

PAPER

Separating minimal valuations, point-continuous valuations, and continuous valuations

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Abstract

We give two concrete examples of continuous valuations on dcpo's to separate minimal valuations, point-continuous valuations, and continuous valuations:

- (1) Let \mathcal{J} be the Johnstone's non-sober dcpo, and μ be the continuous valuation on \mathcal{J} with $\mu(U) = 1$ for nonempty Scott opens U and $\mu(U) = 0$ for $U = \emptyset$. Then, μ is a point-continuous valuation on \mathcal{J} that is not minimal.
- (2) Lebesgue measure extends to a measure on the Sorgenfrey line \mathbb{R}_ℓ . Its restriction to the open subsets of \mathbb{R}_ℓ is a continuous valuation λ . Then, its image valuation $\bar{\lambda}$ through the embedding of \mathbb{R}_ℓ into its Smyth powerdomain $\mathcal{Q}\mathbb{R}_\ell$ in the Scott topology is a continuous valuation that is not point-continuous.

We believe that our construction $\bar{\lambda}$ might be useful in giving counterexamples displaying the failure of the general Fubini-type equations on dcpo's.

Keywords: Minimal valuations; point-continuous valuations; continuous valuations; Fubini equations

1. Introduction

Continuous valuations on topological spaces are analogs of measures on measurable spaces. In domain theory, continuous valuations on dcpo's with the Scott topology are employed by computer scientists and mathematicians to give denotational meanings to probabilistic programming languages. This line of work dates back to Jones and Plotkin (1989) and Jones (1990). Indeed, in her PhD thesis, Jones developed the theory of valuations and used the valuations monad \mathcal{V} on the category **DCPO** of dcpo's and Scott-continuous maps to give denotational semantics to probabilistic programming languages.

While the valuations monad on the category **DCPO** enjoys many nice properties, for example, this monad is a strong monad and dcpo-enriched, it is unknown whether it is a commutative monad on the same category. As a result, it would be difficult, using the valuations monad, to establish the so-called contextual equivalence between programs that only differ in the order of sampling random variables. To combat this problem, the authors in Jia et al. (2021) constructed submonads of the valuations monad \mathcal{V} that are commutative on the category of dcpo's. Among their construction, there is a least submonad of the valuations monad that consists of which

we call *minimal valuations*. Minimal valuations are these continuous valuations that are in the d-closure of the simple valuations; precisely, they consist of directed suprema of simple valuations, directed suprema of directed suprema of simple valuations, and so forth, transfinitely. Every minimal valuation is a point-continuous valuation in the sense of Heckmann (1996) and Jia et al. (2021). Heckmann (1996) proved that the class of point-continuous valuations on space X form the sobrification of the space of simple valuations on X , both in the so-called weak topology. Every point-continuous valuation is a continuous valuation (Heckmann, 1995, Proposition 3.1).

It is relatively easy to see that on general topological spaces, minimal valuations form a strictly smaller class than that of point-continuous valuations. However, it is unknown whether the same is true on dcpo's with the Scott topology. The first example in this paper clarifies the difference between minimal valuations and point-continuous valuations on dcpo's. Concretely, we consider the well-known Johnstone's non-sober dcpo \mathcal{J} and the "constant-1 valuation" μ on \mathcal{J} defined by $\mu(U) = 1$ if U is non-empty and $\mu(\emptyset) = 0$. We show that every bounded continuous valuation on \mathcal{J} can be written as a sum of some discrete valuation and a scalar multiple of μ . This enables us to conclude that every continuous valuation on \mathcal{J} is actually point-continuous. Moreover, we prove that the continuous valuation μ is not in the d-closure of simple valuations; hence, it serves as an example that separates minimal valuations from point-continuous valuations. This example is included in Section 3.

Similar to the difference between minimal valuations and point-continuous valuations, continuous valuations that are not point-continuous can be easily found on topological spaces. For example, Lebesgue measure, restricted to the usual opens of reals, is a continuous valuation that is not point-continuous. However, it has been unknown whether point-continuous valuations differ from continuous valuation on dcpo's since 1996. The second goal of this note is to give an example of a continuous valuation on a dcpo that is not point-continuous. In order to find such an example, one is tempted to find the simplest possible example, and typically to find a continuous valuation that takes only two values, 0 or 1, and hoping that it would not be point-continuous. However, we notice that such a strategy cannot work, as we will see in Section 4.1. Hence, we will have to work a bit more. We show how one can build certain continuous valuations on the Sorgenfrey line \mathbb{R}_ℓ in Section 4.2, including one based on Lebesgue measure λ . We study the compact subsets of \mathbb{R}_ℓ in Section 4.3, as a preparation to studying the dcpo $\mathcal{Q}\mathbb{R}_\ell$ of compact subsets of \mathbb{R}_ℓ under reverse inclusion, and showing that the natural map from \mathbb{R}_ℓ to $\mathcal{Q}\mathbb{R}_\ell$ is a subspace embedding in Section 4.4. We transport Lebesgue measure λ along this embedding, and we will show that the resulting continuous valuation $\bar{\lambda}$ on $\mathcal{Q}\mathbb{R}_\ell$ fails to be point-continuous in Section 4.5. We believe that our construction $\bar{\lambda}$ might be useful in giving counterexamples displaying the failure of the general Fubini-type equations on dcpo's, which is a long-standing open problem in domain theory. More detailed discussion about this part is included in the concluding remarks.

2. Preliminaries

We use standard concepts and notations from topology, measure theory, and domain theory. The reader is referred to Abramsky and Jung (1994), Goubault-Larrecq (2015), and Gierz et al. (2003) for topology and domain theory, and to Royden (1988) for measure theory.

2.1 Valuations

On a topological space X , a valuation ν is a map from the set $\mathcal{O}X$ of opens of X to the extended reals $\overline{\mathbb{R}}_+$, satisfying *strictness* ($\nu(\emptyset) = 0$), *monotonicity* ($U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$), and *modularity* ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). A valuation ν on X is called *continuous* if it is Scott-continuous from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$, and it is called *bounded* if $\nu(X) < \infty$. Continuous valuations are ordered in the *stochastic order*: $\nu_1 \leq \nu_2$ if and only if $\nu_1(U) \leq \nu_2(U)$ for all opens U of X . The set of all continuous valuations on X , which we denote as $\mathcal{V}X$, is a dcpo in the stochastic order. Canonical examples of continuous valuations on X include *Dirac valuations* δ_x for $x \in X$, where $\delta_x(U) = 1$

if $x \in U$ and $\delta_x(U) = 0$ if $x \notin U$. As a dcpo, $\mathcal{V}X$ is closed under suprema, it also is closed under scalar multiplication and sum, for $v_i \in \mathcal{V}X$ and $r_i \in [0, \infty[$, $i = 1, \dots, n$, the sum $\sum_{i=1}^n r_i v_i$ which is defined by $(\sum_{i=1}^n r_i v_i)(U) = \sum_{i=1}^n r_i v_i(U)$ also is in $\mathcal{V}X$. For $r_i \in [0, \infty[$ and $x_i \in X$, $i = 1, \dots, n$, the finite sum $\sum_{i=1}^n r_i \delta_{x_i}$ is called a *simple valuation* on X . The set of all simple valuations on X is denoted by SX . Valuations of the form $\sum_{i=1}^\infty r_i \delta_{x_i} = \sup_{n \in \mathbb{N}} \sum_{i=1}^n r_i \delta_{x_i}$ are called *discrete valuations*. The smallest sub-dcpo of $\mathcal{V}X$ that contains SX (hence all discrete valuations on X) is denoted by $\mathcal{M}X$, and every valuation in $\mathcal{M}X$ is called *minimal*. It is easy to see that for each minimal valuation v , v is either a simple valuation, or a directed supremum of simple valuations, or a directed supremum of directed suprema of simple valuations \dots , transfinitely. A valuation v on a space X is *point-continuous* if and only if for every open subset U , for every real number r such that $0 \leq r < v(U)$, there is a finite subset A of U such that $v(V) > r$ for every open neighborhood V of A . Minimal valuations are point-continuous Jia et al. (2021), and point-continuous valuations are continuous valuations (Heckmann, 1995, Proposition 3.1).

2.2 Ring of sets

We will need the notion of *Boolean ring of sets* (*ring of sets* for short). On a set X , a ring of sets on X is a lattice of sets consisting of subsets of X that also is closed under relative complements. For a topological space X , the set $\mathcal{O}X$ of all opens of X is a lattice of sets, and the ring of sets generated by $\mathcal{O}X$ is the intersection of all rings of sets on X that contain $\mathcal{O}X$, and it is denoted by $\mathcal{A}(\mathcal{O}X)$.

Lemma 1. (Gierz et al., 2003, Lemma IV-9.2) *Let X be a topological space. For each set A in $\mathcal{A}(\mathcal{O}X)$, the ring of sets generated by open sets of X , A is of the form of a finite disjoint union $\bigsqcup_{i=1}^n U_i \setminus V_i$, where U_i and V_i are open subsets of X . One can also stipulate that $V_i \subseteq U_i$ for each i .*

For open subsets U and V of X , the set difference $U \setminus V$ is called a *crescent*. Since $U \setminus V = U \setminus (U \cap V)$, in the sequel, when we write a set $A \in \mathcal{A}(\mathcal{O}X)$ as $\bigsqcup_{i=1}^n U_i \setminus V_i$, we always assume that $V_i \subseteq U_i$ for each i .

Lemma 2. (Heckmann, 1996, Section 3.3) *Let X be a topological space and μ be a bounded continuous valuation on X . For each set $A = \bigsqcup_{i=1}^n U_i \setminus V_i$ in $\mathcal{A}(\mathcal{O}X)$, define $\mu_A = \sum_{i=1}^n \mu|_{U_i}^{V_i}$, where for open sets U, V , and W , $\mu|_U^V(W) = \mu(W \cap U) - \mu(W \cap V \cap U)$. Then, μ_A is a bounded continuous valuation for each $A \in \mathcal{A}(\mathcal{O}X)$. In particular, $\mu_{U \setminus V} = \mu|_U^V$. Note that $\mu_U = \mu|_U^\emptyset$ for every open subset U .*

Note that it is possible that for each $A \in \mathcal{A}(\mathcal{O}X)$, A can be written as a disjoint union $\bigsqcup_{i=1}^n U_i \setminus V_i$ or $\bigsqcup_{j=1}^m U_j \setminus V_j$. However, when $\bigsqcup_{i=1}^n U_i \setminus V_i = \bigsqcup_{j=1}^m U_j \setminus V_j$, we will always have that $\sum_{i=1}^n \mu|_{U_i}^{V_i} = \sum_{j=1}^m \mu|_{U_j}^{V_j}$. This validates the definition of μ_A in the previous lemma.

Lemma 3. *For two disjoint sets $A, B \in \mathcal{A}(\mathcal{O}X)$, $\mu_{A \cup B} = \mu_A + \mu_B$. This implies that $\mu_A \leq \mu_B$ when $A \subseteq B$.*

Proof. From the above remark and straightforward computation. □

Lemma 4. *Let X be a T_0 topological space and μ be a bounded continuous valuation on X . If $\{a\}$ is in $\mathcal{A}(\mathcal{O}X)$, then there exist opens U, V with $V \subseteq U$ and $U \setminus V = \{a\}$. Moreover, in this case $\mu_{\{a\}} = r_a \delta_a$, where $r_a = \mu_{\{a\}}(X) = \mu(U) - \mu(V) = \mu|_{U \setminus V}$.*

Proof. The first assertion is obvious.
 For the second assertion, since X is T_0 , we only need to prove that for each open O , $\mu_a(O)$ is equal to r_a if $a \in O$ and to 0 if $a \notin O$. Assume that U and V are open subsets of X with $V \subseteq U$ and

$U \setminus V = \{a\}$. For an open subset O , if $a \in O$, then $O \cap U \setminus O \cap V = U \setminus V = \{a\}$, which implies that $\mu_a(O) = \mu|_U^V(O) = \mu|_{U \cap O}^{V \cap O}(O) = \mu|_{U \cap O}^{V \cap O}(U) = \mu_a(U)$. If a is not in O , then $O \cap U = O \cap V$. Hence, $\mu_a(O) = 0$. □

3. Point-Continuous Valuations Need Not be Minimal Valuations

In this section, we built a point-continuous valuation μ on the well-known Johnstone’s non-sober dcpo, which is not a minimal valuation.

3.1 Valuations on \mathbb{N}_{cof}

Let \mathbb{N}_{cof} be the topological space of natural numbers equipped with the co-finite topology. It is easy to verify that the the map $\beta: \mathcal{O}\mathbb{N}_{\text{cof}} \rightarrow \overline{\mathbb{R}}_+$ defined by:

$$\beta(U) = \begin{cases} 1, & U \subseteq \mathbb{N}_{\text{cof}} \text{ is open and non-empty;} \\ 0, & U \text{ is empty} \end{cases}$$

is a bounded continuous valuation on \mathbb{N}_{cof} .

Lemma 5. *For each $i \in \mathbb{N}_{\text{cof}}$, the set $\{i\}$ is closed. Hence, it is in the ring of sets generated by the co-finite topology on \mathbb{N}_{cof} .*

Proof. Straightforward. □

Proposition 6. *Let ν be a bounded continuous valuation on \mathbb{N}_{cof} . Then, there exists a discrete valuation α on \mathbb{N}_{cof} and nonnegative real number r such that $\nu = \alpha + r\beta$.*

Proof. We let $\alpha = \sum_{i \in \mathbb{N}} \nu_{\{i\}}$. By Lemma 5 and Lemma 2, each $\nu_{\{i\}}$ is a continuous valuation. Since \mathbb{N}_{cof} is a T_0 topological space, it follows from Lemma 4 that α is a discrete valuation. Now, we define the map:

$$\nu^*: \mathcal{O}\mathbb{N}_{\text{cof}} \rightarrow \mathbb{R}_+ :: U \mapsto \nu(U) - \alpha(U).$$

We proceed to show that ν^* is a multiple of β , that is, there exists some $r \in [0, \infty[$ such that $\nu^* = r\beta$.

First, for each $n \in \mathbb{N}$, $\nu(U) - \sum_{i=1}^n \nu_{\{i\}}(U) = \nu_{\mathbb{N} \setminus \{1,2,\dots,n\}}(U)$ is nonnegative, hence $\nu^*(U)$, which is the infimum of $\nu(U) - \sum_{i=1}^n \nu_{\{i\}}(U)$, $n \in \mathbb{N}$, indeed takes values in \mathbb{R}_+ .

Second, for non-empty open sets U and V with $V \subseteq U$, we prove that $\nu^*(U) = \nu^*(V)$. Because V is co-finite, we know that $U \setminus V$ is a finite set, which we denote by F . Then, we know

$$\begin{aligned} \nu^*(U) &= \nu(U) - \sum_{i=1}^{\infty} \nu_{\{i\}}(U) && \text{definition of } \nu^* \\ &= \nu_U(U) - \sum_{i=1}^{\infty} \nu_{\{i\}}(U) && \nu_U(U) = \nu(U \cap U) \\ &= \nu_V(U) + \sum_{i \in F} \nu_{\{i\}}(U) - \sum_{i=1}^{\infty} \nu_{\{i\}}(U) && \text{by Lemma 3 and } V \cup F = U \\ &= \nu_V(V) - \sum_{i=1}^{\infty} \nu_{\{i\}}(V) && \text{by Lemma 4} \\ &= \nu^*(V). && \text{definition of } \nu^* \end{aligned}$$

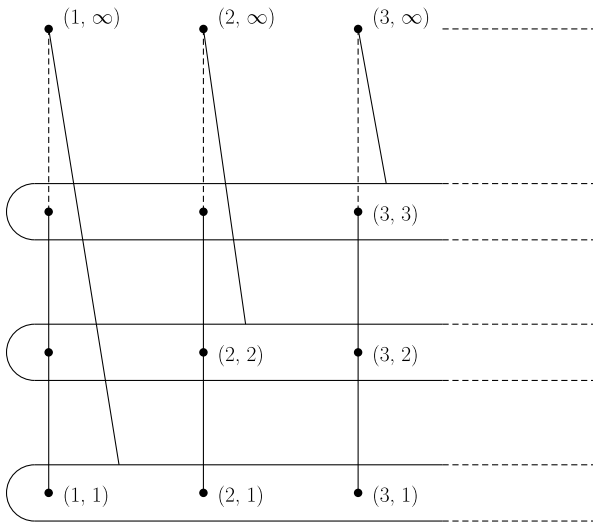


Figure 1. Johnstone's non-sober dcpo \mathcal{J} .

Now for general non-empty opens U and V , we have $v^*(U) = v^*(U \cap V) = v^*(V)$. Let $r = v^*(X)$, then we know that r is nonnegative from above and that $v^* = r\beta$ (hence, v^* also is a continuous valuation).

Finally, we conclude the proof by the fact that $v = \alpha + v^* = \alpha + r\beta$. □

3.2 Valuations on Johnstone's non-sober dcpo \mathcal{J}

In 1980, Johnstone gave the first dcpo which is not sober in the Scott topology (Johnstone, 1981). This dcpo, which we denote by \mathcal{J} , serves as a basic building block in several counterexamples in domain theory (Ho et al., 2018; Isbell, 1982). In this subsection, we will use it to construct a continuous valuation that is not minimal.

Definition 7. (The dcpo \mathcal{J}). Let \mathbb{N} be the set of natural numbers and $\mathcal{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. The order on \mathcal{J} is defined by $(a, b) \leq (c, d)$ if and only if either $a = c$ and $b \leq d$, or $d = \infty$ and $b \leq c$.

The structure of \mathcal{J} is depicted in Figure 1.

We will use the following convention throughout this subsection when we reason about \mathcal{J} .

- M denotes the set of all maximal points of \mathcal{J} , that is, $M = \{(i, \infty) \mid i \in \mathbb{N}\}$;
- $M_k = \{(i, \infty) \mid k < i\}$, and $M_{k,l} = \{(i, \infty) \mid k < i \leq l\}$ for $k, l \in \mathbb{N}$ and $k < l$;
- N denotes the set $\mathcal{J} \setminus M$; elements in N are of finite height;
- L_i denotes the set of points of \mathcal{J} which are at Level i , for each $i \in \mathbb{N}$, that is $L_i = \{(j, i) \mid j \in \mathbb{N}\}$;
- C_i denotes the set of points of \mathcal{J} which are in Column i , that is $C_i = \{(i, j) \mid j \in \mathbb{N} \cup \{\infty\}\}$.
- D_i denotes the set $\bigcup_{j \leq i} C_j$, i.e., D_i consists of elements in the first i -many columns.

The dcpo \mathcal{J} in the Scott topology is a non-sober topological space, and the set M of maximal points of \mathcal{J} equipped with the relative Scott topology is homeomorphic to \mathbb{N}_{cof} .

Let v be an arbitrary bounded continuous valuation on \mathcal{J} . We are going to show that v can be written as a sum of a discrete valuation θ and $r\mu$, where r is a nonnegative real number and in this section, μ is reserved for the fixed valuation on \mathcal{J} that takes value 1 on non-empty Scott-opens, and 0 on the empty set:

$$\mu(U) = \begin{cases} 1, & U \neq \emptyset; \\ 0, & U = \emptyset. \end{cases}$$

Note that μ on \mathcal{J} is an analog of β on \mathbb{N}_{cof} . Indeed, the pushforward image of β along the canonical topological embedding $n \mapsto (n, \infty)$ of \mathbb{N}_{cof} into \mathcal{J} is exactly the valuation μ . Since θ and $r\mu$ are point-continuous (direct verification), by proving that ν is a sum of some discrete θ and $r\mu$, we infer that all (not necessarily bounded) continuous valuations on \mathcal{J} are point-continuous by using a trick due to Heckmann (1996, Theorem 4.2).

Theorem 8. *Every bounded continuous valuation ν on \mathcal{J} is point-continuous. Moreover, there exist a discrete valuation θ and nonnegative real number r such that $\nu = \theta + r\mu$.*

We prove this theorem by a series of results. First, we give a lemma that will be used a few times. It is a slight generalization of Heckmann (1996, Proposition 3.2).

Lemma 9. *Let μ be a bounded continuous valuation on a space X , and ν be a monotonic map from $\mathcal{O}X$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that $\nu(\emptyset) = 0$. If $\mu + \nu$ is a continuous valuation, then so is ν .*

Proof. Since ν is monotonic and $\nu(\emptyset) = 0$, ν actually takes its values in $\overline{\mathbb{R}}_+$. It is clear that $\nu = (\mu + \nu) - \mu$ is modular, using the fact that μ is bounded for the subtraction to make sense. The only challenge is Scott-continuity. Let $(U_i)_{i \in I}$ be any directed family of open subsets of X , and U be its union. We have $\nu(U) \geq \sup_{i \in I} \nu(U_i)$ by monotonicity. In order to prove the reverse inequality, we consider any $a < \nu(U)$, and we show that $a \leq \nu(U_i)$ for some $i \in I$. Since μ is bounded, $\mu(U) + a < (\mu + \nu)(U)$, and since $\mu + \nu$ is a continuous valuation, there is an $i \in I$ such that $\mu(U) + a < (\mu + \nu)(U_i)$. Then $a < \mu(U_i) - \mu(U) + \nu(U_i) \leq \nu(U_i)$. □

Lemma 10. *For each element $a \in N$ (using the convention after Definition 7), the singleton $\{a\}$ is a crescent. Thus, for each $a \in N$, $\{a\}$ is in the ring of sets generated by Scott-opens of \mathcal{J} .*

Proof. For each $a \in N$, if a is at Level n , that is $a = (j, n)$ for some $j \in \mathbb{N}$, then $\{a\}$ can be written as $(\{a\} \cup \uparrow L_{n+1}) \setminus \uparrow L_{n+1}$. The proof is done since both $\{a\} \cup \uparrow L_{n+1}$ and $\uparrow L_{n+1}$ are Scott-open in \mathcal{J} . □

Proposition 11. *For each Scott-open subset U of \mathcal{J} , let $\nu^*(U) = \nu(U) - \sum_{a \in N} \nu_{\{a\}}(U)$. Then, ν^* is a bounded continuous valuation on \mathcal{J} .*

Proof. Since for each $a \in N$, $\{a\}$ is a crescent by Lemma 10, $\nu_{\{a\}}$ is a continuous valuation. Hence, $\nu_{\{a\}}(U)$ makes sense for each $a \in N$.

By Lemma 9, we only need to prove that ν^* is well defined and order-preserving. Note that N is a countable set. We index elements in N by natural numbers by letting $N = \{a_1, a_2, \dots, a_n, \dots\}$. Since for each open set U , $\sum_{i=1}^n \nu_{\{a_i\}}(U) = \nu_{\{a_1, \dots, a_n\}}(U) \leq \nu_{\mathcal{J}}(U) = \nu(U)$ and $\nu(U)$ is bounded, it means that for each U the sequence $\nu_{\{a_i\}}(U)$, $n = 1, \dots, n, \dots$ is commutatively summable. Hence, ν^* is well defined and takes values in \mathbb{R}_+ .

For monotonicity of ν^* , we let U be Scott-open and compute as follows:

$$\begin{aligned} \nu^*(U) &= \nu(U) - \sum_{a \in N} \nu_{\{a\}}(U) \\ &= \nu(U) - \sum_{i=1}^{\infty} \nu_{\{a_i\}}(U) \\ &= \lim_{n \rightarrow \infty} \left(\nu(U) - \sum_{i=1}^n \nu_{\{a_i\}}(U) \right) \\ &= \inf_{n \in \mathbb{N}} (\nu_{\mathcal{J}}(U) - \nu_{\{a_1, \dots, a_n\}}(U)) \\ &= \inf_{n \in \mathbb{N}} \nu_{\mathcal{J} \setminus \{a_1, \dots, a_n\}}(U). \end{aligned}$$

Since for each $n \in \mathbb{N}$, $v_{\mathcal{J} \setminus \{a_1, \dots, a_n\}}$ is a continuous valuation, therefore a nonnegative order-preserving map. Hence, the pointwise infimum v^* of $v_{\mathcal{J} \setminus \{a_1, \dots, a_n\}}$, $n \in \mathbb{N}$ also is order-preserving. \square

Lemma 12. *For any two Scott-open subsets U, V of \mathcal{J} with $M \cap U = M \cap V$, $v^*(U) = v^*(V)$.*

Proof. Without loss of generality, we assume that $U \subseteq V$. Let $V_n = U \cup (V \cap D_n)$ (see the convention after Definition 7). Since $M \cap U = M \cap V$, V_n is Scott-open for each n . Moreover, $V = \bigcup_{n=1}^\infty V_n$.

Note that for every n , $V_n \setminus U = (V \cap D_n) \setminus U$ is a finite subset of N , again by the fact that $M \cap U = M \cap V$. Then, we know that

$$\begin{aligned} v^*(V_n) &= v^*(U) + v^*(V_n) - v^*(U) \\ &= v^*(U) + (v(V_n) - v(U)) - \left(\sum_{a \in N} v_{\{a\}}(V_n) - \sum_{a \in N} v_{\{a\}}(U) \right) \\ &= v^*(U) + v_{V_n \setminus U}(V_n) - \left(\sum_{a \in N} v_{\{a\}}(V_n) - \sum_{a \in N} v_{\{a\}}(U) \right) \\ &= v^*(U) + \sum_{a \in V_n \setminus U} v_{\{a\}}(V_n) - \sum_{a \in N \cap (V_n \setminus U)} v_{\{a\}}(V_n) \\ &= v^*(U). \end{aligned}$$

Hence, by Scott-continuity of v^* (Proposition 11), we know that $v^*(V) = v^*(\bigcup_{n \in \mathbb{N}} V_n) = \sup_{n \in \mathbb{N}} v^*(V_n) = v^*(U)$. \square

We consider M as a subspace of \mathcal{J} with the Scott topology and define a continuous valuation v^∞ on M by stipulating for each open subset W of M that $v^\infty(W) = v^*(U_W)$, where U_W is the largest Scott-open subset of \mathcal{J} with $M \cap U_W = W$. We now use Lemma 12 to prove that v^∞ is indeed a bounded continuous valuation on M . This is a consequence of (Goubault-Larrecq, 2021, Proposition 5.2), and we provide a direct proof here nevertheless.

Lemma 13. *The map v^∞ is a bounded continuous valuation on M equipped with the relative Scott topology from \mathcal{J} .*

Proof. It is easy to see that $v^\infty(\emptyset) = 0$.

For modularity, let W_1 and W_2 be two open subsets of M . Then we have

$$\begin{aligned} v^\infty(W_1) + v^\infty(W_2) &= v^*(U_{W_1}) + v^*(U_{W_2}) && \text{definition of } v^\infty \\ &= v^*(U_{W_1 \cup W_2}) + v^*(U_{W_1 \cap W_2}) && v^* \text{ is a valuation} \\ &= v^*(U_{W_1 \cup W_2}) + v^*(U_{W_1 \cap W_2}) && \text{by Lemma 12} \\ &= v^\infty(W_1 \cup W_2) + v^\infty(W_1 \cap W_2). && \text{definition of } v^\infty \end{aligned}$$

Now we prove that v^∞ is Scott-continuous. Let $W_i, i \in I$ be a directed family of open subsets of M and $W = \bigcup_{i \in I} W_i$. We first note that $U_{W_i}, i \in I$ also form a directed family of Scott-opens in \mathcal{J} . Hence, we have

$$\begin{aligned} v^\infty(W) &= v^*\left(U_{\bigcup_{i \in I} W_i}\right) && \text{definition of } v^\infty \\ &= v^*\left(\bigcup_{i \in I} U_{W_i}\right) && \text{by Lemma 12} \end{aligned}$$

$$\begin{aligned}
 &= \sup_{i \in I} v^*(U_{W_i}) && v^* \text{ is Scott-continuous} \\
 &= \sup_{i \in I} v^\infty(W_i). && \text{definition of } v^\infty
 \end{aligned}$$

Finally, boundedness of v^∞ is clear since v^* is. □

We are now ready to prove Theorem 8.

Proof of Theorem 8. The space M in the relative Scott topology from \mathcal{J} is homeomorphic to \mathbb{N}_{cof} . For example, one of the (infinitely many) homeomorphisms between them is the map sending $n \in \mathbb{N}_{\text{cof}}$ to $(n, \infty) \in M$. It then follows from Proposition 6 that v^∞ can be written as $\alpha + r\beta$ where α is a discrete valuation on M , r is a nonnegative real number, and β is the valuation on M defined as $\beta(W) = 1$ if $W \subseteq M$ is non-empty, and $\beta(W) = 0$ if $W = \emptyset$. For each Scott-open subset U of \mathcal{J} , we have

$$v^*(U) = v^\infty(U \cap M) = \alpha(U \cap M) + r\beta(U \cap M).$$

Now we define $\theta'(U) = \alpha(U \cap M)$. Since α is a discrete valuation on M , it is obvious that θ' is a discrete valuation on \mathcal{J} . Hence, by combining Proposition 11 and the fact that $\mu(U) = \beta(U \cap M)$, we know that for each Scott open subset U of \mathcal{J} ,

$$v(U) = \sum_{a \in \mathbb{N}} v_{\{a\}}(U) + v^*(U) = \sum_{a \in \mathbb{N}} v_{\{a\}}(U) + \theta'(U) + r\mu(U).$$

Finally, we let $\theta = \theta' + \sum_{a \in \mathbb{N}} v_{\{a\}}$ and the proof is complete. □

As a corollary to Theorem 8, we have the following:

Corollary 14. *Every continuous valuation on \mathcal{J} is point-continuous.*

Proof. It is proved in Heckmann (1996, Theorem 4.2) that every continuous valuation on a T_0 topological space can be written as a directed supremum of bounded continuous valuations. By Theorem 8, we know that on \mathcal{J} every bounded continuous valuation is of the form $\theta + r\mu$, which is obviously point-continuous. Hence, every continuous valuation on \mathcal{J} , as a directed supremum of point-continuous valuations, also is point-continuous (Heckmann, 1996, Section 3.2, Item (5)). □

3.3 Non-minimality of μ on \mathcal{J}

In this subsection, we show that the valuation μ on \mathcal{J} , defined in the last subsection, is not a minimal valuation.

As we know from Theorem 8, every bounded continuous valuation v on \mathcal{J} is of the form $\theta + r\mu$. We first prove that that for any $r > 0$, $\theta + r\mu$ can not be written as a supremum of discrete valuations. Without loss of generality, in the sequel, we assume that the total mass of $\theta + r\mu$ is 1, that is, $\theta(\mathcal{J}) + r\mu(\mathcal{J}) = \theta(\mathcal{J}) + r = 1$.

We will need the following extension result for discrete valuations, which is a special case of Alvarez-Manilla et al. (2000, Theorem 4.1).

Lemma 15. *Let θ be a discrete valuation on a dcpo D with $\theta(D) < \infty$. Then, θ has a unique extension to a measure, which we again denote as θ , on the Borel σ -algebra of $(D, \sigma D)$, where σD is the Scott topology on D . In fact, if $\theta = \sum_{i \in \mathbb{N}} r_i \delta_{x_i}$ with $\sum_{i \in \mathbb{N}} r_i < \infty$, then the value of its measure extension on a Borel subset B of D is just $\sum_{i/x_i \in B} r_i$. □*

Lemma 10 says that for each $a \in N \subseteq \mathcal{J}$, the singleton $\{a\}$ is a crescent; hence, $\{a\}$ is a Borel subset of \mathcal{J} . Actually, every subset of \mathcal{J} is a Borel subset. To see this, we only need to show that $\{(i, \infty)\}$ is a Borel subset in \mathcal{J} for each $i \in \mathbb{N}$, since \mathcal{J} is countable. Indeed, $\{(i, \infty)\} = \downarrow(i, \infty) \setminus \bigcup_{j \in \mathbb{N}} \downarrow L_j$, and for $i, j \in \mathbb{N}$, $\downarrow(i, \infty)$ and $\downarrow L_j$ are Scott-closed. Now if θ is a discrete valuation on \mathcal{J} , then it makes sense to apply θ to any subset of \mathcal{J} , viewing that θ is a measure defined on the Borel σ -algebra generated by Scott-opens on \mathcal{J} .

Proposition 16. *Let θ be a discrete valuation on \mathcal{J} with total mass $1 - r$ and $r > 0$. Then, $\theta + r\mu$ is not a directed supremum of discrete valuations.*

Proof. Suppose, for the sake of contradiction, that there exists a directed family $\{\theta_a\}_{a \in A}$ of discrete valuations with supremum $\sup_{a \in A} \theta_a = \theta + r\mu$.

Since $(\theta + r\mu)(\mathcal{J}) = 1$, there exists a discrete valuation $\theta_a, a \in A$ such that $\theta_a(\mathcal{J}) > 1 - \frac{r}{4}$. Note that $\sup_{i \in \mathbb{N}} D_i = \mathcal{J}$, where $D_i = \bigcup_{j \leq i} C_j$ consists of points in the first i -many columns (see Section 3.2), and also that D_i is a Borel subset of \mathcal{J} for each $i \in \mathbb{N}$. Since θ_a extends to a measure (Lemma 15), we know that there exists a $k \in \mathbb{N}$ such that $\theta_a(D_k) > 1 - \frac{r}{4}$.

Obviously, $\downarrow D_k$ is Scott-closed and $U_k = \mathcal{J} \setminus \downarrow D_k$ is a non-empty Scott-open subset of \mathcal{J} , hence $(\theta + r\mu)(U_k) \geq r > \frac{3r}{4}$. Again since $\theta + r\mu = \sup_{a \in A} \theta_a$, there exists $\theta_b > \theta_a$ such that $\theta_b(U_k) > \frac{3r}{4}$.

Note that the set $\uparrow D_k$ is a filtered intersection of countably many Scott-open subsets of \mathcal{J} , for example, one can write $\uparrow D_k = \bigcap_{i \in \mathbb{N}} O_i$, where for each i , $O_i = \mathcal{J} \setminus \downarrow\{(k + 1, i), (k + 2, i), \dots\}$. Then, we know that

$$\theta_b(\uparrow D_k) = \inf_{i \in \mathbb{N}} \theta_b(O_i) \geq \inf_{i \in \mathbb{N}} \theta_a(O_i) = \theta_a(\uparrow D_k) \geq \theta_a(D_k) > 1 - \frac{r}{4}.$$

Since $M_k = \{(i, \infty) \mid k < i\}$ is a Borel subset of \mathcal{J} and it equals to $\uparrow D_k \cap U_k$, hence by inclusion-exclusion of θ_b we know

$$\theta_b(M_k) + \theta_b(\mathcal{J}) \geq \theta_b(U_k) + \theta_b(\uparrow D_k) > 1 + \frac{r}{2},$$

from which it follows that $\theta_b(M_k) > \frac{r}{2}$. Since $\bigcup_{l/l > k} M_{k,l} = M_k$ ($M_{k,l} = \{(i, \infty) \mid k < i \leq l\}$), there exists a big enough $l > k$ such that $\theta_b(M_{k,l}) > \frac{r}{2}$.

Now consider $U_l = \mathcal{J} \setminus \downarrow D_l$. Since $(\theta + r\mu)(U_l) \geq r > \frac{3r}{4}$, we find a $\theta_c, c \in A$ such that $\theta_c \geq \theta_b$ and $\theta_c(U_l) > \frac{3r}{4}$. Meanwhile, note that $D_k \subseteq D_l$, similar to the reasoning above we will have that $\theta_c(\uparrow D_l) \geq \theta_b(\uparrow D_l) \geq \theta_b(\uparrow D_k) > 1 - \frac{r}{4}$ and that $\theta_c(M_l) > \frac{r}{2}$. Then there exists a large enough natural number $m > l$ such that $\theta_c(M_{l,m}) > \frac{r}{2}$.

We claim that $\theta_c(\mathcal{J}) > r$. Indeed, $\theta_c(\mathcal{J}) \geq \theta_c(M_{k,m}) = \theta_c(M_{k,l}) + \theta_c(M_{l,m}) \geq \theta_b(M_{k,l}) + \theta_c(M_{l,m}) > r$. The second to last inequality comes from the fact that $\theta_c \geq \theta_b$ and $M_{k,l}$ is a filtered intersection of countably many Scott-open subsets of \mathcal{J} : one may take such a filtered family as $\{\mathcal{J} \setminus \downarrow F \mid F \subseteq_{\text{fin}} M \setminus M_{k,l}\}$.

Next, we consider the Scott-open set $U_m = \mathcal{J} \setminus \downarrow D_m$ and then find $\theta_d \geq \theta_c$ with $\theta_d(\mathcal{J}) > \frac{3r}{2}$, and so forth. We repeat the above process N times, where N is a natural number satisfying that $N \times \frac{r}{2} > 1$, to find a discrete valuation θ_z in the directed family $\{\theta_a\}_{a \in A}$. By the construction, we would know that $\theta_z(\mathcal{J}) > 1$. However, this is impossible since $\theta_z \leq \theta + r\mu$ and $(\theta + r\mu)(\mathcal{J}) = 1$. So our assumption of the existence of the directed family $\{\theta_a\}_{a \in A}$ of discrete valuations with $\sup_{a \in A} \theta_a = \theta + r\mu$ must have been wrong, and we finish the proof. \square

Theorem 17. *Let θ be a discrete valuation on \mathcal{J} with total mass $1 - r$ and $r > 0$. Then, $\theta + r\mu$ is not a minimal valuation.*

Proof. We know that the set of minimal valuations (bounded by 1) on \mathcal{J} is obtained by taking directed suprema of simple valuations (bounded by 1), directed suprema of directed suprema

of simple valuations (bounded by 1), and so forth, transfinitely. However, from Theorem 8 and Proposition 16, at each of these steps we only obtain discrete valuations. Hence, by transfinite induction, $\theta + r\mu$ is not a minimal valuation for $r > 0$. \square

The following result, as promised, is then obvious:

Corollary 18. *The valuation μ on \mathcal{J} is point-continuous but not minimal.*

4. Continuous Valuations Need Not be Point-Continuous

In the last section, we have seen that on *dcpo*'s not all point-continuous valuations are minimal. In this section, we present another separation result: on *dcpo*'s, not all continuous valuations are point-continuous, a result which is known on general topological spaces, but unknown on *dcpo*'s with the Scott topology.

4.1 What are non-point-continuous valuations like?

Tix showed that, on a sober space X , the bounded continuous valuations that take only finitely many values are the simple valuations, namely the finite linear combinations $\sum_{i=1}^n a_i \delta_{x_i}$, where each coefficient a_i is in \mathbb{R}_+ and $x_i \in X$ (Tix, 1995, Satz 2.2). Hence, they are all point-continuous.

We will extend that result slightly in this subsection. We first recall that a non-empty subset A of a space X is *irreducible* if $A \subseteq B \cup C$, for closed subsets B and C of X , implies that $A \subseteq B$ or $A \subseteq C$. X is *sober* if every irreducible closed subset C of X is the closure of some unique singleton subset of X .

Lemma 19. *Let X be a topological space. For every irreducible closed subset C of X , let $\epsilon_C: \mathcal{O}X \rightarrow \mathbb{R}_+$ map every open subset U of X to 1 if U intersects C , to 0 otherwise. Then ϵ_C is a point-continuous valuation. If $C = \downarrow x$ for some point x , then $\epsilon_C = \delta_x$.*

Proof. It is clear that ϵ_C is strict. In order to show point-continuity, we assume $0 \leq r < \epsilon_C(U)$. We note that C must intersect U , say at x , and that $0 \leq r < 1$. Let $A \stackrel{\text{def}}{=} \{x\}$. For every open neighborhood V of A , V intersects C (at x), so $\epsilon_C(V) = 1 > r$.

In order to show modularity, we observe that, since C is irreducible, for all open subsets U and V of X , C intersects both U and V if and only if C intersects $U \cap V$. Hence, $\epsilon_C(U \cap V) = \min(\epsilon_C(U), \epsilon_C(V))$. Since C intersects U or V if and only if C intersects $U \cup V$, we have $\epsilon_C(U \cup V) = \max(\epsilon_C(U), \epsilon_C(V))$. Now for all real numbers a and b , $\max(a, b) + \min(a, b) = a + b$, so $\epsilon_C(U \cap V) + \epsilon_C(U \cup V) = \epsilon_C(U) + \epsilon_C(V)$.

When $C = \downarrow x$, for every $U \in \mathcal{O}X$ we have $\epsilon_C(U) = 1$ if and only if C intersects U , if and only if $x \in U$, if and only if $\delta_x(U) = 1$. \square

For every continuous valuation ν on a space X , let $\text{Val}(\nu)$ denote the set of nontrivial values $\{\nu(U) \mid U \in \mathcal{O}X, \nu(U) \neq 0, +\infty\}$ taken by ν .

Lemma 20. *Let ν be a continuous valuation on a space X , with the property that $\text{Val}(\nu)$ has a least element r . Then there is an irreducible closed subset C of X such that:*

- (1) $\nu' \stackrel{\text{def}}{=} \nu - r\epsilon_C$ is a continuous valuation;
- (2) $\text{Val}(\nu') \subseteq \{\nu - r \mid \nu \in \text{Val}(\nu), \nu \neq r\}$.

Proof. Let U_r be any open subset of X such that $\nu(U_r) = r \neq 0$. We consider the family \mathcal{U} of open subsets U of X such that $\nu(U \cap U_r) = 0$, or equivalently $\nu(U \cap U_r) < r$. That family contains the

empty set, and for any two elements U, V of \mathcal{U} , we have $v((U \cup V) \cap U_r) \leq v(U \cap U_r) + v(V \cap U_r) = 0$, so $U \cup V$ is in \mathcal{U} . It follows that \mathcal{U} is directed. Let U_* be the union of all the elements of \mathcal{U} . Then, $v(U_* \cap U_r) = \sup_{U \in \mathcal{U}} v(U \cap U_r) = 0$, so U_* is in \mathcal{U} . It follows that U_* is the largest element of \mathcal{U} . In particular, for every $U \in \mathcal{O}X$, $v(U \cap U_r) = 0$ if and only if $U \subseteq U_*$.

We define C as the complement of U_* . Since $v(U_r) \neq 0$, U_r is not included in U_* , so C intersects U_r . In particular, C is non-empty. For any two open subsets U and V of X that intersect C , we claim that $U \cap V$ also intersects C . This will show that C is irreducible. By assumption, neither U nor V is included in U_* , so $v(U \cap U_r) \geq r$ and $v(V \cap U_r) \geq r$. It follows that $v((U \cap V) \cap U_r) = v(U \cap U_r) + v(V \cap U_r) - v((U \cup V) \cap U_r) \geq 2r - r = r$, using modularity and the inequality $v((U \cup V) \cap U_r) \leq v(U_r) = r$. We conclude that $U \cap V$ is not included in U_* , hence intersects C .

(1) Let $v' \stackrel{\text{def}}{=} v - r\epsilon_C$. We first verify that v' is monotonic. Let $U \subseteq V$ be two open subsets of X . If $\epsilon_C(U) = \epsilon_C(V)$, then $v'(U) \leq v'(V)$. Therefore, let us assume that $\epsilon_C(U) \neq \epsilon_C(V)$, hence necessarily $\epsilon_C(U) = 0, \epsilon_C(V) = 1$. As a consequence, $v(U \cap U_r) = 0$ and $v(V \cap U_r) \geq r$. Then:

$$\begin{aligned} v'(U) &= v(U) = v(U \cup U_r) + v(U \cap U_r) - v(U_r) \\ &\quad \text{by modularity, and since } v(U_r) < +\infty \\ &= v(U \cup U_r) - r \\ &\leq v(V \cup U_r) - r \\ &= v(V) + v(U_r) - v(V \cap U_r) - r \\ &\quad \text{by modularity, and since } v(V \cap U_r) \leq v(U_r) < +\infty \\ &\leq v(V) + v(U_r) - r - r = v'(V). \end{aligned}$$

In particular, since $v'(\emptyset) = 0$, v' takes its values in $\overline{\mathbb{R}}_+$. It then follows from Lemma 9 that v' is a continuous valuation.

(2) Let $V' \stackrel{\text{def}}{=} \{v - r \mid v \in \text{Val}(v), v \neq r\}$. For every $U \in \mathcal{O}X$, if U intersects C then $v'(U) = v(U) - r$; hence if $v'(U) \neq 0, +\infty$, then $v'(U) \in V'$. Otherwise, $U \subseteq U_*$, so $v(U \cap U_r) = 0$, and therefore $v'(U) = v(U) = v(U \cup U_r) + v(U \cap U_r) - v(U_r) = v(U \cup U_r) - r$, using modularity and the fact that $v(U_r) = r < +\infty$. Hence if $v'(U) \neq 0, +\infty$, then $v'(U)$ is in V' . \square

Proposition 21. *Let X be a topological space. The bounded continuous valuations v on X that take only finitely many values are exactly the finite linear combinations $\sum_{i=1}^n a_i \epsilon_{C_i}$, where each C_i is irreducible closed and $a_i \in \mathbb{R}_+ \setminus \{0\}$.*

Proof. That $\sum_{i=1}^n a_i \epsilon_{C_i}$ only takes finitely many values is obvious. We prove the converse implication by induction on the number n of nonzero values taken by v . If $n = 0$, then v is the zero valuation. Otherwise, let r be the least nonzero value taken by v . We find C and v' as in Lemma 20. Since $\text{Val}(v')$ has one less element than $\text{Val}(v)$, we can apply the induction hypothesis, allowing us to conclude. \square

Proposition 22. *Every continuous valuation v on a topological space X that takes only finitely many distinct values is point-continuous.*

Proof. If v is bounded, then by Proposition 21, v is of the form $\sum_{i=1}^n a_i \epsilon_{C_i}$, where each C_i is irreducible closed and $a_i \in \mathbb{R}_+$, and ϵ_{C_i} is point-continuous. Any linear combination of point-continuous valuations is point-continuous (Heckmann, 1995, Section 3.2), so v is point-continuous.

Hence, we concentrate on the case where $v(X)$ is equal to $+\infty$. Let s be the greatest finite value that v takes. Consider the family \mathcal{S} of all opens U such that $v(U) \leq s$. For any $U, V \in \mathcal{S}$,

$v(U \cup V) = v(U) + v(V) - v(U \cap V)$, which is obviously a finite value. Hence, $U \cup V$ is in S . This implies that the family S is directed. Let U_s be the directed union of S , and C_∞ be the complement of U_s . We notice that $v(U_s) = \sup_{U \in S} v(U) = s$, and hence that U_s is a proper subset of X .

The collection \mathcal{F} of simple valuations $\sum_{i=1}^n a_i \delta_{x_i}$ such that every x_i is in C_∞ is directed: it is non-empty because C_∞ is non-empty, and any two elements $\sum_{i=1}^n a_i \delta_{x_i}$ and $\sum_{i=1}^n b_i \delta_{x_i}$ (which we take over the same set of points x_i , without loss of generality) have an upper bound, such as $\sum_{i=1}^n \max(a_i, b_i) \delta_{x_i}$. Let μ be the supremum of \mathcal{F} . Since the family of point-continuous valuations is closed under directed suprema (Heckmann, 1995, Section 3.2 (5)), μ is point-continuous.

We compute μ explicitly. For every open subset U of X , if $U \subseteq U_s$, namely if U and C_∞ are disjoint, then the value of any element $\sum_{i=1}^n a_i \delta_{x_i}$ of \mathcal{F} on U is zero, so $\mu(U) = 0$. Otherwise, let us pick an element x from $U \cap C_\infty$. Then $a\delta_x$ is in \mathcal{F} for every $a \in \mathbb{R}_+$, so $\mu(U) \geq a\delta_x(U) = a$ for every $a \in \mathbb{R}_+$, from which it follows that $\mu(U) = +\infty$.

Let $v|_{U_s}$ be the restriction of v to U_s , namely the map $V \mapsto v(U_s \cap V)$. This is a continuous valuation (Heckmann, 1995, Section 3.3). It is bounded by construction of U_s and takes only finitely many values. Therefore, as we have already seen, Proposition 21 entails that $v|_{U_s}$ is point-continuous.

We now observe that $v = v|_{U_s} + \mu$. For every $U \in \mathcal{O}X$, if $U \subseteq U_s$, then $v|_{U_s}(U) + \mu(U) = v(U \cap U_s) + 0 = v(U)$. Otherwise, $v|_{U_s}(U) + \mu(U) = v|_{U_s}(U) + (+\infty) = +\infty = v(U)$. Being a sum of two point-continuous valuations, v is point-continuous. □

4.2 The Sorgenfrey line

Let \mathbb{R} be the set of real numbers, with its usual metric topology.

The *Sorgenfrey line* \mathbb{R}_ℓ has the same set of points as \mathbb{R} , but its topology is generated by the half-open intervals $[a, b[$, $a < b$ (Sorgenfrey, 1947). The topology of \mathbb{R}_ℓ is finer than that of \mathbb{R} . \mathbb{R}_ℓ is a zero-dimensional, first-countable space (Goubault-Larrecq, 2013, Exercices 4.1.34, 4.7.17). It is paracompact hence T_4 , and Choquet-complete hence a Baire space (Goubault-Larrecq, 2013, Exercices 6.3.32, 7.6.11). \mathbb{R}_ℓ is not locally compact, as every compact subset of \mathbb{R}_ℓ has empty interior (Goubault-Larrecq, 2013, Exercise 4.8.5). In fact, \mathbb{R}_ℓ is not even consonant (Bouziad, 1996; Costantini and Watson, 1996). Although it is first-countable, \mathbb{R}_ℓ is not second-countable (Goubault-Larrecq, 2013, Exercise 6.3.10).

A *hereditarily Lindelöf space* is a space in which every family $(U_i)_{i \in I}$ of open subsets has a countable subfamily with the same union, or equivalently a space whose subspaces are all Lindelöf. Every second-countable space is hereditarily Lindelöf, but \mathbb{R}_ℓ is a counterexample to the reverse implication, as the following folklore result demonstrates.

Proposition 23. \mathbb{R}_ℓ is hereditarily Lindelöf.

Proof. Let us denote by U the interior of any set U in the topology of \mathbb{R} (not \mathbb{R}_ℓ).

Let $(U_i)_{i \in I}$ be any family of open subsets of \mathbb{R}_ℓ , $U \stackrel{\text{def}}{=} \bigcup_{i \in I} U_i$ and $U' \stackrel{\text{def}}{=} \bigcup_{i \in I} U_i$. Since \mathbb{R} is second-countable, hence hereditarily Lindelöf, there is a countable subset J of I such that $U' = \bigcup_{i \in J} U_i$.

We claim that $U \setminus U'$ is countable. For each point $x \in U \setminus U'$, x is in some U_i , hence in some basic open set $[a, b[\subseteq U_i$. Then x is also included in the smaller basic open set $[x, b[$. Let us write b as $x + \delta_x$, with $\delta_x > 0$. We observe that $]x, x + \delta_x[$ is included in U_i , hence in U' .

Let x and y be two points of $U \setminus U'$, with $x < y$. If $[x, x + \delta_x[$ and $[y, y + \delta_y[$ intersect, then y is in $[x, x + \delta_x[$, and since $x \neq y$, it follows that y is in $]x, x + \delta_x[$. That is impossible, since $]x, x + \delta_x[$ is included in U' and y is not in U' .

Therefore, for any two distinct points x and y of $U \setminus U'$, $[x, x + \delta_x[$ and $[y, y + \delta_y[$ are disjoint. We pick one rational number q_x in each set $[x, x + \delta_x[$ with $x \in U \setminus U'$: then $x \neq y$ implies $q_x \neq q_y$, and therefore $U \setminus U'$ is countable.

Let us now pick an index $i_x \in I$ such that $x \in U_{i_x}$, one for each $x \in U \setminus U'$. Then $U = U' \cup (U \setminus U') \subseteq U' \cup \bigcup_{x \in U \setminus U'} U_{i_x} \subseteq U$, so $(U_i)_{i \in I \cup \{i_x | x \in U \setminus U'\}}$ is a countable subfamily with the same union as our original family $(U_i)_{i \in I}$. □

Lemma 24. \mathbb{R} and \mathbb{R}_ℓ have the same Borel σ -algebra.

Proof. Let Σ be the Borel σ -algebra of \mathbb{R} , and Σ_ℓ be that of \mathbb{R}_ℓ . Clearly, $\Sigma \subseteq \Sigma_\ell$.

In the converse direction, we first claim that every open subset U of \mathbb{R}_ℓ is in Σ . For every $x \in U$, the union of all the intervals included in U and containing x is convex, hence is an interval I_x . Let a_x be its lower end and b_x be its upper end. We note that b_x cannot be in I_x , otherwise b_x would be in some basic open subset $[b_x, b_x + \delta_x[$ included in U , which would allow us to form a strictly larger interval included in U and containing x . Hence, I_x is equal to $[a_x, b_x[$ (or to $]a_x, b_x]$. In any case, $a_x < b_x$, so I_x contains a rational number q_x . Since the intervals I_x are pairwise disjoint, there are no more such intervals than there are rational numbers. Moreover, $\bigcup_{x \in U} I_x$ is equal to U . It follows that U is a countable union of pairwise disjoint intervals and is therefore in Σ .

Since Σ_ℓ is the smallest σ -algebra containing the open subsets of \mathbb{R}_ℓ , we conclude that $\Sigma_\ell \subseteq \Sigma$. □

Given a topological space X , a Borel measure on X is τ -smooth if and only if its restriction to the lattice $\mathcal{O}X$ of open subsets of X is a continuous valuation. Adamski showed that a space X is hereditarily Lindelöf if and only if every Borel measure on X is τ -smooth (Adamski, 1977, Theorem 3.1).

Proposition 25. For every Borel measure μ on \mathbb{R} , the restriction of μ to the open subsets of \mathbb{R}_ℓ is a continuous valuation.

Proof. By Lemma 24, μ is also a Borel measure on \mathbb{R}_ℓ . We then apply Adamski's theorem, thanks to Proposition 23. □

Corollary 26. The restriction λ of the Lebesgue measure on \mathbb{R} to $\mathcal{O}\mathbb{R}_\ell$ is a continuous valuation. □

4.3 The compact subsets of \mathbb{R}_ℓ

We recall that every compact subset Q of \mathbb{R}_ℓ has empty interior. For completeness, we give the proof here. Let us assume a compact subset Q of \mathbb{R}_ℓ with non-empty interior. Q contains a basic open set $[a, b[$ with $a < b$. $[a, b[$ is not only open, but also closed, since it is the complement of the open set $] - \infty, a[\cup [b, +\infty[$. (Those two sets are open, being equal to $\bigcup_{m \in \mathbb{N}} [a - m - 1, a - m[$ and to $\bigcup_{n \in \mathbb{N}} [b + n, b + n + 1[$, respectively). Being closed in a compact set, $[a, b[$ is compact. But the open cover $([a, b - \epsilon]_{\epsilon \in]0, b-a[}$ of $[a, b[$ has no finite subcover: contradiction.

We use the following folklore result. The fact that every compact subset of \mathbb{R}_ℓ is countable is the only thing we will need to know in later sections, together with Corollary 32 below, but we think that giving a complete characterization of the compact subsets of \mathbb{R}_ℓ is interesting in its own right and may help one understand better what they look like.

Lemma 27. Every compact subset Q of \mathbb{R}_ℓ is countable, bounded, and is well founded in the ordering \geq .

Proof. For every point x of Q , the family of open sets $] - \infty, x - \epsilon[$ ($\epsilon > 0$) plus $[x, +\infty[$ is an open cover of Q (in fact, of the whole of \mathbb{R}_ℓ), hence contains a finite subcover. It follows that Q is contained in $] - \infty, x - \epsilon_x[\cup [x, +\infty[$ for some $\epsilon_x > 0$. Equivalently, Q contains no point in $[x - \epsilon_x, x[$.

From this, we deduce that $[x - \epsilon_x, x[$ and $[y - \epsilon_y, y[$ are disjoint for any two distinct points x and y of Q . Indeed, without loss of generality, let us assume that $y < x$. Since y is in Q , it is not in $[x - \epsilon_x, x[$, so $y > x$ (which is impossible), or $y \leq x - \epsilon_x$. Then, $[y - \epsilon_y, y[$ lies entirely to the left of $[x - \epsilon_x, x[$ and does not intersect it.

Next, for each $x \in Q$, we pick a rational number q_x in $[x - \epsilon_x, x[$. By the disjointness property we just proved, the map $x \in Q \mapsto q_x$ is injective, so Q is countable.

Since Q is compact in \mathbb{R}_ℓ and the topology of \mathbb{R}_ℓ is finer than that of $\overline{\mathbb{R}}_+$, Q is also compact in \mathbb{R} , hence is bounded.

Let us imagine that Q contains an infinite increasing sequence $r_0 < r_1 < \dots < r_n < \dots$. Let also $r \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} r_n$. Then the open sets $] - \infty, r_0[$, $[r_0, r_1[$, \dots , $[r_n, r_{n+1}[$, \dots , and $[r, +\infty[$ (if $r < +\infty$, otherwise we ignore the last interval) form an open cover of Q without a finite subcover. It follows that Q is well founded in the ordering \geq . □

We now give a few seemingly lesser known results. Given a compact subset Q of \mathbb{R}_ℓ , Q is a chain, namely a totally ordered subset of (\mathbb{R}, \geq) . Notice that we use the reverse ordering \geq : Lemma 27 tells us that Q is even a well-founded chain. A chain A in a poset (P, \sqsubseteq) is a *subdcpo* of P if and only if for every non-empty (equivalently, directed) subset D of A , the supremum of D exists in P and is in A .

Lemma 28. *Every compact subset Q of \mathbb{R}_ℓ is a subdcpo of (\mathbb{R}, \geq) : for every non-empty subset D of Q , $\inf D$ is in Q . If Q is non-empty, then Q has a least element in the usual ordering \leq .*

Proof. We first need to note that a net $(x_i)_{i \in I, \sqsubseteq}$ converges to x in \mathbb{R}_ℓ if and only if x_i tends to x from the right, namely: for every $\epsilon > 0$, $x \leq x_i < x + \epsilon$ for i large enough (Goubault-Larrecq, 2013, Exercise 4.7.6).

Let D be any non-empty subset of Q , and let $r \stackrel{\text{def}}{=} \inf D$. By Lemma 27, Q is bounded, so r is a well-defined real number. Since Q is well founded, D is isomorphic to a unique ordinal β , and we can write the elements of D as r_α , $\alpha < \beta$, in such a way that for all $\alpha, \alpha' < \beta$, $\alpha \leq \alpha'$ if and only if $r_\alpha \geq r_{\alpha'}$. The net $(r_\alpha)_{\alpha < \beta, \leq}$ then converges to r from the right. Since \mathbb{R}_ℓ is T_2 , its compact subset Q is closed, so r is in Q .

If Q is non-empty, we can take $D \stackrel{\text{def}}{=} Q$. Then $\inf D \in Q$ is the least element of Q . □

Another way of expressing Lemma 28 together with the well-foundedness property of Lemma 27 is to say the following. The fact that β_Q is not a limit ordinal is due to the fact that Q must be empty or have a least element, namely that β_Q must be equal to 0 or have a largest element.

Lemma 29. *For every compact subset Q of \mathbb{R}_ℓ , (Q, \geq) is order-isomorphic to a unique ordinal β_Q ; β_Q is not a limit ordinal, and the order-isomorphism is a Scott-continuous map from β_Q to (Q, \geq) .*

Lemma 30. *Let β be a non-limit ordinal, and $f: \beta \rightarrow (\mathbb{R}, \geq)$ be a Scott-continuous map. The image $\text{Im } f$ of f is compact in \mathbb{R}_ℓ .*

Proof. Since β is a non-limit ordinal, it is a dcpo. Its finite elements are the ordinals $\alpha < \beta$ that are not limit ordinals, and then it is easy to see that β is an algebraic domain. It is also a complete lattice, and every algebraic complete lattice is Lawson-compact (Gierz et al., 2003, Corollary III.1.11).

We claim that f is continuous from β , with its Lawson topology, to \mathbb{R}_ℓ . Let $[a, b[$ be any subbasic open subset of \mathbb{R}_ℓ . We aim to show that $f^{-1}([a, b[)$ is Lawson-open. If $f^{-1}(] - \infty, a[)$ is empty, then $f^{-1}([a, b[) = f^{-1}(] - \infty, b[)$, which is Scott-open since f is Scott-continuous; in

particular, it is Lawson-open. So let us assume that $f^{-1}(] - \infty, a[)$ is non-empty. Since β is well founded, $f^{-1}(] - \infty, a[)$ has a least element α . Then $f^{-1}(] - \infty, a[) = \uparrow\alpha$, and therefore $f^{-1}([a, b[) = f^{-1}(] - \infty, b[) \setminus f^{-1}(] - \infty, a[) = f^{-1}(] - \infty, b[) \setminus \uparrow\alpha$ is Lawson-open.

Since β is compact in the Lawson topology, it follows that $\text{Im } f$ is compact in \mathbb{R}_ℓ . □

In particular, we obtain the following converse to Lemma 28.

Lemma 31. *Every well-founded subdcpo Q of (\mathbb{R}, \geq) is compact in \mathbb{R}_ℓ .*

Proof. Q is order-isomorphic to a unique ordinal β_Q through some map $f: \beta_Q \rightarrow (Q, \geq)$. Since Q has a least element, β_Q has a largest element, so β_Q cannot be a limit ordinal. Let D be any non-empty family D in β_Q . Since Q is a subdcpo of (\mathbb{R}, \geq) , $\inf f(D)$ is an element $f(\alpha)$ of Q . Since f is an order-isomorphism, α is the supremum of D , and therefore f is Scott-continuous. We can now apply Lemma 30. □

Corollary 32. *Let x be any real number and $x_0 > x_1 > \dots > x_n \dots$ be any decreasing sequence of real numbers such that $\inf_{n \in \mathbb{N}} x_n = x$. Then $\{x_0, x_1, \dots, x_n, \dots\} \cup \{x\}$ is compact in \mathbb{R}_ℓ .* □

Together, Lemmas 27, 29, and 31 imply the following.

Theorem 33. *The compact subsets of \mathbb{R}_ℓ are exactly the well-founded subdcpos of (\mathbb{R}, \geq) . They are all countable.* □

Remark 34. In general, any well-founded chain in (\mathbb{R}, \geq) is countable. Conversely, for every countable ordinal β , there is a well-founded chain of (\mathbb{R}, \geq) that is order-isomorphic to β . This is proved by induction on β , using the fact that every countable ordinal has countable cofinality, namely is the supremum of countably many strictly lower countable ordinals. All this is folklore and is left as an exercise.

Remark 35. A space X is *consonant* if and only if, for every Scott-open subset \mathcal{U} of $\mathcal{O}X$, for every $U \in \mathcal{U}$, there is a compact saturated subset Q of X such that $U \in \blacksquare Q \subseteq \mathcal{U}$, where $\blacksquare Q$ is the set of open neighborhoods of Q . We now have enough to give an elementary proof that \mathbb{R}_ℓ is not consonant (Bouziad, 1996; Costantini and Watson, 1996). Let us pick any real number $r > 0$. By Corollary 26, $\mathcal{U} \stackrel{\text{def}}{=} \lambda^{-1}(]r, +\infty[)$ is a Scott-open subset of $\mathcal{O}\mathbb{R}_\ell$. For every compact (saturated) subset Q of \mathbb{R}_ℓ , Q is countable. Let us write Q as $\{x_n \mid n \in I\}$, where I is some subset of \mathbb{N} . Then $V \stackrel{\text{def}}{=} \bigcup_{n \in I} [x_n, x_n + r/2^{n+1}[$ is an open neighborhood of Q such that $\lambda(V) \leq \sum_{n \in I} r/2^{n+1} \leq r$, so V is not in \mathcal{U} . It follows that no set of the form $\blacksquare Q$ is included in \mathcal{U} .

4.4 The dcpo $\mathcal{Q}\mathbb{R}_\ell$

Given any topological space X , we can form the set $\mathcal{Q}X$ of all non-empty compact saturated subsets of X . $\mathcal{Q}X$ is a poset under the reverse inclusion ordering \supseteq called the *Smyth powerdomain* of X . When X is well-filtered, this is a dcpo, where suprema of directed families $(Q_i)_{i \in I}$ are their intersection $\bigcap_{i \in I} Q_i$ (Goubault-Larrecq, 2013, Proposition 8.3.25). This is notably the case when $X = \mathbb{R}_\ell$, since \mathbb{R}_ℓ is T_2 , hence sober, hence well-filtered. Note also that, in that case, every subset is saturated, so we may safely omit “saturated” from the description of elements of $\mathcal{Q}\mathbb{R}_\ell$.

There are at least two topologies of interest on $\mathcal{Q}X$. One is the Scott topology on the poset $(\mathcal{Q}X, \supseteq)$. Another one is the *upper Vietoris topology*, whose basic open sets are the sets $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid U \subseteq Q\}$.

$\{Q \in \mathcal{Q}X \mid Q \subseteq U\}$, where U ranges over the open subsets of X . When X is well-filtered (e.g., if $X = \mathbb{R}_\ell$), $\square U$ is Scott-open, and hence the Scott topology is finer than the upper Vietoris topology.

The set $\text{Max } \mathcal{Q}\mathbb{R}_\ell$ of maximal points of $\mathcal{Q}\mathbb{R}_\ell$ consists of the one-element compact sets $\{x\}$, $x \in \mathbb{R}$. By equating them with x , we equate the set $\text{Max } \mathcal{Q}\mathbb{R}_\ell$ with \mathbb{R}_ℓ . This allows us to write $U \cap \mathbb{R}_\ell$ for any subset U of $\mathcal{Q}\mathbb{R}_\ell$.

Lemma 36. *For every Scott-open subset U of $\mathcal{Q}\mathbb{R}_\ell$, $U \cap \mathbb{R}_\ell$ is open in \mathbb{R}_ℓ .*

Proof. Let x be an arbitrary point of $U \cap \mathbb{R}_\ell$. We claim that $U \cap \mathbb{R}_\ell$ contains an interval $[x, x + \epsilon[$ for some $\epsilon > 0$. We reason by contradiction, and we assume that every interval $[x, x + \epsilon[$ contains a point outside $U \cap \mathbb{R}_\ell$.

We use this to build a sequence of points $x_0 > x_1 > \dots > x_n > \dots \geq x$ as follows. Since $U \cap \mathbb{R}_\ell$ does not contain $[x, x + 1[$, there is a point x_0 in $[x, x + 1[$ that is not in $U \cap \mathbb{R}_\ell$. This cannot be x , since x is in $U \cap \mathbb{R}_\ell$. Hence, $\min(x_0, x + 1/2) > x$. Since $U \cap \mathbb{R}_\ell$ does not contain $[x, \min(x_0, x + 1/2)[$, there is a point x_1 in $[x, \min(x_0, x + 1/2)[$ that is not in $U \cap \mathbb{R}_\ell$. Again, x_1 is different from x . This allows us to build the interval $[x, \min(x_1, x + 1/4)[$, and as before, this must contain a point x_2 outside $U \cap \mathbb{R}_\ell$. By induction, this allows us to define points x_n outside $U \cap \mathbb{R}_\ell$ such that $x_{n+1} \in [x, \min(x_n, x + 1/2^{n+1})[$. In particular, $x_0 > x_1 > \dots > x_n > \dots \geq x$. Also, $\inf_{n \in \mathbb{N}} x_n = x$.

For each $n \in \mathbb{N}$, let Q_n be the set $\{x_m \mid m \geq n\} \cup \{x\}$. This is compact in \mathbb{R}_ℓ by Corollary 32.

It is clear that Q_n is non-empty, and that $\bigcap_{n \in \mathbb{N}} Q_n = \{x\}$, which is in U . Since U is Scott-open, some Q_n must be in U . This implies that $\{x_n\}$, which is included, hence above Q_n in $\mathcal{Q}\mathbb{R}_\ell$, is also in U . Therefore, x_n is in $U \cap \mathbb{R}_\ell$; but that is impossible, since all the points x_n were built so as to lie outside $U \cap \mathbb{R}_\ell$. □

A *dcpo model* of a T_1 space X is a dcpo P such that $\text{Max } P$, the subset of maximal elements of P with the subspace topology, from the Scott topology on P , is homeomorphic to X . The following is a special case of Corollary 2.12 of He et al. (2019), which says that for every T_1 , first-countable and well-filtered space, $\mathcal{Q}X$ is a dcpo model of X .

Theorem 37. *$\mathcal{Q}\mathbb{R}_\ell$ is a dcpo model of \mathbb{R}_ℓ .*

Proof. By Lemma 36, every open subset in the subspace topology on $\text{Max } \mathcal{Q}\mathbb{R}_\ell$ is open in \mathbb{R}_ℓ . Conversely, for every open subset U of \mathbb{R}_ℓ , $\square U$ is a Scott-open subset of $\mathcal{Q}\mathbb{R}_\ell$ whose intersection with \mathbb{R}_ℓ is equal to U , so U is open in the subspace topology on $\text{Max } \mathcal{Q}\mathbb{R}_\ell$. □

Remark 38. Actually, one can expect more in this case. Xu and Yang (2021) proved that for a first-countable well-filtered space X , in which each compact saturated subset has countable minimal elements (in the specialization order), the Scott and the upper Vietoris topologies coincide on $\mathcal{Q}X$. As we have seen from above that \mathbb{R}_ℓ does satisfy these properties, hence the Scott and the upper Vietoris topologies coincide on $\mathcal{Q}\mathbb{R}_\ell$. This is slightly stronger than Theorem 37. We speak in full of the development of Theorem 37 in order to keep this note self-contained.

4.5 A continuous, non-point-continuous valuation on $\mathcal{Q}\mathbb{R}_\ell$

By Proposition 25, every Borel measure μ on \mathbb{R}_ℓ defines a continuous valuation by restriction to $\mathcal{O}\mathbb{R}_\ell$, and we again write that continuous valuation as μ . By Theorem 37, the map $x \mapsto \{x\}$ is a topological embedding of \mathbb{R}_ℓ into $\mathcal{Q}\mathbb{R}_\ell$. The image of μ by that embedding is a continuous valuation $\bar{\mu}$ on $\mathcal{Q}\mathbb{R}_\ell$. Explicitly, we have

$$\bar{\mu}(U) \stackrel{\text{def}}{=} \mu(U \cap \mathbb{R}_\ell) \tag{1}$$

for every Scott-open subset U of $\mathcal{Q}\mathbb{R}_\ell$.

Theorem 39. *Let μ be any Borel measure on \mathbb{R}_ℓ with the property that there is an open subset U of \mathbb{R}_ℓ such that $0 < \mu(U) < +\infty$, and $\underline{\mu}(\{x\}) = 0$ for every $x \in U$. Then the continuous valuation $\bar{\mu}$ is not point-continuous. In particular, $\bar{\lambda}$ is not point-continuous.*

Proof. Since $\mu(U) > 0$ and since $\bar{\mu}(\square U) = \mu(U)$, we can find a real number r such that $0 < r < \bar{\mu}(\square U)$.

We will show that, for every finite subset $A \stackrel{\text{def}}{=} \{Q_1, \dots, Q_m\}$ of $\square U$, there is an open neighborhood \mathcal{V} of A such that $\bar{\mu}(\mathcal{V}) \leq r$.

We make the following preliminary claim (*): for every $x \in U$, for every $a > 0$, there is an $\epsilon > 0$ such that $[x, x + \epsilon[\subseteq U$ and $\mu([x, x + \epsilon[) \leq a$. Indeed, it is a general property of measures that $\mu(\bigcap_{n \in \mathbb{N}}^\downarrow E_n) = \inf_{n \in \mathbb{N}} \mu(E_n)$ for any decreasing family of measurable subsets E_n such that $\mu(E_n) < +\infty$ for at least one n . Hence, $\mu(\{x\}) = \mu(\bigcap_{n \in \mathbb{N}}^\downarrow [x, x + \epsilon_0/2^n[) = \inf_{n \in \mathbb{N}} \mu([x, x + \epsilon_0/2^n[)$, where $\epsilon_0 > 0$ is chosen so that $[x, x + \epsilon_0[\subseteq U$. Since $\mu(\{x\}) = 0$, there is an $n \in \mathbb{N}$ such that $\mu([x, x + \epsilon_0/2^n[< a$.

Let $s > 0$ be such that $ms \leq r$. For every $i \in \{1, \dots, m\}$, Q_i is countable (Lemma 27), so let us write it as $\{x_{i0}, x_{i1}, \dots\}$. (We allow for infinite repetitions of elements in order not to have to make a special case when Q_i is finite.) For every $i \in \{1, \dots, m\}$, for every $j \in \mathbb{N}$, we use (*) to find a number $\epsilon_{ij} > 0$ such that $\mu([x_{ij}, x_{ij} + \epsilon_{ij}[) \leq s/2^{j+1}$. Let $V_i \stackrel{\text{def}}{=} \bigcup_{j \in \mathbb{N}} [x_{ij}, x_{ij} + \epsilon_{ij}[$. We note that $\mu(V_i) \leq \sum_{j \in \mathbb{N}} \mu([x_{ij}, x_{ij} + \epsilon_{ij}[) = \sum_{j \in \mathbb{N}} s/2^{j+1} = s$. We now define V as $\bigcup_{i=1}^m V_i$. Then $\mu(V) \leq \sum_{i=1}^m \mu(V_i) \leq ms \leq r$.

Clearly, Q_i is included in V_i , hence in V , for every $i \in \{1, \dots, m\}$, so A is included in $\square V$. We define \mathcal{V} as $\square V$. Then $\bar{\mu}(\mathcal{V}) = \mu(V) \leq r$. □

4.6 More remarks on $\mathcal{Q}\mathbb{R}_\ell$

In Lyu and Jia (2019, Theorem 3.1), Lyu and the second author showed that a space X is locally compact if and only if that $\mathcal{Q}X$ is core-compact in the upper Vietoris topology. It is easy to see that \mathbb{R}_ℓ is not locally compact as the interior of each compact set is empty. So $\mathcal{Q}\mathbb{R}_\ell$ is not core-compact in the upper Vietoris topology. By Remark 38, we have

Proposition 40. *$\mathcal{Q}\mathbb{R}_\ell$ is not core-compact in its Scott topology.*

\mathbb{R}_ℓ exhibits a diverse mix of pleasant and unpleasant properties, and so does $\mathcal{Q}\mathbb{R}_\ell$. While Proposition 40 would be on the unpleasant side, the following shows more regularity.

Proposition 41. *$\mathcal{Q}\mathbb{R}_\ell$ is sober.*

Proof. Theorem 3.13 of Heckmann and Keimel (2013) states that, for any topological space X , X is sober if and only if $\mathcal{Q}X$ is sober in the upper Vietoris topology. Since \mathbb{R}_ℓ is T_2 , it is sober. By Remark 38, the upper Vietoris topology coincides with the Scott topology on $\mathcal{Q}\mathbb{R}_\ell$, so $\mathcal{Q}\mathbb{R}_\ell$ is sober. □

In particular, $\mathcal{Q}\mathbb{R}_\ell$ is well-filtered, something we can rederive in another way.

Proposition 42. *For every well-filtered, coherent space, $\mathcal{Q}X$ is a meet-continuous dcpo inf-semilattice, which is well-filtered and coherent in its Scott topology. Hence, $\mathcal{Q}\mathbb{R}_\ell, \mathcal{Q}^2\mathbb{R}_\ell, \dots$ all are meet-continuous dcpo inf-semilattices, which are well-filtered and coherent in their Scott topologies.*

Proof. Let X be a well-filtered, coherent space. In a well-filtered space, filtered intersections $\bigcap_{i \in I} Q_i$ of compact saturated subsets are compact saturated, and are non-empty if all the sets Q_i are non-empty. Hence, $\mathcal{Q}X$ is a dcpo.

Given any two elements Q and Q' of $\mathcal{Q}X$, their infimum is $Q \cup Q'$. Since directed suprema are filtered intersections, and intersections commute with binary unions, $\mathcal{Q}\mathbb{R}_\ell$ is a meet-continuous inf-semilattice.

For every dcpo P , let its *lifting* P_\perp be P plus a fresh element \perp below all elements of P . It is an easy exercise to show that the Scott-open subsets of P_\perp are those of P , plus P_\perp itself, and that the compact saturated subsets of P_\perp are those of P plus P_\perp . It follows that P is well-filtered if and only if P_\perp is.

A bounded-complete dcpo is one in which every (upper) bounded family has a least upper bound, or equivalently in which every non-empty family has a greatest lower bound. $\mathcal{Q}X$ is not bounded-complete in general, since the empty family has no least upper bound unless X is compact.

However, $(\mathcal{Q}X)_\perp$ is bounded-complete: the least upper bound of the empty set is \perp , and the least upper bound of any non-empty set A bounded by some element Q_0 of $\mathcal{Q}X$ is $\bigcap A$, which is compact because \mathbb{R}_ℓ is T_2 , and non-empty because it contains Q_0 .

Corollary 3.2 of Xi and Lawson (2017) shows that every bounded-complete dcpo is well-filtered in its Scott topology. Hence, $(\mathcal{Q}X)_\perp$ is well-filtered, and therefore $\mathcal{Q}X$ is well-filtered as well.

Lemma 3.1 of Jia et al. (2016) states that any well-filtered dcpo X in which $\uparrow x \cap \uparrow y$ is compact saturated for all $x, y \in X$ is coherent. For all $Q, Q' \in \mathcal{Q}X$, $\uparrow Q \cap \uparrow Q' = \uparrow(Q \cap Q')$, and $Q \cap Q'$ is again in $\mathcal{Q}X$ since X is coherent. In particular, $\uparrow Q \cap \uparrow Q'$ is a compact saturated subset of $\mathcal{Q}X$. Therefore $\mathcal{Q}X$ is coherent.

Finally, \mathbb{R}_ℓ is T_2 , hence (sober hence) well-filtered, and coherent, so we may apply the above to $X \stackrel{\text{def}}{=} \mathbb{R}_\ell$, then to $X \stackrel{\text{def}}{=} \mathcal{Q}\mathbb{R}_\ell$, and so on. □

5. Concluding Remarks

We have given two concrete examples (Corollary 18, Theorem 39) on dcpo's to separate minimal valuations, point-continuous valuations, and continuous valuations, showing these three classes of valuations differ from each other.

In Jia et al. (2021), the Fubini-type equation:

$$\int_{x \in D} \int_{y \in E} h(x, y) dv d\xi = \int_{y \in E} \int_{x \in D} h(x, y) d\xi dv \tag{2}$$

is established when either ν or ξ is point-continuous, where D and E are dcpo's and $h: D \times E \rightarrow \overline{\mathbb{R}}_+$ are Scott-continuous. (For a definition of the integration, see Jones (1990).) This is crucial in proving that Heckmann's point-continuous valuations monad is commutative over **DCPO**. However, it is unknown whether the Equation (2) holds for general continuous valuations ν and ξ , a crucial question in establishing commutativity of the valuations monad \mathcal{V} on **DCPO**. The aforementioned result in Jia et al. (2021) entails that any valuations ν and ξ that possibly fail Equation (2) must also fail to be point-continuous. Hence, if one aims to find examples on dcpo's to invalidate Equation (2), the valuations in desire must be non-point-continuous valuations. For the first time, we have given a continuous valuation $\bar{\lambda}$ that is not point-continuous on dcpo's, but more non-point-continuous valuations (of different types from those in Theorem 39) are needed before they are sent to test Equation (2).

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