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Maximizing weighted sums of binomial coefficients using generalized continued fractions

S.P. Glasby*
Center for the Mathematics of Symmetry and Computation, University of Western Australia, Perth 6009, Australia
(Stephen.Glasby@uwa.edu.au)

G.R. Paseman
Sheperd Systems, UC Berkeley, USA (sheperdsystems@gmail.com)

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Let m, $r \in \mathbb{Z}$ and $\omega \in \mathbb{R}$ satisfy $0 \leqslant r \leqslant m$ and $\omega \geqslant 1$. Our main result is a generalized continued fraction for an expression involving the partial binomial sum $s_m(r) = \sum_{i=0}^{r} \binom{m}{i}$. We apply this to create new upper and lower bounds for $s_m(r)$ and thus for $g_{\omega,m}(r) = \omega^{-r} s_m(r)$. We also bound an integer $r_0 \in \{0, 1, \ldots, m\}$ such that $g_{\omega,m}(0) < \cdots < g_{\omega,m}(r_0 - 1) \leqslant g_{\omega,m}(r_0)$ and $g_{\omega,m}(r_0) > \cdots > g_{\omega,m}(m)$. For real $\omega \geqslant \sqrt{3}$ we prove that $r_0 \in \{\lfloor \frac{m+2}{\omega+1} \rfloor, \lfloor \frac{m+2}{\omega+1} \rfloor + 1\}$, and also $r_0 = \lfloor \frac{m+2}{\omega+1} \rfloor$ for $\omega \in \{3, 4, \ldots\}$ or $\omega = 2$ and $3 \nmid m$.

Keywords: partial sum; binomial coefficients; continued fraction; bounds

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1. Introduction

Given a real number $\omega \geqslant 1$ and integers m, r satisfying $0 \leqslant r \leqslant m$, set

$$s_m(r) := \sum_{i=0}^{r} \binom{m}{i} \quad \text{and} \quad g(r) = g_{\omega,m}(r) := \omega^{-r} s_m(r), \tag{1.1}$$

where the binomial coefficient $\binom{m}{i}$ equals $\prod_{k=1}^{i} \frac{m-k+1}{k}$ for $i > 0$ and $\binom{m}{0} = 1$. The weighted binomial sum $g_{\omega,m}(r)$ and the partial binomial sum $s_m(r) = g_{1,m}(r)$ appear in many formulas and inequalities, e.g. the cumulative distribution function $2^{-m} s_m(r)$ of a binomial random variable with $p = q = \frac{1}{2}$ as in remark 5.3, and the Gilbert–Varshamov bound [6, Theorem 5.2.6] for a code $C \subseteq \{0, 1\}^n$. Partial sums of binomial coefficients are found in probability theory, coding theory, group theory, and elsewhere. As $s_m(r)$ cannot be computed exactly for most values of r, it is desirable for certain applications to find simple sharp upper and lower bounds for

* Corresponding author.

$s_m(r)$. Our interest in bounding $2^{-r}s_m(r)$ was piqued in [4] by an application to Reed–Muller codes RM(m, r), which are linear codes of dimension $s_m(r)$.

Our main result is a generalized continued fraction $a_0 + \mathcal{K}_{i=1}^{r}\frac{b_i}{a_i}$ (using Gauss' Kettenbruch notation) for $Q := \frac{(r+1)}{s_m(r)}\binom{m}{r+1}$. From this we derive useful approximations to Q, $2 + \frac{Q}{r+1}$, and $s_m(r)$, and with these find a maximizing input r_0 for $g_{\omega,m}(r)$.

The *jth tail* of the generalized continued fraction $\mathcal{K}_{i=1}^{r}\frac{b_i}{a_i}$ is denoted by $\mathcal{T}_j$ where

$$\mathcal{T}_j := \mathop{\mathcal{K}}_{i=j}^{r}\frac{b_i}{a_i} = \cfrac{b_j}{a_j + \cfrac{b_{j+1}}{a_{j+1} + \cfrac{b_{j+2}}{\ddots\, a_{r-1} + \cfrac{b_r}{a_r}}}} = \frac{b_j}{a_j + \mathcal{T}_{j+1}} \quad \text{and } 1 \leqslant j \leqslant r. \tag{1.2}$$

If $\mathcal{T}_j = \frac{B_j}{A_j}$, then $\mathcal{T}_j = \frac{b_j}{a_j+\mathcal{T}_{j+1}}$ shows $b_jA_j - a_jB_j = \mathcal{T}_{j+1}B_j$. By convention we set $\mathcal{T}_{r+1} = 0$.

It follows from $\binom{m}{r-i} = \binom{m}{r}\prod_{k=1}^{i}\frac{r-k+1}{m-r+k}$ that $x^i\binom{m}{r} \leqslant \binom{m}{r-i} \leqslant y^i\binom{m}{r}$ for $0 \leqslant i \leqslant r$ where $x := \frac{1}{m}$ and $y := \frac{r}{m-r+1}$. Hence $\frac{1-x^{r+1}}{1-x}\binom{m}{r} \leqslant s_m(r) \leqslant \frac{1-y^{r+1}}{1-y}\binom{m}{r}$. These bounds are close if $\frac{r}{m}$ is near 0. If $\frac{r}{m}$ is near $\frac{1}{2}$ then better approximations involve the Berry–Esseen inequality [7] to estimate the binomial cumulative distribution function $2^{-m}s_m(r)$. The cumulative distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$ is used in remark 5.3 to show that $|2^{-m}s_m(r) - \Phi(\frac{2r-m}{\sqrt{m}})| \leqslant \frac{0.4215}{\sqrt{m}}$ for $0 \leqslant r \leqslant m$ and $m \neq 0$. Each binomial $\binom{m}{i}$ can be estimated using Stirling's approximation as in [10, p. 2]: $\binom{m}{i} = \frac{C_i^m}{\sqrt{2\pi p(1-p)m}}\left(1 + O\left(\frac{1}{m}\right)\right)$ where $C_i = \frac{1}{p^p(1-p)^{1-p}}$ and $p = p_i = i/m$. However, the sum $\sum_{i=0}^{r}\binom{m}{i}$ of binomials is harder to approximate. The preprint [11] discusses different approximations to $s_m(r)$.

Sums of binomial coefficients modulo prime powers, where i lies in a congruence class, can be studied using number theory, see [5, p. 257]. Theorem 1.1 below shows how to find excellent rational approximations to $s_m(r)$ via generalized continued fractions.

THEOREM 1.1. *Fix $r, m \in \mathbb{Z}$ where $0 \leqslant r \leqslant m$ and recall that $s_m(r) = \sum_{i=0}^{r}\binom{m}{i}$.*

(a) *If $b_i = 2i(r+1-i)$, $a_i = m - 2r + 3i$ for $0 \leqslant i \leqslant r$, then*

$$Q := \frac{(r+1)\binom{m}{r+1}}{s_m(r)} = a_0 + \mathcal{K}_{i=1}^{r}\frac{b_i}{a_i}.$$

(b) *If $1 \leqslant j \leqslant r$, then $\mathcal{T}_j = R_j/R_{j-1} > 0$ where the sum $R_j := 2^j j!\sum_{k=0}^{r-j}\binom{r-k}{j}\binom{m}{k}$ satisfies $b_jR_{j-1} - a_jR_j = R_{j+1}$. Also, $(m-r)\binom{m}{r} - a_0R_0 = R_1$.*

Since $s_m(m) = 2^m$, it follows that $s_m(m-r) = 2^m - s_m(r-1)$ so we focus on values of r satisfying $0 \leqslant r \leqslant \lfloor\frac{m}{2}\rfloor$. Theorem 1.1 allows us to find a sequence of successively sharper upper and lower bounds for Q (which can be made arbitrarily

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tight), the coarsest being $m-2r \leqslant Q \leqslant m-2r+\frac{2r}{m-2r+3}$ for $1 \leqslant r < \frac{m+3}{2}$, see proposition 2.3 and corollary 2.4.

The fact that the tails $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are all positive is unexpected as b_i/a_i is negative if $\frac{m+3i}{2} < r$. This fact is crucial for approximating $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$, see theorem 1.3. Theorem 1.1 implies that $\mathcal{T}_1\mathcal{T}_2\cdots\mathcal{T}_r = R_r/R_0$. Since $R_0 = s_m(r)$, $R_r = 2^r r!$, $\mathcal{T}_j = \frac{b_j}{a_j+\mathcal{T}_{j+1}}$ and $\prod_{j=1}^{r} b_j = 2^r(r!)^2$, the surprising factorizations below follow *c.f.* remark 2.1.

Corollary 1.2. *We have* $s_m(r)\prod_{j=1}^{r}\mathcal{T}_j = 2^r r!$ *and* $r!s_m(r) = \prod_{j=1}^{r}(a_j+\mathcal{T}_{j+1})$.

Suppose that $\omega > 1$ and write $g(r) = g_{\omega,m}(r)$. We extend the domain of $g(r)$ by setting $g(-1) = 0$ and $g(m+1) = \frac{g(m)}{\omega}$ in keeping with (1.1). It is easy to prove that $g(r)$ is a *unimodal* function *c.f.* [2, § 2]. Hence there exists some $r_0 \in \{0, 1, \ldots, m\}$ that satisfies

$$g_{\omega,m}(-1) < \cdots < g_{\omega,m}(r_0-1) \leqslant g_{\omega,m}(r_0) \quad \text{and}$$
$$g_{\omega,m}(r_0) > \cdots > g_{\omega,m}(m+1). \tag{1.3}$$

As $g(-1) < g(0) = 1$ and $\left(\frac{2}{\omega}\right)^m = g(m) > g(m+1) = \frac{2^m}{\omega^{m+1}}$, both chains of inequalities are non-empty. The chains of inequalities (1.3) serve to define r_0.

We use theorem 1.1 to show that r_0 is commonly close to $r' := \lfloor\frac{m+2}{\omega+1}\rfloor$. We always have $r' \leqslant r_0$ (by lemma 3.3) and though $r_0 - r'$ approaches $\frac{m}{2}$ as ω approaches 1 (see remark 4.4), if $\omega \geqslant \sqrt{3}$ then $0 \leqslant r_0 - r' \leqslant 1$ by the next theorem.

Theorem 1.3. *If* $\omega \geqslant \sqrt{3}$, $m \in \{0, 1, \ldots\}$ *and* $r' := \lfloor\frac{m+2}{\omega+1}\rfloor$, *then* $r_0 \in \{r', r'+1\}$, *that is*

$$g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \textit{and} \quad g(r'+1) > g(r'+2) > \cdots > g(m).$$

Sharp bounds for Q seem powerful: they enable short and elementary proofs of results that previously required substantial effort. For example, our proof in [4, Theorem 1.1] for $\omega = 2$ of the formula $r_0 = \lfloor\frac{m}{3}\rfloor + 1$ involved a lengthy argument, and our first proof of theorem 1.4 below involved a delicate induction. By this theorem there is a unique maximum, namely $r_0 = r' = \lfloor\frac{m+2}{\omega+1}\rfloor$ when $\omega \in \{3, 4, 5, \ldots\}$ and $\omega \neq m+1$, *c.f.* remark 4.2. In particular, strict inequality $g_{\omega,m}(r'-1) < g_{\omega,m}(r')$ holds.

Theorem 1.4. *Suppose that* $\omega \in \{3, 4, 5, \ldots\}$ *and* $r' = \lfloor\frac{m+2}{\omega+1}\rfloor$. *Then*

$$g_{\omega,m}(0) < \cdots < g_{\omega,m}(r'-1) \leqslant g_{\omega,m}(r') > g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m),$$

with equality if and only if $\omega = m+1$.

Our motivation was to analyse $g_{\omega,m}(r)$ by using estimates for Q given by the generalized continued fraction in theorem 1.1. This gives tighter estimates than the method involving partial sums used in [4]. The plots of $y = g_{\omega,m}(r)$ for $0 \leqslant r \leqslant m$ are highly asymmetrical if $\omega - 1$ and m are small. However, if m is large the plots exhibit an 'approximate symmetry' about the vertical line $r = r_0$ (see figure 1).

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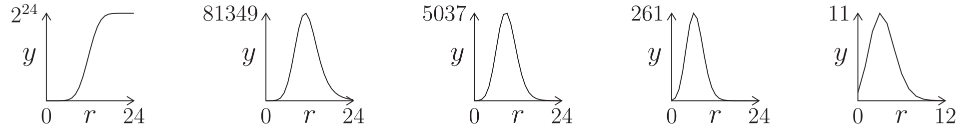

Figure 1. Plots of $y = g_{\omega,24}(r)$ for $0 \leqslant r \leqslant 24$ with $\omega \in \{1, \frac{3}{2}, 2, 3\}$, and $y = g_{3,12}(r)$

Our observation that r_0 is close to r' for many choices of ω was the starting point of our research.

Byun and Poznanović [2, Theorem 1.1] compute the maximizing input, call it r^*, for the function $f_{m,a}(r) := (1+a)^{-r}\sum_{i=0}^{r}\binom{m}{i}a^i$ where $a \in \{1, 2, \ldots\}$. Their function equals $g_{\omega,m}(r)$ only when $\omega = 1 + a = 2$. Some of their results and methods are similar to those in [4] which studied the case $\omega = 2$. They prove that $r^* = \lfloor\frac{a(m+1)+2}{2a+1}\rfloor$ provided $m \notin \{3, 2a+4, 4a+5\}$ or $(a, m) \neq (1, 12)$ when $r^* = \lfloor\frac{a(m+1)+2}{2a+1}\rfloor - 1$.

In Section 2 we prove theorem 1.1 and record approximations to our generalized continued fraction expansion. When m is large, the plots of $y = g_{\omega,m}(r)$ are reminiscent of a normal distribution with mean $\mu \approx \frac{m}{\omega+1}$. Section 3 proves key lemmas for estimating r_0, and applies theorem 1.1 to prove theorem 1.4. Non-integral values of ω are considered in Section 4 where theorem 1.3 is proved. In Section 5 we estimate the maximum height $g(r_0)$ using elementary methods and estimations, see lemma 5.1. A 'statistical' approximation to $s_m(r)$ is given in remark 5.3, and it is compared in remark 5.4 to the 'generalized continued fraction approximations' of $s_m(r)$ in proposition 2.3.

2. Generalized continued fraction formulas

In this section we prove theorem 1.1, namely that $Q := \frac{r+1}{s_m(r)}\binom{m}{r+1} = a_0 + \mathcal{T}_1$ where $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r}\frac{b_i}{a_i}$. The equality $s_m(r) = \frac{r+1}{a_0+\mathcal{T}_1}\binom{m}{r+1}$ is noted in corollary 2.2.

A version of theorem 1.1(a) was announced in the SCS2022 Poster room, created to run concurrently with vICM 2022, see [9].

Proof of theorem 1.1. Set $R_{-1} = Q\, s_m(r) = (r+1)\binom{m}{r+1} = (m-r)\binom{m}{r}$ and

$$R_j = 2^j j! \sum_{k=0}^{r-j}\binom{r-k}{j}\binom{m}{k} \quad \text{for } 0 \leqslant j \leqslant r+1.$$

Clearly $R_0 = s_m(r)$, $R_{r+1} = 0$ and $R_j > 0$ for $0 \leqslant j \leqslant r$. We will prove in the following paragraph that the quantities R_j, $a_j = m - 2r + 3j$, and $b_j = 2j(r+1-j)$ satisfy the following $r+1$ equations, where the first equation (2.1) is atypical:

$$R_{-1} - a_0 R_0 = R_1, \tag{2.1}$$

$$b_j R_{j-1} - a_j R_j = R_{j+1} \quad \text{where } 1 \leqslant j \leqslant r. \tag{2.2}$$

Assuming (2.2) is true, we prove by induction that $\mathcal{T}_j = R_j/R_{j-1}$ holds for $r+1 \geqslant j \geqslant 1$. This is clear for $j = r+1$ since $\mathcal{T}_{r+1} = R_{r+1} = 0$. Suppose

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that $1 \leqslant j \leqslant r$ and $\mathcal{T}_{j+1} = R_{j+1}/R_j$ holds. We show that $\mathcal{T}_j = R_j/R_{j-1}$ holds. Using (2.2) and $R_j > 0$ we have $b_j R_{j-1}/R_j - a_j = R_{j+1}/R_j = \mathcal{T}_{j+1}$. Hence $R_j/R_{j-1} = b_j/(a_j + \mathcal{T}_{j+1}) = \mathcal{T}_j$, completing the induction. Equation (2.1) gives $Q = R_{-1}/R_0 = a_0 + R_1/R_0 = a_0 + \mathcal{T}_1$ as claimed. Since $R_j > 0$ for $0 \leqslant j \leqslant r$, we have $\mathcal{T}_j = R_j/R_{j-1} > 0$ for $1 \leqslant j \leqslant r$. This proves the first half of theorem 1.1(b), and the recurrence $\mathcal{T}_j = b_j/(a_j + \mathcal{T}_{j+1})$ for $1 \leqslant j \leqslant r$, proves part (a).

We now show that (2.1) holds. The identity $R_0 = 2^0 0! \sum_{k=0}^{r} \binom{m}{k} = s_m(r)$ gives

$$\begin{aligned} R_{-1} - a_0 R_0 &= (r+1)\binom{m}{r+1} - (m-2r)\sum_{i=0}^{r}\binom{m}{i} \\ &= (r+1)\binom{m}{r+1} - \sum_{i=0}^{r}(-i + m - i - 2r + 2i)\binom{m}{i} \\ &= \sum_{i=0}^{r}\left[(i+1)\binom{m}{i+1} - (m-i)\binom{m}{i}\right] + 2\sum_{i=0}^{r-1}(r-i)\binom{m}{i}. \end{aligned}$$

As $(i+1)\binom{m}{i+1} = (m-i)\binom{m}{i}$, we get $R_{-1} - a_0 R_0 = 2\sum_{k=0}^{r-1}\binom{r-k}{1}\binom{m}{k} = R_1$.

We next show that (2.2) holds. In order to simplify our calculations, we divide by $C_j := 2^j j!$. Using $(j+1)\binom{r-k}{j+1} = (r-k-j)\binom{r-k}{j}$ gives

$$\begin{aligned} \frac{R_{j+1}}{C_j} &= \sum_{k=0}^{r-j-1} 2(j+1)\binom{r-k}{j+1}\binom{m}{k} \\ &= \sum_{k=0}^{r-j} 2(r-k-j)\binom{r-k}{j}\binom{m}{k} \\ &= \sum_{k=0}^{r-j+1} (j-k)\binom{r-k}{j}\binom{m}{k} - \sum_{k=0}^{r-j}(k-2r+3j)\binom{r-k}{j}\binom{m}{k} \end{aligned}$$

noting that the term with $k = r-j+1$ in the first sum is zero as $\binom{j-1}{j} = 0$. Using the abbreviation $L = \sum_{k=0}^{r-j}(k-2r+3j)\binom{r-k}{j}\binom{m}{k}$ and using the identity $j\binom{r-k}{j} = (r+1-j-k)\binom{r-k}{j-1}$ gives

$$\begin{aligned} \frac{R_{j+1}}{C_j} &= \sum_{k=0}^{r-j+1}\left[(r+1-j-k)\binom{r-k}{j-1} - k\binom{r-k}{j}\right]\binom{m}{k} - L \\ &= \sum_{k=0}^{r-j+1}\left[(r+1-j)\binom{r-k}{j-1} - k\binom{r-k}{j-1} - k\binom{r-k}{j}\right]\binom{m}{k} - L \\ &= \sum_{k=0}^{r-j+1}\left[(r+1-j)\binom{r-k}{j-1} - k\binom{r-k+1}{j}\right]\binom{m}{k} - L. \end{aligned}$$

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However, $k\binom{m}{k} = (m-k+1)\binom{m}{k-1}$, and therefore,

$$\sum_{k=0}^{r-j+1} k\binom{r-k+1}{j}\binom{m}{k} = \sum_{k=1}^{r-j+1}(m-k+1)\binom{r-k+1}{j}\binom{m}{k-1}$$
$$= \sum_{\ell=0}^{r-j}(m-\ell)\binom{r-\ell}{j}\binom{m}{\ell}.$$

Thus

$$\begin{aligned}\frac{R_{j+1}}{C_j} &= \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}(m-k)\binom{r-k}{j}\binom{m}{k} - L\\ &= \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}(m-k+k-2r+3j)\binom{r-k}{j}\binom{m}{k}\\ &= \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}\overbrace{(m-2r+3j)}^{a_j}\binom{r-k}{j}\binom{m}{k}\\ &= \frac{\overbrace{2j(r-j+1)}^{b_j}\,2^{j-1}(j-1)!}{C_j}\sum_{k=0}^{r-j+1}\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}a_j\binom{r-k}{j}\binom{m}{k}\end{aligned}$$

Hence $\frac{R_{j+1}}{C_j} = \frac{b_jR_{j-1}}{C_j} - \frac{a_jR_j}{C_j}$ for $1 \leqslant j \leqslant r$. When $j = r$, our convention gives $R_{r+1} = 0$. This proves part (b) and completes the proof of part (a). □

REMARK 2.1. View m as an indeterminant, so that $r!s_m(r)$ is a polynomial in m over $\mathbb{Z}$ of degree r. The factorization $r!s_m(r) = \prod_{j=1}^{r}(a_j + \mathcal{T}_{j+1})$ in corollary 1.2 involves the rational functions $a_j + \mathcal{T}_{j+1}$. However, theorem 1.1(b) gives $\mathcal{T}_{j+1} = \frac{R_{j+1}}{R_j}$, so that $a_j + \mathcal{T}_{j+1} = \frac{a_jR_j+R_{j+1}}{R_j} = \frac{b_jR_{j-1}}{R_j}$. This determines the numerator and denominator of the rational function $a_j + \mathcal{T}_{j+1}$, and explains why we have $\prod_{j=1}^{r}(a_j + \mathcal{T}_{j+1}) = \frac{R_0}{R_r}\prod_{j=1}^{r} b_j = r!s_m(r)$. This is different from, but reminiscent of, the ratio p_{j+1}/p_j described on p. 26 of [8]. ⋄

COROLLARY 2.2. *If* $r, m \in \mathbb{Z}$ *and* $0 < r < m$, *then*

$$s_m(r) := \sum_{i=0}^{r}\binom{m}{i} = \frac{(r+1)\binom{m}{r+1}}{m-2r+\mathcal{T}_1} \quad \textit{where } \mathcal{T}_1 = \mathcal{K}_{i=1}^{r}\frac{2i(r+1-i)}{m-2r+3i} > 0.$$

If $r = 0$, then $s_m(r) = \frac{(r+1)\binom{m}{r+1}}{m-2r+\mathcal{T}_1}$ is true, but $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r}\frac{2i(r+1-i)}{m-2r+3i} = 0$.

We will need some additional tools such as proposition 2.3 and corollary 2.4 below in order to prove theorem 1.3.

Since $s_m(m-r) = 2^m - s_m(r-1)$ approximating $s_m(r)$ for $0 \leqslant r \leqslant m$ reduces to approximating $s_m(r)$ for $0 \leqslant r \leqslant \lfloor \frac{m}{2} \rfloor$. Hence the hypothesis $r < \frac{m+3}{2}$ in proposition 2.3 and corollary 2.4 is not too restrictive. Proposition 2.3 generalizes [8, Theorem 3.3].

Let $\mathcal{H}_j := \mathcal{K}_{i=1}^{j} \frac{b_i}{a_i}$ denote the *jth head* of the fraction $\mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$, where $\mathcal{H}_0 = 0$.

PROPOSITION 2.3. *Let $b_i = 2i(r+1-i)$ and $a_i = m-2r+3i$ for $0 \leqslant i \leqslant r$. If $r < \frac{m+3}{2}$, then $a_0 + \mathcal{H}_r = \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ can be approximated using the following chain of inequalities*

$$a_0 + \mathcal{H}_0 < a_0 + \mathcal{H}_2 < \cdots < a_0 + \mathcal{H}_{2\lfloor r/2\rfloor} < a_0 + \mathcal{H}_{2\lfloor (r-1)/2\rfloor+1} < \cdots < a_0 + \mathcal{H}_3 < a_0 + \mathcal{H}_1.$$

Proof. Note that r equals either $2\lfloor r/2\rfloor$ or $2\lfloor (r-1)/2\rfloor + 1$, depending on its parity.

We showed in the proof of theorem 1.1 that $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} = a_0 + \mathcal{H}_r = a_0 + \mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$. Since $r < \frac{m+3}{2}$, we have $a_i > 0$ and $b_i > 0$ for $1 \leqslant i \leqslant r$ and hence $\frac{b_i}{a_i} > 0$. A straightforward induction (which we omit) depending on the parity of r proves that $\mathcal{H}_0 < \mathcal{H}_2 < \cdots < \mathcal{H}_{2\lfloor r/2\rfloor} < \mathcal{H}_{2\lfloor (r-1)/2\rfloor+1} < \cdots < \mathcal{H}_3 < \mathcal{H}_1$. For example, if $r = 3$, then

$$\mathcal{H}_0 = 0 < \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2}} < \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3}}} < \frac{b_1}{a_1} = \mathcal{H}_1.$$

proves $\mathcal{H}_0 < \mathcal{H}_2 < \mathcal{H}_3 < \mathcal{H}_1$ as the tails are positive. Adding a_0 proves the claim. □

In asking whether $g_{\omega,m}(r)$ is a unimodal function, it is natural to consider the ratio $g_{\omega,m}(r+1)/g_{\omega,m}(r)$ of successive terms. This suggests defining

$$t(r) = t_m(r) := \frac{s_m(r+1)}{s_m(r)} = 1 + \frac{\binom{m}{r+1}}{s_m(r)} = 1 + \frac{Q}{r+1}. \tag{2.3}$$

We will prove in lemma 3.1 that $t(r)$ is a strictly decreasing function that determines when $g_{\omega,m}(r)$ is increasing or decreasing, and $t_m(r_0 - 1) \geqslant \omega > t_m(r_0)$ determines r_0.

COROLLARY 2.4. *We have $m - 2r \leqslant \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ for $r \geqslant 0$, and*

$$\frac{(r+1)\binom{m}{r+1}}{s_m(r)} \leqslant m - 2r + \frac{2r}{m-2r+3} \quad \text{for} \quad 0 \leqslant r < \frac{m+3}{2}.$$

Hence $\frac{m+2}{r+1} \leqslant t_m(r) + 1$ for $r \geqslant 0$, and

$$\frac{m+2}{r+1} \leqslant t_m(r) + 1 \leqslant \frac{m+2}{r+1} + \frac{2r}{(r+1)(m-2r+3)} \quad \textit{for } 0 \leqslant r < \frac{m+3}{2}.$$

Also $\frac{m+2}{r+1} < t_m(r) + 1$ for $r > 0$, and the above upper bound is strict for $1 < r < \frac{m+3}{2}$.

Table 1. *Values of* $g_{w,m}(r)$

r	0	1	2	3	$\cdots$	$m-2$	$m-1$	m
$g_{\omega,m}(r)$	1	$\frac{m+1}{\omega}$	$\frac{m^2+m+2}{2\omega^2}$	$\frac{m^3+5m+6}{6\omega^3}$	$\cdots$	$\frac{2^m-m-1}{\omega^{m-2}}$	$\frac{2^m-1}{\omega^{m-1}}$	$(\frac{2}{\omega})^m$

Proof. We proved $Q = \frac{(r+1)\binom{m}{r}}{s_m(r)} = (m-2r) + \mathcal{K}_{i=1}^{r} \frac{2i(r+1-i)}{m-2r+3i}$ in theorem 1.1. Hence $m - 2r = \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $r = 0$ and $m - 2r < \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $1 \leqslant r < \frac{m+3}{2}$ by proposition 2.3. Clearly $m - 2r < 0 \leqslant \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $\frac{m+3}{2} \leqslant r \leqslant m$. Similarly $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} = m - 2r + \frac{2r}{m-2r+3}$ if $r = 0, 1$, and again proposition 2.3 shows that $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} < m - 2r + \frac{2r}{m-2r+3}$ if $1 < r < \frac{m+3}{2}$. The remaining inequalities (and equalities) follow similarly since $t_m(r) + 1 = 2 + \frac{\binom{m}{r+1}}{s_m(r)}$ and $2 + \frac{m-2r}{r+1} = \frac{m+2}{r+1}$. $\square$

3. Estimating the maximizing input r_0

Fix $\omega > 1$. In this section we consider the function $g(r) = g_{\omega,m}(r)$ given by (1.1). As seen in table 1, it is easy to compute $g(r)$ if r is near 0 or m. For m large and r near 0, we have 'sub-exponential' growth $g(r) \approx \frac{m^r}{r!\omega^r}$. Similarly for r near m, we have exponential decay $g(r) \approx \frac{2^m}{\omega^r}$. The middle values require more thought.

On the other hand, the plots $y = g(r)$, $0 \leqslant r \leqslant m$, exhibit a remarkable visual symmetry when m is large. The relation $s_m(m-r) = 2^m - s_m(r-1)$ and the distorting scale factor of ω^{-r} shape the plots. The examples in figure 1 show an approximate left–right symmetry about a maximizing input $r \approx \frac{m}{\omega+1}$. It surprised the authors that in many cases there exists a simple exact formula for the maximizing input (it is usually unique as corollary 3.2 suggests). In figure 1 we have used different scale factors for the y-axes. The maximum value of $g_{\omega,m}(r)$ varies considerably as ω varies (*c.f.* lemma 5.1), so we scaled the maxima (rounded to the nearest integer) to the same height.

LEMMA 3.1. *Recall that* $g(r) = \omega^{-r} s_m(r)$ *by* (1.1) *and* $t(r) = \frac{s_m(r+1)}{s_m(r)}$ *by* (2.3).

(a) $t(r-1) > t(r) > \frac{m-r}{r+1}$ *for* $0 \leqslant r \leqslant m$ *where* $t(-1) := \infty$;

(b) $g(r) < g(r+1)$ *if and only if* $t(r) > \omega$;

(c) $g(r) \leqslant g(r+1)$ *if and only if* $t(r) \geqslant \omega$;

(d) $g(r) > g(r+1)$ *if and only if* $\omega > t(r)$;

(e) $g(r) \geqslant g(r+1)$ *if and only if* $\omega \geqslant t(r)$;

(f) *if* $\omega > 1$ *then some* $r_0 \in \{0, \ldots, m\}$ *satisfies* $t(r_0 - 1) \geqslant \omega > t(r_0)$, *and this condition is equivalent to*

$$g(0) < \cdots < g(r_0 - 1) \leqslant g(r_0) \quad \text{and} \quad g(r_0) > \cdots > g(m).$$

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Proof. (a) We prove, using induction on r, that $t(r-1) > t(r) > \binom{m}{r+1}/\binom{m}{r}$ holds for $0 \leqslant r \leqslant m$. These inequalities are clear for $r = 0$ as $\infty > m+1 > m$. For real numbers α, β, γ, $\delta > 0$, we have $\alpha\delta - \beta\gamma > 0$ if and only if $\frac{\alpha}{\beta} > \frac{\alpha+\gamma}{\beta+\delta} > \frac{\gamma}{\delta}$; that is, the mediant $\frac{\alpha+\gamma}{\beta+\delta}$ of $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$ lies strictly between $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$. If $0 < r \leqslant m$, then by induction

$$t(r-1) > \frac{\binom{m}{r}}{\binom{m}{r-1}} = \frac{m-r+1}{r} > \frac{m-r}{r+1} = \frac{\binom{m}{r+1}}{\binom{m}{r}}.$$

Applying the 'mediant sum' to $t(r-1) = \frac{s_m(r)}{s_m(r-1)} > \frac{\binom{m}{r+1}}{\binom{m}{r}}$ gives

$$\frac{s_m(r)}{s_m(r-1)} > \frac{s_m(r) + \binom{m}{r+1}}{s_m(r-1) + \binom{m}{r}} = \frac{s_m(r+1)}{s_m(r)} = t(r) > \frac{\binom{m}{r+1}}{\binom{m}{r}}.$$

Therefore $t(r-1) > t(r) > \binom{m}{r+1}/\binom{m}{r} = \frac{m-r}{r+1}$ completing the induction, and proving (a).

(b,c,d,e) The following are equivalent: $g(r) < g(r+1)$; $\omega s_m(r) < s_m(r+1)$; and $\omega < t(r)$. The other claims are proved similarly by replacing $<$ with $\leqslant$, $>$, $\geqslant$.

(f) Observe that $t(m) = \frac{s_m(m+1)}{s_m(m)} = \frac{2^m}{2^m} = 1$. By part (a), the function $y = t(r)$ is decreasing for $-1 \leqslant r \leqslant m$. Since $\omega > 1$, there exists an integer $r_0 \in \{0, \dots, m\}$ such that $\infty = t(-1) > \cdots > t(r_0 - 1) \geqslant \omega > t(r_0) > \cdots > t(m) = 1$. By parts (b,c,d,e) an equivalent condition is $g(0) < \cdots < g(r_0 - 1) \leqslant g(r_0)$ and $g(r_0) > \cdots > g(m)$. □

The following is an immediate corollary of lemma 3.1(f).

Corollary 3.2. *If $t(r_0 - 1) > \omega$, then the function $g(r)$ in (1.1) has a unique maximum at r_0. If $t(r_0 - 1) = \omega$, then $g(r)$ has two equal maxima, one at $r_0 - 1$ and one at r_0.*

As an application of theorem 1.1 we show that the largest maximizing input r_0 for $g_{\omega,m}(r)$ satisfies $\lfloor \frac{m+2}{\omega+1} \rfloor \leqslant r_0$. There are at most two maximizing inputs by corollary 3.2.

Lemma 3.3. *Suppose that $\omega > 1$ and $m \in \mathbb{Z}$, $m \geqslant 0$. If $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$, then*

$$g(-1) < g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \text{and} \quad g(-1) < \cdots < g(r'-1) < g(r')$$

if $r' > 1$ or $\omega \neq m+1$.

Proof. The result is clear when $r' = 0$. If $r' = 1$, then $r' \leqslant \frac{m+2}{\omega+1}$ gives $\omega \leqslant m+1$ or $g(0) \leqslant g(1)$. Hence $g(0) < g(1)$ if $\omega \neq m+1$. Suppose that $r' > 1$. By lemma 3.1(c,f) the chain $g(0) < \cdots < g(r')$ is equivalent to $g(r'-1) < g(r')$, that is $t(r'-1) > \omega$. However, $t(r'-1) + 1 > \frac{m+2}{r'}$ by corollary 2.4 and $r' \leqslant \frac{m+2}{\omega+1}$ implies $\frac{m+2}{r'} \geqslant \omega + 1$. Hence $t(r'-1) + 1 > \omega + 1$, so that $t(r'-1) > \omega$ as desired. □

Proof of theorem 1.4. Suppose that $\omega \in \{3, 4, \dots\}$. Then $g(0) < \dots < g(r'-1) \leqslant g(r')$ by lemma 3.3 with strictness when $\omega \neq m+1$. If $\omega = m+1$,

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then $r' = \lfloor \frac{m+2}{\omega+1} \rfloor = 1$ and $g(0) = g(1)$ as claimed. It remains to show that $g(r') > g(r'+1) > \cdots > g(m)$. However, we need only prove that $g(r') > g(r'+1)$ by lemma 3.1(f), or equivalently $\omega > t(r')$ by lemma 3.1(d).

Clearly $\omega \geqslant 3$ implies $r' \leqslant \frac{m+2}{\omega+1} \leqslant \frac{m+2}{4}$. As $0 \leqslant r' < \frac{m+3}{2}$, corollary 2.4 gives

$$\frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)} \geqslant t(r') + 1.$$

Hence $\omega + 1 > t(r') + 1$ holds if $\omega + 1 > \frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)}$. Since $\omega + 1$ is an integer, we have $m + 2 = r'(\omega + 1) + c$ where $0 \leqslant c \leqslant \omega$. It follows from $0 \leqslant r' \leqslant \frac{m+2}{4}$ that $\frac{2r'}{m-2r'+3} < 1$. This inequality and $m + 2 \leqslant r'(\omega + 1) + \omega$ gives

$$m + 2 + \frac{2r'}{m-2r'+3} < r'(\omega+1) + \omega + 1 = (r'+1)(\omega+1).$$

Thus $\omega + 1 > \frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)} \geqslant t(r') + 1$, so $\omega > t(r')$ as required. □

Remark 3.4. The proof of theorem 1.4 can be adapted to the case $\omega = 2$. If $m + 2 = 3r' + c$ where $c \leqslant \omega - 1 = 1$, then $\frac{2r'}{m-2r'+3} = \frac{2r'}{r'+c+1} < 2$, and if $c = \omega = 2$, then a sharper $\mathcal{H}_2$-bound must be used. This leads to a much shorter proof than [**4**, Theorem 1.1]. ⋄

4. Non-integral values of ω

In this section, we prove that the maximum value of $g(r)$ is $g(r')$ or $g(r'+1)$ if $\omega \geqslant \sqrt{3}$. Before proving this result (theorem 1.3), we shall prove two preliminary lemmas.

Lemma 4.1. *Suppose that* $\omega > 1$ *and* $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$. *If* $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$, *then*

$$g(-1) < g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \textit{and} \quad g(r'+1) > g(r'+2) > \cdots > g(m).$$

Proof. It suffices, by lemma 3.1(f) and lemma 3.3 to prove that $g(r'+1) > g(r'+2)$. The strategy is to show $\omega > t(r'+1)$, that is $\omega + 1 > t(r'+1) + 1$. However, $\omega + 1 > \frac{m+2}{r'+1}$, so it suffices to prove that $\frac{m+2}{r'+1} \geqslant t(r'+1) + 1$. Since $r' + 1 \leqslant \frac{m+2}{\sqrt{3}+1} < \frac{m+2}{2}$, we can use corollary 2.4 and just prove that $\frac{m+2}{r'+1} \geqslant \frac{m+2}{r'+2} + \frac{2r'+2}{(r'+2)(m-2r'+1)}$. This inequality is equivalent to $\frac{m+2}{r'+1} \geqslant \frac{2r'+2}{m-2r'+1}$. However, $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$, so we need only show that $\sqrt{3} + 1 \geqslant \frac{2(r'+1)}{m-2r'+1}$, or equivalently $m - 2r' + 1 \geqslant (\sqrt{3} - 1)(r' + 1)$. This is true since $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$ implies $m - 2r' + 1 \geqslant (\sqrt{3} - 1)r' + \sqrt{3} > (\sqrt{3} - 1)(r' + 1)$. □

Remark 4.2. The strict inequality $g(r'-1) < g(r')$ holds by lemma 3.3 if $r' > 1$ or $\omega \neq m + 1$. It holds vacuously for $r' = 0$. Hence adding the additional hypothesis that $\omega \neq m + 1$ if $r' = 1$ to lemma 4.1 (and theorem 1.3), we may conclude that the inequality $g(r'-1) \leqslant g(r')$ is strict.

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REMARK 4.3. In lemma 4.1, the maximum can occur at $r'+1$. If $\omega = 2.5$ and $m = 8$, then $r' = \lfloor \frac{10}{3.5} \rfloor = 2$ and $\frac{m+2}{r'+1} = \frac{10}{3} \geqslant \sqrt{3}+1$ however $g_{2.5,8}(2) = \frac{740}{125} < \frac{744}{125} = g_{2.5,8}(3)$. ◇

REMARK 4.4. The gap between r' and the largest maximizing input r_0 can be arbitrarily large if ω is close to 1. For $\omega > 1$, we have $r' = \lfloor \frac{m+2}{\omega+1} \rfloor < \frac{m+2}{2}$. If $1 < \omega \leqslant \frac{1}{1-2^{-m}}$, then $g(m-1) \leqslant g(m)$, so $r_0 = m$. Hence $r_0 - r' > \frac{m-2}{2}$.

REMARK 4.5. Since $r' \leqslant \lfloor \frac{m+2}{\omega+1} \rfloor < r'+1$, we see that $r'+1 \approx \frac{m+2}{\omega+1}$, so that $\frac{m+2}{r'+1} \approx \omega+1$. Thus lemma 4.1 suggests that if $\omega \gtrsim \sqrt{3}$, then $g_{\omega,m}(r)$ may have a maximum at r' or $r'+1$. This heuristic reasoning is made rigorous in theorem 1.3. ◇

REMARK 4.6. Theorem 1.1 can be rephrased as $t_m(r) = \frac{s_m(r+1)}{s_m(r)} = \frac{m-r+1}{r+1} + \frac{\mathcal{K}_m(r)}{r+1}$ where

$$\mathcal{K}_m(r) = \mathop{\mathcal{K}}_{i=1}^{r} \frac{2i(r+1-i)}{m-2r+3i} = \cfrac{2r}{m-2r+3+\cfrac{4r-4}{m-2r+6+\cfrac{6r-12}{\ddots\, m+r-3+\cfrac{2r}{m+r}}}}. \tag{4.1}$$

The following lemma repeatedly uses the expression $\omega > t_m(r+1)$. This is equivalent to $\omega > \frac{m-r}{r+2} + \frac{\mathcal{K}_m(r+1)}{r+2}$, that is $(\omega+1)(r+2) > m+2+\mathcal{K}_m(r+1)$. ◇

LEMMA 4.7. *Let $m \in \{0, 1, \ldots\}$ and $r' = \lfloor \frac{m+2}{\omega+1} \rfloor$. If any of the following three conditions are met, then $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds:* (a) *$\omega \geqslant 2$, or* (b) *$\omega \geqslant \frac{1+\sqrt{97}}{6}$ and $r' \neq 2$, or* (c) *$\omega \geqslant \sqrt{3}$ and $r' \notin \{2, 3\}$.*

Proof. The conclusion $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds trivially if $r'+1 \geqslant m$. Suppose henceforth that $r'+1 < m$. Except for the excluded values of r', ω, we will prove that $g_{\omega,m}(r'+1) > g_{\omega,m}(r'+2)$ holds, as this implies $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ by lemma 3.1(f). Hence we must prove that $\omega > t_m(r'+1)$ by lemma 3.1(d).

Recall that $r' \leqslant \frac{m+2}{\omega+1} < r'+1$. If $r' = 0$, then $m+2 < \omega+1$, that is $\omega > m+1 > t(1)$ as desired. Suppose now that $r' = 1$. There is nothing to prove if $m = r'+1 = 2$. Assume that $m > 2$. Since $m+2 < 2(\omega+1)$, we have $2 < m < 2\omega$. The last line of remark 4.6 and (4.1) give the desired inequality:

$$\omega > \frac{m}{2} \geqslant \frac{m-1}{3} + \frac{4}{3\left(m-1+\frac{4}{m+2}\right)} = t_m(2).$$

In summary, $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds for all $\omega > 1$ if $r' \in \{0, 1\}$.

We next prove $g_{\omega,m}(r'+1) > g_{\omega,m}(r'+2)$, or equivalently $\omega > t_m(r'+1)$ for r' large enough, depending on ω. We must prove that $(\omega+1)(r'+2) > m+2+$

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$\mathcal{K}_m(r'+1)$ by remark 4.6. Writing $m+2=(\omega+1)(r'+\varepsilon)$ where $0\leqslant\varepsilon<1$, our goal, therefore, is to show $(\omega+1)(2-\varepsilon)>\mathcal{K}_m(r'+1)$. Using (4.1) gives

$$\mathcal{K}_m(r'+1)=\frac{2(r'+1)}{m-2(r'+1)+3+\mathcal{T}}=\frac{2(r'+1)}{(\omega+1)(r'+\varepsilon)-2(r'+1)+1+\mathcal{T}}$$

where $\mathcal{T}>0$ by theorem 1.1 as $r'>0$. Rewriting the denominator using

$$(\omega+1)(r'+\varepsilon)-2(r'+1)=(\omega-1)(r'+1)-(\omega+1)(1-\varepsilon),$$

our goal $(\omega+1)(2-\varepsilon)>\mathcal{K}_m(r'+1)$ becomes

$$(\omega+1)(2-\varepsilon)\left[(\omega-1)(r'+1)-(\omega+1)(1-\varepsilon)+1+\mathcal{T}\right]>2(r'+1).$$

Dividing by $(2-\varepsilon)(r'+1)$ and rearranging gives

$$(\omega^2-1)+\frac{(\omega+1)(1+\mathcal{T})}{r'+1}>\frac{2}{2-\varepsilon}+\frac{(\omega+1)^2(1-\varepsilon)}{r'+1}.$$

This inequality may be written $(\omega^2-1)+\lambda>\frac{2}{2-\varepsilon}+\mu(1-\varepsilon)$ where $\lambda=\frac{(\omega+1)(1+\mathcal{T})}{r'+1}>0$ and $\mu=\frac{(\omega+1)^2}{r'+1}>0$. We view $f(\varepsilon):=\frac{2}{2-\varepsilon}+\mu(1-\varepsilon)$ as a function of a real variable ε where $0\leqslant\varepsilon<1$. However, $f(\varepsilon)$ is concave as the second derivative $f''(\varepsilon)=\frac{4}{(2-\varepsilon)^3}$ is positive for $0\leqslant\varepsilon<1$. Hence the maximum value occurs at an end point: either $f(0)=1+\mu$ or $f(1)=2$. Therefore, it suffices to prove that $(\omega^2-1)+\lambda>\max\{2,1+\mu\}$.

If $2\geqslant 1+\mu$, then the desired bound $(\omega^2-3)+\lambda>0$ holds as $\omega\geqslant\sqrt{3}$. Suppose now that $2<1+\mu$. We must show $(\omega^2-1)+\lambda>1+\mu$, that is $\omega^2-2>\mu-\lambda=\frac{(\omega+1)(\omega-\mathcal{T})}{r'+1}$. Since $\mathcal{T}>0$, a stronger inequality (that implies this) is $\omega^2-2\geqslant\frac{(\omega+1)\omega}{r'+1}$. The (equivalent) quadratic inequality $r'\omega^2-\omega-2(r'+1)\geqslant 0$ in ω is true provided $\omega\geqslant\frac{1+\sqrt{1+8r'(r'+1)}}{2r'}$. This says $\omega\geqslant 2$ if $r'=2$, and $\omega\geqslant\frac{1+\sqrt{97}}{6}$ if $r'=3$. If $r'\geqslant 4$, we have

$$\frac{1+\sqrt{1+8r'(r'+1)}}{2r'}=\frac{1}{2r'}+\sqrt{\frac{1}{4(r')^2}+2\left(1+\frac{1}{r'}\right)}\leqslant\frac{1}{8}+\sqrt{\frac{1}{64}+\frac{5}{2}}<\sqrt{3}.$$

The conclusion now follows from the fact that $2>\frac{1+\sqrt{97}}{6}>\sqrt{3}$. □

Proof of theorem 1.3. By lemma 4.1 it suffices to show that $g(r'+1)>g(r'+2)$ holds when $r'+1<m$ and $\omega\geqslant\sqrt{3}$. By lemma 4.7(a), we can assume that $\sqrt{3}\leqslant\omega<2$ and $r'\in\{2,3\}$. For these choices of ω and r', we must show that $\omega>t_m(r'+1)$ by lemma 3.1 for all permissible choices of m. Since $(\omega+1)r'\leqslant m+2<(\omega+1)(r'+1)$, when $r'=2$ we have $5<2(\sqrt{3}+1)\leqslant m+2<9$ so that $4\leqslant m\leqslant 6$. However, $t_m(3)$ equals $\frac{16}{15}$, $\frac{31}{26}$, $\frac{19}{14}$ for these values of m. Thus $\sqrt{3}>t_m(3)$ holds as desired. Similarly, if $r'=3$, then $8<3(\sqrt{3}+1)\leqslant m+2<12$ so that $7\leqslant m\leqslant 9$. In this case $t_m(4)$ equals $\frac{40}{33}$, $\frac{219}{163}$, $\frac{191}{128}$ for these values of m. In each case $\sqrt{3}>t_m(4)$, so the proof is complete. □

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Remark 4.8. We place remark 4.4 in context. The conclusion of theorem 1.3 remains true for values of ω smaller than $\sqrt{3}$ and not 'too close to 1' and m is 'sufficiently large'. Indeed, by adapting the proof of lemma 4.7 we can show there exists a sufficiently large integer d such that $m > d^4$ and $\omega > 1 + \frac{1}{d}$ implies $g(r' + d) > g(r' + d + 1)$. This shows that $r' \leqslant r_0 \leqslant r' + d$, so $r_0 - r' \leqslant d$. We omit the technical proof of this fact. ◇

Remark 4.9. The sequence, $a_0 + \mathcal{H}_1, \ldots, a_0 + \mathcal{H}_r$ terminates at $\frac{r+1}{s_m(r)}\binom{m}{r+1}$ by theorem 1.1. We will not comment here on *how quickly* the alternating sequence in proposition 2.3 converges when $r < \frac{m+3}{2}$. If $r = m$, then $a_0 = -m$ and $\frac{m+1}{s_m(m+1)}\binom{m}{m+1} = 0$, so theorem 1.1 gives the curious identity $\mathcal{H}_m = \mathcal{K}_{i=1}^{m} \frac{2i(m+1-i)}{3i-m} = m$. If ω is less than $\sqrt{3}$ and 'not too close to 1', then we believe that r_0 is approximately $\lfloor \frac{m+2}{\omega+1} + \frac{2}{\omega^2-1} \rfloor$, *c.f.* remark 4.8.

5. Estimating the maximum value of $g_{\omega,m}(r)$

In this section we relate the size of the maximum value $g_{\omega,m}(r_0)$ to the size of the binomial coefficient $\binom{m}{r_0}$. In the case that we know a formula for a maximizing input r_0, we can readily estimate $g_{\omega,m}(r_0)$ using approximations, such as [**10**], for binomial coefficients.

Lemma 5.1. *The maximum value $g_{\omega,m}(r_0)$ of $g_{\omega,m}(r)$, $0 \leqslant r \leqslant m$, satisfies*

$$\frac{1}{(\omega-1)\omega^{r_0}}\binom{m}{r_0+1} < g_{\omega,m}(r_0) \leqslant \frac{1}{(\omega-1)\omega^{r_0-1}}\binom{m}{r_0}.$$

Proof. Since $g(r_0)$ is a maximum value, we have $g(r_0 - 1) \leqslant g(r_0)$. This is equivalent to $(\omega - 1)s_m(r_0 - 1) \leqslant \binom{m}{r_0}$ as $s_m(r_0) = s_m(r_0 - 1) + \binom{m}{r_0}$. Adding $(\omega - 1)\binom{m}{r_0}$ to both sides gives the equivalent inequality $(\omega - 1)s_m(r_0) \leqslant \omega\binom{m}{r_0}$. This proves the upper bound.

Similar reasoning shows that the following are equivalent: (a) $g(r_0) > g(r_0 + 1)$; (b) $(\omega - 1)s_m(r_0) > \binom{m}{r_0+1}$; and (c) $g_{\omega,m}(r_0) > \frac{1}{(\omega-1)\omega^{r_0}}\binom{m}{r_0+1}$. □

In theorem 1.3 the maximizing input r_0 satisfies $r_0 = r' + d$ where $d \in \{0, 1\}$. In such cases when r_0 and d are known, we can bound the maximum $g_{\omega,m}(r_0)$ as follows.

Corollary 5.2. *Set $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$ and $k := m + 2 - (\omega + 1)r'$. Suppose that $r_0 = r' + d$ and $G = \frac{1}{(\omega-1)\omega^{r_0-1}}\binom{m}{r_0}$. Then*

$$0 \leqslant k < \omega + 1, d \geqslant 0 \quad \text{and} \quad 1 - \frac{1 + d - \frac{d+2-k}{\omega}}{r_0 + 1} < \frac{g_{\omega,m}(r_0)}{G} \leqslant 1.$$

Proof. By lemma 3.3, $r_0 = r' + d$ where $d \in \{0, 1, \ldots\}$. Since $r' = \lfloor \frac{m+2}{\omega+1} \rfloor$, we have $m + 2 = (\omega + 1)r' + k$ where $0 \leqslant k < \omega + 1$. The result follows from lemma 5.1 and

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$m = (\omega+1)(r_0-d)+k-2$ as $\binom{m}{r_0+1} = \frac{m-r_0}{r_0+1}\binom{m}{r_0}$ and $\frac{m-r_0}{r_0+1}$ equals

$$\frac{\omega(r_0-d)-d+k-2}{r_0+1} = \omega - \frac{\omega+\omega d+d+2-k}{r_0+1} = \omega\left(1-\frac{1+d+\frac{d+2-k}{\omega}}{r_0+1}\right).$$

□

The following remark is an application of the Chernoff bound, *c.f.* [**11**, Section 4]. Unlike theorem 1.1, it requires the cumulative distribution function $\Phi(x)$, which is a non-elementary integral, to approximate $s_m(r)$. It seems to give better approximations only for values of r near $\frac{m}{2}$, see remark 5.4.

Remark 5.3. We show how the Berry–Esseen inequality for a sum of binomial random variables can be used to approximate $s_m(r)$. Let $B_1, \ldots, B_m$ be independent identically distributed binomial variables with a parameter p where $0<p<1$, so that $P(B_i=1)=p$ and $P(B_i=0)=q:=1-p$. Let $X_i := B_i - p$ and $X := \frac{1}{\sqrt{mpq}}(\sum_{i=1}^m X_i)$. Then

$$E(X_i)=E(B_i)-p=0, \quad E(X_i^2)=pq, \quad \text{and} \quad E(|X_i|^3)=pq(p^2+q^2).$$

Hence $E(X)=\frac{1}{\sqrt{mpq}}(\sum_{i=1}^m E(X_i))=0$ and $E(X^2)=\frac{1}{mpq}(\sum_{i=1}^m E(X_i^2))=1$. By [**7**, Theorem 2] the Berry–Esseen inequality applied to X states that

$$\begin{aligned}|P(X\leqslant x)-\Phi(x)| &\leqslant \frac{Cpq(p^2+q^2)}{(pq)^{3/2}\sqrt{m}} \\ &= \frac{C(p^2+q^2)}{\sqrt{mpq}} \quad \text{for all } m\in\{1,2,\ldots\} \text{ and } x\in\mathbb{R},\end{aligned}$$

where the constant $C:=0.4215$ is close to the lower bound $C_0=\frac{10+\sqrt{3}}{6\sqrt{2\pi}}=0.4097\cdots$ and $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt=\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$ is the cumulative distribution function for standard normal distribution.

Writing $B=\sum_{i=1}^m B_i$ we have $P(B\leqslant b)=\sum_{i=0}^{\lfloor b\rfloor}\binom{m}{i}p^iq^{m-i}$ for $b\in\mathbb{R}$. Thus $X=\frac{B-mp}{\sqrt{mpq}}$ and $x=\frac{b-mp}{\sqrt{mpq}}$ satisfy

$$\left|P(B\leqslant b)-\Phi\left(\frac{b-mp}{\sqrt{mpq}}\right)\right| \leqslant \frac{C(p^2+q^2)}{\sqrt{mpq}} \quad \text{for all } m\in\{1,\,2,\,\ldots\} \text{ and } b\in\mathbb{R}.$$

Setting $p=q=\frac{1}{2}$, and taking $b=r\in\{0,\,1,\,\ldots,\,m\}$ shows

$$\left|2^{-m}s_m(r)-\Phi\left(\frac{2r-m}{\sqrt{m}}\right)\right| \leqslant \frac{0.4215}{\sqrt{m}} \quad \text{for } m\in\{1,2,\ldots\}.$$

Remark 5.4. Let $a_0+\mathcal{H}_k$ be the generalized continued fraction approximation to $\frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ suggested by theorem 1.1, where $\mathcal{H}_k := \mathcal{K}_{i=1}^k \frac{b_i}{a_i}$, and k is the depth of

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Table 2. *Upper bounds for* $|e_{m,r,k}|$ *and* $E_{m,r}$ *for* $m = 10^4$

r	$\lvert e_{m,r,3}\rvert$	$\lvert e_{m,r,5}\rvert$	$\lvert e_{m,r,21}\rvert$	$E_{m,r}$
1000	2.3×10^{-17}	6.6×10^{-25}	5.7×10^{-79}	1
4500	1.3×10^{-7}	2.5×10^{-10}	7.1×10^{-27}	0.018
5000	0.93	0.86	0.24	0.008

.

the generalized continued fraction. We compare the following two quantities:

$$e_{m,r,k} := 1 - \frac{(r+1)\binom{m}{r+1}}{(a_0 + \mathcal{H}_k)s_m(r)} \quad \text{and} \quad E_{m,r} := \left|1 - \frac{2^m \Phi(\frac{2r-m}{\sqrt{m}})}{s_m(r)}\right| \leqslant \frac{0.4215 \cdot 2^m}{\sqrt{m}\, s_m(r)}.$$

The sign of $e_{m,r,k}$ is governed by the parity of k by proposition 2.3. We shall assume that $r \leqslant \frac{m}{2}$. As $\frac{2^m}{s_m(r)}$ ranges from 2^m to about 2 as r ranges from 0 to $\lfloor \frac{m}{2} \rfloor$, it is clear that the upper bound for $E_{m,r}$ will be huge unless r satisfies $\frac{m-\varepsilon}{2} \leqslant r \leqslant \frac{m}{2}$ where ε is 'small' compared to m. By contrast, the computer code [**3**] verifies that the same is true for $E_{m,r}$, and shows that $|e_{m,r,k}|$ is small, even when k is tiny, when $0 \leqslant r < \frac{m-\varepsilon}{2}$, see table 2. Hence the 'generalized continued fraction' approximation to $s_m(r)$ is complementary to the 'statistical' approximation, as shown in table 2. The reader can extend table 2 by running the code [**3**] written in the Magma [**1**] language, using the online calculator http://magma.maths.usyd.edu.au/calc/, for example.

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