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RENEWAL PROCESSES, POPULATION DYNAMICS, AND UNIMODULAR TREES

FRANÇOIS BACCELLI,* ** AND
ANTONIO SODRE,* *** *The University of Texas at Austin*

Abstract

Based on a simple object, an i.i.d. sequence of positive integer-valued random variables $\{a_n\}_{n\in\mathbb{Z}}$, we introduce and study two random structures and their connections. First, a population dynamics, in which each individual is born at time n and dies at time $n+a_n$. This dynamics is that of a D/GI/∞ queue, with arrivals at integer times and service times given by $\{a_n\}_{n\in\mathbb{Z}}$. Second, the directed random graph T^f on $\mathbb{Z}$ generated by the random map $f(n)=n+a_n$. Assuming only that $\mathrm{E}[a_0]<\infty$ and $\mathrm{P}[a_0=1]>0$, we show that, in steady state, the population dynamics is regenerative, with one individual alive at each regeneration epoch. We identify a unimodular structure in this dynamics. More precisely, T^f is a unimodular directed tree, in which $f(n)$ is the parent of n. This tree has a unique bi-infinite path. Moreover, T^f splits the integers into two categories: ephemeral integers, with a finite number of descendants of all degrees, and successful integers, with an infinite number. Each regeneration epoch is a successful individual such that all integers less than it are its descendants of some order. Ephemeral, successful, and regeneration integers form stationary and mixing point processes on $\mathbb{Z}$.

Keywords: Population dynamics; queueing system; unimodular random graph; eternal family tree; branching process

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1. Introduction

To each integer n we assign a positive random integer a_n. Then, n is mapped a_n units to the right. Given a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ supporting the random sequence $\{a_n\}_{n\in\mathbb{Z}}$, we consider the function $f:\Omega\times\mathbb{Z}\to\mathbb{Z}$ defined by $f(\omega, n)=n+a_n(\omega)$. We assume the sequence $\{a_n\}_{n\in\mathbb{Z}}$ is i.i.d. We study two objects arising from the random map f: a population dynamics and a directed random graph. When there is no chance of ambiguity, we omit ω in our notation.

In the population dynamics, individual n is *born* at time n and *dies* at time $f(n)$. More precisely, the lifespan of individual n is assumed to be $[n, n+a_n)$. Let $\boldsymbol{N}:=\{N_n\}_{n\in\mathbb{Z}}$ be the discrete-time random process of the number of individuals alive at time n, or simply the *population process*. The process $\boldsymbol{N}$ can be seen as the number of customers at arrival epochs in a D/GI/∞ queue, namely a queueing system with one arrival at every integer time and i.i.d. integer-valued service times, all distributed like a_0. As a new arrival takes place at each integer,

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* Postal address: Department of Mathematics, The University of Texas at Austin, 2515 Speedway, RLM 8.100, Austin, TX 78712-1202, USA.
** Email address: baccelli@math.utexas.edu
*** Email address: antonio.sodre@gmail.com

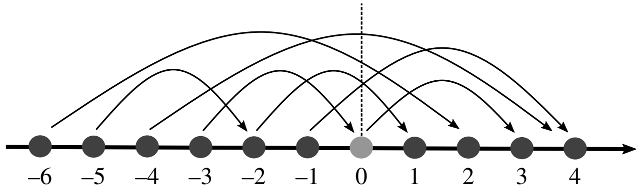

FIGURE 1: The population process. Here, $a_{-6} = 8$, $a_{-5} = 3$, $a_{-4} = 8$, $a_{-3} = 3$, $a_{-2} = 3$, and $a_{-1} = 5$. Assuming $a_{-i} < i$ for all $i > 6$, there are four edges crossing 0. Individuals -6, -4, -2, and -1 are still alive at time 0. Hence, $N_0 = 5$, as it includes the individual born at 0.

the population never goes extinct or, equivalently, the queue is never empty. Nonetheless, as we shall see, when $\mathrm{E}[a_0] < \infty$ the stationary population process is regenerative. To visualize the number of individuals alive at time 0 we count the edges that cross over 0 *and* add the individual born at time 0 (Figure 1).

By letting $V^f = \mathbb{Z}$ and $E^f = \{(n, f(n)) : n \in \mathbb{Z}\}$, the random map f also induces a random directed graph T^f. Further, assuming the distribution of a_0 is aperiodic, we will show that T^f is a directed tree.

We interpret T^f as a family tree and connect it to the population process $\boldsymbol{N}$. In order to gain insight, we draw a parallel with the classical age-dependent branching process. Such a process is built upon a lifetime distribution L on $\mathbb{R}^+$ and an offspring distribution O on $\mathbb{N}$. In this classical model, the first individual is born at time 0 and has a lifetime l sampled from L. When the first individual dies at l, it is replaced by its offspring, whose cardinality is sampled from O. From then on, each individual statistically behaves as the first one, with all individuals having independent lifetimes and offspring cardinalities (see e.g. [10]).

Our family tree, in which individuals are indexed by $\mathbb{Z}$, is obtained by declaring that the individual $f(n)$ is the parent of n, $f^2(n)$ is its grandparent, and so on. In terms of interpretation, this requires that we look at the population dynamics in reverse time. In 'reverse time', individual n 'dies' at time n (note that it is the only one to die at that time) and was 'born earlier', namely at time $n + a_n$. Since individual m is born at time n if $f(n) = m$, the set of children of m is $f^{-1}(m)$, the set of its grandchildren is $f^{-2}(m)$, and so on. As in the age-dependent branching process, each individual has exactly one parent, but may have no children, and the death time of an individual coincides with the birth time of its children. Also notice that $f^{-1}(n) \subset (-\infty, n-1]$. That is, as in the natural enumerations used in branching processes, the children of individual n have 'larger' indices than that of individual n (recall that individuals are enumerated using their 'death times'). Hence, each individual is born at the death of its parent and dies 'after' its parent. Figure 2 illustrates the relation between the population process and T^f. However, our age-dependent family tree is far from being that of the age-dependent branching process discussed above. In particular, there is no first individual: our family tree is *eternal* [3]. More importantly, it lacks the independence properties of branching processes. In particular, the offspring cardinalities of different individuals are dependent.

In the age-dependent branching process described above, if we set $L = 1$ we recover the Bienaymé–Galton–Watson process. Despite the fact that the building block of both our model and the Bienaymé–Galton–Watson model is just a sequence of i.i.d. random variables, the two models are quite different. In the former, the i.i.d. random variables define the offspring cardinalities. In the latter, they define the lifespans of individuals.

Moreover, in our model, the expected offspring cardinality of a typical individual is one. Hence, we follow the standard terminology of branching processes and say the population

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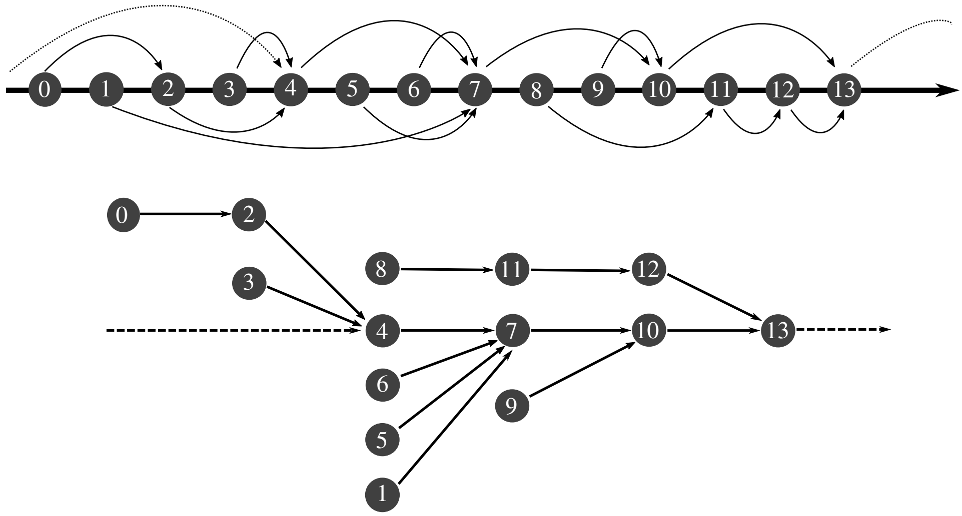

FIGURE 2: From the population process to the family tree. The top figure depicts the f dynamics running on $\mathbb{Z}$. The curved black lines represent individual lifespans. For example, individual 7 lives three units of time. The bottom figure depicts the tree representation of the dynamics on $\mathbb{Z}$, with the directed edges representing parenthood. There, 7 is the child of 10, and has four children: 1, 4, 5, and 6. Individual 8, for example, has no children.

process is *critical*. In branching processes, the critical case leads to extinction unless there is no variability at all. In contrast, in our model there is no chance extinction regardless of the distribution of a_0 (Section 3).

The fact that when $\mathrm{P}[a_0 = 1] > 0$ and $\mathrm{E}[a_0] < \infty$ T^f, rooted at 0, is a unimodular directed tree, allows us to complement the classical queueing and regenerative analysis by structural observations that we believe to be new:

1. this tree is two ended;
2. an integer (individual) is either successful or ephemeral depending on whether the number of its descendants (pre-images by f) of all orders is infinite or finite almost surely (a.s.);
3. the set of successful (resp. ephemeral) integers forms a stationary point process;
4. there is a stationary thinning of the successful point process consisting of individuals such that all individuals born after one of them are its descendants; these are called original ancestors;
5. each individual has a finite number of cousins of all orders.

These structural observations are completed by closed-form expressions for the intensities of the point processes in question.

The last interpretation of the family tree pertains to renewal theory. One can see $n, f(n), f^2(n), \ldots$ as a renewal process on the integers with interarrivals distributed like a_0 and starting from time n. The graph $T^f = (V^f, E^f)$ can hence be seen as the *mesh* of all such renewal processes. By mesh, we mean that the renewal process of n merges with that of m at the first time when the orbits $\{f^p(n)\}_{n\in\mathbb{N}}$ and $\{f^q(m)\}_{m\in\mathbb{N}}$ meet.

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Section 2 is dedicated to the study of the population process. Section 3 investigates the family tree. Section 4 contains our final remarks.

2. Population and queue dynamics

In this section we study the population process

$$N_n = \#\{\text{all } m \in \mathbb{Z} \text{ such that } m < n \text{ and } f(m) > n\} + 1, \quad n \in \mathbb{Z}. \tag{2.1}$$

Section 2.1 introduces our definitions and notation. In Section 2.2 we show that the population process is regenerative with independent cycles. In Section 2.3 an explicit formula for the moment-generating function of N_0 is given and it is indicated that this random variable is always light-tailed, regardless of the distribution of a_0. Finally, in Section 2.4 we work out the case in which a_0 follows a geometric distribution and the population process is consequently Markovian. The notions and proof techniques in this section are classical and details are kept to a minimum.

2.1. Definitions and assumptions

Let $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_n\}_{n\in\mathbb{Z}})$ be the underlying probability space supporting all random elements discussed in this paper, endowed with a discrete flow. We assume P is preserved by θ_n, i.e. for all n,

$$\mathrm{P} \circ \theta_n^{-1} = \mathrm{P}. \tag{2.2}$$

A random integer-valued discrete sequence $\boldsymbol{W} = \{W_n\}_{n\in\mathbb{Z}}$ defined on Ω is compatible with the flow $\{\theta_n\}_{n\in\mathbb{Z}}$ if $W_n(\omega) = W_0(\theta_n\omega)$ for all $n \in \mathbb{Z}$. Notice that, given (2.2), if a process $\boldsymbol{W}$ is compatible with $\{\theta_n\}_{n\in\mathbb{Z}}$ then it is strictly stationary. All integer-valued discrete sequences considered here are $\{\theta_n\}_{n\in\mathbb{Z}}$-compatible (or, for short, stationary).

In particular, since $\boldsymbol{a} := \{a_n\}_{n\in\mathbb{Z}}$ is stationary, so is $\boldsymbol{N} := \{N_n\}_{n\in\mathbb{Z}}$, assuming the population process starts at $-\infty$.

Consider a stationary integer-valued discrete sequence $\{U_n\}_{n\in\mathbb{Z}}$ in which U_n equals 0 or 1. Let $\{k_n\}_{n\in\mathbb{Z}}$, with

$$\cdots < k_{-1} < k_0 \le 0 < k_1 < k_2 < \cdots, \tag{2.3}$$

be the sequence of times at which $U_n = 1$. A simple stationary point process (henceforth *s.s.p.p.*) on $\mathbb{Z}$ is then a random counting measure $\Phi(\cdot) = \sum_{n\in\mathbb{Z}} \delta_{k_n}(\cdot)$, where $\delta_{k_n}(\cdot)$ is the Dirac measure at k_n. We often identify Φ with the sequence $\{k_n\}_{n\in\mathbb{Z}}$, writing $k_n \in \Phi$, whenever $\Phi(\{k_n\}) = 1$.

The intensity of a *s.s.p.p.*, denoted by λ_Φ, is given by $\mathrm{P}[0 \in \Phi]$.

Let Φ be a *s.s.p.p.* on $\mathbb{Z}$ such that $\lambda_\Phi > 0$ and let $\mathrm{P}_\Phi[\cdot] := \mathrm{P}[\cdot \,|0 \in \Phi]$. Then P_Φ is the *Palm probability* of Φ. We denote by E_Φ the expectation operator of P_Φ. Consider the operator θ_{k_1}, i.e.

$$\Phi \circ \theta_{k_1} := \{k_{n+1} - k_1\}_{n\in\mathbb{Z}}.$$

Since θ_{k_1} is a bijective map on $\Omega_0 := \{k_0(\omega) = 0\}$, the following holds [5].

Lemma 2.1. *The operator θ_{k_1} preserves the Palm probability P_Φ.*

A *s.s.p.p.* Ψ on $\mathbb{Z}$ is an *integer-valued renewal process* if $\{k_n - k_{n-1}\}_{n\in\mathbb{Z}}$ is i.i.d. under P_Ψ.

Finally, we say that a stationary process $\boldsymbol{R} := \{R_n\}_{n\in\mathbb{Z}}$ is *regenerative* if there exists an integer-valued renewal process $\Psi = \{k_n\}_{n\in\mathbb{Z}}$ such that under P_Ψ $(\{k_n - k_{n-1}\}_{n>j}, \{R_n\}_{n\geq k_j})$ is independent of $\{k_n - k_{n-1}\}_{n\leq j}$ for all $j \in \mathbb{Z}$ and its distribution does not depend on j. Moreover, $\boldsymbol{R}$ is *regenerative with independent cycles* if $\boldsymbol{R}$ is regenerative and $\{R_n\}_{n<0}$ is independent of $\{R_n\}_{n\geq 0}$ under P_Ψ. We call $\{k_n\}_{n\in\mathbb{Z}}$ the regeneration epochs of $\boldsymbol{R}$.

For most of this work, unless otherwise stated, we assume $\{a_n\}_{n\in\mathbb{Z}}$ is i.i.d., $\mathrm{E}[a_0] < \infty$, and $\mathrm{P}[a_0 = 1] > 0$.

Remark 2.1. Since randomness in this model comes from the sequence $\{a_n\}_{n\in\mathbb{Z}}$ and θ_1 preserves P, by assuming $\{a_n\}_{n\in\mathbb{Z}}$ is i.i.d. there is no loss of generality in assuming that $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_n\}_{n\in\mathbb{Z}})$ is strongly mixing, i.e.

$$\lim_{n\to\infty} \mathrm{P}[A \cap \theta_{\pm n} B] = \mathrm{P}[A]\,\mathrm{P}[B] \quad \text{forall } A, B \in \mathcal{F}.$$

2.2. General case: main results

Theorem 2.1. *Let* $\Psi^{\mathrm{o}} := \{m \in \mathbb{Z} : N_m = 1\}$*. Then,* Ψ^{o} *is an integer-valued renewal process with intensity* $\lambda^{\mathrm{o}} := \lambda_{\Psi^{\mathrm{o}}} = \prod_{i=1}^{\infty} \mathrm{P}[a_0 \leq i] > 0$.

The atoms of Ψ^{o} are called *original ancestors* as all individuals born after any of them are necessarily its descendants in the family tree T^f studied in Section 3.

Proof of Theorem 2.1. By (2.1), $\mathrm{P}[N_0 = 1] = \mathrm{P}[\cap_{j=1}^{\infty} a_{-j} \leq j]$. Then, as the sequence $\{a_n\}_{n\in\mathbb{Z}}$ is i.i.d.,

$$\mathrm{P}[N_0 = 1] = \prod_{j=1}^{\infty} \mathrm{P}[a_0 \leq j].$$

Since $\mathrm{P}[a_0 = 1] > 0$, none of the elements of the above product equals 0. Then, as $\mathrm{E}[a_0] < \infty$, $\prod_{j=1}^{\infty} \mathrm{P}[a_0 \leq j] = \mathrm{P}[N_0 = 1] > 0$ (see Appendix A, Lemma A.1). The stationarity of N implies that, for all $n \in \mathbb{Z}$, $\mathrm{P}[N_n = 1] = \mathrm{P}[N_0 = 1] > 0$.

Since $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_n\}_{n\in\mathbb{Z}})$ is strongly mixing (Remark 2.1), it is ergodic. Therefore, by Birkhoff's pointwise ergodic theorem, for all measurable functions $g : \Omega \to \mathbb{R}^+$ such that $\mathrm{E}[g] < \infty$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \sum_{i=1}^{\pm n} g \circ \theta_{\pm i} = \mathrm{E}[g] \quad \text{P-a.s.}$$

Let $g = \mathbf{1}\{N_0 = 1\}$. Then

$$\lim_{n\to\pm\infty} \frac{1}{n} \sum_{i=1}^{\pm n} \mathbf{1}\{N_0 = 1\} \circ \theta_{\pm i} = \mathrm{P}[N_0 = 1] > 0 \quad \text{P-a.s.}$$

Hence, there exists a.s. a subsequence of distinct integers $\Psi^{\mathrm{o}} := \{k_n^{\mathrm{o}}\}_{n\in\mathbb{Z}}$ satisfying (2.3) such that $N_{k_n^{\mathrm{o}}} = 1$ for all $n \in \mathbb{Z}$. Since $\{a_n\}_{n\in\mathbb{Z}}$ is stationary, Ψ^{o} is a *s.s.p.p.* That Ψ^{o} is a renewal process is proved in Appendix A, Proposition A.1. □

In order to show that N is a stationary regenerative process with respect to Ψ^{o} with independent cycles, we rely on the following lemma.

Lemma 2.2. *Under* $\mathrm{P}_{\Psi^{\mathrm{o}}}$, $\{a_n\}_{n<0}$ *is independent of* $\{a_n\}_{n\geq 0}$.

Proof. Let $g, f : \Omega \to \mathbb{R}^+$ be two measurable, continuous, bounded functions. Using the fact that the event $\{k_0^{\mathrm{o}} = 0\}$ is equal, by definition, to $\{N_0 = 1\} = \{\cap_{i>0} a_{-i} \leq i\}$, we have

$$\begin{aligned}\mathrm{E}_{\Psi^{\mathrm{o}}}[g(\{a_n\}_{n<0}) f(\{a_n\}_{n\geq 0})] &= \mathrm{E}\,[g(\{a_n\}_{n<0}) f(\{a_n\}_{n\geq 0}) \mid k_0^{\mathrm{o}} = 0] \\ &= \mathrm{E}\,[g(\{a_n\}_{n<0}) f(\{a_n\}_{n\geq 0}) \mid \cap_{i>0}\, a_{-i} \leq i].\end{aligned}$$

Now, as $\{a_n\}_{n\geq 0}$ is independent of $\{a_n\}_{n<0}$ under P,

$$\begin{aligned}&\mathrm{E}\,[g(\{a_n\}_{n<0}) f(\{a_n\}_{n\geq 0}) \mid \cap_{i>0}\, a_{-i} \leq i] \\ &= \mathrm{E}\,[g(\{a_n\}_{n<0}) \mid \cap_{i>0}\, a_{-i} \leq i]\, \mathrm{E}\,[f(\{a_n\}_{n\geq 0}) \mid \cap_{i>0}\, a_{-i} \leq i] \\ &= \mathrm{E}_{\Psi^{\mathrm{o}}}[g(\{a_n\}_{n<0})]\, \mathrm{E}_{\Psi^{\mathrm{o}}}[f(\{a_n\}_{n\geq 0})],\end{aligned}$$

completing the proof. □

Corollary 2.1. *The population process* $\boldsymbol{N}$ *is a stationary regenerative process with respect to* Ψ^{o} *with independent cycles.*

Proof. Given $k_0^{\mathrm{o}} = 0$, i.e. under $\mathrm{P}_{\Psi^{\mathrm{o}}}$, we have $N_0 = 1$. Then $(\{k_n^{\mathrm{o}} - k_{n-1}^{\mathrm{o}}\}_{n>0}, \{N_n\}_{n\geq 0})$ is a function of $\{a_n\}_{n\geq 0}$, while $\{k_n^{\mathrm{o}}\}_{n\leq 0}$ is a function of $\{a_n\}_{n<0}$. Then, by Lemma 2.2, $(\{k_n - k_{n-1}\}_{n>0}, \{N_n\}_{n\geq 0})$ is independent of $\{k_n - k_{n-1}\}_{n\leq 0}$. By the same reasoning as in Lemma 2.2, $\{a_n\}_{n\geq k_j}$ is independent of $\{a_n\}_{n<k_j}$ for all $j \in \mathbb{Z}$. It follows that $(\{k_n - k_{n-1}\}_{n>j}, \{N_n\}_{n\geq k_j})$ is independent of $\{k_n - k_{n-1}\}_{n\leq j}$ for all j. We conclude that $\{N_n\}_{n\in\mathbb{Z}}$ is a regenerative process.

The independence of the random vectors $\{a_n\}_{k_j<n\leq k_{j+1}}$ for $j \in \mathbb{Z}$, which holds by the same reasoning used in Lemma 2.2, implies that $\{N_n\}_{n\in\mathbb{Z}}$ is regenerative with independent cycles. □

Remark 2.2. When we do not assume that $\mathrm{P}\,[a_0 = 1] > 0$, let $\underline{m} > 1$ be the smallest integer such that $\mathrm{P}\,[a_0 = \underline{m}] > 0$. Then $\mathrm{P}\,[N_0 < \underline{m}] = 0$, while $\mathrm{P}\,[N_0 = \underline{m}] = \prod_{j=\underline{m}}^{\infty} \mathrm{P}^0[a_0 \leq j] > 0$. Proceeding as in the proof of Theorem 2.1, we conclude that there exists an integer-valued renewal process $\tilde{\Phi}$ such that $k_n \in \tilde{\Phi}$ if and only if $N_{k_n} = \underline{m}$.

2.3. Analytical properties of N_n

We now turn to some analytical properties of $\boldsymbol{N}$. Some of our results are adapted from the literature on GI/GI/∞ queues, i.e. queues with an infinite number of servers, and independently distributed arrival and service times. Proofs can be found in Appendix A. The results in this subsection do not depend on the assumption $\mathrm{P}\,[a_0 = 1] > 0$.

The following proposition can be extended to the case in which $\{a_n\}_{n\in\mathbb{Z}}$ is stationary rather than just i.i.d.

Proposition 2.1. *For all* $n \in \mathbb{Z}$, $\mathrm{E}\,[a_0] = \mathrm{E}\,[N_n]$.

Proposition 2.1 is Little's law for the infinite server queue, i.e. the expected number of individuals being served equals the arrival rate multiplied by the expected service time given that there is an arrival at the origin a.s. (see e.g. [2]). This shows that the mean number of customers in steady state is finite if and only if the expected service time is finite.

A general formula for the moment-generating function of the number of customers in steady state in a GI/GI/∞ queue can be found in [9]. We limit ourselves to showing that N_0 is a light-tailed random variable regardless of the distribution of a_0. The latter property has the following

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intuitive basis: N_0 is large only if several realizations of $\{a_n\}_{n<0}$ are large as well. Another way to get an intuition for this result is if we let the arrival times be exponentially distributed with parameter λ. Then one can show that N_0 has a Poisson distribution with parameter $\lambda \mathrm{E}\,[a_0]$, and it is light-tailed regardless of the tail of a_0.

Proposition 2.2. *The moment-generating function of* N_0 *is given by*

$$\mathrm{E}\,[\mathrm{e}^{tN_0}] = \mathrm{e}^t \prod_{i=1}^{\infty} (\mathrm{e}^t \mathrm{P}\,[a_0 > i] + \mathrm{P}\,[a_0 \le i]) \quad \textit{for all } t \in \mathbb{R}. \tag{2.4}$$

Moreover, $\mathrm{E}\,[\mathrm{e}^{tN_0}] < \infty$ *for all* $t \in \mathbb{R}$.

2.4. Geometric marks

In general, the process $\boldsymbol{N}$ is not Markovian. However, it is when the marks are geometrically distributed. Let us assume a_0 is geometrically distributed supported on $\mathbb{N}_+$ with parameter s. Let $r = 1 - s$. Then, due to the memoryless property, $\boldsymbol{N}$ is a time-homogeneous, aperiodic, irreducible Markov chain with state space $\{1, 2, 3, \ldots\}$, whose transition matrix is given by

$$\mathrm{P}\,[N_n = n | N_{n-1} = k] = \binom{k}{n-1} r^{n-1} s^{k-n+1}, \qquad 1 \le n \le k+1. \tag{2.5}$$

Proposition 2.3. *In the geometric case, the probability-generating function of* N_n *at steady state observes the functional relation*

$$G(z) = zG(sz + r). \tag{2.6}$$

Proof. Let $\{\tilde{N}_n\}_{n\ge 0}$ be the population process starting at 0 and $G_m(z)$, $z \in [0, 1]$, the probability-generating function of $\tilde{N}_m$. Using (2.5),

$$\mathrm{E}\,[z^{\tilde{N}_n} \mid \tilde{N}_{n-1} = k] = \sum_{n=1}^{k+1} \binom{k}{n-1} r^{n-1} s^{k-n+1} z^n = z \sum_{n'=0}^{k} \binom{k}{n'} (rz)^{n'} s^{k-n'},$$

where $n' = n - 1$. So,

$$\mathrm{E}\,[z^{\tilde{N}_n} \mid \tilde{N}_{n-1} = k] = z(rz + s)^k.$$

Hence,

$$G_m(z) = \sum_{k=1}^{m} \mathrm{P}\,[\tilde{N}_{m-1} = k](z(rz + s)^k) = zG_{m-1}(rz + s).$$

Letting $m \to \infty$ we get (2.6). □

Using (2.6) we can easily compute the moments of N_0. For example, by differentiating both sides and setting $z = 1$ we get

$$\mathrm{E}\,[N_0] = \frac{1}{s},$$

which is the mean of a_0, as expected. Proceeding the same way, the second moment is given by

$$\mathrm{E}\,[N_0^2] = \frac{2r}{s(1 - r^2)}.$$

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Remark 2.3. By (2.4), and noticing that $\mathrm{P}[a_i > i] = r^i$, we get

$$G(z) = z \prod_{i=1}^{\infty} (zr^i + 1 - r^i).$$

By Lemma A.1 in Appendix A, $\prod_{i=1}^{\infty} (zr^i + 1 - r^i)$ converges for all $z \in \mathbb{R}$ as $\sum_{i=1}^{\infty} r^i < \infty$.

3. The eternal family tree

In this section we study the directed graph $T^f = (V^f, G^f)$, where $V^f = \mathbb{Z}$ and $E^f = \{(n, f(n)) : n \in \mathbb{Z}\}$. In Section 3.1 we show that T^f is an infinite tree containing a unique bi-infinite path. The indexes of the nodes on this bi-infinite path form a *s.s.p.p.* on $\mathbb{Z}$ with positive intensity. We derive this result by exploiting the fact that T^f is an *eternal family tree*, i.e. the out-degrees of all vertices are exactly one [3].

In the following two sections we delve deeper into the genealogy of T^f. In Section 3.2 we give the basic properties of certain *s.s.p.p.*s derived from T^f. First, the process of integers forming the bi-infinite path: each integer in it is called *successful*, since its lineage (the set of its descendants) has infinite cardinality a.s. Second, we consider the process coming from the complement of the bi-infinite path: each integer in it is called *ephemeral*, since its lineage is finite a.s. Third, we consider the process of original ancestors defined in Theorem 2.1, which is a subprocess of the first.

In Section 3.3 we look at the set of direct ephemeral descendants of a successful integer n, whose path to n on T^f consists only of ephemerals. The conservation law that defines unimodular networks allows us to establish probabilistic properties of the set of direct ephemeral descendants and the set of cousins of a typical successful node.

3.1. The global properties of T^f

Theorem 3.1. *The directed random graph T^f is a tree with a unique bi-infinite path for which the corresponding nodes, when mapped to $\mathbb{Z}$, form a s.s.p.p. with positive intensity.*

In order to prove Theorem 3.1 we resort to recent results on dynamics on unimodular networks [3]. In Appendix B we present a brief review of definitions and properties of unimodular networks that we use. First, we notice that the directed graph $G = (V, E)$, where $V = \mathbb{Z}$ and $E = \{(n, n+1) : n \in \mathbb{Z}\}$, rooted at 0, in which each node n is assigned a mark a_n, is a locally finite unimodular random network. Second, we notice that f is a translation-invariant dynamics on this network; more precisely, it is a *covariant vertex-shift* (see Appendix B, Definition B.2).

Define the *connected component* of n as

$$C(n) = \{m \in \mathbb{Z} \text{ s.t. } \exists\, i, j \in \mathbb{N} \text{ with } f^i(n) = f^j(m)\}.$$

Proposition 3.1. *The directed random graph T^f has only one connected component.*

Proof. Consider the process of original ancestors, Ψ^o as defined in Theorem 2.1, and let $m \in \Psi^o$. Then, for every $n < m$, $n \in C(m)$. Hence, the result follows from the fact that Ψ^o is a *s.s.p.p.* consisting of an infinite number of points a.s. □

Let

$$D(n) = \{m \in \mathbb{Z} \text{ s.t. } \exists \quad j \in \mathbb{N} \text{ with } f^i(m) = n\} \tag{3.1}$$

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denote the set of *descendants* of n. Also let

$$L(n) = \{m \in \mathbb{Z} \text{ s.t. } \exists\, j \in \mathbb{N} \text{ with } f^j(m) = f^j(n)\}$$

denote the set of *cousins* of n of all degrees (this set is referred to as the *foil* of n in [3]).

We further subdivide $D(n)$ and $L(n)$ in terms:

$$D_i(n) = \{m \in \mathbb{Z} \text{ s.t. } f^i(m) = n\}, \quad i \geq 0$$

and

$$L_i(n) = \{m \in \mathbb{Z} \text{ s.t. } f^i(m) = f^i(n)\}, \quad i \geq 0.$$

So $D_i(n)$ is the set of descendants of degree i of n and $L_i(n)$ the set of cousins of degree i of n.

Lower-case letters denote the cardinalities of the above sets. So $c(n)$ is the cardinality of $C(n)$, $d_i(n)$ is the cardinality of $D_i(n)$, and so on. Moreover, $d_\infty(n)$ denotes the weak limit of $d_i(n)$ (if such a limit exists).

It is shown in [3] that each connected component of a graph generated by the action of a covariant vertex-shift on a unimodular network, $C(n)$, falls within one of the following three categories:

Class **F/F**: $c(n) < \infty$ and for all $v \in C(n)$, $l(v) < \infty$. In this case $C(n)$ has a unique cycle.

Class **I/F**: $c(n) = \infty$ and for all $v \in C(n)$, $l(v) < \infty$. In this case, $C(n)$ is a tree containing a unique bi-infinite path. Moreover, the bi-infinite path forms a *s.s.p.p.* on $\mathbb{Z}$ with positive intensity.

Class **I/I**: $c(n) = \infty$ and for all $v \in C(n)$, $l(v) = \infty$. In this case, $C(n)$ is a one-ended tree such that $d_\infty(v) = 0$ for all $v \in C(n)$.

Notice that the dynamics f precludes the connected component $\mathbb{Z}$ of being of class **F/F**. Theorem 3.1 follows from proving that, in our case, $C(0)$ is of class **I/F**. We rely on the following lemma derived from the results found in [3].

Lemma 3.1. *A connected component $C(m)$ is of class **I/I** if and only if for all $n \in C(m)$,* $\mathrm{P}[d(n) = \infty] = 0$.

Proof of Theorem 3.1. For $k \in \Psi^o$, we have $d(k) = \infty$ a.s., and consequently $C(k)$ is of class **I/F**. Since there is a unique component, the result follows. □

The next proposition concerns the case in which it is not necessarily assumed that $\mathrm{P}[a_0 = 1] > 0$.

Proposition 3.2. *Let $d = gcd\{n \in \mathbb{Z} : \mathrm{P}[a_n > 0] > 0\}$, i.e. assume that the distribution of a_0 has period d. Then T^f has d connected components. Each connected component is a tree with a unique bi-infinite path. Hence, when $d > 1$, T^f is a forest.*

Proof. First we deal with the aperiodic case, i.e. $d = 1$. Consider the processes $\{\boldsymbol{V}^{(m)}\}_{m\in\mathbb{Z}}$ in which

$$v_n^{(m)} = \begin{cases} 1 & \text{if there exists } j \geq 0 \text{ s.t. } f^j(m) = n, \\ 0 & \text{otherwise.} \end{cases}$$

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Since, for $j>0$, $f^j(m)=a_{f^{j-1}(m)}+f^{j-1}(m)$, we have, for all $m\in\mathbb{Z}$, that $\boldsymbol{V}^{(m)}$ is a renewal process starting from m, in which the distribution of the inter-arrivals is the same as the distribution of a_0. Let

$$\tau_{(0,m)}=\inf\{n\geq 0: v_n^{(m)}=v_n^{(0)}=1\}.$$

Then $m\in C(0)$ if and only if $\tau_{(0,m)}<\infty$ a.s.

Now let $\boldsymbol{U}^{(0)}$ and $\boldsymbol{U}^{(m)}$ be two independent renewal processes, which are also independent of $\{\boldsymbol{V}^{(m)}\}_{m\in\mathbb{Z}}$. We assume the first arrival of $\boldsymbol{U}^{(m)}$ happens at m and that of $\boldsymbol{U}^{(0)}$ happens at 0. The distributions of the inter-arrivals of $\boldsymbol{U}^{(0)}$ and $\boldsymbol{U}^{(m)}$ are the same as the distribution of a_0.

More precisely, let $\{a'_n\}_{n\geq 0}$ and $\{a''_n\}_{n\geq 0}$ be two independent sequences of i.i.d. random variables, also independent of $\{a_n\}_{n\in\mathbb{Z}}$, in which $a'_0\overset{\text{D}}{=}a''_0\overset{\text{D}}{=}a_0$. Then $\boldsymbol{U}^{(0)}=\{u_n^{(0)}\}_{n\geq 0}$ and $\boldsymbol{U}^{(m)}=\{u_n^{(m)}\}_{n\geq m}$, where

$$u_n^{(0)}=\begin{cases}1 & \text{if there exists } j\geq 0 \text{ s.t. } \sum_{i=0}^{j}a'_i=n,\\ 0 & \text{otherwise}\end{cases}$$

and

$$u_n^{(m)}=\begin{cases}1 & \text{if there exists } j\geq 0 \text{ s.t. } m+\sum_{i=0}^{j}a''_i=n,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that $\boldsymbol{V}^{(0)}\overset{\text{D}}{=}\boldsymbol{U}^{(0)}$ and $\boldsymbol{V}^{(m)}\overset{\text{D}}{=}\boldsymbol{U}^{(m)}$.

Define $\sigma_{(0,m)}=\inf\{n\geq 0: u_n^{(m)}=u_n^{(0)}=1\}$. We will show that $\sigma_{(0,m)}\overset{\text{D}}{=}\tau_{(0,m)}$. For any $n>0$ and $p,q\in\mathbb{N}$, let

$$\begin{aligned}\mathcal{A}(p,q;n)=\{&(k_0,\dots,k_p)\in\mathbb{N}^p,(l_0,\dots,l_p)\in\mathbb{N}^q:0=k_0<k_1<\cdots<k_p=n,\\ &m=l_0<l_1<cdots<l_q=n,\ \text{ and } \{k_1,\dots,k_{p-1}\}\cap\{l_1,\dots,l_{q-1}\}=\emptyset\}.\end{aligned}$$

Then

$$\{\tau_{(0,m)}=n\}=\bigcup_{p,q\in\mathbb{N}}\ \bigcup_{\substack{(k_0,\dots,k_p,l_0,\dots,l_q)\\ \in\mathcal{A}(p,q;n)}}\ \bigcap_{\substack{i\in\{1,\dots,p\}\\ j\in\{1,\dots,q\}}}\{f^i(0)=k_i,f^j(m)=l_j\}$$

and

$$\{\sigma_{(0,m)}=n\}=\bigcup_{p,q\in\mathbb{N}}\ \bigcup_{\substack{(k_0,\dots,k_p,l_0,\dots,l_q)\\ \in\mathcal{A}(p,q;n)}}\ \bigcap_{\substack{i\in\{1,\dots,p\}\\ j\in\{1,\dots,q\}}}\{a'_{i-1}=k_i-k_j,a''_{j-1}=l_j-l_{j-1}\}.$$

By the definition of f, and since $\{a_n\}_{n\in\mathbb{Z}}$ is i.i.d., we have

$$\begin{aligned}&\mathrm{P}[\tau_{(0,m)}=n]\\ &=\sum_{p,q\in\mathbb{N}}\ \sum_{\substack{(k_0,\dots,k_p,l_0,\dots,l_q)\\ \in\mathcal{A}(p,q;n)}}\ \prod_{\substack{i\in\{1,\dots,p\}\\ j\in\{1,\dots,q\}}}\mathrm{P}[a_{k_{i-1}}=k_i-k_{i-1},a_{l_{j-1}}=l_j-l_{j-1}]\\ &=\sum_{p,q\in\mathbb{N}}\ \sum_{\substack{(k_0,\dots,k_p,l_0,\dots,l_q)\\ \in\mathcal{A}(p,q;n)}}\ \prod_{\substack{i\in\{1,\dots,p\}\\ j\in\{1,\dots,q\}}}\mathrm{P}[a_0=k_i-k_{i-1}]\,\mathrm{P}[a_0=l_j-l_{j-1}].\end{aligned}\tag{3.2}$$

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Similarly,

$$
\begin{aligned}
&\mathrm{P}\,[\sigma_{(0,m)} = n] \\
&= \sum_{p,q\in\mathbb{N}} \sum_{\substack{(k_0,\ldots,k_p,l_0,\ldots,l_q)\\ \in\mathcal{A}(p,q;n)}} \prod_{\substack{i\in\{1,\ldots,p\}\\ j\in\{1,\ldots,q\}}} \mathrm{P}\,[a'_{i-1} = k_i - k_{i-1}]\,\mathrm{P}\,[a''_{j-1} = l_j - l_{j-1}] \\
&= \sum_{p,q\in\mathbb{N}} \sum_{\substack{(k_0,\ldots,k_p,l_0,\ldots,l_q)\\ \in\mathcal{A}(p,q;n)}} \prod_{\substack{i\in\{1,\ldots,p\}\\ j\in\{1,\ldots,q\}}} \mathrm{P}\,[a_0 = k_i - k_{i-1}]\,\mathrm{P}\,[a_0 = l_j - l_{j-1}].
\end{aligned} \tag{3.3}
$$

From (3.2) and (3.3), we conclude that for all $n \geq 0$, $\mathrm{P}\,[\sigma_{(0,m)} = n] = \mathrm{P}\,[\tau_{(0,m)} = n]$. Therefore, $\tau_{(0,m)} \overset{\mathrm{D}}{=} \sigma_{(0,m)}$.

It is well known that $\mathrm{P}\,[\sigma_{(0,m)} < \infty] = 1$ (see e.g. [7]). This result is an incarnation of Doeblin's coupling for independent discrete renewal processes. Therefore $\mathrm{P}\,[\tau_{(0,m)} < \infty] = 1$ as well, so $m \in C(0)$ a.s. As this result holds for any $m \in \mathbb{Z}$, there exists a unique connected component.

Next we will show that the unique connected component is of class $\mathbf{I}/\mathbf{F}$, so T^f is a tree with a unique bi-infinite path. From Proposition 2.1 we know $N_0 < \infty$. Assume $N_0 = q$ and let $\{n_1, \ldots, n_q\}$ be the set of individuals alive at time 0. This set of individuals must have, jointly, an infinite number of descendants, i.e. $\sum_{i=1}^{q} d(n_i) = \infty$. If not, since there is a unique connected component, we would have $N_0 = \infty$, a contradiction. Consequently, there exists at least one individual alive at time 0 having an infinite number of descendants. We conclude, invoking Proposition 3.1, that $C(0)$ is of class $\mathbf{I}/\mathbf{F}$.

When $d > 1$, $C(i) = i + d\mathbb{Z}$ for $i \in \{0, \ldots, d-1\}$, so there are d connected components. That $C(i) \subset i + d\mathbb{Z}$ follows from the periodicity of a_0. For any $m \in i + d\mathbb{Z}$, the coupling argument used in the aperiodic case applies, so $m \in C(i)$. Therefore $i + d\mathbb{Z} \subset C(i)$ for $i \in \{0, \ldots, d-1\}$.

Since (i) there is at least one individual alive at time 0 which belongs to $d\mathbb{Z}$ (namely, individual 0), (ii) $N_0 < \infty$, and (iii) the number of connected components is finite, we conclude that there is at least one individual alive at time 0 which both belongs to $d\mathbb{Z}$ and has an infinite number of descendants in $d\mathbb{Z}$. It then follows that $C(0)$ is of class $\mathbf{I}/\mathbf{F}$, again from Proposition 3.1. The same reasoning implies that $C(i)$ is of class $\mathbf{I}/\mathbf{F}$ for all $i \in \{1, \ldots, d-1\}$, completing the proof. □

Definition 3.1. (*Diagonally invariant functions.*) A measurable function $h : \Omega \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is said to be diagonally invariant if $h(\theta_k(\omega), m, n) = h(\omega, m+k, n+k)$ for all $k, m, n \in \mathbb{Z}$.

We notice that T^f, rooted at 0, is itself a unimodular network [3]. Unimodularity is characterized by the fact that such a network satisfies the *mass transport principle* (see Appendix B). In our setting, the mass transport principle takes the following form: for all diagonally invariant functions h (Definition 3.1),

$$\mathrm{E}\Bigg[\sum_{n\in\mathbb{Z}} h(n, 0)\Bigg] = \mathrm{E}\Bigg[\sum_{n\in\mathbb{Z}} h(0, n)\Bigg].$$

3.2. Successful and ephemeral individuals, and original ancestors

From the analysis of the population process and the shape of T^f, we learned that f defines three *s.s.p.p.*s on $\mathbb{Z}$ related to the genealogy of T^f:

Φ^{s}: the set of successful individuals, consisting of individuals $n \in \mathbb{Z}$ having an infinite number of descendants in T^f.

Ψ^{o}: the set of original ancestors (defined in Section 2), consisting of all individuals $n \in \mathbb{Z}$ such that for all $m < n$, m is a descendant of n in T^f. Clearly, $\Psi^{\text{o}} \subset \Phi^{\text{s}}$.

Φ^{e}: the set of ephemeral individuals, consisting of individuals $n \in \mathbb{Z}$ which have a finite number of descendants in T^f. Clearly, $\Phi^{\text{e}} \cup \Phi^{\text{s}} = \mathbb{Z}$.

We now look at the basic properties of these processes. In what follows we let $\text{E}^{\text{s}} := \text{E}_{\Phi^{\text{s}}}$, i.e. E^{s} is the expectation operator of the Palm probability of Φ^s. In the same vein, $\text{E}^{\text{e}} := \text{E}_{\Phi^{\text{e}}}$ and $\text{E}^{\text{o}} := \text{E}_{\Psi^{\text{o}}}$.

Proposition 3.3. *Let λ^{s} be the intensity of Φ^{s}. Then*

$$\lambda^{\text{s}} = \frac{1}{\text{E}\,[a_0]}.$$

It follows that the intensity of Φ^{e}, λ^{e}, equals $1 - \frac{1}{\text{E}\,[a_0]}$.

Proof. Let $S_0 = 0$ and $S_n = a_1 + a_2 + \cdots + a_n$. Consider the associated renewal sequence $\boldsymbol{U} = \{u_k\}_{k \geq 0}$ defined as

$$u_k = \text{P}\,[S_n = k \text{ for some } n \geq 0],$$

so u_k is the probability that k is hit at some renewal epoch S_n. Since the distribution of $\{a_n\}_{n \in \mathbb{Z}}$ is aperiodic, as $\text{P}\,[a_0 = 1] > 0$, $u_k \to \frac{1}{\text{E}\,[a_0]}$ as $k \to \infty$, P-a.s. As $\lim_{k \to \infty} u_k = \text{P}\,[0 \in \Psi^{\text{s}}] = \lambda^{\text{s}}$ and $\lambda^{\text{s}} + \lambda^{\text{e}} = 1$, the result follows. □

Remark 3.1. As $\prod_{i=1}^{\infty} \text{P}\,[a_0 \leq i] \leq \frac{1}{\text{E}\,[a_0]}$, $\lambda^{\text{o}} \leq \lambda^{\text{s}}$, as expected.

We know from [3] that $\text{E}\,[d_n(0)] = 1$, i.e. the expected number of descendants of all degrees of a typical integer is one. This result follows from the mass transport principle. The process of successful individuals is locally supercritical, while the process of ephemeral individuals is locally subcritical, as the next proposition shows.

Proposition 3.4. *Assume* $\text{P}\,[a_0 = 1] \in (0, 1)$. *Then, for all* $n \geq 1$, $\text{E}^{\text{s}}[d_n(0)] > 1$, *while* $\text{E}^{\text{e}}[d_n(0)] < 1$.

Proof. By the law of total probability and the definition of P^{s} and P^{e},

$$\text{E}^{\text{s}}[d_n(0)]\lambda^{\text{s}} + \text{E}^{\text{e}}[d_n(0)]\lambda^{\text{e}} = \text{E}\,[d_n(0)] = 1.$$

Since a successful individual has at least one descendant of degree n a.s., $\text{P}\,[a_0 = 1] < 1$, and $\lambda^{\text{e}} = 1 - \lambda^{\text{s}}$, it follows that $\text{E}^{\text{s}}[d_n(0)] > 1$ and $\text{E}^{\text{e}}[d_n(0)] < 1$. □

3.3. Cousins and direct ephemeral descendants

Given a successful node n, we say that an ephemeral individual m is a *direct ephemeral descendant* of n if n is the first successful individual in the ancestry lineage of m. The set of direct ephemeral descendants of n is given by

$$D^{\text{e}}(n) = \{m \in D(n) \cap \Phi^{\text{e}} : \text{for the smallest } k > 0 \text{ s.t. } f^k(m) = n \ \in \Phi^{\text{s}}\},$$

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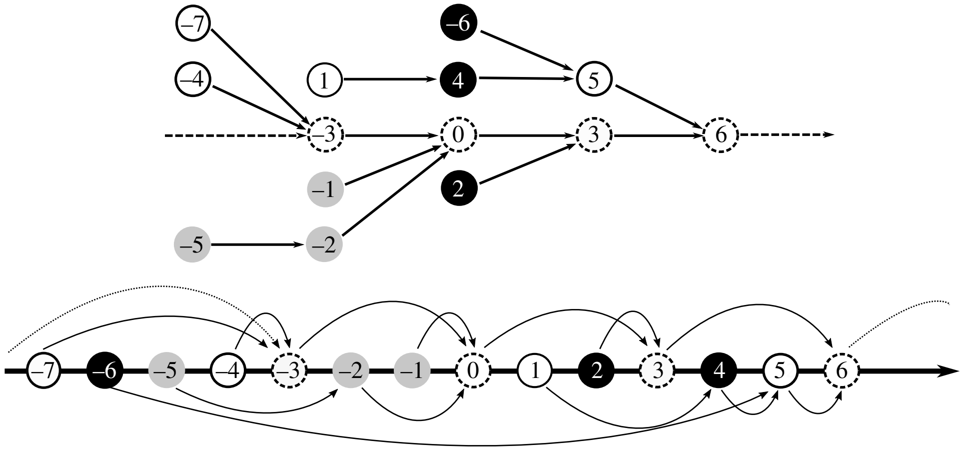

FIGURE 3: The direct ephemeral descendants and the cousins of 0. Above we have a realization of T^f and below the corresponding representation on $\mathbb{Z}$. Here, -3, 0, 3, and 6 belong to the bi-infinite path (denoted by discs with dashed boundaries). The grey discs represent the individuals that are the direct ephemeral descendants of 0, while the black ones are cousins. Individual 0 has two ephemeral children (nodes -1 and -2), one successful (node -3), and one ephemeral grandchild (node -5). It has a first-degree cousin (node 2) and two second-degree cousins (nodes -6 and 4). While any descendant of 0 must be to the left of it on $\mathbb{Z}$, cousins can be either to the left or the right.

where $D(n)$ is the set of descendants of all degrees of n (Equation (3.1)). By Theorem 3.1, the cardinality of $D^{\mathrm{e}}(n)$, denoted by $d^{\mathrm{e}}(n)$, is finite. Moreover, for $m \neq n \in \Psi^{\mathrm{s}}$, $D^{\mathrm{e}}(n) \cap D^{\mathrm{e}}(m) = \emptyset$.

Definition 3.2. (*Direct ephemeral descendants partition.*) Let

$$\tilde{D}^{\mathrm{e}}(n) := D^{\mathrm{e}}(n) \cup \{n\}$$

be the directed ephemeral tree rooted at $n \in \Phi^{\mathrm{s}}$. The *direct ephemeral descendant partition* is

$$\mathcal{P}^D := \{\tilde{D}^{\mathrm{e}}(n) : n \in \Phi^{\mathrm{s}}\}.$$

By convention, any individual n is a cousin of itself (i.e. n is a 0-degree cousin of itself). Moreover, if $m \neq n$ both belong to Ψ^{s}, then $L(n) \cap L(m) = \emptyset$, as either m is a descendant or an ancestor of n. In other words, for $n \in \Phi^{\mathrm{s}}$, $L(n) \backslash \{n\} \subset \Phi^{\mathrm{e}}$. Hence we get the following partition.

Definition 3.3. (*Successful cousin partition.*) The cousin partition of $\mathbb{Z}$ is

$$\mathcal{P}^L := \{L(n) : n \in \Phi^{\mathrm{s}}\}.$$

Figure 3 illustrates $\tilde{D}^{\mathrm{e}}(0)$ and $L(0)$. For all $n \in \Psi^{\mathrm{e}}$ and $j > 0$, let $d_j^{\mathrm{e}}(n) = \#\{d^{\mathrm{e}}(n) \cap D_j(n)\}$ be the number of directed ephemeral descendants of degree j of n. By construction, the following equality holds for all $j > 0$ P^{s}-a.s. (see Figure 4):

$$l_j(0) = d_j^{\mathrm{e}}(k_j^{\mathrm{s}}), \tag{3.4}$$

so that

$$l(0) = \sum_{j=1}^{\infty} d_j^{\mathrm{e}}(k_j^{\mathrm{s}}) + 1.$$

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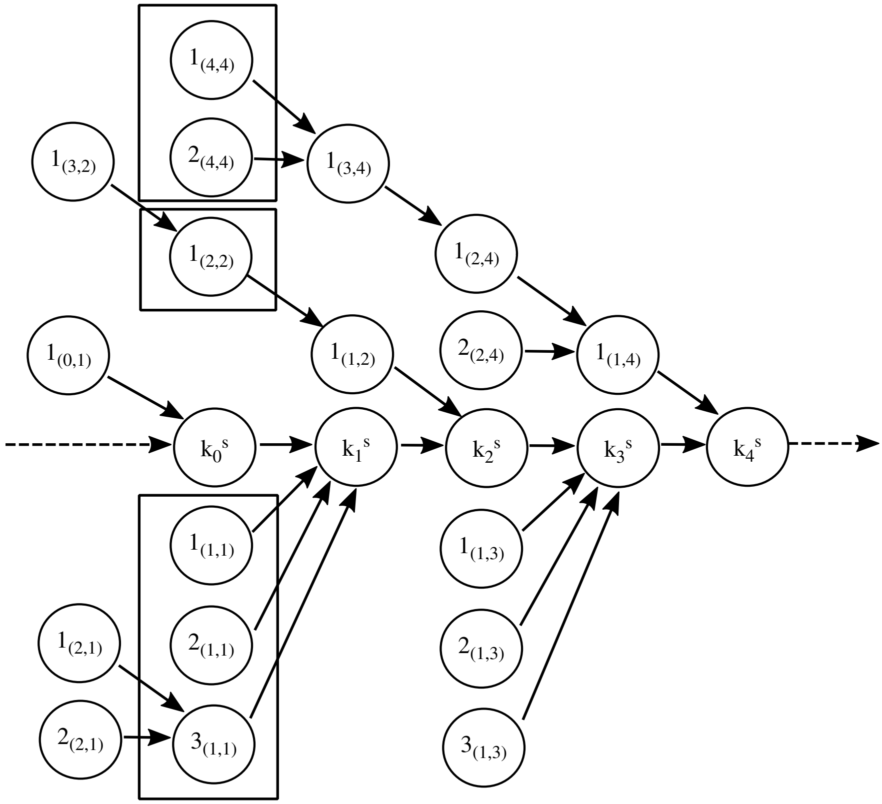

FIGURE 4: Cousins and the direct ephemeral descendants trees. The integers $\{k_i^{\mathrm{s}}\}_{i=1}^4$ represent the successful individuals. The notation $I_{(n,m)}$ reads 'individual I of the nth layer of the the direct ephemeral descendant tree (d.e.d.t.) of k_m^{s}'. For example, $2_{(2,4)}$ is the second individual in the second layer of the d.e.d.t. of k_4^{s}. Each box contains the cousins of k_0^{s} of different degrees. The lower box contains the first-degree cousins, the middle box contains the second-degree cousins, and the top box contains the fourth-degree cousins (there are no third-degree cousins). Equivalently, the lower box contains all elements of the first layer of the d.e.d.t. of k_1^{s}, the middle box contains all elements of the second layer of the d.e.d.t. of k_2^{s}, and the top box contains all elements of the fourth layer of the d.e.d.t. of k_4^{s}. As the d.e.d.t. of k_3^{s} has no third layer, k_0^{s} has no third-degree cousins.

Proposition 3.5. *For all* $j \geq 1$ *and* $q \geq 0$, $\mathrm{P}^{\mathrm{s}}[d_j^{\mathrm{e}}(0) = q] = \mathrm{P}^{\mathrm{s}}[l_j(0) = q]$.

Proof. From (3.4), for all $j \geq 1$ and $q \geq 0$,

$$\mathrm{P}^{\mathrm{s}}[l_j(0) = q] = \mathrm{P}^{\mathrm{s}}[d_j^{\mathrm{e}}(k_j^{\mathrm{s}}) = q]. \tag{3.5}$$

As $\theta_{k_j^{\mathrm{s}}}$ preserves P^{s},

$$\mathrm{P}^{\mathrm{s}}[d_j^{\mathrm{e}}(k_j^{\mathrm{s}}) = q] \circ \theta_{k_j^{\mathrm{s}}}^{-1} = \mathrm{P}^{\mathrm{s}}[d_j^{\mathrm{e}}(0) = q], \quad \text{for all } q \geq 1. \tag{3.6}$$

We then get the result by combining Equations (3.5) and (3.6). □

Proposition 3.6. *Given* $0 \in \Phi^{\mathrm{e}}$, *let* $n^{(d)}$ *be the unique random successful individual such that* $0 \in \tilde{D}^{\mathrm{e}}(n^{(d)})$. *In the same way, let* $n^{(l)}$ *be the unique successful individual such that* $0 \in L(n^{(l)})$.

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Then, for any diagonally invariant function g (Definition 3.1),

$$\lambda^s \mathrm{E}^s\Bigg[\sum_{n\in\tilde{D}^e(0)} g(0, n)\Bigg] = \lambda^s \mathrm{E}^s[g(0, 0)] + \lambda^e \mathrm{E}^e[g(n^{(d)}, 0)] \tag{3.7}$$

and

$$\lambda^s \mathrm{E}^s\Bigg[\sum_{n\in L(0)} g(0, n)\Bigg] = \lambda^s \mathrm{E}^s[g(0, 0)] + \lambda^e \mathrm{E}^e[g(n^{(l)}, 0)]. \tag{3.8}$$

Proof. Equation (3.7) follows from applying the mass transport principle to the diagonally invariant function

$$h_1(\omega, 0, n) := \mathbf{1}\{0 \in \Phi^s(\omega)\}\mathbf{1}\{n \in \tilde{D}^e(0)(\omega)\}g(\omega, 0, n),$$

while (3.8) follows from applying the mass transport principle to the diagonally invariant function

$$h_2(\omega, 0, n) := \mathbf{1}\{0 \in \Phi^s(\omega)\}\mathbf{1}\{n \in L(0)(\omega)\}g(\omega, 0, n).$$

□

Corollary 3.1. *The following holds:*

$$\mathrm{E}^s[\tilde{d}^e(0)] = \mathrm{E}^s[l(0)] = \mathrm{E}\,[a_0], \tag{3.9}$$

where $\tilde{d}^e(0)$ is the cardinality of $\tilde{D}^e(0)$. Moreover,

$$\frac{\mathrm{E}^s[\sum_{n\in\tilde{D}^e(0)} |n|]}{\mathrm{E}^e[|n^{(d)}|]} = \frac{\mathrm{E}^s[\sum_{n\in L(0)} |n|]}{\mathrm{E}^e[|n^{(l)}|]} = \mathrm{E}\,[a_0] - 1. \tag{3.10}$$

Proof. The results follow from choosing particular diagonally invariant functions in Proposition 3.6. Let $g(0, n) \equiv 1$. Then (3.9) holds as $\lambda^s + \lambda^e = 1$ and $\lambda^s = \frac{1}{\mathrm{E}\,[a_0]}$. Set $g(0, n) = |n|$. Again, using (3.7),

$$\lambda^s \mathrm{E}^s\Bigg[\sum_{n\in\tilde{D}^e(0)} |n|\Bigg] = \lambda^s \mathrm{E}^s[0] + \lambda^e \mathrm{E}^e[|n^{(d)}|] = \lambda^e \mathrm{E}^e[|n^{(d)}|].$$

Hence,

$$\mathrm{E}^e[|n|^{(d)}] = \left(\frac{\mathrm{E}\,[a_0]}{\mathrm{E}\,[a_0] - 1}\right)\frac{1}{\mathrm{E}\,[a_0]}\mathrm{E}^s\Bigg[\sum_{n\in\tilde{D}^e(0)} |n|\Bigg]$$
$$= \frac{\mathrm{E}^s[\sum_{n\in\tilde{D}^e(0)} |n|]}{\mathrm{E}\,[a_0] - 1}.$$

Following the same steps using (3.8), we recover (3.10). □

4. Final remarks

When $\mathrm{E}\,[a_0] = \infty$ the structure of the family random graph changes radically. A rich set of open questions for future research then arises, requiring different approaches to the ones

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used here. Proposition 3.3 still holds, and consequently the intensity of the point process of successful individuals is 0. Combined with the classification theorem of [3], this implies that the connected component of 0 is of class **I/I**, namely, it is an eternal family tree with one end, in which all individuals have a finite lineage and an infinite number of cousins of all degrees.

Also, not only does $\mathrm{E}[N_n]=\infty$, but also $N_n=\infty$ a.s. for all $n\in\mathbb{Z}$. That $\mathrm{E}[N_n]=\infty$ simply follows from Little's law (Proposition 2.1). Moreover,

$$\mathrm{P}[N_0\le m]=\mathrm{P}\Bigg[\bigcup_{(q_1,\ldots,q_{m-1})\in\mathbb{N}^{m-1}}\ \bigcap_{j\notin(q_1,\ldots,q_{m-1})}\{a_{-j}\le j\}\Bigg].$$

As $\mathrm{E}[a_0]=\infty$, for any $(q_1,\ldots,q_{m-1})\in\mathbb{N}^{m-1}$,

$$\prod_{j\notin(q_1,\ldots,q_{m-1})\in\mathbb{N}^{m-1}}\mathrm{P}[a_{-j}\le j]=0,$$

from which follows $\mathrm{P}[N_0\le m]=0$. Hence, $N_0=\infty$ a.s., which implies $N_n=\infty$ for all $n\in\mathbb{Z}$.

The unimodular structure of T^f is preserved when $\mathrm{E}[a_0]=\infty$, and consequently the model remains critical, i.e. the expected value of the offspring cardinality of a typical individual is one. From [3], it also follows that, despite $d(n)<\infty$ a.s. for all $n\in\mathbb{Z}$, $\mathrm{E}[d(n)]=\infty$. This result is obtained by applying the mass transport principle.

We conclude with a remark on the population dynamics interpretation of the model discussed in the core of this work, namely when $\mathrm{E}[a_0]<\infty$. Our model is concerned with a dynamics in which the population is infinitely often close to extinction, as the original ancestor point process shows. In connection with this, we find it interesting to mention that there is some genetic and archaeological evidence that the human population was close to extinction several times in the distant past ([6] and [8]).

Appendix A. Section 2 proofs

Proposition A.1. *The s.s.p.p. of original ancestors constructed in Theorem 2.1 is a renewal process.*

Proof. Notice that, for all m,

$$\mathrm{P}_{\Psi^o}[(k_1^o-k_0^o)=m]=\mathrm{P}[k_1^o=m|k_0^o=0]=:y_m.$$

Now, for any $n\in\mathbb{Z}$, as θ_{k_n} preserves P_{Ψ^o},

$$\begin{aligned}\mathrm{P}_{\Psi^o}[(k_{n+1}^o-k_n^o)=m]&=\mathrm{P}_{\Psi^o}[(k_{n+1}^o-k_n^o)=m]\circ\theta_{k_n^o}^{-1}\\&=\mathrm{P}_{\Psi^o}[(k_1^o-k_0^o)=m]=y_m.\end{aligned}$$

Hence $\{k_{n+1}^o-k_n^o\}_{n\in\mathbb{Z}}$ is identically distributed.

Now let $i_0<i_1<i_2<\cdots<i_q$ be a finite set of integers. Then, for $m_0,\ldots,m_q\in\mathbb{N}$,

$$\begin{aligned}&\mathrm{P}_{\Psi^o}[(k_{i_0+1}^o-k_{i_0}^o)=m_0,(k_{i_2+1}^o-k_{i_2}^o)=m_1,\ldots,(k_{i_q+1}^o-k_{i_q}^o)=m_q]\\&\quad=\mathrm{P}_{\Psi^o}[(k_{i_0-i_q+1}^o-k_{i_0-i_q}^o)=m_0,(k_{i_1-i_q+1}^o-k_{i_1-i_q}^o)=m_1,\ldots,k_1^o=m_q].\end{aligned}$$

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Now, under $\mathrm{P}_{\Psi^{\mathrm{o}}}$ the realization of k_1^{o} is a function of $\{a_n\}_{n\geq 0}$, and for all $i<0$, $(k_i^{\mathrm{o}}-k_{i-1}^{\mathrm{o}})$ is a function of $\{a_n\}_{n<0}$. Hence, as $\{a_n\}_{n\geq 0}$ is independent of $\{a_n\}_{n<0}$ (Lemma 2.2),

$$\begin{aligned}
&\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_0-i_q+1}-k^{\mathrm{o}}_{i_0-i_q})=m_0,\,(k^{\mathrm{o}}_{i_1-i_q+1}-k^{\mathrm{o}}_{i_1-i_q})=m_1,\,\ldots,\,k^{\mathrm{o}}_1=m_q]\\
&\quad=\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_0-i_q+1}-k^{\mathrm{o}}_{i_0-i_q})=m_0,\,(k^{\mathrm{o}}_{i_1-i_q+1}-k^{\mathrm{o}}_{i_1-i_q})=m_1,\,\ldots,\\
&\qquad\qquad(k^{\mathrm{o}}_{i_{q-1}-i_q+1}-k^{\mathrm{o}}_{i_{q-1}-i_q})=m_{q-1}]\,\mathrm{P}_{\Psi^{\mathrm{o}}}[k^{\mathrm{o}}_1=m_q]\\
&\quad=\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_0+1}-k^{\mathrm{o}}_{i_0})=m_0,\,(k^{\mathrm{o}}_{i_2+1}-k^{\mathrm{o}}_{i_2})=m_1,\,\ldots,\,(k^{\mathrm{o}}_{i_{q-1}+1}-k^{\mathrm{o}}_{i_{q-1}})=m_{q-1}]\\
&\qquad\times\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_q+1}-k^{\mathrm{o}}_{i_q})=m_q].
\end{aligned}$$

By continuing to apply the same reasoning, we conclude that

$$\begin{aligned}
&\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_0+1}-k^{\mathrm{o}}_{i_0})=m_0,\,(k^{\mathrm{o}}_{i_2+1}-k^{\mathrm{o}}_{i_2})=m_1,\,\ldots,\,(k^{\mathrm{o}}_{i_q+1}-k^{\mathrm{o}}_{i_q})=m_q]\\
&\quad=\prod_{j=0}^{q}\mathrm{P}_{\Psi^{\mathrm{o}}}[(k^{\mathrm{o}}_{i_j+1}-k^{\mathrm{o}}_{i_j})=m_j].
\end{aligned}$$

Therefore, $\{k^{\mathrm{o}}_{n+1}-k^{\mathrm{o}}_n\}_{n\in\mathbb{Z}}$ is also an independent sequence. □

Proof of Proposition 2.1. Let $Z_{n,m}=\mathbf{1}\{f(m)>n\}$. Notice that

$$N_n=\sum_{-\infty<m<n}Z_{n,m}+1. \tag{A.1}$$

For $m\leq 0$, let $M_m=\sum_{m<l<0}Z_{0,l}+1$, so $\lim_{m\to-\infty}M_m=N_0$ a.s. Then, as $a_m\overset{\mathrm{D}}{=}a_0$ for all $m\in\mathbb{Z}$,

$$\begin{aligned}
\mathrm{E}[M_m]&=\sum_{m\leq l<0}\mathrm{P}[a_{-l}>l]+1\\
&=\sum_{l=1}^{m}\mathrm{P}[a_0>l]+\sum_{l=1}^{m}\mathrm{P}[a_0=l]+\sum_{l>m}\mathrm{P}[a_0=l]\\
&=\sum_{l=1}^{m}\mathrm{P}[a_0\geq l]+\sum_{l>m}\mathrm{P}[a_0=l].
\end{aligned}$$

Letting $m\to\infty$, by monotone convergence we get $\mathrm{E}[N_0]=\mathrm{E}[a_0]$. □

Lemma A.1. *Consider the infinite product $\prod_{i=1}^{\infty}(1\pm b_i)$, where $b_i\geq 0$ for all i. Then if $\sum_{i=1}^{\infty}b_i$ converges (resp. diverges to infinity), then $\prod_{i=1}^{\infty}(1\pm b_i)$ also converges to a non-negative number (resp. diverges).*

For a proof, see e.g. [4].

Proof of Proposition 2.2. Using (A.1), we can compute the moment-generating function of N_0 as follows. For $t\in\mathbb{R}$:

$$\begin{aligned}
\mathrm{E}[\mathrm{e}^{tN_0}]=\mathrm{E}[\mathrm{e}^{t}\mathrm{e}^{t\sum_{i=1}^{\infty}Z_{0,-i}}]&=\mathrm{e}^{t}\prod_{i=1}^{\infty}\mathrm{E}[\mathrm{e}^{tZ_{0,-i}}]\\
&=\mathrm{e}^{t}\prod_{i=1}^{\infty}(\mathrm{e}^{t}\mathrm{P}[a_0>i]+\mathrm{P}[a_0\leq i]).
\end{aligned}$$

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Thus, $\mathrm{E}[\mathrm{e}^{tN_0}] = \mathrm{e}^t \prod_{i=1}^{\infty} (b_i + 1)$, where $b_i = \mathrm{P}[a_0 > i](\mathrm{e}^t - 1)$. As $\mathrm{E}[a_0] < \infty$, $\sum_{i=1}^{\infty} b_i < \infty$, and therefore by Lemma A.1, $\mathrm{E}[\mathrm{e}^{tN_0}] < \infty$. □

Appendix B. Dynamics on unimodular networks

We review the necessary concepts on unimodular networks dynamics used in this paper. We borrow the concepts of this section from [3] and [1], which contain a more complete treatment of the subject.

Definition B.1. (*Locally finite rooted networks.*) A network is a quadruple (G, Ξ, u_1, u_2) where:

- $G = (V, E)$ is a (multi-)graph;
- Ξ is a complete separable metric space;
- $u_1 : V \to \Xi$ (the element $u_1(v)$ is called the mark of the vertex v);
- $u_2 : \{(v, e) : v \in V, e \in E, v \sim e\} \to \Xi$ (the element $u_2(v, e)$ is called the mark of the pair (v, e)).

If G has a distinguished vertex, we call the network rooted. A network is locally finite if all its vertices have finite degrees. We also consider the case in which G has two distinguished vertices.

An isomorphism between graphs G and G' is a bijection that preserves the (direct) vertices. An isomorphism between rooted networks also preserves its marks and maps the distinguished vertex (or vertices) of G to G'.

Let $\mathcal{G}_*$ (resp. $\mathcal{G}_{**}$) be the space of all isomorphism classes of locally finite rooted (resp. with two distinct vertices) networks and $\mathcal{B}(\mathcal{G}_*)$ (resp. $\mathcal{B}(\mathcal{G}_{**})$) its Borel-$\sigma$ algebra. An element of $\mathcal{G}_*$ (resp. $\mathcal{G}_{**}$) is denoted by $[G, o]$ (resp. $[G, o, o']$). We denote the set of vertices of any representative of the isomorphism class $[G, o]$ (or $[G, o, o']$) by V.

A random rooted (resp. with two distinct vertices) network is a measurable mapping from a general probability space $(\Omega, \mathcal{F}, \mathrm{P})$ to $(\mathcal{G}_*, \mathcal{B}(\mathcal{G}_*))$ (resp. $(\mathcal{G}_{**}, \mathcal{B}(\mathcal{G}_*))$). We denote a random network by $[\mathbf{G}, \mathbf{o}]$ (resp. $[\mathbf{G}, \mathbf{o}, \mathbf{o}']$).

A locally finite network is unimodular if for all measurable functions $g : \mathcal{G}_{**} \to \mathbb{R}^{\geq}$:

$$\mathrm{E}\Bigg[\sum_{v \in V} g[\mathbf{G}, \mathbf{o}, v]\Bigg] = \mathrm{E}\Bigg[\sum_{v \in V} g[\mathbf{G}, v, \mathbf{o}]\Bigg].$$

This definition does not depend on the choice of representative of $[\mathbf{G}, \mathbf{o}, \mathbf{v}]$.

Definition B.2. A *covariant vertex-shift* is a map X_G which associates with each unimodular network a function $x_G : V \to V$ such that

- x_G is covariant under isomorphisms and
- the function $[G, o, o'] \to \mathbf{1}\{x_G(o) = o'\}$ is measurable on $\mathcal{G}_{**}$.

From vertex-shift X_G acting on $[G, o, o']$ we let $G^X = (V^X, E^X)$ be the graph such that $V^X = V$ and $E^X = \{v, x_G(v)\}_{v \in V}$. We then define the following subsets of V^X.

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Definition B.3. (*Connected component.*) The *connected component* of v under the action of vertex-shift x_G is given by

$$C(v) := \{w \in V \text{ s.t. } \exists\, i, j \in \mathbb{N} \text{ with } x_G^i(v) = x_G^j(w)\}.$$

Definition B.4. (*Foil.*) The *foil* of v is defined as

$$L(v) = \{w \in V \text{ s.t. } \exists\, j \in \mathbb{N} \text{ with } x_G^j(w) = x_G^j(v)\}.$$

We let $c(v)$ (resp. $l(v)$) denote the cardinality of $C(v)$ (resp. $L(v)$). In [3], it is shown that each connected component of G^X, in particular $C(o)$, falls within one of the following categories:

Class **F/F**: $c(o) < \infty$ and for all $v \in C(o)$, $l(v) < \infty$.

Class **I/F**: $c(o) = \infty$ and for all $v \in C(o)$, $l(v) < \infty$.

Class **I/I**: $c(o) = \infty$ and for all $v \in C(o)$, $l(v) = \infty$.

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