Nagoya Math. J., **243** (2021), 1–10
DOI 10.1017/nmj.2019.28

INVARIANCE OF CERTAIN PLURIGENERA FOR SURFACES IN MIXED CHARACTERISTICS

ANDREW EGBERT and CHRISTOPHER D. HACON

Abstract. We prove the deformation invariance of Kodaira dimension and of certain plurigenera and the existence of canonical models for log surfaces which are smooth over an integral Noetherian scheme S.

§1. Introduction

If $f : X \to T$ is a smooth projective morphism of complex quasi-projective varieties, then by a celebrated theorem of Siu [13], [14], it is known that the plurigenera of the fibers $P_m(X_t) := h^0(mK_{X_t})$ are independent of the point $t \in T$. This result (and its generalizations to log pairs) is a fundamental fact of great importance in higher dimensional birational geometry. It plays a fundamental role in the construction of moduli spaces of varieties of log general type. Unluckily, over an algebraically closed field of characteristic $p > 0$, this result does not generalize even to families of surfaces over a curve (or a discrete valuation ring (DVR)). In [12], it is shown that P_1 is not deformation invariant for Enriques surfaces in characteristic 2. In [11], it is shown that, in fact, the deformation invariance of plurigenera does not hold for certain elliptic surfaces, and in [15], there are examples of smooth families of surfaces of general type over any DVR of mixed characteristic for which P_1 is nonconstant (and, in fact, its value can jump by an arbitrarily big amount). On the positive side, in [11], it is shown that if $X \to \mathrm{Spec}(R)$ is a smooth family of surfaces over a DVR in positive or mixed characteristic, then one can run the minimal model program (MMP) for X (over an extension of R). As a consequence of this, it is observed that $\kappa(X_K) = \kappa(X_k)$, where k is the residue field and K is the fraction field of R. It should be noted that the minimal model program is established for semistable families of surfaces in positive or mixed characteristic (see [9]), for log canonical surfaces over excellent base schemes (see [19]) and for 3 folds over a field k of characteristic $p \geqslant 7$ (see [8] and [3]). In this paper (Theorems 3.1 and 3.4), we generalize the result of Katsura and Ueno to log surfaces (smooth over a DVR) and we show the deformation invariance of certain plurigenera.

THEOREM 1.1. *Let (X, B) be a klt pair which is log smooth, projective of dimension 2 over an irreducible integral Noetherian scheme S, then $\kappa(K_{X_s} + B_s)$ is independent of $s \in S$. If, moreover, $K_X + B$ is big over S, then there exists an integer $m_0 > 0$ such that for any positive integer $m \in m_0\mathbb{N}$, we have*

$$h^0(m(K_{X_s} + B_s)) = h^0(m(K_{X_{s'}} + B_{s'})) \quad \forall s, s' \in S$$

and the log canonical model for (X, B) over S exists.

Received November 22, 2017. Revised August 22, 2019. Accepted August 26, 2019.
2010 Mathematics subject classification. Primary 14J10, 14E30.
The second author was partially supported by NSF research grants no: DMS-1300750, DMS-1265285, by a grant from the Simons Foundation; Award Number: 256202 and by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University. The authors are grateful to an anonymous referee for many useful suggestions.

When S is the spectrum of a DVR, we obtain a more precise result (see Theorems 3.1 and 3.4). The strategy is to reduce the proof of the above theorem to the case when (X_k, B_k) is terminal and $\mathbf{B}(K_{X_k}+B_k)$ contains no components of the support of B_k. In this case, we observe that the steps of a $K_{X_k}+B_k$ MMP are also steps of a K_{X_k} MMP, and we are thus able to deduce the result from [11].

REMARK 1.2. Many results and techniques in this paper were developed in the first author's Ph.D. thesis [4].

§2. Preliminaries

Let X be a normal quasi-projective variety over an algebraically closed field k and $\mathrm{WDiv}(X)$ the group of Weil divisors. If $B=\sum b_iB_i \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ is a $\mathbb{Q}$-divisor on X, then $\lfloor B\rfloor=\sum\lfloor b_i\rfloor B_i$, where $\lfloor b_i\rfloor=\max\{n\in\mathbb{Z}|n\leqslant b_i\}$. We denote $\{B\}=B-\lfloor B\rfloor$ and $|B|=|\lfloor B\rfloor|+\{B\}$, where

$$|\lfloor B\rfloor|=\{D\in \mathrm{WDiv}(X)|D\geqslant 0,\ D-\lfloor B\rfloor=(f),\ f\in K(X)\}.$$

The **stable base locus** of B is $\mathbf{B}(D)=\bigcap_{m\in\mathbb{N}}\mathrm{Bs}(mD)$. Let (X,B) be a **pair** so that X is normal, $0\leqslant B$ is a $\mathbb{Q}$-divisor and K_X+B is $\mathbb{Q}$-Cartier. If $\nu: X'\to X$ is a proper birational morphism, then we write $K_{X'}+B_{X'}=\nu^*(K_X+B)$. We say that (X,B) is **Kawamata log terminal or klt** (resp. **terminal**) if for any proper birational morphism $\nu: X'\to X$, we have $\lfloor B_{X'}\rfloor\leqslant 0$ (resp. $B_{X'}\leqslant \nu_*^{-1}B+E$, where E denotes the reduced exceptional divisor). We let $\mathbf{M}_B$ be the b-divisor defined by the sum of the strict transform of B and the exceptional divisors (over X). We refer the reader to [10] and [2] for the standard definitions of the minimal model program including extremal rays, flipping and divisorial contractions, running a minimal model program with scaling, log terminal and weak log canonical models.

THEOREM 2.1. *Let (X,B) be a two-dimensional projective klt pair over an algebraically closed field k. Then*

$$\overline{NE}(X)=\overline{NE}(X)_{K_X+B\geqslant 0}+\sum_{i\in I}\mathbb{R}_{\geqslant 0}C_i,$$

where I is countable, $(K_X+B)\cdot C_i<0$ and C_i is rational. If H is an ample $\mathbb{Q}$-divisor on X, then the set $\{i\in I|(K_X+B+H)\cdot C_i\leqslant 0\}$ is finite.

Proof. See [17, 3.13, 3.15] and [10, 3.7]. ∎

LEMMA 2.2. *Let X be a surface over an algebraically closed field k and (X,B) a projective klt pair. If R is a K_X+B negative extremal ray, then there exists a proper morphism $f: X\to X'$ such that $f_*\mathcal{O}_X=\mathcal{O}_{X'}$ and f contracts a curve $C\subset X$ if and only if $[C]=R$.*

Proof. See [17, 3.21] and [10, 3.7]. ∎

THEOREM 2.3. *Let X be a projective surface over an algebraically closed field k. Assume that (X,B) is klt. Then*

(1) *The ring $R(K_X+B)=\bigoplus_{m\geqslant 0}H^0(m(K_X+B))$ is finitely generated.*
(2) *If K_X+B is pseudo-effective, then $\kappa(K_X+B)\geqslant 0$ and there exists a minimal model $\nu: X\to X'$ such that $K_{X'}+B'=\nu_*(K_X+B)$ is semiample. If we write $K_X+B=\nu^*(K_{X'}+B')+F$, then $F\geqslant 0$ is ν-exceptional and we have $F=N_\sigma(K_X+B)$.*

(3) *If H is an ample $\mathbb{Q}$-divisor which is general in $N^1(X/X')$ and K_X+B+H is nef, then the MMP with scaling of H yields a sequence of rational numbers $1\geqslant\lambda_1>\lambda_2>\cdots>\lambda_n\geqslant 0$ and divisorial contractions $X=X_0\to X_1\to X_2\to\cdots\to X_n$, where $X_i=\operatorname{Proj}R(K_X+B+\lambda_i H)$ and $K_{X_i}+B_i+tH_i$ is ample for $\lambda_i\geqslant t>\lambda_{i+1}$. If $\lambda_n=0$, then $X\to X_n$ is a K_X+B minimal model and if $\lambda_n>0$, then $X_n\to Z$ is a K_X+B Mori fiber space.*

Proof. (1) and (2) follow immediately from [17]. To see (3), we proceed by induction. Assume that we have constructed $X\to X_1\to\cdots\to X_i$ and assume that $K_{X_i}+B_i+\lambda_i H_i$ is ample where B_i and H_i denote the pushforwards of B and H. Let $\lambda_{i+1}:=\inf\{t>0|K_{X_i}+B_i+tH_i \text{ is nef}\}$. It is easy to see that $0\leqslant\lambda_{i+1}<\lambda_i$ and $K_{X_i}+B_i+tH_i$ is ample for $\lambda_i\geqslant t>\lambda_{i+1}$. If $\lambda_{i+1}=0$, then $K_{X_i}+B_i$ is nef and we have the required K_X+B minimal model. Otherwise, by Theorem 2.1, there exists a $K_{X_i}+B_i+\lambda_{i+1}H_i$-trivial and $K_{X_i}+B_i$ negative extremal ray C_i. Let $\nu_i:X_i\to X_{i+1}$ be the corresponding contraction. If $\dim X_{i+1}<2$, then we have the required K_X+B Mori fiber space. Otherwise, $X_i\to X_{i+1}$ is a divisorial contraction. Since H is general in $N^1(X/X')$, H_i is general in $N^1(X_i/X')$ and hence $NE(X_i)_{K_{X_i}+B_i+\lambda_{i+1}H_i=0}=[C_i]$. It follows that $K_{X_i}+B_i+\lambda_{i+1}H_i=\nu_i^*(K_{X_{i+1}}+B_{i+1}+\lambda_{i+1}H_{i+1})$, where $K_{X_{i+1}}+B_{i+1}+\lambda_{i+1}H_{i+1}$ is ample. ∎

PROPOSITION 2.4. *Let X be a projective surface over an algebraically closed field k. Assume that (X,B) is a klt pair and $\nu:X'\to X$ is a proper birational morphism such that (X',B') is terminal where $K_{X'}+B'=\nu^*(K_X+B)$. Let $\Theta=B'-B'\wedge N_\sigma(K_{X'}+B')$ and $\phi':X'\to X'_M$ the minimal model for (X',Θ). If $\phi:X\to X_M$ is the minimal model for (X,B), then the rational map $\mu:X'_M\to X_M$ is a morphism and $K_{X'_M}+\phi'_*\Theta=\mu^*(K_{X_M}+\phi_*B)$. If $\kappa(K_X+B)=1$ and B is big over* Proj $R(K_X+B)$*, then Θ is big over* Proj $R(K_{X'}+\Theta)$.

Proof. Consider the morphism $\psi:X'\to X_M$. Since $K_X+B=\phi^*(K_{X_M}+\phi_*B)+E$ where $E\geqslant 0$ is ϕ-exceptional, then $K_{X'}+B'=\psi^*(K_{X_M}+\phi_*B)+\nu^*E$ where $K_{X_M}+\phi_*B$ is nef and ν^*E is effective and ψ exceptional. It follows that $N_\sigma(K_{X'}+B')=\nu^*E$ and so $K_{X'}+\Theta=\psi^*(K_{X_M}+\phi_*B)+E'$, where $0\leqslant E'\leqslant\nu^*E$. In particular, $N_\sigma(K_{X'}+\Theta)=E'$ and so the divisors contracted by ϕ' are precisely the divisors contained in $\operatorname{Supp}(E')$. Thus, $X'\to X_M$ factors through ϕ'. We have $K_{X'_M}+\phi'_*\Theta=\mu^*(K_{X_M}+\phi_*B)+\phi'_*E'$ where $\mu_*(\phi'_*E')\leqslant\phi_*E=0$ and hence ϕ'_*E is μ exceptional. By the negativity lemma, it follows that $\phi'_*E'=0$.

Note that since $H^0(m(K_X+B))\cong H^0(m(K_{X'}+\Theta))$ for all $m\geqslant 0$, it follows that $Z:=\operatorname{Proj}R(K_X+B)=\operatorname{Proj}R(K_{X'}+\Theta)$. We have $\dim Z=\kappa(K_X+B)=1$. The bigness of B over Z is equivalent to $B\cdot X_z>0$ for general $z\in Z$. But then

$$\Theta\cdot X'_z=\mu_*\phi'_*\Theta\cdot(X_M)_z=\phi_*B\cdot(X_M)_z=B\cdot X_z>0$$

and so Θ is big over Z. ∎

Consider now X a smooth projective scheme over an integral Noetherian scheme S and let $f:X\to S$ be the structure morphism. We say that a pair (X,B) is log smooth over S if X is smooth over S and B is an effective $\mathbb{R}$-divisor whose support is simple normal crossings over S so that X is étale over $\mathbb{A}^n_S$ and some choice of local coordinates on $\mathbb{A}^n_S$ pulls back to a parameter system $t_1,\ldots,t_n$ on X and $\operatorname{Supp}(B)=\{t_1\ldots t_r=0\}$ for some $0\leqslant r\leqslant n$. We refer the reader to [16, Section 01V4] for a discussion of smooth morphisms.

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

In particular, each strata of the support of B is smooth over S. We say that a log smooth pair $(X, B=\sum b_iB_i)$ is klt iff $0\leqslant b_i<1$ and $(X, B=\sum b_iB_i)$ is terminal iff $0\leqslant b_i<1$ and $b_i+b_j<1$ if $i\neq j$ and $B_i\cap B_j\neq\emptyset$.

In what follows, R will denote a DVR with residue field k and fraction field K. Let X be an integral Noetherian scheme over $\operatorname{Spec}(R)$ and $f:X\to\operatorname{Spec}(R)$ the structure morphism, then we let $X_K=X\times_{\operatorname{Spec}(R)}\operatorname{Spec}(K)$ be the generic fiber and $X_k=X\times_{\operatorname{Spec}(R)}\operatorname{Spec}(k)$ be the special fiber. As usual, we say that two Cartier divisors on X are numerically equivalent $L_1\equiv_R L_2$ iff $(L_1-L_2)\cdot C=0$ for any curve C contained in a fiber X_K or X_k. Note that it suffices to check this on the special fiber X_k. We then let

$$N^1(X/R)=(\{\text{Cartier divisors } L \text{ on } X\}/\equiv_R)\otimes_{\mathbb{Z}}\mathbb{R},$$
$$N_1(X/R)=(\{\text{curves on } X_k\}/\equiv_R)\otimes_{\mathbb{Z}}\mathbb{R}.$$

$NE(X/R)\subset N_1(X/R)$ is the closed cone spanned by effective curves. Note that the natural map $N_1(X/R)\to N^1(X_k)$ is injective and the dual map $N_1(X_k)\to N_1(X/R)$ is surjective and so is the induced map $NE(X_k)\to NE(X/R)$.

LEMMA 2.5. *Let $f:X\to\operatorname{Spec}(R)$ be a smooth projective morphism from a smooth variety to a DVR and L a line bundle on X, then*

(1) *L is ample if and only if $L_k:=L|_{X_k}$ is ample, and*
(2) *L is nef if and only if $L_k:=L|_{X_k}$ is nef.*

Proof. Clearly, if L is ample or nef, then so is L_k. It is well known that ampleness is an open condition and so if L_k is ample, then so is L. Finally, if L_k is nef and H is ample, then L_k+tH_k is ample for any $t>0$ so that $L+tH$ is ample and hence L is nef. ∎

LEMMA 2.6. *Let $f:X\to\operatorname{Spec}(R)$ be a flat projective morphism from a variety to a DVR and (X,B) a log pair. Then (X,B) is log smooth over* $\operatorname{Spec}(R)$ *if and only if (X_k,B_k) is log smooth.*

Proof. See [16, Section 01V4]. ∎

LEMMA 2.7. *Let (X,B) be a log pair which is log smooth over* $\operatorname{Spec}(R)$*, where R is a DVR. If $R\subset\tilde R$ is an inclusion of DVR's, then $(X_{\tilde R},B_{\tilde R})$ is log smooth over* $\operatorname{Spec}(\tilde R)$*. If (X,B) is terminal (resp. klt), then so is $(X_{\tilde R},B_{\tilde R})$.*

Proof. Since smoothness is preserved by base change, it follows that $(X_{\tilde R},B_{\tilde R})$ is log smooth over $\operatorname{Spec}(\tilde R)$. The pair (X,B) is klt (resp. terminal) if and only if the coefficients of B are <1 (resp. the coefficients of B are <1 and if two components intersect, then the sum of the coefficients is <1). The lemma now follows since if there are two intersecting components of $B_{\tilde R}$, then there are two intersecting components of B (with the same coefficients). ∎

THEOREM 2.8. (Katsura–Ueno [11]) *Let $f:X\to\operatorname{Spec}(R)$ be an algebraic space which is smooth, proper and two-dimensional over* $\operatorname{Spec}(R)$*, where R is a DVR with algebraically closed residue field k and field of fractions K. If X_k contains a -1 curve $e\subset X_k$, then there exists a DVR $\tilde R\supset R$ with residue field k and fraction field $\tilde K$ and a surjective proper morphism $\pi:X_{\tilde R}\to\tilde Y$ over* $\operatorname{Spec}(\tilde R)$ *where $\tilde Y\to\operatorname{Spec}(\tilde R)$ is smooth, proper, and two-dimensional, π_k contracts the -1 curve $e\subset X_k$ and $\pi_K:X_{\tilde K}\to\tilde Y_{\tilde K}$ is also a contraction of a -1 curve.*

§3. Main result

In this section, we will prove Theorem 1.1. We begin by showing that a more general version of this result holds when $S=\operatorname{Spec}(R)$ is the spectrum of a DVR and then we will deduce the general case.

THEOREM 3.1. *Let (X,B) be a klt pair which is log smooth, projective of dimension 2 over $S=\operatorname{Spec}(R)$, where R is a DVR with residue field k and fraction field K. If K_X+B is $\mathbb{Q}$-Cartier, then $\kappa(K_{X_k}+B_k)=\kappa(K_{X_K}+B_K)$ and if either $\kappa(K_{X_k}+B_k)\neq 1$ or $\kappa(K_{X_k}+B_k)=1$ and B_k is big over $\operatorname{Proj} R(K_{X_k}+B_k)$, then there exists an integer $m_0>0$ such that for any integer $m\in m_0\mathbb{N}$, we have*

$$h^0(m(K_{X_k}+B_k))=h^0(m(K_{X_K}+B_K)).$$

Proof. Consider an inclusion of DVR's $R\subset\tilde{R}$. If $\tilde{k}$ and $\tilde{K}$ denote the residue field and the fraction field of $\tilde{R}$, then $h^0(m(K_{X_k}+B_k))=h^0(m(K_{X_{\tilde{k}}}+B_{\tilde{k}}))$ and $h^0(m(K_{X_K}+B_K))=h^0(m(K_{X_{\tilde{K}}}+B_{\tilde{K}}))$. Note also that if $\tilde{X}=X\times_{\operatorname{Spec}(R)}\operatorname{Spec}(\tilde{R})$ and $\tilde{B}=B\times_{\operatorname{Spec}(R)}\operatorname{Spec}(\tilde{R})$, then $(\tilde{X},\tilde{B})$ is log smooth over $\tilde{R}$ and $\tilde{X}_{\tilde{k}}\cong X_k\times_{\operatorname{Spec}(k)}\operatorname{Spec}(\tilde{k})$. Thus, we are free to replace $X\to R$ by $\tilde{X}\to\tilde{R}$. In particular, we may assume that k is algebraically closed.

If $h^0(m(K_{X_k}+B_k))=0$, then, by semicontinuity, $h^0(m(K_{X_K}+B_K))=0$. Therefore, the theorem holds trivially in the case $\kappa(K_{X_k}+B_k)=-\infty$. Thus, we may assume that $\kappa(K_{X_k}+B_k)\geqslant 0$.

CLAIM 3.2. *The theorem holds under the additional assumption that (X_k,B_k) is terminal and no component of the support of B_k is contained in $\mathbf{B}(K_{X_k}+B_k)$.*

Proof. Since k is algebraically closed, then by the Cone Theorem (Theorem 2.1),

$$\overline{NE}(X_k)=\overline{NE}(X_k)_{K_{X_k}+B_k\geqslant 0}+\sum_{i\in I}\mathbb{R}_{\geqslant 0}C_i,$$

where I is countable, $(K_{X_k}+B_k)\cdot C_i<0$ and C_i is rational.

Suppose that one of the above curves C_i is contained in the support of B_k, then since $\kappa(K_{X_k}+B_k)\geqslant 0$ and $(K_{X_k}+B_k)\cdot C_i<0$, we have $C_i\subset\mathbf{B}(K_{X_k}+B_k)$, which we have assumed is impossible.

Note then that C_i is not contained in the support of B_k and thus $C_i\cdot B_k\geqslant 0$ and so $K_{X_k}\cdot C_i<0$. It follows that if C_i spans a $K_{X_k}+B_k$-negative extremal ray, then it also spans a K_{X_k}-negative extremal ray and so it can be contracted by a divisorial contraction of a -1 curve $X_k\to X'_k$. In particular, X'_k is also a smooth surface. Thus, we may assume that C_i is a -1 curve. By Theorem 2.8 (after extending R), we may assume that there is a morphism $X\to X'$ of smooth surfaces over R such that $X_K\to X'_K$ also contracts a -1 curve.

We now run an MMP by contracting a sequence of K_X+B-negative curves as above. Let $\nu:X\to\bar{X}$ be the induced morphism of smooth surfaces over $\operatorname{Spec}(R)$. We may assume that $X_K\to\bar{X}_K$ and $X_k\to\bar{X}_k$ are given by a finite sequence of contractions of -1 curves such that the exceptional locus of $X_k\to\bar{X}_k$ contains no components of B_k. Then $(\bar{X}_k,\bar{B}_k)$ is terminal and $K_{X_k}+B_k=\nu_k^*(K_{\bar{X}_k}+\bar{B}_k)+F_k$, where $B_k=\nu_{k,*}^{-1}\bar{B}_k$ and $B_k\wedge F_k=0$. In particular, $\mathbf{B}(K_{X_k}+B_k)=\mathbf{B}(\nu_k^*(K_{\bar{X}_k}+\bar{B}_k))+F_k$. Suppose that $C\subset\bar{X}_k$ is contained in $\mathbf{B}(K_{\bar{X}_k}+\bar{B}_k)\cap\operatorname{Supp}(\bar{B}_k)$, then $\nu_*^{-1}C\subset\mathbf{B}(K_{X_k}+B_k)\cap\operatorname{Supp}(B_k)$ which is impossible.

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

Therefore, if $K_{\bar{X}_k}+\bar{B}_k$ is not nef, we can continue to contract -1 curves. Since each contraction reduces the Picard number of the central fiber X_k by one, this procedure must terminate after finitely many steps. We may therefore assume that $K_{\bar{X}_k}+\bar{B}_k$ is semiample. In particular, $K_{\bar{X}_k}+\bar{B}_k$ is nef and hence so is $K_{\bar{X}}+\bar{B}$ (see Lemma 2.5).

Suppose now that $\nu(K_{X_k}+B_k)=2$. In this case, $K_{\bar{X}_k}+\bar{B}_k$ is nef and big and $m_0(K_{\bar{X}_k}+\bar{B}_k)$ is Cartier for some $m_0>0$. We may write

$$km_0(K_{\bar{X}_k}+\bar{B}_k)=K_{\bar{X}_k}+\lceil (m_0-1)(K_{\bar{X}_k}+\bar{B}_k))\rceil+(k-1)m_0(K_{\bar{X}_k}+\bar{B}_k)$$

so that by [18, 2.6] $h^i(m(K_{\bar{X}_k}+\bar{B}_k))=0$ for all sufficiently big integers $m\in m_0\mathbb{N}$ and all $i>0$. Replacing m_0 by an appropriate multiple, this condition holds for all $m\in m_0\mathbb{N}$. By semicontinuity, we also have $h^i(m(K_{\bar{X}_K}+\bar{B}_K))=0$ for all $m\in m_0\mathbb{N}$ and $i>0$. The result now follows from cohomology and base change.

Suppose that $\kappa(K_{X_k}+B_k)=0$. Then we have $K_{\bar{X}_k}+\bar{B}_k\sim_{\mathbb{Q}}0$. By Lemma 2.5, it follows that $\pm(K_{\bar{X}_K}+\bar{B}_K)$ is nef and hence that $K_{\bar{X}_K}+\bar{B}_K\equiv 0$. By [17, 1.2], $K_{\bar{X}_K}+\bar{B}_K\sim_{\mathbb{Q}}0$. Thus, there exists an integer $m_0>0$ such that $m_0(K_{\bar{X}_K}+\bar{B}_K)\sim 0$ and $m_0(K_{\bar{X}_k}+\bar{B}_k)\sim 0$. Thus, $h^0(m(K_{X_K}+B_K))=h^0(m(K_{X_k}+B_k))$ for all $m\geqslant 0$ divisible by m_0.

Suppose that $\kappa(K_{X_k}+B_k)=1$. Since $K_{\bar{X}_k}+\bar{B}_k$ is nef, so is $K_{\bar{X}_K}+\bar{B}_K$. In particular, $\kappa(K_{\bar{X}_K}+\bar{B}_K)\geqslant 0$ and, thus, by semicontinuity, we have $\kappa(K_{\bar{X}_K}+\bar{B}_K)\in\{0,1\}$. Let H be a sufficiently ample divisor on $\bar{X}$. Then $(K_{\bar{X}_K}+\bar{B}_K)\cdot H_K=(K_{\bar{X}_k}+\bar{B}_k)\cdot H_k>0$ so $K_{\bar{X}_K}+\bar{B}_K\not\equiv 0$. Therefore, $\kappa(K_{\bar{X}_K}+\bar{B}_K)=1$.

Finally, suppose that $\kappa(K_{X_k}+B_k)=1$ and B_k is big over Proj $R(K_{X_k}+B_k)$. Note that $\bar{B}_k$ is also big over Proj $R(K_{\bar{X}_k}+\bar{B}_k)$ and hence $\bar{B}_k+K_{\bar{X}_k}+\bar{B}_k$ is big. Thus, we may write $\bar{B}_k+K_{\bar{X}_k}+\bar{B}_k\sim_{\mathbb{Q}}\bar{A}_k+\bar{E}_k$, where $\bar{A}_k$ is ample and $\bar{E}_k$ is effective. For any rational number $0<\epsilon\ll 1$, the pair $(\bar{X}_k,\Delta_k=(1-\epsilon)\bar{B}_k+\epsilon\bar{E}_k)$ is Kawamata log terminal and so the corresponding multiplier ideal sheaf is trivial $\mathcal{J}(\Delta_k)=\mathcal{O}_{\bar{X}_k}$. If $L=N=m(K_{\bar{X}_k}+\bar{B}_k)$, then N is nef and not numerically equivalent to zero while

$$L-(K_{\bar{X}_k}+\Delta_k)\sim_{\mathbb{Q}}(m-1-\epsilon)(K_{\bar{X}_k}+\bar{B}_k)+\epsilon\bar{A}_k$$

is ample, and so by [18, 0.3] and [10, 2.70], $H^i(\mathcal{O}_{\bar{X}_k}(m(l+1)(K_{\bar{X}_k}+\bar{B}_k)))=0$ for $i>0$ and $l\gg 0$. By semicontinuity, $H^i(\mathcal{O}_{\bar{X}_K}(m(l+1)(K_{\bar{X}_K}+\bar{B}_K)))=0$ for $i>0$ and $l\gg 0$ and hence $h^0(\mathcal{O}_{\bar{X}_k}(m(l+1)(K_{\bar{X}_k}+\bar{B}_k)))=h^0(\mathcal{O}_{\bar{X}_K}(m(l+1)(K_{\bar{X}_K}+\bar{B}_K)))$. □

We will now consider the general case. Since (X,B) is log smooth over R, there is a sequence of blowups along strata of $\mathbf{M}_B$ say $\nu: X'\to X$ such that $K_{X'}+B'=\nu^*(K_X+B)$ is terminal and, in particular, $B'\geqslant 0$ and (X',B') is log smooth. Since $R(K_{X'_k}+B'_k)\cong R(K_{X_k}+B_k)$ is finitely generated, $N_\sigma(K_{X'_k}+B'_k)$ is a $\mathbb{Q}$-divisor and hence so is

$$\Theta_k:=B'_k-(B'_k\wedge N_\sigma(K_{X'_k}+B'_k)).$$

Note that $R(K_{X'_k}+\Theta_k)\cong R(K_{X'_k}+B'_k)$, (X'_k,Θ_k) is terminal and no component of Θ_k is contained in $\mathbf{B}(K_{X'_k}+\Theta_k)$ [[6, 2.8.3] and [7, 2.4]]. Let Θ be the unique $\mathbb{Q}$-divisor supported on B' such that $\Theta|_{X'_k}=\Theta_k$. We remark that if $\kappa(K_{X_k}+B_k)=1$ and B_k is big over Proj $R(K_{X_k}+B_k)$, then by Proposition 2.4, Θ_k is big over Proj $R(K_{X'_k}+\Theta_k)$. By Claim 3.2, it follows that $\kappa(K_{X'_K}+\Theta_K)=\kappa(K_{X'_k}+\Theta_k)$ and there exists an integer $m_0>0$ such that

$$h^0(m(K_{X'_K}+\Theta_K))=h^0(m(K_{X'_k}+\Theta_k))\quad\forall m\in m_0\mathbb{N}.$$

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

By semicontinuity, we then have

$$h^0(m(K_{X_k}+B_k)) \geqslant h^0(m(K_{X_K}+B_K)) \geqslant h^0(m(K_{X'_K}+B'_K)) \geqslant h^0(m(K_{X'_K}+\Theta_K)) = h^0(m(K_{X'_k}+\Theta_k)) = h^0(m(K_{X_k}+B_k))$$

and hence $h^0(m(K_{X_k}+B_k)) = h^0(m(K_{X_K}+B_K))$. The equality $\kappa(K_{X_k}+B_k) = \kappa(K_{X_K}+B_K)$ follows similarly. ∎

COROLLARY 3.3. *Let (X,B) be a klt pair which is log smooth, projective of dimension 2 over a DVR R with residue field k of characteristic $p>0$ and fraction field K. If K_X+B is $\mathbb{Q}$-Cartier and either $\kappa(K_{X_k}+B_k)\in\{0,2\}$ or $\kappa(K_{X_k}+B_k)=1$ and B_k is big over $\operatorname{Proj} R(K_{X_k}+B_k)$, then $R(K_X+B)$ is finitely generated.*

Proof. By Theorem 2.3, $R(K_{X_k}+B_k)$ is finitely generated and hence there is a positive integer m such that

$$R(m(K_{X_k}+B_k))$$

is generated in degree 1, that is, by $H^0(m(K_{X_k}+B_k))$. By Theorem 3.1, after replacing m by a multiple, we may assume that $m(K_X+B)$ is Cartier and

$$H^0(m(K_X+B)) \to H^0(m(K_{X_k}+B_k))$$

is surjective. Therefore, the induced map

$$S^k H^0(m(K_X+B)) \to S^k H^0(m(K_{X_k}+B_k)) \to H^0(mk(K_{X_k}+B_k))$$

is surjective. By Nakayama's lemma,

$$S^k H^0(m(K_X+B)) \to H^0(mk(K_X+B))$$

is surjective and so $R(m(K_X+B))$ is finitely generated. ∎

THEOREM 3.4. *Let (X,B) be a klt pair which is log smooth, projective of dimension 2 over a DVR R with residue field k and fraction field K. If K_X+B is $\mathbb{Q}$-Cartier, then (after possibly extending R) we may run a K_X+B MMP over R which is given by a sequence of divisorial contractions and terminates with a K_X+B minimal model $X\to\bar{X}$ over R or a K_X+B Mori fiber space over R.*

Proof. After extending R, we may assume that k is algebraically closed. Suppose that H is ample and let

$$\tau = \inf\{t\geqslant 0 | \kappa(K_{X_k}+B_k+tH_k)\geqslant 0\}.$$

Pick $1\gg \tau'-\tau>0$, then by Theorem 2.3 and its proof, $\operatorname{Proj} R(K_{X_k}+B_k+\tau' H_k)$ is the minimal model of (X_k, B_k+tH_k) for $\tau'\geqslant t\geqslant\tau$. Let $\nu_k: X'_k\to X_k$ be a terminalization of (X_k,B_k) given by a sequence of blowups along strata of $\mathbf{M}_{B_k}$, $K_{X'_k}+B'_k=\nu_k^*(K_{X_k}+B_k)$, $H'_k=\nu_k^*H_k$ and

$$\Theta_k = B'_k - B'_k\wedge N_\sigma(K_{X'_k}+B'_k+\tau' H'_k).$$

If $X'_k\to X'_{1,k}\to\cdots\to X'_{n,k}$ is a MMP for $K_{X'_k}+\Theta_k+\tau' H'_k$, then $K_{X'_{n,k}}+\Theta_{n,k}+\tau' H'_{n,k}$ is semiample and induces a morphism $\nu_{n,k}: X'_{n,k}\to X_{n,k} := \operatorname{Proj} R(K_{X'_k}+\Theta_k+\tau' H'_k)$. By Proposition 2.4, $R(K_{X'_k}+\Theta_k+\tau' H'_k)\cong R(K_{X_k}+B_k+\tau' H_k)$, and so the induced birational map $X_k\to X_{n,k}$ is in fact the morphism corresponding to the minimal model

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

of $K_{X_k}+B_k+\tau H_k$. If $\tau=0$, then $X_k\to X_{n,k}$ is a $K_{X_k}+B_k$ minimal model and if $\tau>0$, then $X_{n,k}\to Z_k=\mathrm{Proj}R(K_{X_k}+B_k+\tau H_k)$ is a $K_{X_k}+B_k$ Mori fiber space.

We claim that the exceptional divisors of $X'_k\to X_{n,k}$ are either contained in the support of $\mathbf{M}_{B_k}$ or in $N_\sigma(K_{X'_k}+\Theta_k+\tau' H'_k)$. To see this, note that the support of $\mathbf{M}_{B_k}$ contains the $X'_k\to X_k$ exceptional divisors and so it suffices to show that the exceptional divisors of $X_k\to X_{n,k}$ are contained in the support of B'_k and $N_\sigma(K_{X'_k}+\Theta_k+\tau' H'_k)$. The exceptional divisors of $X_k\to X_{n,k}$ are given by the support of $N_\sigma(K_{X_k}+B_k+\tau' H_k)$. The strict transforms of divisors in $N_\sigma(K_{X_k}+B_k+\tau' H_k)$ are divisors in $N_\sigma(K_{X'_k}+B'_k+\tau' H'_k)$ and hence in $N_\sigma(K_{X'_k}+\Theta_k+\tau' H'_k)$ plus some divisors supported on B'_k. Thus, the claim holds.

By the proof of Theorem 3.1, there is a sequence of divisorial contractions of smooth varieties $X'\to X'_1\to\cdots\to X'_n$ extending the MMP $X'_k\to X'_{1,k}\to\cdots\to X'_{n,k}$ which induces contractions of -1 curves on $X_{i,k}$ and $X_{i,K}$. It follows that if P_k is an exceptional prime divisor of $X'_k\to X_{n,k}$, then there is a prime divisor $P\subset X'$ such that $P_k=P|_{X'_k}$. To see this, note that either P_k is a component of $\mathbf{M}_{B_k}$ and hence we may take P as the corresponding component of $\mathbf{M}_B$ or P_k is a component of $N_\sigma(K_{X'_k}+\Theta_k+\tau' H_k)$ and hence the exceptional divisor for some divisorial contraction $X'_{i,k}\to X'_{i+1,k}$. We can then pick P to be the exceptional divisor of $X'_i\to X'_{i+1}$.

Therefore, all $X'_k\to X_{n,k}$ exceptional divisors extend to divisors on X' and hence $N^1(X')\to N^1(X'_k/X_{n,k})$ is surjective and so $N^1(X)\to N^1(X_k/X_{n,k})$ is also surjective.

We now replace H by a sufficiently ample $\mathbb{Q}$-divisor on X which is general in $N^1(X)$. Since H_k is general in $N^1(X_k/X_{n,k})$, by Theorem 2.3, running the minimal model program with scaling of H_k, we obtain a sequence of rational numbers $\lambda_1>\lambda_2>\cdots>\lambda_n=\tau$ and divisorial contractions $X_{i,k}\to X_{i+1,k}$ such that $X_{i,k}=\mathrm{Proj}(R(K_{X_k}+B_k+tH_k))$ for $\lambda_i\geqslant t>\lambda_{i+1}$ where we let $X_k=X_{0,k}$ and $\lambda_0=1$. By Corollary 3.3, $R(K_X+B+\lambda_i H)$ is finitely generated over R. Let $X\dashrightarrow X_i=\mathrm{Proj}_R(R(K_X+B+\lambda_i H))$ be the induced rational map. We claim that

(1) X_i is normal and $\mathbb{Q}$-factorial, (X_i,B_i) is klt,
(2) $(X_i,B_i)_k=(X_{i,k},B_{i,k})$,
(3) $K_{X_i}+B_i+tH_i$ is ample for $\lambda_i\geqslant t>\lambda_{i+1}$ and
(4) $K_{X_i}+B_i+\lambda_{i+1}H_i$ is semiample and induces a divisorial contraction $X_i\to X_{i+1}$.

We will prove this by induction. Clearly, the statements $(1-3)_{i=0}$ hold and $(4)_{i=-1}$ is vacuous. We will prove that $(1-3)_i$ and $(4)_{i-1}$ hold imply that $(1-3)_{i+1}$ and $(4)_i$ hold.

Since $R(K_X+B+\lambda_{i+1}H)\cong R(K_{X_i}+B_i+\lambda_{i+1}H_i)$ and $K_{X_{i,k}}+B_{i,k}+\lambda_{i+1}H_{i,k}$ is semiample, by Theorem 3.1, it follows that $K_{X_i}+B_i+\lambda_{i+1}H_i$ is semiample (over R) and hence $|m(K_{X_i}+B_i+\lambda_{i+1}H_i)|$ defines a morphism $\mu_i:X_i\to X_{i+1}$ for $m>0$ sufficiently divisible which extends the morphism $\mu_{i,k}:X_{i,k}\to X_{i+1,k}$. Since $\mu_{i,k}$ is the divisorial contraction of a prime divisor P_k which extends to a prime divisor P on X_i, it follows that $X_i\to X_{i+1}$ is a divisorial contraction and so $(4)_i$ holds.

To show $(1)_{i+1}$, first observe that since $X_{i+1,k}$ is normal, so is X_{i+1}. By what we have seen above, $K_{X_{i+1}}+B_{i+1}+\lambda_{i+1}H_{i+1}$ is $\mathbb{Q}$-Cartier and $\mu_i^*(K_{X_{i+1}}+B_{i+1}+\lambda_{i+1}H_{i+1})=K_{X_i}+B_i+\lambda_{i+1}H_i$. Since $(X_i,B_i+\lambda_{i+1}H_i)$ is klt, it follows that $(X_{i+1},B_{i+1}+\lambda_{i+1}H_{i+1})$ is klt. Therefore, to show that (X_{i+1},B_{i+1}) is klt, it suffices to show that X_{i+1} is $\mathbb{Q}$-factorial.

Let D_{i+1} be a divisor on X_{i+1}, we wish to show that D_{i+1} is $\mathbb{Q}$-Cartier. We may assume that the support of D_{i+1} does not contain $X_{i+1,k}$. Let D_k be the pull back of $D_{i+1,k}$ to X_k. Fix $0<\epsilon\ll 1$. Since $N^1(X)\to N^1(X_k/X_{n,k})$ is surjective, we may pick a $\mathbb{Q}$-divisor G on X

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

such that $G_k \sim_{\mathbb{Q}} \lambda_{i+1} H_k + \epsilon D_k$. Since $0 < \epsilon \ll 1$, it follows that G_k is ample and $X_k \to X_{i,k}$ is a sequence of $K_{X_k} + B_k + G_k$ negative divisorial contractions. It then follows that G is ample (over R) and $X \to X_i$ is a sequence of $K_X + B + G$ negative divisorial contractions. Note that by assumption, $K_{X_{i,k}} + B_{i,k} + G_{i,k} = \mu^*_{i,k}(K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k})$. Here,

$$K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k} \sim_{\mathbb{Q}} K_{X_{i+1,k}} + B_{i+1,k} + \lambda_{i+1} H_{i+1,k} + \epsilon D_{i+1,k}$$

is ample. Since $R(K_{X_k} + B_k + G_k) \cong R(K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k})$, by Theorem 3.1,

$$H^0(m(K_{X_{i+1}} + B_{i+1} + G_{i+1})) \to H^0(m(K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k}))$$

is surjective for $m > 0$ sufficiently divisible. Since $K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k}$ is ample (and, in particular, $\mathbb{Q}$-Cartier), we may assume that for any $x \in X_{i+1,k}$, there exists a global section $s_{i+1,k} \in H^0(m(K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k}))$ which generates the line bundle $\mathcal{O}_{X_{i+1,k}}(m(K_{X_{i+1,k}} + B_{i+1,k} + G_{i+1,k}))$ locally at x. Let $s_{i+1} \in H^0(m(K_{X_{i+1}} + B_{i+1} + G_{i+1}))$ be a lift of $s_{i+1,k}$ so that $s_{i+1}|_{X_{i+1,k}} = s_{i+1,k}$. It follows that $\mathcal{O}_{X_{i+1}}(m(K_{X_{i+1}} + B_{i+1} + G_{i+1}))$ is generated by s_{i+1} locally at x, and hence it is Cartier on a neighborhood of $x \in X$. Thus, $K_{X_{i+1}} + B_{i+1} + G_{i+1}$ is $\mathbb{Q}$-Cartier, and hence so is $D_{i+1} = \frac{1}{\epsilon}(G_{i+1} - H_{i+1})$. This concludes the proof that $(1)_{i+1}$ holds.

$(2)_{i+1}$ follows immediately from what we have observed above. To see $(3)_{i+1}$, note that $K_{X_{i+1,k}} + B_{i+1,k} + tH_{i+1,k}$ is ample for $\lambda_{i+1} \leqslant t < \lambda_{i+2}$ and apply Lemma 2.5.

If $\tau = 0$, then after finitely many steps, we have obtained a minimal model of (X, B) over $\mathrm{Spec}(R)$. Otherwise, there is a Mori fiber space $X_{n,k} \to Z_k$. By Theorem 3.1 and Corollary 3.3, $X_{n,k} \to Z_k$ extends to a morphism $X_n \to Z$ which is $K_X + B$ negative. ∎

Proof of Theorem 1.1. The independence of $\kappa(K_{X_s} + B_s)$ for $s \in S$ is an immediate consequence of Theorem 3.1; however, the statement regarding the log plurigenera $h^0(m(K_{X_s} + B_s))$ is more subtle as the integer m_0 given in Theorem 3.1 (with $R = \mathcal{O}_{s,S}$) may depend on the point $s \in S$. Note, however, that it easily follows that the volumes $\mathrm{vol}(K_{X_s} + B_s)$ are independent of $s \in S$.

Assume now that $\mathrm{vol}(K_{X_s} + B_s) > 0$. By [1, Theorem 7.7] (see also [5]), the corresponding canonical models (X_s^{lc}, B_s^{lc}) belong to a bounded family and, in particular, there is an integer $m > 0$ and finitely many degree-2 polynomials $P_1, \ldots, P_l \in \mathbb{Q}[x]$ such that for all $s \in S$, $m(K_{X_s^{lc}} + B_s^{lc})$ is Cartier, $R(m(K_{X_s^{lc}} + B_s^{lc}))$ is generated in degree 1 and for every $k > 0$,

$$h^0(mk(K_{X_s^{lc}} + B_s^{lc})) = \chi(mk(K_{X_s^{lc}} + B_s^{lc})) = P_j(k)$$

for some $1 \leqslant j \leqslant l$. Let $\eta \in S$ be the generic point. Since

$$h^0(mk(K_{X_s^{lc}} + B_s^{lc})) = h^0(mk(K_{X_s} + B_s)) = h^0(mk(K_{X_\eta} + B_\eta))$$

for all $k > 0$ sufficiently divisible, it follows that we may assume that $P_1 = P_2 = \cdots = P_l$ and so $h^0(mk(K_{X_s} + B_s))$ is constant for all $k > 0$. But then, for any $k > 0$, $f_*\mathcal{O}_X(mk(K_X + B))$ is locally free and $f_*\mathcal{O}_X(mk(K_X + B)) \to H^0(mk(K_{X_s} + B_s))$ is surjective for any $s \in S$, where $f : X \to S$ is the given morphism. Since $S^k H^0(m(K_{X_s} + B_s)) \to H^0(mk(K_{X_s} + B_s))$ is surjective for any $k > 0$, it follows from Nakayama's lemma that

$$S^k f_*\mathcal{O}_X(m(K_X + B)) \to f_*\mathcal{O}_X(mk(K_X + B))$$

https://doi.org/10.1017/nmj.2019.28 Published online by Cambridge University Press

is surjective for every $k>0$ and so $R(m(K_X+B))$ is finitely generated over S. The canonical model of (X,B) over S is then given by

$$\mathrm{Proj}_{\mathcal{O}_S}\left(\bigoplus_{k\geqslant 0} f_*\mathcal{O}_X(mk(K_X+B))\right). \qquad \square$$

References

[1] V. Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5**(6) (1994), 779–810.
[2] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23**(2) (2010), 405–468.
[3] C. Birkar, *Existence of flips and minimal models for 3-folds in char p*, Ann. Sci. Éc. Norm. Supér. (4) **49**(1) (2016), 169–212.
[4] P. A. Egbert, Log minimal models for arithmetic threefolds. Ph.D. Thesis, The University of Utah, 2016, 65 pp. ISBN: 978-1369-12426-2, ProQuest LLC.
[5] C. D. Hacon and S. Kovács, *On the boundedness of slc surfaces of general type*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19**(1) (2019), 191–215.
[6] C. D. Hacon, J. McKernan and C. Xu, *Boundedness of moduli of varieties of general type*, J. Eur. Math. Soc. (JEMS) **20**(4) (2018), 865–901.
[7] C. D. Hacon and C. Xu, *Existence of log canonical closures*, Invent. Math. **192**(1) (2013), 161–195.
[8] C. D. Hacon and C. Xu, *On the three dimensional minimal model program in positive characteristic*, J. Amer. Math. Soc. **28**(3) (2015), 711–744.
[9] Y. Kawamata, *Semistable minimal models of threefolds in positive or mixed characteristic*, J. Algebraic Geom. **3**(3) (1994), 463–491.
[10] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[11] T. Katsura and K. Ueno, *On elliptic surfaces in characteristic p*, Math. Ann. **272**(3) (1985), 291–330.
[12] W. Lang, *On Enriques surfaces in characteristic p. I*, Math. Ann. **265**(1) (1983), 45–65.
[13] Y.-T. Siu, *Invariance of plurigenera*, Invent. Math. **134**(3) (1998), 661–673.
[14] Y.-T. Siu, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, in *Complex Geometry (Göttingen, 2000)*, Springer, Berlin, 2002, 223–277.
[15] J. Suh, *Plurigenera of general type surfaces in mixed characteristic*, Compos. Math. **144**(5) (2008), 1214–1226.
[16] *The Stacks project*, https://stacks.math.columbia.edu, 2019.
[17] H. Tanaka, *Minimal models and abundance for positive characteristic log surfaces*, Nagoya Math. J. **216** (2014), 1–70.
[18] H. Tanaka, *The X-method for klt surfaces in positive characteristic*, J. Algebraic Geom. **24**(4) (2015), 605–628.
[19] H. Tanaka, *Minimal model program for excellent surfaces*, Ann. Inst. Fourier (Grenoble) **68**(1) (2018), 345–376.

Andrew Egbert
Department of Mathematics
University of Utah
Salt Lake City, UT 84112, USA
AndrewEg2@outlook.com

Christopher D. Hacon
Department of Mathematics
University of Utah
Salt Lake City, UT 84112, USA
hacon@math.utah.edu