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# AN EXAMPLE REGARDING KALTON'S PAPER 'ISOMORPHISMS BETWEEN SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS'

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*Abstract* The paper alluded to in the title contains the following striking result: Let $I$ be the unit interval and $\Delta$ the Cantor set. If $X$ is a quasi Banach space containing no copy of $c_0$ which is isomorphic to a closed subspace of a space with a basis and $C(I, X)$ is linearly homeomorphic to $C(\Delta, X)$, then $X$ is locally convex, i.e., a Banach space. We will show that Kalton result is sharp by exhibiting non-locally convex quasi Banach spaces $X$ with a basis for which $C(I, X)$ and $C(\Delta, X)$ are isomorphic. Our examples are rather specific and actually, in all cases, $X$ is isomorphic to $C(K, X)$ if $K$ is a metric compactum of finite covering dimension.

*Keywords:* spaces of vector-valued continuous functions; quasi Banach spaces

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The paper alluded to in the title contains the following striking result [8, Theorem 6.4]: Let $I$ be the unit interval and $\Delta$ the Cantor set. If $X$ is a quasi Banach space containing no copy of $c_0$ which is isomorphic to a closed subspace of a space with a basis and $C(I, X)$ is linearly homeomorphic to $C(\Delta, X)$, then $X$ is locally convex, i.e., a Banach space.

Here $C(K, X)$ denotes the space of continuous functions $F : K \longrightarrow X$. When $K$ is a compact space and $X$ a quasi Banach space $C(K, X)$ is also a quasi Banach space under the quasinorm $\|F\| = \sup\{\|F(t)\| : t \in K\}$.

When $X$ is a Banach space, the isomorphic theory of the spaces $C(K, X)$ is somehow oversimplified by Miljutin theorem (the spaces $C(K) = C(K, \mathbb{R})$ for $K$ uncountable and metrizable are all mutually isomorphic) and, above all, by Grothendieck's identity $C(K, X) = C(K)\check{\otimes}_\varepsilon X$ which implies that the isomorphic type of the Banach space $C(K, X)$ depends only on those of $C(K)$ and $X$. The situation for quasi Banach spaces is more thrilling and actually some seemingly innocent questions remain open: Is $C(I, \ell_p)$ isomorphic to $C(I^2, \ell_p)$? Is $C(I, L_p)$ isomorphic to $C(\Delta, L_p)$? These appear as Problems 7.2 and 7.3 at the end of [8]. Problem 7.1, namely if $C(K) \otimes X$ (the subspace of functions whose range is contained in some finite-dimensional subspace of $X$) is always dense in $C(K, X)$, was posed by Klee and is connected with quite serious mathematics. While it seems to be widely open for quasi Banach spaces $X$, the answer is negative for $F$-spaces (complete linear metric spaces) as shown by Cauty's celebrated example [4] (see also [9]) and affirmative for locally convex spaces. See Waelbroeck [13, Section 8] for a discussion on Klee's *density* problem.

The aim of this short note is much more modest: we will show that Kalton's result is sharp by exhibiting non-locally convex quasi Banach spaces $X$ with a basis for which $C(I, X)$ and $C(\Delta, X)$ are isomorphic. Our examples are rather specific and actually, in all cases, $X$ is isomorphic to $C(K, X)$ if $K$ is a metric compactum of finite covering dimension.

Recall that the (Lebesgue) covering dimension of a (not necessarily compact) topological space $K$ is the smallest number $n \geq 0$ such that every open cover admits a refinement in which every point of $K$ lies in the intersection of no more than $n+1$ sets of the refinement.

A quasi Banach space $X$ has the $\lambda$-approximation property ($\lambda$-AP) if for every $x_1, \ldots, x_n \in X$ (or in some dense subset) there is a finite-rank operator $T$ on $X$ such that $\|T\| \leq \lambda$ and $\|x_i - Tx_i\| < \varepsilon$. We say that $X$ has the bounded approximation property (BAP) if it has the $\lambda$-AP for some $\lambda \geq 1$.

We end these preliminaries by recalling that a $p$-norm, where $0 < p \leq 1$, is a quasinorm satisfying the inequality $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ and that every quasinormed space has an equivalent $p$-norm for some $0 < p \leq 1$, so says the Aoki–Rolewicz theorem.

**Lemma.** *If $K$ has finite covering dimension and $X$ has the BAP, then $C(K, X)$ has the BAP.*

**Proof.** We first observe that if $K$ has finite covering dimension *or* $X$ has the BAP, then $C(K) \otimes X$ is dense in $C(K, X)$. The part concerning the BAP is obvious; the other part is a result by Shuchat [12, Theorem 1].

Given $g \in C(K)$ and $x \in X$, we denote by $g \otimes x$ the function $t \longmapsto g(t)x$. Since every function in $C(K) \otimes X$ can be written as a finite sum $\sum_i g_i \otimes x_i$ with $g_i \in C(K)$, $x_i \in X$ (which justifies our notation, see [12, Proposition 1]), it suffices to see that there is a constant $\Lambda$ such that, given $f_1, \ldots, f_m \in C(K)$, $y_1, \ldots, y_m \in X$ and $\varepsilon > 0$, there is a finite-rank operator $T$ on $C(K, X)$ such that $\|T\| \leq \Lambda$ and $\|f_i \otimes y_i - T(f_i \otimes y_i)\| < \varepsilon$. As $\varepsilon$ is arbitrary, there is no loss of generality in assuming that $\|f_i\| = \|y_i\| = 1$ for $1 \leq i \leq m$.

Take an open cover $U_1, \ldots, U_r$ of $K$ such that for every $i$, $j$, one has $|f_i(s) - f_i(t)| < \varepsilon$ for all $s, t \in U_j$. Put $n = \dim(K)$ and take a refinement $V_1, \ldots, V_k$ so that each point of $K$ lies in no more than $n+1$ of those sets. Finally, let $\phi_1, \ldots, \phi_k$ be a partition of unity of $K$ subordinate to $V_1, \ldots, V_k$.

For each $j$, pick $t_j \in V_j$ and define an operator $L$ on $C(K, X)$ by letting $L(F) = \sum_{j \leq k} \phi_j \otimes F(t_j)$, that is, $(LF)(t) = \sum_{j \leq k} \phi_j(t) F(t_j)$. Let us estimate $\|L\|$ assuming $X$ is $p$-normed: one has

$$\|L(F)\| = \sup_{t \in K} \left\| \sum_{j \leq k} \phi_j(t) F(t_j) \right\|,$$

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but for each $t \in K$ the sum has no more than $n+1$ nonzero summands, so

$$\left\|\sum_{j\leq k}\phi_j(t)F(t_j)\right\| \leq \left(\sum_{j\leq k}\phi_j(t)^p\|F(t_j)\|^p\right)^{1/p} \leq \|F\|\left(\sum_{j\leq k}\phi_j(t)^p\right)^{1/p}$$
$$\leq \|F\|\|\mathbf{I}:\ell_1^{n+1}\longrightarrow \ell_p^{n+1}\| = (n+1)^{1/p-1}\|F\|$$

We claim that $\|f_i\otimes y_i - L(f_i\otimes y_i)\| \leq \varepsilon$ for all $i$. We have $L(f_i\otimes y_i)(t) = \sum_{j\leq k} f_i(t_j)\phi_j(t)y_i$, hence

$$\|f_i\otimes y_i - L(f_i\otimes y_i)\| = \left\|\sum_{j\leq k} f_i\phi_j - \sum_{j\leq k} f_i(t_j)\phi_j\right\|\|y_i\| = \left\|\sum_{j\leq k} f_i\phi_j - \sum_{j\leq k} f_i(t_j)\phi_j\right\|.$$

For each $j$ and each $t\in K$ one has $|f_i(t)\phi_j(t) - f_i(t_j)\phi_j(t)| \leq \varepsilon\phi_j(t)$: this is obvious if $t\notin V_j$ since in this case $\phi_j(t)=0$, while for $t\in V_j$ we have $|f_i(t)-f_i(t_j)|\leq\varepsilon$ by our choice of $V_1,\ldots,V_k$ and thus

$$\left|\sum_{j\leq k} f_i(t)\phi_j(t) - \sum_{j\leq k} f_i(t_j)\phi_j(t)\right| \leq \varepsilon\sum_{j\leq k}\phi_j(t) = \varepsilon$$

holds for all $t\in K$; consequently, we have

$$\left\|\sum_{j\leq k} f_i\phi_j - \sum_{j\leq k} f_i(t_j)\phi_j\right\| \leq \varepsilon.$$

Let $R$ be a finite-rank operator on $X$ such that $\|y_i - R(y_i)\| < \varepsilon$, with $\|R\|\leq\lambda$, where $\lambda$ is the 'approximation constant' of $X$, and define $T$ on $C(K, X)$ by $(TF)(t) = R((LF)(t))$. Clearly, $T$ has finite-rank since for an elementary tensor $f\otimes x$ one has

$$T(f\otimes x) = \sum_{j\leq k} f(t_j)\phi_j\otimes R(x).$$

Finally, let us estimate $\|f_i\otimes y_i - T(f_i\otimes y_i)\|$. Write

$$f_i\otimes y_i - T(f_i\otimes y_i) = f_i\otimes y_i - f_i\otimes R(y_i) + f_i\otimes R(y_i) - \sum_{j\leq k} f_i(t_j)\phi_j\otimes R(y_i)$$

and then

$$\|f_i\otimes y_i - T(f_i\otimes y_i)\|^p \leq \|f_i\otimes y_i - f_i\otimes R(y_i)\|^p + \left\|f_i\otimes R(y_i) - \sum_{j\leq k} f_i(t_j)\phi_j\otimes R(y_i)\right\|^p$$
$$= \|f_i\|^p\|y_i - R(y_i)\|^p + \left\|f_i - \sum_{j\leq k} f_i(t_j)\right\|^p\|R(y_i)\|^p \leq \varepsilon^p + \varepsilon^p\lambda^p,$$

so that $C(K, X)$ has the BAP with constant at most $\lambda(n+1)^{1/p-1}$. □

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The proof raises the question of whether the lemma is true for, say, the Hilbert cube $I^\omega$.

The other ingredient we need is a *complementably universal* space for the BAP. A separable $p$-Banach space is complementably universal for the BAP if it has the BAP and contains a complemented copy of each separable $p$-Banach space with the BAP. The existence of such spaces (one for each $0 < p < 1$) was first mentioned by Kalton himself in [6, Theorem 4.1(b)]. A complete proof appears in the *related issues* of [1]. In any case, it easily follows from the Pełczyński decomposition method that any two separable $p$-Banach spaces complementably universal for the BAP are isomorphic, so let us denote by $\mathscr{K}_p$ the isomorphic type of such specimens and observe that since each separable $p$-Banach space with the BAP is complemented in one with a basis, it follows that $\mathscr{K}_p$ does have a basis. Needless to say, $\mathscr{K}_p$ is not locally convex since it contains a complemented copy of $\ell_p$.

**Corollary.** *If $K$ is a (non-empty) metrizable compactum of finite covering dimension, then $C(K, \mathscr{K}_p)$ is linearly homeomorphic to $\mathscr{K}_p$. In particular, $C(I, \mathscr{K}_p)$ and $C(\Delta, \mathscr{K}_p)$ are linearly homeomorphic although $\mathscr{K}_p$ is not locally convex.*

**Proof.** This clearly follows from the lemma since $C(K, \mathscr{K}_p)$ is separable, has the BAP and contains $\mathscr{K}_p$ complemented as the subspace of constant functions. □

We do not know of any other non-locally convex quasi Banach space $X$ for which $C(I, X)$ and $C(\Delta, X)$ are isomorphic, apart from the obvious ones arising as direct sums of $\mathscr{K}_p$ and Banach spaces lacking the BAP. An obvious candidate is the $p$-Gurariy space, introduced by Kalton in [7, Theorem 4.3] and further studied in [2]. Note that if $X$ is a quasi Banach space isomorphic to $X \oplus F$, with $F$ finite dimensional and $C(I, X)$ and $C(\Delta, X)$ are not isomorphic then neither are $C(I, X \oplus c_0)$ and $C(\Delta, X \oplus c_0)$.

It's time to leave. Perhaps the most important question regarding the general topological properties of quasi Banach spaces is to know whether every quotient operator $Q : Z \longrightarrow X$ (acting between quasi Banach spaces) admits a continuous section, namely a continuous $\sigma : X \longrightarrow Z$ such that $Q \circ \sigma = \mathbf{I}_X$. More generally, let us say that $f \in C(K, X)$ lifts through $Q$ if there is $F \in C(K, Z)$ such that $f = F \circ Q$. Now, given $0 < p < 1$, a quotient operator between $p$-Banach spaces $Q : Z \longrightarrow X$ and a compactum $K$, consider the following statements:

(1) $Q$ admits a continuous section.

(2) Every continuous $f : K \longrightarrow X$ has a lifting to $Z$.

(3) $C(K) \otimes X$ is dense in $C(K, X)$.

Clearly, (1) $\Longrightarrow$ (2): set $F = \sigma \circ f$, where $\sigma$ is the hypothesized section of $Q$. Besides, if (1) is true for some quotient map $\ell_p(J) \longrightarrow X$ then so it is for every $Q$. Similarly, if (2) is true for a given $K$ for some quotient map $\ell_p(J) \longrightarrow X$, then it is true for any quotient map onto $X$ and (3) holds.

Following (badly) Klee [10, Section 2], let us say that the pair $(K, X)$ is *admissible* if (3) holds, that $K$ is admissible if (3) holds for every quasi Banach space $X$ and that $X$

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is admissible if (3) holds for every compact $K$. We do not know whether the $p$-Gurariy spaces are admissible or not.

We have mentioned Shuchat's result that every compactum of finite covering dimension is admissible. Actually one can prove that (2) holds for any $Q$ if $\dim(K) < \infty$. This indeed follows from Michael's [11, Theorem 1.2] but a simpler proof can be given using Shuchat's result, the argument of the proof of the lemma, and the open mapping theorem. Since every metrizable compactum is the continuous image of $\Delta$, this implies that for every compact subset $S \subset X$, there is a compact subset $T \subset Z$ such that $Q[T] = S$.

Long time ago, Riedrich proved that the spaces $L_p$ are admissible for $0 \leq p < 1$; see [3, 5] for more general results that cover all modular function spaces. We do not know if *the* quotient map $\ell_p \longrightarrow L_p$ has a continuous section or satisfies (2) for arbitrary compact $K$ and $0 < p < 1$.

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