_{\text{ext}}]$-cobordism between $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}[\zeta^1 \tilde{\kappa}_{ij}]$ defined by a 1-cocycle $(e^{\tilde{T}_{ij}}) \in Z^1(\tilde{\mathcal{U}}, \mathfrak{I}_{\tilde{Z}} \mathcal{X}_{\mathcal{F}[e^{T_{ij}}_{\text{ext}}], \tilde{Z}})$. From equations (19) and (23), we have $\mathcal{N} \overset{\mathrm{G}_0}{\simeq} \tilde{\mathcal{M}}[\zeta^1 \tilde{\kappa}_{ij}]$. Hence, we obtain a final cobordism between $\mathcal{N}$ and $\mathcal{M}$:

$$\mathcal{N} \overset{\mathrm{G}_0}{\simeq} \tilde{\mathcal{M}}[e^{\tilde{T}_{ij}}], \quad \tilde{\mathcal{M}} \overset{\mathrm{G}_0}{\simeq} \mathcal{M}[e^{T_{ij}}_{\text{ext}}].$$

Now, if the cobordisms are not elementary, one can suppose them to be decomposed in sequences of elementary cobordisms of the same length. Indeed, we have

$$\begin{aligned}
\mathcal{M}_1 = \mathcal{M}_1^1 &\overset{\mathcal{F}_1^1, Z_1^1}{\dashrightarrow} \mathcal{M}_1^2 \cdots \mathcal{M}_1^{p-1} \overset{\mathcal{F}_1^{p-1}, Z_1^{p-1}}{\dashrightarrow} \mathcal{M}_1^p \overset{\mathrm{G}_0}{\simeq} \mathcal{N}_1, \\
\mathcal{M}_2 = \mathcal{M}_2^1 &\overset{\mathcal{F}_2^1, Z_2^1}{\dashrightarrow} \mathcal{M}_2^2 \cdots \mathcal{M}_2^{p-1} \overset{\mathcal{F}_2^{p-1}, Z_2^{p-1}}{\dashrightarrow} \mathcal{M}_2^p \overset{\mathrm{G}_0}{\simeq} \mathcal{N}_2, \\
&\vdots \\
\mathcal{M}_N = \mathcal{M}_N^1 &\overset{\mathcal{F}_N 1, Z_1^1}{\dashrightarrow} \mathcal{M}_N^2 \cdots \mathcal{M}_N^{p-1} \overset{\mathcal{F}_N^{p-1}, Z_N^{p-1}}{\dashrightarrow} \mathcal{M}_N^p \overset{\mathrm{G}_0}{\simeq} \mathcal{N}_N,
\end{aligned}$$

where $\mathcal{F}_l^k$ and Z_l^k, $l \geq 1$ naturally refer to the foliations and the crosses induced by the successive cobordisms. Let $(e^{T_{ij}^{k,l}})$ be the cocycle defining the elementary cobordism $\mathcal{M}_l^k \overset{\mathcal{F}_l^k, Z_l^k}{\dashrightarrow} \mathcal{M}_l^{k+1}$, where $(T_{ij}^{k,l})$ belongs to $\mathcal{Z}^1(\mathcal{U}_l^k, \mathfrak{I}_{Z_l^k} \mathcal{X}_{\mathcal{F}_l^k, Z_l^k})$. For $1 \leq k \leq p-2$, we consider the trees $\dot{\mathcal{M}}^k$ defined by $\dot{\mathcal{M}}^1 = \mathcal{M}$ and

$$\dot{\mathcal{M}}^{k+1} = \dot{\mathcal{M}}^k [e^{T_{ij}^{k,l}}{}_{\text{ext}}]$$

with

$$(e^{T_{ij}^{k,l}})_{\text{ext}} = \begin{cases} \text{Id} & \text{on } U_l^{k,l} \cap U_0^{k,l}, \\ e^{T_{ij}^{k,l}} & \text{elsewhere.} \end{cases}$$

Here, $\mathcal{U}^{k,l}$ refers to the distinguished covering induced by the successive gluings, $\mathcal{U}^{k+1,l} = \mathcal{U}^{k,l}[e^{T_{ij}^{k,l}}_{\text{ext}}]$. In view of the construction, the previous relations give us a general $\mathcal{F}$-cobordism between $\mathcal{M}$ and $\dot{\mathcal{M}}^{p-1}$. Moreover, for all l the tree $\dot{\mathcal{M}}_l^{p-1}$ (restriction of the tree $\dot{\mathcal{M}}^{p-1}$ over the singularity l) is isomorphic to $\mathcal{M}_l^{p-1}$ and the foliation $\dot{\mathcal{F}}_l^{p-1}$

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obtained by successive cobordisms is isomorphic to $\mathcal{F}_l^{p-1}$. Hence, to complete the proof of Proposition 2.1, one must solve the elementary case from $\dot{\mathcal{M}}^{p-1}$ to $\mathcal{N}$, which has already been done.

2.2. *The second kind case.* In this subsection, we want to extend the Proposition 2.1 to a bigger and more natural class of foliations than the generalized curves.

A reduced singularity of foliation with a linear part of the form $x\,dy$ admits a formal normal form of type [**5**]

$$x^{p+1}\,dy - y(1+\lambda x^p)\,dx.$$

The separatrix $x = 0$ is called the strong invariant curve. If some singularities of such a kind appear in the reduction process, most of the previous results are false, except when these kinds of singularities are in a specific position. The notion of foliation of the second kind was introduced in [**11**] to cover that possibility. This class of foliations admits the same properties as generalized curves and provides an efficient description of local formal moduli spaces.

Definition 2.4. $\mathcal{F}$ is of second kind if:
(1) $\mathcal{F}$ is non-dicritical;
(2) each singularity of the divisor is a reduced singularity of the foliation with two non-vanishing eigenvalues; and
(3) each singularity of the foliation on the regular part of the divisor is either of the above kind or has $x\,dy$ for the linear part (in the latter case, one asks the strong invariant curve to be the germ defined by the divisor).

With obvious notation, we consider the sheaf $\widehat{\mathfrak{X}}_{S,Z}$ over $\mathcal{D}$ of germs of formal vector fields that are tangent to the divisor, the separatrix and the cross. We denote by $\widehat{\mathfrak{X}}_{\mathcal{F},Z}$ the subsheaf of vector fields tangent to the foliation and the cross. In [**9**], one can find the following criterion for a foliation to be of second kind.

PROPOSITION 2.4. *The foliation $\mathcal{F}$ is of second kind if and only if the sequence of sheaves*

$$0 \longrightarrow \widehat{\mathfrak{X}}_{\mathcal{F},Z} \longrightarrow \widehat{\mathfrak{X}}_{S,Z} \xrightarrow{E^*\omega(\cdot)} \widehat{\mathcal{O}}(f\circ E) \longrightarrow 0$$

is exact.

It is not difficult to see that from the above exact sequence every previous construction and argument can be repeated in the formal context under the second kind hypothesis. More precisely, one can obtain the following result.

PROPOSITION 2.5. *Let $\mathcal{F}$ be a foliation of the second kind on a marked tree $\mathcal{M}$ and Z an adapted cross. Let $\mathcal{N}$ be in* $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$. *There exists a finite sequence of* 1*-cocyles* $(\widehat{T}_{ij}^k)_{k=1\ldots N}$ *such that*

$$\mathcal{N} \overset{\widehat{G}_0}{\simeq} \mathcal{M}[e^{\widehat{T}_{ij}^1}][e^{\widehat{T}_{ij}^2}][\cdots][e^{\widehat{T}_{ij}^N}].\dagger$$

† Here, the isomorphism is related to the category $\widehat{\mathrm{Glu}}_0(\mathcal{M}, Z, \mathcal{U})$. This notation refers to the same one but in a convergent context. The transposition to a formal context is easy.

Here, $(\widehat{T}_{ij}^p)$ *is a* 1*-cocycle with values in* $\widehat{\mathfrak{X}}_{\widehat{\mathcal{F}}_p,\widehat{Z}_p}$*: this is the sheaf over the divisor of the tree* $\mathcal{M}[e^{\widehat{T}_{ij}^1}][e^{\widehat{T}_{ij}^2}][\cdots][e^{\widehat{T}_{ij}^{p-1}}]$ *whose fiber is the space of the germ of a formal vector field tangent to the following formal foliation and formal cross:*

$$\widehat{\mathcal{F}}_p = \widehat{\mathcal{F}}[e^{\widehat{T}_{ij}^1}][e^{\widehat{T}_{ij}^2}][\cdots][e^{\widehat{T}_{ij}^{p-1}}], \quad \widehat{Z}_p = Z[e^{\widehat{T}_{ij}^1}][e^{\widehat{T}_{ij}^2}][\cdots][e^{\widehat{T}_{ij}^{p-1}}].$$

From this formal construction, one can go back to the convergent context using the following lemma: let $\mathcal{M}$ be foliated by a convergent foliation $\mathcal{F}$ of the second kind and let $(\widehat{T}_{ij})$ be in $Z^1(\mathcal{U}, \mathfrak{I}_Z\widehat{\mathfrak{X}}_{\mathcal{F},Z})$.

LEMMA 2.4. *There exists* $(T_{ij}^c) \in Z^1(\mathcal{U}, \mathfrak{I}_Z\mathfrak{X}_{\mathcal{F},Z})$ *such that* $\mathcal{M}[\widehat{T}_{ij}] \overset{\widehat{G}_0}{\simeq} \mathcal{M}[T_{ij}^c]$.

Proof. Since $\mathcal{F}$ is convergent, there exists a convergent vector field T_{ij} tangent to the foliation and a formal series $\widehat{\phi}_{ij}$ such that $\widehat{T}_{ij} = \widehat{\phi}_{ij}T_{ij}$. For any integer n, we consider the cocyle (T_{ij}^c) in $\mathfrak{X}_{\mathcal{F},Z}$ defined by $T_{ij}^c = \phi_{ij}^n T_{ij}$, where ϕ_{ij}^n refers to a representative function of the n-jet of $\widehat{\phi}_{ij}$. We are going to show that, for n big enough, the gluings associated with $\widehat{T}_{ij}$ and T_{ij}^c are formally equivalent in $\widehat{\mathrm{Glu}}_0(\mathcal{M}, Z, \mathcal{U})$. Let us consider the 1-cocycle

$$\delta_{ij}(x, t, s) = (e^{(t-s)\widehat{T}_{ij}} \circ e^{(s)T_{ij}^c}, t, s).$$

We construct the equisingular unfolding

$$\mathcal{M}_{\overline{\mathbb{D}}^2} = \bigcup_i U_i \times \overline{\mathbb{D}}^2 /_{(x,t,s)\sim\delta_{ij}(x,t,s)},$$

which admits a projection $\mathcal{M}_{\overline{\mathbb{D}}^2} \xrightarrow{\Pi} \overline{\mathbb{D}}^2$. This manifold is the neighborhood of a divisor $\mathcal{D}_{\overline{\mathbb{D}}^2}$. Thanks to local triviality of the unfolding along $U_i \times \overline{\mathbb{D}}^2$, one can find a family $\{X_i\}$ of vector fields tangent to the foliation such that in $U_i \times \overline{\mathbb{D}}^2$ we have $T\Pi(X_i) = \partial/\partial s$. The 1-cocycle $(X_i - X_j)$ takes its values in the sheaf over $\mathcal{D}_{\overline{\mathbb{D}}}$ of germs of vector fields tangent to the foliation, vertical—$T\Pi(X) = 0$—and vanishing at order n along $\mathcal{D}_{\overline{\mathbb{D}}}$. We denote by $\widehat{\mathfrak{X}}_{\mathcal{F},Z,n}^{\overline{\mathbb{D}}^2}$ the latter sheaf. Since $\widehat{\mathfrak{X}}_{\mathcal{F},Z,1}^{\overline{\mathbb{D}}^2}$ is coherent, one can find the following property in [**3**].

SUB-LEMMA 2.2. *For n big enough, the map*

$$H^1(\widehat{\mathfrak{X}}_{\mathcal{F},Z,n}^{\overline{\mathbb{D}}^2}) \longrightarrow H^1(\widehat{\mathfrak{X}}_{\mathcal{F},Z,1}^{\overline{\mathbb{D}}^2})$$

is trivial.

Hence, there exists a 0-cocycle (Y_i) in $\widehat{\mathfrak{X}}_{\mathcal{F},Z,1}^{\overline{\mathbb{D}}^2}$ such that

$$X_i - X_j = Y_i - Y_j.$$

Therefore, $X = X_i - Y_i$ is a global formal vector field such that $T\Pi(X) = \partial/\partial s$. The biholomorphism $\phi(x, t, s) = (e^{(s)X}(x, t, 0), t, s)$ formally conjugates the equisingular unfoldings $\mathcal{M}_{\overline{\mathbb{D}}^2}$ and $\mathcal{M}_{\overline{\mathbb{D}}\times\{0\}} \times \overline{\mathbb{D}}$. By restricting them along the diagonal, one can see that $\mathcal{M}[\widehat{T}_{ij}]$ and $\mathcal{M}[T_{ij}^c]$ are conjugated. □

By applying the previous lemma to each elementary cobordism in Proposition 2.5, one can prove the cobordism result for a foliation of the second kind.

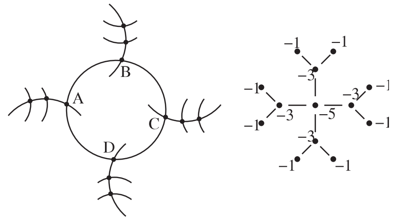

FIGURE 2. A tree and its dual tree.

PROPOSITION 2.6. *Let $\mathcal{F}$ be a foliation of the second kind on a marked tree $\mathcal{M}$ and Z a cross adapted to $\mathcal{F}$. Any element in* $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$ *is $\mathcal{F}$-cobordant to $\mathcal{M}$.*

3. *Existence theorem*

In this section, we use Proposition 2.6 to establish the existence theorem. Let $\mathcal{F}$ be a foliation of the second kind on a marked tree $(\mathcal{M}, \mathcal{D}, \Sigma_{\mathcal{F}})$ and $(\mathcal{M}', \mathcal{D}', \Sigma')$ be any marked tree.

THEOREM 3.1. *If the marked weighted dual trees of $\mathcal{M}$ and $\mathcal{M}'$ are topologically equivalent then there exists a foliation $\mathcal{F}'$ on $\mathcal{M}'$ such that $\mathcal{F}$ and $\mathcal{F}'$ are linked by an equisingular unfolding, which respects the marking.*

We focus on the proof of the above statement.

3.1. *First step: cocyle transformation.* Since the marked dual trees of $\mathcal{M}$ and $\mathcal{M}'$ are conjugated, one can find in [**12**] the following result. There exists an equisingular unfolding of $\mathcal{F}$, which leads to a foliation defined on a tree with a divisor biholomorphic to the divisor of $\mathcal{M}'$. Hence, one can suppose that the divisors $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic.

Remark 3.1. One has to notice that the divisor could be isomorphic without the neighborhoods being the same. For example, consider any tree $\mathcal{M}$ topologically equivalent to the one in Figure 2. If one fixes the position of the four points A, B, C and D on the projective line of self-intersection -5, there remain eight degrees of freedom to determine $\mathcal{M}$, which correspond to the four pairs of singular points on the components of self-intersection -3. Two such trees have isomorphic divisor. Now, let ϕ be in a germ of the automorphism of $(\mathbb{C}^2, 0)$, which leaves invariant the four points A, B, C and D when lifted up to the first projective line: the first jet of ϕ is necessarily a *homothetie* and the action of the lifting up of ϕ on any projective line of self-intersection -3 is completely determined by the second jet of ϕ. Finally, this provides seven degrees of freedom on ϕ. Therefore, there exist trees topologically equivalent to $\mathcal{M}$ with isomorphic divisors, which are not analytically equivalent to $\mathcal{M}$.

Let us denote by h a biholomorphism between the divisors. For each component D of $\mathcal{D}$, we consider a fibration π_D transversal to D

$$\pi_D : T(D) \longrightarrow D$$

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where $T(D)$ is a fixed tubular neighborhood of D such that the traces of the transversal components of $\mathcal{D}$ are fibers of π_D. Such a fibration can be obtained as follows: assume that the component D appears in the blowing-up process by blowing up a point c. Consider the radial foliation $\mathcal{R}_c$ generated by the vector field $x\partial_x + y\partial_y$ where (x, y) are some adapted coordinates near the point c. The checked fibration is the restriction on $T(D)$ of the foliation $\mathcal{R}_c$ blown up by the remaining blowing-up process. We make the same construction over the components of $\mathcal{D}'$. An easy computation in coordinates allows us to show the following lemma.

LEMMA 3.1. *There exists a family of maps* $(H_D)_{D\in\mathrm{Comp}(\mathcal{D})}$ *such that the following diagram is commutative*

$$\begin{array}{ccc} T(D) & \xrightarrow{H_D} & T(h(D)) \\ {\scriptstyle \pi_D}\downarrow & & \downarrow{\scriptstyle \pi_{h(D)}} \\ D & \xrightarrow{h_{|D}} & h(D) \end{array}$$

We denote by $\mathrm{Comp}(\mathcal{D})^{\check{2}}$ the set $\{(D, D') \in \mathrm{Comp}(D)^2 \mid D \cap D' \neq \emptyset\}$. Let us consider the germ of the manifold defined by

$$\tilde{\mathcal{M}} = \coprod_{D\in\mathrm{Comp}(\mathcal{D})} T(D)/(x \sim H_D{}^{-1} \circ H_{D'}x)_{(D,D')\in\mathrm{Comp}(\mathcal{D})^{\check{2}}}.$$

One has to notice that the above gluing is different from the gluing introduced in previous sections. Here, coverings are made of tubular neighborhoods which intersect each other along polydisks, whereas open sets of distinguished covering are finer and intersect each other along a torus. Nevertheless, the tree $\tilde{\mathcal{M}}$ is a neighborhood of some divisor $\tilde{\mathcal{D}}$. The tree $(\tilde{\mathcal{M}}, \tilde{\mathcal{D}}, \tilde{\Sigma})$ is a marked tree conjugated to $(\mathcal{M}', \mathcal{D}', \Sigma')$; indeed, the family (H_D) induces a biholomorphism adapted to markings.

In order to apply Proposition 2.1, we build a tree $\dot{\mathcal{M}}$ such that $\dot{\mathcal{M}}$ verifies the existence theorem and $\tilde{\mathcal{M}}$ belongs to $\mathrm{Glu}_0(\dot{\mathcal{M}}, Z, \mathcal{U})$ for a suitable cross and covering.

Let us denote by $s_{DD'}$ the intersection of D and D' and let $\phi_{DD'} := H_D{}^{-1} \circ H_{D'}$.

LEMMA 3.2. *There exist families* $(\Delta_{DD'})_{(D,D')\in\mathrm{Comp}(\mathcal{D})^{\check{2}}}$ *and* $(\phi_D)_{D\in\mathrm{Comp}(\mathcal{D})}$ *of automorphism germs such that:*

- $\Delta_{DD'}$ *is defined near* $s_{DD'}$*, fixes this point and leaves invariant each local leaf;*
- ϕ_D *is defined along* D *and fixes each point of* D*; and*
- *the germ of automorphism* $\phi_{D'}^{-1} \circ \Delta_{DD'} \circ \phi_{DD'} \circ \phi_D$ *is tangent to the identity at* $s_{DD'}$.

Proof. Let us consider the standard metric d on the dual weighted graph of $\mathcal{M}$ and fix one vertex $D_0 \in \mathrm{Comp}(\mathcal{D})$. We define the subgraph $\mathbb{A}_n^*$, $n \geq 0$ whose vertices are at a distance smaller than n from D_0. The graphs $\mathbb{A}_n^*$ are connected and cover the whole $\mathcal{M}$ graph. One can now consider the restricted family $(\phi_{DD'}^n)$ defined by

$$\phi_{DD'}^n = \phi_{DD'}, \qquad DD' \text{ is an edge of } \mathbb{A}_n^*.$$

By induction, we establish the result on the subfamily $(\phi_{DD'}^n)$. For $n = 0$, the result is obvious and one can choose for ϕ_{D_0} the identity automorphism. Let us suppose the result

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is true for n: more precisely, we have two families $\Delta_{DD'}$ and ϕ_D as in the lemma such that for any edge DD' of $\mathbb{A}_n^*$,

$$\phi_{D'}^{-1} \circ \Delta_{DD'} \circ \phi_{DD'} \circ \phi_D$$

is tangent to the indentity at $s_{DD'}$. Let $D_i D_j$ be an edge of $\mathbb{A}_{n+1}^* \backslash \mathbb{A}_n^*$ such that D_i is a vertex of $\mathbb{A}_n^*$ and D_j a vertex of $\mathbb{A}_{n+1}^*$. One has to define ϕ_{D_j} and $\Delta_{D_i D_j}$ in an efficient way in order to conclude the result. Now, one can easily prove the following general results.

SUB-LEMMA 3.2.

(1) *Let s be a singular point of $\mathcal{D}$ and $\alpha \in \mathbb{C}^*$. There exists a germ of an automorphism Δ defined near s and leaving invariant each local leaf such that, in adapted coordinates, the tangent map at s is*

$$T_s \Delta = \begin{pmatrix} \alpha & 0 \\ 0 & * \end{pmatrix}.$$

(2) *For any β in $\mathbb{C}^*$, any component of D and any $c \in D$, there exists a germ of an automorphism ϕ defined in the neighborhood of D fixing each point of D such that, in adapted coordinates (assuming the second coordinates to be transversal to the divisor), the tangent map at c is*

$$T_c \phi = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

In adapted coordinates, the tangent map of the automorphism $\phi_{D_i D_j} \circ \phi_{D_i}$ is $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. In view of the previous lemma, there exists a germ of an automorphism $\Delta_{D_i D_j}$ defined near $s_{D_i D_j}$, leaving invariant each local leaf and a germ of an automorphism ϕ_{D_j} defined in the neighborhood of D_j fixing each point of D_j such that

$$T_{s_{D_i D_j}} (\Delta_{D_i D_j}^{-1} \circ \phi_{D_j}) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Then, $T_{s_{D_i D_j}} (\phi_{D_j}^{-1} \circ \Delta_{D_i D_j} \circ \phi_{D_i D_j} \circ \phi_{D_i}) = \mathrm{Id}$. Since the dual tree does not have a cycle, one can repeat the same construction for all edges of $\mathbb{A}_{n+1}^* \backslash \mathbb{A}_n^*$. Hence, the lemma is proved for $\mathbb{A}_{n+1}^*$. □

3.2. *Second step: a fine distinguished covering.* With the family $\{\Delta_{DD'}\}$ of the previous lemma, one can define

$$\dot{\mathcal{M}} = \coprod_{D \in \mathrm{Comp}(\mathcal{D})} T(D) \Big/ (x \sim \Delta_{DD'}^{-1} x).$$

The tree $\dot{\mathcal{M}}$ is a neighborhood of some divisor $\dot{\mathcal{D}}$. One can find in [**1**] a precise description of germs leaving invariant each local leaf of a reduced singularity with two non-vanishing eigenvalues. These can be written as $(x, y) \mapsto \phi_X^{t(x,y)}$ where $t(x, y)$ is a function, X a germ of a vector field tangent to the foliation and ϕ_X^t the flow of X at time t. In particular, this description ensures that $\dot{\mathcal{M}}$ admits a foliation $\dot{\mathcal{F}}$ linked to $\mathcal{F}$ by an equisingular unfolding. Basically, the unfolding exists because there is a suitable isotropy from $(x, y) \mapsto \phi_X^{t(x,y)}$ to the identity defined by $(s, x, y) \mapsto \phi_X^{st(x,y)}$.

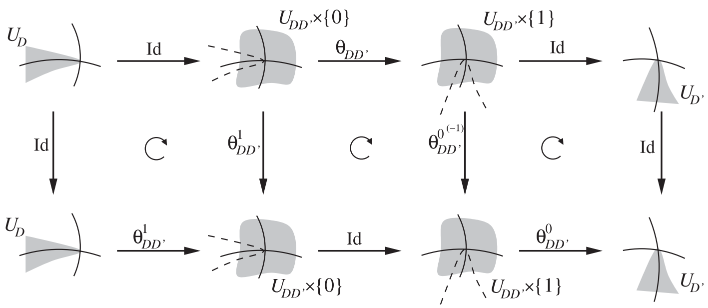

FIGURE 3. Finer distinguished covering.

One notices that we have the isomorphism

$$\tilde{\mathcal{M}} \simeq \coprod_{\dot{D}\in \mathrm{Comp}(\dot{\mathcal{D}})} T(\dot{D}) \Big/ (x \sim \Delta_{DD'} \circ \phi_{DD'} x).$$

Let us consider $\theta_{DD'} := {\phi_{D'}}^{-1} \circ \Delta_{DD'} \circ \phi_{DD'} \circ \phi_D$. One can see that

$$\coprod_{\dot{D}\in \mathrm{Comp}(\dot{\mathcal{D}})} T(\dot{D}) \Big/ (x \sim \theta_{DD'} x) \simeq \tilde{\mathcal{M}}.$$

We shall make a last transformation on $\theta_{DD'}$ in order to obtain a cocycle with values in $\mathcal{G}^0_Z$; the proof of the following result is not difficult.

SUB-LEMMA 3.3. *Let θ be a germ of an automorphism tangent to the identity at $0 \in \mathbb{C}^2$, letting the axes $\{x = 0\}$ and $\{y = 0\}$ be invariant. The germ θ admits a decomposition of the form*

$$\theta = \theta^0 \circ \theta^1, \quad \theta^0 = \begin{pmatrix} x + x^2(\cdots) \\ y + xy(\cdots) \end{pmatrix}, \quad \theta^1 = \begin{pmatrix} x + xy(\cdots) \\ y + y^2(\cdots) \end{pmatrix}.$$

For each $D, D' \in \mathrm{Comp}(\mathcal{D})^{\check{2}}$, we decompose $\theta_{DD'}$ in $\theta_{DD'} = \theta^0_{DD'} \circ \theta^1_{DD'}$. Let us consider a distinguished covering $\mathcal{U} = (U_i)$ of $\mathcal{D}$ finer than the tubular covering $(T(\dot{D}))_{D\in \mathrm{Comp}(\mathcal{D})}$ defined by

$$\begin{cases} U_D = T(D) \setminus \{\text{Singular points of } \mathcal{D}\} & \text{for } D \in \mathrm{Comp}(\mathcal{D}), \\ U_{DD'} = T(D) \cap T(D') & \text{for } (D, D') \in \mathrm{Comp}(\mathcal{D})^{\check{2}}. \end{cases}$$

Using the disjoint union $\mathfrak{U}$ defined by

$$\mathfrak{U} = \coprod_{D\in \mathrm{Comp}(\mathcal{D})} U_D \coprod_{(D,D')\in \mathrm{Comp}(\mathcal{D})^{\check{2}}} U_{DD'} \times \{0\} \coprod_{(D,D')\in \mathrm{Comp}(\mathcal{D})^{\check{2}}} U_{DD'} \times \{1\},$$

we consider the commutative diagram of Figure 3. The two horizontal lines of this diagram describe two different ways to glue together the open sets of $\mathfrak{U}$. The glued manifolds are both trees equal to

$$\tilde{\mathcal{M}} \quad \text{and} \quad \dot{\mathcal{M}}\Big[\{\theta^{\epsilon}_{DD'}\}_{\substack{\epsilon = 0,\,1 \\ (D,\,D') \in \mathrm{Comp}(\mathcal{D})^{\check{2}}}}\Big],$$

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respectively. Since the diagram commutes, the vertical arrows provide an isomorphism between them. Let Z be any suitable cross on $\dot{\mathcal{M}}$ adapted to $\dot{\mathcal{F}}$. Since the cocyle $\{\theta^{\epsilon}_{DD'}\}_{\substack{\epsilon=0,1\\(D,D')\in\mathrm{Comp}(\mathcal{D})^{\check{2}}}}$ belongs to $\mathcal{G}^0_Z$, the tree $\tilde{\mathcal{M}}$ belongs to $\mathrm{Glu}_0(\dot{\mathcal{M}}, Z, \mathcal{U})$. In view of the previous results, $\tilde{\mathcal{M}}$ is $\dot{\mathcal{F}}$-cobordant to $\dot{\mathcal{M}}$. Hence, $\mathcal{M}' \simeq \tilde{\mathcal{M}}$ admits a foliation $\mathcal{F}'$ linked to $\mathcal{F}$ by an equisingular unfolding. □

4. *Cobordism class*

Let $\mathcal{F}$ be a non-dicritical generalized curve and $S = \mathrm{Sep}(\mathcal{F})$ its separatrix. A foliation $\mathcal{F}'$ is said to be cobordant to $\mathcal{F}$ when there is an equisingular unfolding in the sense of [**8**] that links $\mathcal{F}$ and $\mathcal{F}'$. We denote by $\mathrm{Cob}(\mathcal{F})$ the set of cobordant foliations. The set $\mathrm{E}(S)$ refers to the equisingularity class of S or, in an equivalent way, the set of curves topologically conjugated to S.

THEOREM 4.1. *The map defined by*

$$\mathcal{F}' \in \mathrm{Cob}(\mathcal{F}) \longmapsto \mathrm{Sep}(\mathcal{F}') \in \mathrm{E}(S)$$

is onto.

In order to prove the above result, one notices that the existence theorem holds for trees that may not be the minimal reduction tree of a foliation. This remark leads us to define the following sequence of blowing-up processes: let $(E^j)_{j=1\ldots h}$ be the blowing-up process of the reduction of the foliation $\mathcal{F}$ (§1.1). We define a sequence of blowing-up processes as follows.

- $\mathfrak{E}_h = (E^j)_{j=1\ldots h}$.
- $\mathfrak{E}_{n+1}$ is the blowing-up process build over $\mathfrak{E}_n$ where S^n and Σ^n are both the set of singularities of $E_n^*\mathcal{F}$, where E_n refers to the total morphism of $\mathfrak{E}_n$ (§1.1).

Proof of Theorem 4.1. Let S' be a curve in the equisingularity class of S. Let E and E' refer to the reduction processes of S and S', respectively. In view of a result in [**2**], since $\mathcal{F}$ is a generalized curve, E is also the reduction of $\mathcal{F}$. Moreover, according to a result of Zariski [**15**], the weighted dual graphs of S and S' are conjugated. In particular, the points of attachment of the S and S' strict transforms are on conjugated components.

Let $\mathcal{H}$ be the foliation $dh = 0$ where h is any reduced equation of S'. For any integer n bigger than the height of E, we consider the sequence E_n and E'_n built as above with $\mathcal{F}$ and $\mathcal{H}$ as respective initial data. For every n, the weighted dual trees associated with E_n and E'_n are topologically equivalent. Theorem 3.1 ensures the existence of a foliation $\mathcal{F}_n$ on the tree of E'_n linked to $E_n^*\mathcal{F}$ by an equisingular unfolding. Let ξ_n be the germ of the separatrix of $\mathcal{F}_n$ at $0 \in \mathbb{C}^2$. One can see that the points of attachment of the strict transforms by E'_n of the curves ξ_n and S' are on the same components. Since $\mathcal{F}_n$ is cobordant to $\mathcal{F}$, $\mathcal{F}_n$ is topologically equivalent to $\mathcal{F}$. In particular, ξ_n is topologically equivalent to S'. For n large enough, the study in [**15**] ensures that ξ_n and S' are in fact analytically conjugated. The image of $\mathcal{F}_n$ by this conjugacy is a holomorphic foliation topologically equivalent to $\mathcal{F}$ admitting S' for a separatrix.

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