Canad. J. Math. 2025, pp. 1–44
http://dx.doi.org/10.4153/S0008414X24000907

The imprimitivity Fell bundle

Anna Duwenig

Abstract. Given a full right-Hilbert C*-module $\mathbf{X}$ over a C*-algebra A, the set $\mathbb{K}_A(\mathbf{X})$ of A-compact operators on $\mathbf{X}$ is the (up to isomorphism) unique C*-algebra that is strongly Morita equivalent to the coefficient algebra A via $\mathbf{X}$. As a bimodule, $\mathbb{K}_A(\mathbf{X})$ can also be thought of as the balanced tensor product $\mathbf{X} \otimes_A \mathbf{X}^{\mathrm{op}}$, and so the latter naturally becomes a C*-algebra. We generalize both of these facts to the world of Fell bundles over groupoids: Suppose $\mathscr{B}$ is a Fell bundle over a groupoid $\mathcal{H}$ and $\mathscr{M}$ is an upper semi-continuous Banach bundle over a principal $\mathcal{H}$-space X. If $\mathscr{M}$ carries a right-action of $\mathscr{B}$ and a sufficiently nice $\mathscr{B}$-valued inner product, then its *imprimitivity Fell bundle* $\mathbb{K}_{\mathscr{B}}(\mathscr{M}) = \mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ is a Fell bundle over the imprimitivity groupoid of X, and it is the unique Fell bundle that is equivalent to $\mathscr{B}$ via $\mathscr{M}$. We show that $\mathbb{K}_{\mathscr{B}}(\mathscr{M})$ generalizes the "higher order" compact operators of Abadie–Ferraro in the case of saturated bundles over groups, and that the theorem recovers results such as Kumjian's Stabilization trick.

1 Introduction

Suppose a groupoid $\mathcal{H}$ acts on the right of a topological space X, meaning that we have a continuous surjection $\sigma\colon X \to \mathcal{H}^{(0)}$ (the *anchor map* of the action) and a continuous map

$$X\,{}_{\sigma}{*}_{r}\,\mathcal{H} := \{(x,h) \in X \times \mathcal{H} : \sigma(x) = r_{\mathcal{H}}(h)\} \to X, \quad (x,h) \mapsto x \triangleleft h. \tag{1.1}$$

Let us further assume that X and $\mathcal{H}$ are locally compact Hausdorff, that the action is free and proper (i.e., X is a *principal (right) $\mathcal{H}$-space*; [21, p. 6]), and that the anchor map σ is an open map. Out of X, we can build a *left* $\mathcal{H}$-space X^{op} as follows: as a topological space, it is just X, but we write its elements with a superscript-op to avoid confusion. Its left action uses the anchor map $\sigma^{\mathrm{op}}\colon x^{\mathrm{op}} \mapsto \sigma(x)$, and the action is given by $h \triangleright x^{\mathrm{op}} = (x \triangleleft h^{-1})^{\mathrm{op}}$ for any h with $s_{\mathcal{H}}(h) = \sigma^{\mathrm{op}}(x^{\mathrm{op}})$. Whenever σ^{op} appears in a subscript, we will drop its superscript and simply write σ. Now consider

$$X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}} = \{(x, y^{\mathrm{op}}) \in X \times X^{\mathrm{op}} : \sigma(x) = \sigma^{\mathrm{op}}(y^{\mathrm{op}})\}.$$

With the subspace topology of the product topology, $X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}}$ is locally compact Hausdorff. For any given $(x, y^{\mathrm{op}}) \in X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}}$, consider the orbit

$$[x, y^{\mathrm{op}}] := \{(x \triangleleft h, h^{-1} \triangleright y^{\mathrm{op}}) : h \in \sigma(x)\mathcal{H}\}$$

Received by the editors March 4, 2024; accepted September 30, 2024.
The work was supported by Methusalem grant METH/21/03 — long term structural funding of the Flemish Government, and an FWO Senior Postdoctoral Fellowship (project number 1206124N).
AMS subject classification: **46L55, 46L05, 22A22.**
Keywords: Fell bundle, groupoid equivalence, upper semi-continuous Banach bundle.

of (x, y^{op}) under the "diagonal" $\mathcal{H}$-action on $X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}}$. Since $\mathcal{H}$ is assumed to have open source map and to act properly on X, it follows that the quotient $X \times_{\mathcal{H}} X^{\mathrm{op}}$ by this equivalence relation is also a locally compact Hausdorff space [28, Proposition 2.18], and likewise is the quotient $X/\mathcal{H}$.

We equip $X \times_{\mathcal{H}} X^{\mathrm{op}}$ with the structure of a groupoid with unit space $X/\mathcal{H}$: its source and range maps are given by

$$s[x, y^{\mathrm{op}}] = [y, y^{\mathrm{op}}] = r[y, z^{\mathrm{op}}],$$

and we can identify $[y, y^{\mathrm{op}}]$ with $y \triangleleft \mathcal{H} \in X/\mathcal{H}$. Given $\xi, \eta \in X \times_{\mathcal{H}} X^{\mathrm{op}}$ with $s(\xi) = r(\eta)$, we can find representatives (x, y^{op}) and (y, z^{op}) of ξ respectively η with matching component "in the middle", and then define their product by $\xi\eta = [x, z^{\mathrm{op}}]$.

The groupoid $X \times_{\mathcal{H}} X^{\mathrm{op}}$ acts on the left of X: we define the anchor map

$$\rho\colon X \to (X \times_{\mathcal{H}} X^{\mathrm{op}})^{(0)} \cong X/\mathcal{H} \quad \text{by} \quad \rho(x) = [x, x^{\mathrm{op}}] \triangleq x \triangleleft \mathcal{H}. \tag{1.2}$$

Note that ρ is open [28, Proposition 2.12] and continuous. If $(\xi, y) \in (X \times_{\mathcal{H}} X^{\mathrm{op}})\,{}_s{*}_\rho\, X$, then $s(\xi) = \rho(y)$ implies that there exists $x \in X$ such that $\xi = [x, y^{\mathrm{op}}]$; this x is unique by freeness of the $\mathcal{H}$-action on X. We can therefore define $\xi \triangleright y = x$, i.e.,

$$X \times_{\mathcal{H}} X^{\mathrm{op}} \curvearrowright X\colon \qquad [x, y^{\mathrm{op}}] \triangleright y = x. \tag{1.3}$$

Remark 1.1 Suppose X is a $(\mathcal{G}, \mathcal{H})$-groupoid equivalence (see [21]) with anchor maps $\sigma\colon X \to \mathcal{H}^{(0)}$ and $\rho\colon X \to \mathcal{G}^{(0)}$; these maps are open by definition. If $\sigma(x) = \sigma(y)$, then there exists a unique element ${}_{\mathcal{G}}^{\;X}\{x \mid y^{\mathrm{op}}\}$ in $\mathcal{G}$ such that $x = {}_{\mathcal{G}}^{\;X}\{x \mid y^{\mathrm{op}}\} \triangleright y$. Indeed, since σ is assumed to induce a homeomorphism $\tilde{\sigma}\colon \mathcal{G}\backslash X \to \mathcal{H}^{(0)}$, the equality $\sigma(x) = \sigma(y)$ means exactly that $\mathcal{G} \triangleright x = \mathcal{G} \triangleright y$, and so freeness of the $\mathcal{G}$-action implies the existence of the unique ${}_{\mathcal{G}}^{\;X}\{x \mid y^{\mathrm{op}}\}$ which transforms y into x. Since the action is proper and ρ is open, the surjective map ${}_{\mathcal{G}}^{\;X}\{\_ \mid \_\}\colon X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}} \to \mathcal{G}$ is continuous and open, and it factors through the quotient $X \times_{\mathcal{H}} X$ of $X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}}$ by the diagonal $\mathcal{H}$-action, yielding a homeomorphism $X \times_{\mathcal{H}} X^{\mathrm{op}} \cong \mathcal{G}$. In fact, equipping $X \times_{\mathcal{H}} X^{\mathrm{op}}$ with the groupoid structure described above, this map is an isomorphism of topological groupoids.

Likewise, if $\rho(x) = \rho(y)$, we write $\{x^{\mathrm{op}} \mid y\}_{\mathcal{H}}^{X}$ for the unique element of $\mathcal{H}$ such that $x \triangleleft \{x^{\mathrm{op}} \mid y\}_{\mathcal{H}}^{X} = y$, and $\{\_ \mid \_\}_{\mathcal{H}}^{X}\colon X^{\mathrm{op}}\,{}_\rho{*}_\rho\, X \to \mathcal{H}$ induces an isomorphism $X^{\mathrm{op}} \times_{\mathcal{G}} X \cong \mathcal{H}$ of topological groupoids. A quick computation shows that the following equalities hold (wherever one side of the equation makes sense):

$${}_{\mathcal{G}}^{\;X}\{x \mid y^{\mathrm{op}}\} \triangleright z = x \triangleleft \{y^{\mathrm{op}} \mid z\}_{\mathcal{H}}^{X} \tag{1.4}$$

$$\begin{aligned} {}_{\mathcal{G}}^{\;X}\{x \mid (y \triangleleft h^{-1})^{\mathrm{op}}\} &= {}_{\mathcal{G}}^{\;X}\{x \triangleleft h \mid y^{\mathrm{op}}\} \\ \{(g^{-1} \triangleright y)^{\mathrm{op}} \mid z\}_{\mathcal{H}}^{X} &= \{y^{\mathrm{op}} \mid g \triangleright z\}_{\mathcal{H}}^{X}. \end{aligned} \tag{1.5}$$

Example 1.2 Any groupoid $\mathcal{H}$ acts freely and properly on $X = \mathcal{H}$; say, on the right. The anchor map is then given by $\sigma = s_{\mathcal{H}}\colon X \to \mathcal{H}^{(0)}$, and the associated imprimitivity groupoid $\mathcal{G} = \mathcal{H} \times_{\mathcal{H}} \mathcal{H}^{\mathrm{op}}$ is isomorphic to $\mathcal{H}$ via $f\colon [h_1, h_2^{\mathrm{op}}] \mapsto h_1 h_2^{-1}$. In particular, the map ${}_{\mathcal{G}}^{\;X}\{\_ \mid \_\}$ becomes the map $X\,{}_s{*}_s\, X^{\mathrm{op}} \to \mathcal{H}, (h_1, h_2) \mapsto h_1 h_2^{-1}$. This isomorphism further turns the anchor map $\rho\colon X \to \mathcal{G}^{(0)}, h \mapsto [h, h^{\mathrm{op}}]$, into the range map $r_{\mathcal{H}}: X \to \mathcal{H}^{(0)}$, and $\{\_ \mid \_\}^{X}_{\mathcal{H}} : X^{\mathrm{op}}\ {}_{r}{*}_{r}\ X \to \mathcal{H}$ is therefore given by $\left\{h_1^{\mathrm{op}} \mid h_2\right\}^{X}_{\mathcal{H}} = h_1^{-1}h_2$.

We arrive at a well-known result.

Lemma 1.3 (motivation; [28, Lemma 2.45, Proposition 2.47]) *Suppose X is a principal $\mathcal{H}$-space. Then $X \times_{\mathcal{H}} X^{op}$ is a locally compact Hausdorff groupoid with open source map that acts freely and properly on the left of X with anchor map $\rho: x \mapsto [x, x^{op}]$. With this structure, X is a $(X \times_{\mathcal{H}} X^{op}, \mathcal{H})$-equivalence of groupoids. Moreover, if X is also a $(\mathcal{G}, \mathcal{H})$-equivalence, then there exists an isomorphism $X \times_{\mathcal{H}} X^{op} \to \mathcal{G}$ of topological groupoids that is uniquely determined by $[x, y^{op}] \mapsto {}^{\;X}_{\mathcal{G}}\{x \mid y^{op}\}$.*

Note that the above in particular states that, in the setting where $\mathcal{G} = X \times_{\mathcal{H}} X^{\mathrm{op}}$, the element ${}^{\;X}_{\mathcal{G}}\{x \mid y^{\mathrm{op}}\}$ of $\mathcal{G}$ for $x, y \in Xu$ is $[x, y^{\mathrm{op}}]$. A reason why one might be interested in groupoid equivalences is the following result due to works by Muhly, Renault, and Williams.

Lemma 1.4 ([30], [21]) *Suppose $\mathcal{G}, \mathcal{H}$ are locally compact Hausdorff groupoids, that $\{\lambda_u\}_{u \in \mathcal{H}^{(0)}}$ is a Haar system on $\mathcal{H}$, and that there exists an equivalence X between $\mathcal{G}$ and $\mathcal{H}$. Then $\mathcal{G}$ also allows a Haar system, and if say $\{\mu_v\}_{v \in \mathcal{G}^{(0)}}$ is any such Haar system, then the groupoid C^*-algebras $C^*_r(\mathcal{G}, \mu)$ and $C^*_r(\mathcal{H}, \lambda)$ are strongly Morita equivalent via an imprimitivity bimodule built as a completion of $C_c(X)$.*

An analogous theorem holds for the C*-algebra of equivalent Fell bundles [22]. Such results are powerful, for example because two strongly Morita equivalent C*-algebras have the same representation theory, K-theory, and lattices of ideals (via the so-called "Rieffel Correspondence"). But not only do the above mentioned results state the existence of a strong Morita equivalence, they also *construct it explicitly*, which allows one to, for example, construct representatives of certain famous classes in KK-theory [3].

There is a statement analogous to Lemma 1.3 for a right Hilbert C*-module $\mathbf{X}$ over a C*-algebra A. In the following, we use the symbol ${}^{\mathbf{X}}_{A}|\mathbf{x}\rangle\langle\mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ to denote the "A-rank-one" operator $\mathbf{X} \to \mathbf{X}$ that maps $\mathbf{z}$ to $\mathbf{x} \cdot \langle \mathbf{y} \mid \mathbf{z}\rangle^{\mathbf{X}}_{A} \in \mathbf{X}$, and we let $\mathbb{K}(\mathbf{X}_A) = \mathbb{K}_A(\mathbf{X})$ denote the C*-algebra generated by these operators; it is an ideal of the A-adjointable operators (see [24, Lemma 2.25]). We get that $\mathbf{X}$ has a *left* $\mathbb{K}_A(\mathbf{X})$-inner product given by[1]

$$
{}^{\;\;\;\mathbf{X}}_{\mathbb{K}_A(\mathbf{X})}\langle \mathbf{x} \mid \mathbf{y}\rangle := {}^{\mathbf{X}}_{A}|\mathbf{x}\rangle\langle\mathbf{y}|. \tag{1.6}
$$

With inner products such as these, we will drop the sub- and/or superscripts whenever there is no ambiguity. Furthermore, we let $\mathbf{X}^{\mathrm{op}}$ be the dual, left-Hilbert C*-module as defined in [24, p. 49], meaning there exists an additive bijection $F: \mathbf{X} \to \mathbf{X}^{\mathrm{op}}$ such that $F(\lambda\mathbf{x}) = \bar{\lambda}F(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}$ and $\lambda \in \mathbb{C}$. The analog of Lemma 1.4 can now be stated as:

Lemma 1.5 (motivation; [24, Propositions 2.21 and 3.8, Lemma 2.25]) *Suppose $\mathbf{X}$ is a full right-Hilbert C^*-module over a C^*-algebra A. Then the set $\mathbb{K}_A(\mathbf{X})$ of*

[1] Hopefully, it is not too confusing that the symbol on the right-hand side of Equation (1.6) carries A in the subscript rather than the cumbersome $\mathbb{K}_A(\mathbf{X})$; the ket–bra notation in place of the bra–ket should be a clear indicator as to where the inner product takes values.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

*A-compact operators on $\mathbf{X}$ is a C^*-algebra with respect to the operator norm, and $\mathbf{X}$ is a $(\mathbb{K}_A(\mathbf{X}), A)$-imprimitivity bimodule. Moreover, if $\mathbf{X}$ is also a (B, A)-imprimitivity bimodule, then there exists a *-isomorphism $\mathbb{K}_A(\mathbf{X}) \to B$ that is uniquely determined by ${}^{\mathbf{X}}_{A}|\mathbf{x}\rangle\langle\mathbf{y}| \mapsto {}^{\mathbf{X}}_{B}\langle\mathbf{x} \mid \mathbf{y}\rangle$.*

Let us elaborate on the analogy to Lemma 1.3: If $\mathbf{Y}$ is another full right-Hilbert C*-module over A, then the map $\mathbf{X} \otimes_A \mathbf{Y}^{\mathrm{op}} \to \mathbb{K}_A(\mathbf{Y}, \mathbf{X})$ determined by $\mathbf{x} \otimes \mathbf{y}^{\mathrm{op}} \mapsto {}^{\mathbf{X}}_{A}|\mathbf{x}\rangle\langle\mathbf{y}|$ is an isomorphism of bi-Hilbert $\mathbb{K}_A(\mathbf{X}) - \mathbb{K}_A(\mathbf{Y})$-bimodules (see Lemma A.2). In particular, $\mathbf{X} \otimes_A \mathbf{X}^{\mathrm{op}} \cong \mathbb{K}_A(\mathbf{X})$ is, in fact, the (up to isomorphism) unique C*-algebra that is equivalent to A via $\mathbf{X}$. (Note that we similarly have $\mathbf{X}^{\mathrm{op}} \otimes_{\mathbb{K}} \mathbf{X} \cong A$ as bi-Hilbert $A - A$-modules via $\mathbf{x}_1^{\mathrm{op}} \otimes \mathbf{x}_2 \mapsto \langle\mathbf{x}_1 \mid \mathbf{x}_2\rangle^{\mathbf{X}}_{A}$.)

The main result of this paper, Theorem 1.6, is an analog of Lemma 1.3 and Lemma 1.5 in the setting of Fell bundles over groupoids. To this end, we will first show that the correct analog of a principal $\mathcal{H}$-space and a full right-Hilbert C*-module over A is a *$\mathscr{B}$-demi-equivalence*, which can be thought of as "half" of a Fell bundle equivalence in the sense of [22, Definition 6.1]. We will then prove:

Theorem 1.6 *Suppose $\mathcal{H}$ is a locally compact Hausdorff groupoid with open source map and X is a principal right $\mathcal{H}$-space with open anchor map. If $\mathscr{B}$ is a saturated Fell bundle over $\mathcal{H}$ and $\mathscr{M}$ is a $\mathscr{B}$-demi-equivalence over X, then there is a saturated Fell bundle $\mathbb{K}(\mathscr{M}_{\mathscr{B}})$ over the imprimitivity groupoid of X that is equivalent to $\mathscr{B}$ via $\mathscr{M}$. Moreover, if $\mathscr{M}$ is also an $(\mathscr{A}, \mathscr{B})$-Fell bundle equivalence, then there exists a Fell bundle isomorphism $\mathbb{K}(\mathscr{M}_{\mathscr{B}}) \to \mathscr{A}$ that is uniquely determined by ${}^{\mathscr{M}}_{\mathscr{B}}|m\rangle\langle n| \mapsto {}^{\mathscr{M}}_{\mathscr{A}}\langle m \mid n\rangle$.*

Many theorems about symmetric imprimitivity [5, 7, 13, 14, 16] and generalized fixed point algebras [8, 10, 25] have appeared over the years and are extending (in various directions) a 1977-result of Green's [11]. In Section 6, we will see in what way our theorem captures some of those result.

The structure of the paper is as follows. After establishing notation and assumptions in Section 1.1, we give the definition of a $\mathscr{B}$-demi-equivalence $\mathscr{M}$ over a principal $\mathcal{H}$-space X (Definition 2.1) and prove some of its basic properties in Section 2. Section 3 is devoted to proving that $\mathscr{M}$ gives rise to a Fell bundle $\mathbb{K}(\mathscr{M}_{\mathscr{B}})$ over the imprimitivity groupoid of X, which we will also denote by $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ for reasons that will become apparent. We prove that this *imprimitivity Fell bundle* is equivalent to $\mathscr{B}$ via $\mathscr{M}$ in Section 4. In Section 5, we prove that $\mathbb{K}(\mathscr{M}_{\mathscr{B}})$ is unique (up to isomorphism), and we see some applications in Section 6. There are two short appendices to establish some background results about Hilbert C*-modules and about upper semi-continuous Banach bundles.

1.1 Assumptions, conventions, and notation

We will denote groupoids using $\mathcal{G}, \mathcal{H}, \mathcal{K} \ldots$, bundles using $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$, and stand-alone Hilbert C*-modules and their elements using $\mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{Y}, \mathbf{z} \in \mathbf{Z} \ldots$. Fibred products are denoted by $\_\, {}_{f}{*}_{g}\, \_$, as defined in (1.1). The algebraic tensor product is $\odot$ and its completion is $\otimes$; we add a subscript for the internal (i.e., balanced) tensor product. Right and left actions on spaces are usually denoted by $\_ \triangleleft \_$ and $\_ \triangleright \_$, respectively, while actions on bundles are denoted by $\_ \prec \_$ and $\_ \succ \_$. (The triangle is always pointing at the object that is being acted on.)

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

We try to use the letter p for the projection map of Fell bundles, while those of general Banach bundles are denoted by q. The fibre over a point h of the base space of a bundle $\mathscr{B} = (B \to \mathcal{H})$ with total space B is denoted $B(h)$; we adopt analogous notation for other bundles. We write $s_{\mathscr{B}}: B \to \mathcal{H}^{(0)}$ for the source map of $\mathcal{H}$ composed with the projection map of $\mathscr{B}$, and we adopt analogous notation for anchor maps $\sigma: X \to \mathcal{H}^{(0)}$.

We use the notion of upper semi-continuous Banach bundles, Fell bundles, actions of Fell bundles on upper semi-continuous Banach bundles, and equivalences of Fell bundles as in [4], Definitions 2.3, 2.9, 2.10, and 2.11, respectively. As done in [22], our Fell bundles are assumed **saturated** in the sense that their fibres are full (and hence imprimitivity) bimodules; we will rarely mention this assumption again. We frequently use the fact that our upper semi-continuous Banach bundles have enough continuous cross sections (see [18, Corollary 2.10], [22, Appendix A]).

▸ **For the remainder of the paper**, we assume that

- $\mathcal{H}$ is a locally compact Hausdorff groupoid with open source map[2] $s_{\mathcal{H}}: \mathcal{H} \to \mathcal{H}^{(0)}$,
- X is a locally compact Hausdorff space and $\sigma: X \to \mathcal{H}^{(0)}$ is a continuous open surjection,
- X is a principal right $\mathcal{H}$-space with anchor map σ, and
- the groupoid $X \times_{\mathcal{H}} X^{\mathrm{op}}$ will be denoted by $\mathcal{G}$, and we will refer to it as the *imprimitivity groupoid* of X as in [28, Lemma 2.45].

In analogy to the notation $\mathcal{H}u = s_{\mathcal{H}}^{-1}(u)$ and $u\mathcal{H} = r_{\mathcal{H}}^{-1}(u)$ for $u \in \mathcal{H}^{(0)}$, we will write $Xu := \sigma^{-1}(u)$. We chose the letter σ since it serves the purpose of a "source" map; for left-actions, we will therefore generally use ρ (as in "range") for the anchor map.

2 Demi-Equivalences

For the upcoming definition, we remind the reader that the anchor map $\rho: X \to \mathcal{G}^{(0)}$ of the left action of $\mathcal{G} = X \times_{\mathcal{H}} X^{\mathrm{op}}$ on the principal $\mathcal{H}$-space X is exactly the quotient map if we identify $\mathcal{G}^{(0)}$ with $X/\mathcal{H}$; see (1.2).

Definition 2.1 (cf. [1, Definition 2.1]) Suppose $\mathscr{B} = (p_{\mathscr{B}}: B \to \mathcal{H})$ is a Fell bundle over the groupoid $\mathcal{H}$, and $\mathscr{M} = (q_{\mathscr{M}}: M \to X)$ is an upper semi-continuous Banach bundle over the principal $\mathcal{H}$-space X. We call $\mathscr{M}$ a *(right) $\mathscr{B}$-demi-equivalence*[3] if there exist maps

$$\sqcup \triangleleft \sqcup: \ M \,{}_{\sigma}{*}_{r} B \longrightarrow M, \qquad (m, b) \longmapsto m \triangleleft b,$$

and

$$\langle \sqcup \mid \sqcup \rangle_{\mathscr{B}}: \ M \,{}_{\rho}{*}_{\rho} M \longrightarrow B, \qquad (m_1, m_2) \longmapsto \langle m_1 \mid m_2 \rangle_{\mathscr{B}},$$

such that the following[4] hold for all appropriately chosen $m_i \in M$ and $b \in B$.

[2] Any locally compact Hausdorff groupoid with a Haar system (and thus any étale groupoid) is an example of a groupoid with open source map [28, Proposition 1.23].

[3] While it would have been nice to call $\mathscr{M}$ a "principal $\mathscr{B}$-bundle" to show the analogy to the groupoid-world concept of a principal $\mathcal{H}$-space, the clash with the existing notions of principal group bundles in differential geometry made that a non-viable option.

[4] In the ensuing list of properties, the DE in "(DEn)" stands for "demi-equivalence".

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

(DE1) $\_\prec\_$ covers the map $\_\triangleleft\_$ in the sense that $q_{\mathscr{M}}(m \prec b) = q_{\mathscr{M}}(m) \triangleleft p_{\mathscr{B}}(b)$;
(DE2) $\langle \_ \mid \_ \rangle_{\mathscr{B}}$ covers the map $\{\_ \mid \_\}^{X}_{\mathcal{H}}$ in the sense that $q_{\mathscr{M}}(m_1) \triangleleft p_{\mathscr{B}}(\langle m_1 \mid m_2 \rangle_{\mathscr{B}}) = q_{\mathscr{M}}(m_2)$;
(DE3) $\langle \_ \mid \_ \rangle_{\mathscr{B}}$ is continuous, and fibrewise sesquilinear (meaning linear in the second and anti-linear in the first coordinate);
(DE4) $\langle m_1 \mid m_2 \prec b \rangle_{\mathscr{B}} = \langle m_1 \mid m_2 \rangle_{\mathscr{B}} b$;
(DE5) $\langle m_1 \mid m_2 \rangle^*_{\mathscr{B}} = \langle m_2 \mid m_1 \rangle_{\mathscr{B}}$;
(DE6) $\langle m \mid m \rangle_{\mathscr{B}} \geq 0$ in the C*-algebra $B(\sigma_{\mathscr{M}}(m))$, and $\langle m \mid m \rangle_{\mathscr{B}} = 0$ only if $m = 0$;
(DE7) the norm $m \mapsto \|\langle m \mid m \rangle_{\mathscr{B}}\|^{1/2}$ agrees with the norm that the upper semi-continuous Banach bundle $\mathscr{M}$ carries; and
(DE8) for each $x \in X$, the linear span of $\{\langle m_1 \mid m_2 \rangle_{\mathscr{B}} : m_i \in M(x)\}$ is dense in $B(\sigma(x))$.

An analogous definition of a left $\mathscr{B}$-demi-equivalence can be made; there, each instance of a range map becomes a source map (and vice versa) and sesquilinearity means that the *first* coordinate is linear. When there is no ambiguity, we will drop the subscript-$\mathscr{B}$ on the inner product; conversely, when there is ample room for ambiguity, we might add a superscript-$\mathscr{M}$.

The reader might have noticed that there are a few natural properties, both algebraic and analytic, that a demi-equivalence should satisfy if it is to be "half" of an equivalence in the sense of Muhly and Williams. Let us show that all those properties actually follow automatically.

Lemma 2.1 (cf. [1, Lemma 2.7]) *Suppose $\mathscr{B} = (p_{\mathscr{B}}: B \to \mathcal{H})$ is a Fell bundle over the groupoid $\mathcal{H}$ and $\mathscr{M} = (q_{\mathscr{M}}: M \to X)$ is a right $\mathscr{B}$-demi-equivalence over the principal right $\mathcal{H}$-space X. Then we have the following, where $(m_1, m_2) \in M\ {}_{\rho}{*}_{\rho}\ M$.*

(DE9) *Each $M(x)$ is a full right-Hilbert C*-module over the C*-algebra $B(\sigma(x))$;*
(DE10) *$\langle m_1 \prec b^* \mid m_2 \rangle_{\mathscr{B}} = b\langle m_1 \mid m_2 \rangle_{\mathscr{B}}$ for all appropriate $b \in B$;*
(DE11) *$\sigma_{\mathscr{M}}(m_1) = r_{\mathscr{B}}(\langle m_1 \mid m_2 \rangle_{\mathscr{B}})$ and $\sigma_{\mathscr{M}}(m_2) = s_{\mathscr{B}}(\langle m_1 \mid m_2 \rangle_{\mathscr{B}})$;*
(DE12) *$\langle m_1 \mid m_2 \rangle_{\mathscr{B}} \langle m_1 \mid m_2 \rangle^*_{\mathscr{B}} \leq \|m_2\|^2 \langle m_1 \mid m_1 \rangle_{\mathscr{B}}$ as elements of the C*-algebra $B(\sigma_{\mathscr{M}}(m_1))$ (Cauchy–Schwarz).*

Moreover, $\mathscr{B}$ acts on the right of $\mathscr{M}$ in the sense of [4, Definition 2.11], i.e.,

(DE13) *$\_\prec\_$ is fibrewise bilinear;*
(DE14) *$(m \prec b) \prec b' = m \prec (bb')$ for all appropriate $b, b' \in B$;*
(DE15) *$\|m \prec b\| \leq \|m\|\,\|b\|$; and*
(DE16) *$\_\prec\_$ is continuous.*

Again, the analogous result for a left demi-equivalence must be rephrased in a way that every range becomes a source etc. With the properties listed in Lemma 2.1, we see that a $\mathscr{B}$-demi-equivalence really satisfies all "one-sided" properties of a Fell bundle equivalence as defined in [22, Definition 6.1]. In Proposition 2.3, we will see which properties are needed for a "two-sided" demi-equivalence to be an equivalence.

Proof of Lemma 2.1 Condition (DE9) is just a restatement of other properties. To be precise, $M(x)$ is a right inner product $B(\sigma(x))$-module in the sense of [24,

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Definition 2.1] because of Conditions (DE3)–(DE6), and it is a Hilbert $B(\sigma(x))$-module because we assumed that the norm with respect to which $M(x)$ is complete coincides with the norm induced by the inner product (Assumption (DE7)). Fullness is exactly Assumption (DE8).

Condition (DE10) follows from (DE5) and (DE4) combined, and (DE11) follows directly from (DE2).

(DE13) Fix $x \in X$ and $m, m_i \in M(x)$. If $b \in B(h)$ for some $h \in \sigma(x)\mathcal{H}$, then

$$\begin{aligned}\langle m \mid (m_1 + \lambda m_2) \prec b\rangle &\overset{\text{(DE4)}}{=} \langle m \mid m_1 + \lambda m_2\rangle b \\ &= (\langle m \mid m_1\rangle + \lambda\langle m \mid m_2\rangle)b \qquad \text{by (DE3) (sesquilinearity).}\end{aligned}$$

Since multiplication in $\mathscr{B}$ is bilinear, we conclude that

$$\langle m \mid (m_1 + \lambda m_2) \prec b\rangle = \langle m \mid m_1 \prec b + \lambda(m_2 \prec b)\rangle$$

for all $m \in M(x)$. Choosing $m = (m_1 + \lambda m_2) \prec b - m_1 \prec b + \lambda(m_2 \prec b)$, it follows from (DE6) that $m = 0$, meaning that $\_ \prec \_$ is linear in the first component. A similar computation, using (DE10), proves linearity in the second component and also (DE14), using associativity of the multiplication of $\mathscr{B}$.

(DE15) We have

$$\begin{aligned}\|m \prec b\|^2 &\overset{\text{(DE7)}}{=} \|\langle m \prec b \mid m \prec b\rangle_{\mathscr{B}}\| \overset{(\dagger)}{=} \|b^*\langle m \mid m\rangle_{\mathscr{B}} b\| \\ &\leq \|b^*\| \, \|\langle m \mid m\rangle_{\mathscr{B}}\| \, \|b\| \overset{\text{(DE7)}}{=} \|m\|^2 \, \|b\|^2,\end{aligned}$$

where $(\dagger)$ follows from (DE4) and (DE10).

(DE16) Suppose (m_i, b_i) is a net in $M\,{}_{\sigma}{*}_{r}\,B$ that converges to (m, b); in particular, $x_i := q_{\mathscr{M}}(m_i)$ converges to $x := q_{\mathscr{M}}(m)$ in X and $h_i := p_{\mathscr{B}}(b_i)$ converges to $h := p_{\mathscr{B}}(b)$ in $\mathcal{H}$. Now choose a continuous section $\mu \in \Gamma_0(X; \mathscr{M})$ of $\mathscr{M}$ with $\mu(x \triangleleft h) = m \prec b$. Since μ and the right $\mathcal{H}$-action $\_ \triangleleft \_$ on X are continuous, we have $\mu(x_i \triangleleft h_i) \to m \prec b$. Using (DE4), (DE10), and sesquilinearity of the inner product, we have

$$\begin{aligned}&\langle m_i \prec b_i - \mu(x_i \triangleleft h_i) \mid m_i \prec b_i - \mu(x_i \triangleleft h_i)\rangle \\ &= b_i^*\langle m_i \mid m_i\rangle b_i - b_i^*\langle m_i \mid \mu(x_i \triangleleft h_i)\rangle \\ &\quad - \langle \mu(x_i \triangleleft h_i) \mid m_i\rangle b_i + \langle \mu(x_i \triangleleft h_i) \mid \mu(x_i \triangleleft h_i)\rangle.\end{aligned}$$

By continuity of the involution and multiplication of $\mathscr{B}$ and by continuity of the inner product on $\mathscr{M}$, the above converges to

$$b^*\langle m \mid m\rangle b - b^*\langle m \mid m \prec b\rangle - \langle m \prec b \mid m\rangle b + \langle m \prec b \mid m \prec b\rangle,$$

which, again by (DE4) and (DE10), is 0. Since

$$\|m_i \prec b_i - \mu(x_i \triangleleft h_i)\|^2 = \|\langle m_i \prec b_i - \mu(x_i \triangleleft h_i) \mid m_i \prec b_i - \mu(x_i \triangleleft h_i)\rangle\|$$

by (DE7), it thus follows from [4, Lemma A.5] that $m_i \prec b_i - \mu(x_i \triangleleft h_i)$ converges in the total space of $\mathscr{M}$ to the zero-element of the Banach space $M(x \triangleleft h)$. Since $\mu(x_i \triangleleft h_i)$ converges to $m \prec b$ and since limits are unique, this implies $m_i \prec b_i \to \mu(x \triangleleft h) = m \prec b$, as claimed.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

The proof of (DE12) is verbatim that for [4, Lemma 4.7] (after translating from the left to the right). ∎

Remark 2.2 In [22, Definition 6.1], neither continuity of the inner product on a Fell bundle equivalence nor Condition (DE7) were explicitly assumed but often invoked. I am unsure whether there is a way to deduce continuity from a combination of the algebraic properties of $\mathscr{M}$ and the topological properties of $\mathscr{B}$, like it was the case for continuity of the action, (DE16). Consequently, I chose to include these conditions as assumptions in the definition.

Next, we show that a two-sided version of a demi-equivalence is an equivalence in the sense of [22, Definition 6.1], provided there is some algebraic compatibility between the two structures.

Proposition 2.3 *Suppose that X is a groupoid equivalence between $\mathcal{G}$ and $\mathcal{H}$ with anchor maps ρ respectively σ, that $\mathscr{A} = (A \to \mathcal{G})$ and $\mathscr{B} = (B \to \mathcal{H})$ are Fell bundles, and that $\mathscr{M} = (M \to X)$ is a left $\mathscr{A}$- and a right $\mathscr{B}$-demi-equivalence. Assume further that*

(1) *the actions of $\mathscr{A}$ and $\mathscr{B}$ on $\mathscr{M}$ commute, i.e., for all $(a, m, b) \in A\,{}_s{*}_\rho\, M\,{}_\sigma{*}_r\, B$, we have $(a \triangleright m) \triangleleft b = a \triangleright (m \triangleleft b)$;*
(2) *for each $x \in X$, $A(\rho(x))$ acts by $B(\sigma(x))$-adjointable operators on $M(x)$, meaning that*

$$\langle a \triangleright m_1 \mid m_2 \rangle_{\mathscr{B}} = \langle m_1 \mid a^* \triangleright m_2 \rangle_{\mathscr{B}}$$

for all $m_i \in M(x)$ and all $a \in A(\rho(x))$;
(3) *for all $(m_1, m_2, m_3) \in M\,{}_\sigma{*}_\sigma\, M\,{}_\rho{*}_\rho\, M$, the inner products on $\mathscr{M}$ satisfy*

$${}_{\mathscr{A}}\langle m_1 \mid m_2 \rangle \triangleright m_3 = m_1 \triangleleft \langle m_2 \mid m_3 \rangle_{\mathscr{B}}.$$

Then $\mathscr{M}$ is an equivalence between $\mathscr{A}$ and $\mathscr{B}$.

Note that Assumption (2) is distinctly asymmetric; it could have been replaced by its counterpart:

(2)' For each $x \in X$, $B(\sigma(x))$ acts by $A(\rho(x))$-adjointable operators on $M(x)$.

Proof First, we will do some sanity checks: Condition (1) makes sense because the actions on X commute, meaning that $(a \triangleright m) \triangleleft b$ and $a \triangleright (m \triangleleft b)$ live over the same fibre of $\mathscr{M}$ by (DE1). Next, let us check that Condition (3) makes sense. Since ${}_{\mathscr{A}}\langle \_ \mid \_ \rangle$ is defined on $M\,{}_\sigma{*}_\sigma\, M$ and $\langle \_ \mid \_ \rangle_{\mathscr{B}}$ on $M\,{}_\rho{*}_\rho\, M$, we can evaluate the shown inner products. By (DE11), we have $\sigma_{\mathscr{M}}(m_2) = r_{\mathscr{B}}(\langle m_2 \mid m_3 \rangle_{\mathscr{B}})$, and since $(m_1, m_2) \in M\,{}_\sigma{*}_\sigma\, M$, we therefore have that m_1 can be acted on by $\langle m_2 \mid m_3 \rangle_{\mathscr{B}}$ on the right; a similar argument shows that m_3 can be acted on by ${}_{\mathscr{A}}\langle m_1 \mid m_2 \rangle$ on the left. Lastly, note that Conditions (DE1) and (DE2) combined with Equation (1.4) show that the elements in question are indeed living in the same fibre of $\mathscr{M}$.

To see that $\mathscr{M}$ is an equivalence, we have to check Properties (FE1), (FE2), and (FE3) in [4, Definition 2.11]. By assumption, $\mathscr{M}$ is an upper semi-continuous Banach bundle over a groupoid equivalence, and as explained in Lemma 2.1, $\_ \triangleright \_$ and $\_ \triangleleft \_$ are actions in the sense of [4, Definition 2.10]. Since they are assumed to commute, we therefore get Condition (FE1). For (FE2), note that the inner products are assumed to be sesquilinear in the correct sense. Furthermore,

(FE2.a) is satisfied by (DE2);
(FE2.b) is satisfied by (DE5);
(FE2.c) is satisfied by (DE4); and
(FE2.d) is exactly Assumption (3).

Lastly, for (FE3), we must check that each $M(x)$ is an imprimitivity bimodule between $A(\rho(x))$ and $B(\sigma(x))$. We know by (DE9) that it is a full left- and right-Hilbert C*-module. By Assumption (3), the left- and right-inner products are compatible and the left-action is by $B(\sigma(x))$-adjointable operators; it remains to show that the right-action is by $A(\rho(x))$-adjointable operators. For $n_i \in M(x)$, we have

$$
\begin{aligned}
{}_{\mathscr{A}}\langle n_1 \mid n_2 \prec b\rangle \succ n_3 &\overset{(3)}{=} n_1 \prec \langle n_2 \prec b \mid n_3\rangle_{\mathscr{B}} \overset{(\mathrm{DE10})}{=} n_1 \prec (b^* \langle n_2 \mid n_3\rangle_{\mathscr{B}})\\
&\overset{(\mathrm{DE14})}{=} (n_1 \prec b^*) \prec \langle n_2 \mid n_3\rangle_{\mathscr{B}} \overset{(3)}{=} {}_{\mathscr{A}}\langle n_1 \prec b^* \mid n_2\rangle \succ n_3.
\end{aligned}
$$

This proves that the element $a := {}_{\mathscr{A}}\langle n_1 \mid n_2 \prec b\rangle - {}_{\mathscr{A}}\langle n_1 \prec b^* \mid n_2\rangle$ of $A(\rho(x))$ "kills" all of $M(x)$. By Assumption (2), a is an adjointable operator with respect to the $B(\sigma(x))$-valued right-inner product. It therefore follows from [24, Remark 2.29] that $a^* a$ and hence a is 0, i.e., ${}_{\mathscr{A}}\langle n_1 \mid n_2 \prec b\rangle = {}_{\mathscr{A}}\langle n_1 \prec b^* \mid n_2\rangle$, so b is an $A(\rho(x))$-adjointable operator. ∎

Remark 2.4 In the setting of Proposition 2.3, it follows from [4, Corollary 4.6] that Condition (3) actually also holds more generally: if $(a, m_1, m_2) \in A\,{}_s{*}_\rho\, M\,{}_\sigma{*}_\sigma\, M$, then $\langle a \succ m_1 \mid m_2\rangle_{\mathscr{B}} = \langle m_1 \mid a^* \succ m_2\rangle_{\mathscr{B}}$ as elements of $B(h)$ where $h = \{q_{\mathscr{M}}(m_1)^{\mathrm{op}} \mid p_{\mathscr{A}}(a)^{-1} \triangleright q_{\mathscr{M}}(m_2)\}^X_{\mathcal{H}}$. In other words, $\mathscr{A}$ is acting on $\mathscr{M}$ by "$\mathscr{B}$-adjointable" operators. This observation foreshadows a connection to [1], where results similar to the main theorem of the paper at hand have appeared. We will do a more in-depth comparison later in Corollary 6.7, but let us already point out some points of distinction: On the one hand, their result is more general in that they consider Fell bundles that are not necessarily separable or saturated; instead of (DE8), they only assume that $\mathscr{M} = (M \to X)$ satisfies

$$\text{(7R)} \qquad \overline{\operatorname{span}}\{\langle M(x) \mid M(x)\rangle_{\mathscr{B}} : x \in Xu\} = B(u).$$

On the other hand, their result is more restrictive in that our topological groupoid $\mathcal{H}$ is replaced by a topological *group*. Moreover, they only consider $X = \mathcal{H}$, which is restrictive even in the case of groups (see Example 6.9).

We have introduced all ingredients for the main theorem which says that any $\mathscr{B}$-demi-equivalence can be rigged (in a unique way) to give a Fell bundle equivalence in the sense of [22, Definition 6.1].

3 The imprimitivity Fell bundle: Existence

▸ **For the remainder of the paper**, we fix a Fell bundle $\mathscr{B} = (p_{\mathscr{B}}\colon B \to \mathcal{H})$ over the groupoid $\mathcal{H}$ and a right $\mathscr{B}$-demi-equivalence $\mathscr{M} = (q_{\mathscr{M}}\colon M \to X)$ over the principal right $\mathcal{H}$-space X.

Suppose that $\mathscr{A}$ and $\mathscr{C}$ are, like $\mathscr{B}$, Fell bundles over locally compact Hausdorff groupoids. In [4, Sections 5 and 6], it was shown that a so-called *hypo-equivalence* $\mathscr{N}_1$ from $\mathscr{A}$ to $\mathscr{B}$ and a hypo-equivalence $\mathscr{N}_2$ from $\mathscr{B}$ to $\mathscr{C}$ can be "multiplied" to yield a hypo-equivalence $\mathscr{N}_1 \otimes_{\mathscr{B}} \mathscr{N}_2$ from $\mathscr{A}$ to $\mathscr{C}$. A careful examination of the proofs shows that not all the structure of hypo-equivalences is needed to construct the upper semi-continuous Banach bundle $\mathscr{N}_1 \otimes_{\mathscr{B}} \mathscr{N}_2$. We will start this section by making this claim more precise.

Given $x, y \in X$ with $u := \sigma(x) = \sigma(y)$, the demi-equivalence $\mathscr{M}$ gives us two full Hilbert C*-modules over the C*-algebra $B(u)$, namely the right-module $M(x)$ and the *left*-module $M(y)^{\mathrm{op}}$. In particular, combining the well-known results mentioned earlier, we get an imprimitivity bimodule between $\mathbb{K}_{B(u)}(M(x))$ and $\mathbb{K}_{B(u)}(M(y)^{\mathrm{op}})$ by taking the balanced tensor product:

$$K(x, y^{\mathrm{op}}) := M(x) \otimes_u M(y)^{\mathrm{op}} \cong \mathbb{K}_{B(u)}\left(M(y), M(x)\right), \tag{3.1}$$

where we write $\_ \otimes_u \_$ as short-hand for $\_ \otimes_{B(u)} \_$. The norm of this Banach space is, on sums of elementary tensors, given by

$$\begin{aligned}
&\left\| \sum_i m_i \otimes n_i^{\mathrm{op}} \right\| \\
&= \left\| \sum_{i,j} |n_i \triangleleft \langle m_i \mid m_j \rangle_{B(u)}^{M(x)} \rangle \langle n_j| \right\|^{1/2} \quad \text{operator norm on } \mathbb{K}_{B(u)}(M(y)) \\
&= \left\| \sum_{i,j} |m_i\rangle \langle m_j \triangleleft \langle n_j \mid n_i \rangle_{B(u)}^{M(y)}| \right\|^{1/2} \quad \text{operator norm on } \mathbb{K}_{B(u)}(M(x)).
\end{aligned} \tag{3.2}$$

We construct an upper semi-continuous Banach bundle over $X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}}$ as follows.

Lemma 3.1 (cf. [4, Lemma 5.2]) *On the set*

$$K = \bigsqcup_{(x, y^{op}) \in X_\sigma *_\sigma X^{op}} K(x, y^{op}),$$

consider all cross-sections of the form

$$\mu \otimes \nu^{op} \colon \quad X\,{}_\sigma{*}_\sigma\, X^{op} \to K, \qquad (x, y^{op}) \mapsto \mu(x) \otimes \nu(y)^{op},$$

for $\mu, \nu \in \Gamma_0(X; \mathscr{M})$. Then there is a unique topology on K making it an upper semi-continuous Banach bundle over $X\,{}_\sigma{*}_\sigma\, X^{op}$ such that all cross-sections $\mu \otimes \nu^{op}$ are continuous.

In the literature [4, 20], the above bundle is denoted $\mathscr{M} \otimes_{\mathscr{B}^{(0)}} \mathscr{M}^{\mathrm{op}}$. But since we fixed $\mathscr{M}$, we will ease notation and instead denote the bundle by $\mathscr{K} = (q_{\mathscr{K}} \colon K \to X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}})$.

Proof Consider

$$\Gamma := \mathrm{span}_{\mathbb{C}}\{\mu \otimes \nu^{\mathrm{op}} : \mu, \nu \in \Gamma_0(X; \mathscr{M})\}.$$

Recall that $\mathscr{M}$ has enough continuous sections, so that for each $m \in M(x)$, there exists $\mu \in \Gamma_0(X; \mathscr{M})$ such that $\mu(x) = m$. In particular, it follows that $\{\gamma(x, y^{\mathrm{op}}) : \gamma \in \Gamma\}$ is

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

dense in $K(x, y^{\text{op}})$. If we can show for an arbitrary element $\gamma = \sum_{i=1}^{k} \mu_i \otimes \nu_i^{\text{op}}$ of Γ that $(x, y^{\text{op}}) \mapsto \|\gamma(x, y^{\text{op}})\|$ is upper semi-continuous, then the claim follows from [12, Corollary 3.7] (see also [9, p. 13.18], [29, Theorem C.25], [4, Remark 2.7]). We have by Equation (3.2)

$$\|\gamma(x, y^{\text{op}})\|^2 = \left\| \sum_{i,j} |\nu_i(y) \dashv \langle \mu_i(x) \mid \mu_j(x)\rangle\rangle\langle \nu_j(y)| \right\|.$$

It was shown in [24, Lemma 2.65] that the $k \times k$-matrix with i, j-entry $\langle \mu_i(x) \mid \mu_j(x)\rangle$ is positive for each $x \in X$, so there exists a matrix $(b_{i,j}(x))_{i,j}$ over $B(\sigma(x))$ such that $\langle \mu_i(x) \mid \mu_j(x)\rangle = \sum_{l=1}^{k} b_{il}(x) b_{jl}(x)^*$. Note that we may choose each $b_{i,j}: x \mapsto b_{i,j}(x)$ to be a *continuous* section of the pullback bundle $\sigma^*(\mathscr{B})$. For $1 \le l \le k$, we let $\nu_l(x, y^{\text{op}}) \coloneqq \sum_{i=1}^{k} \nu_i(y) \dashv b_{il}(x)$. By continuity of $\dashv$, of each ν_i and b_{il}, and of summation in $\mathscr{M}$, each ν_l is continuous. Moreover,

$$\begin{aligned} \|\gamma(x, y^{\text{op}})\|^2 &= \left\| \textstyle\sum_{l=1}^{k} |\nu_l(x, y^{\text{op}})\rangle\langle \nu_l(x, y^{\text{op}})| \right\| \\ &= \left\| \textstyle\sum_{l=1}^{k} \langle \nu_l(x, y^{\text{op}}) \mid \nu_l(x, y^{\text{op}})\rangle \right\| \qquad \text{by Lemma A.1.} \end{aligned}$$

Since the $\mathscr{B}$-valued inner product on $\mathscr{M}$, summation in $\mathscr{B}$, and each ν_l are continuous, and the norm on $\mathscr{B}$ is upper semi-continuous, we conclude that $(x, y^{\text{op}}) \mapsto \|\gamma(x, y^{\text{op}})\|$ is upper semi-continuous. ∎

Remark 3.2 The proof of [4, Lemma 5.4] shows that, if we have two convergent nets $m_\lambda \to m$ and $n_\lambda \to n$ in M with $\sigma_{\mathscr{M}}(m_\lambda) = \sigma_{\mathscr{M}}(n_\lambda)$, then $m_\lambda \otimes n_\lambda^{\text{op}} \to m \otimes n^{\text{op}}$ in K.

Implicitly, Lemma 3.1 is making use of the "dual" bundle to the $\mathscr{B}$-demi-equivalence $\mathscr{M}$: given a fibre $M(x)$ of $\mathscr{M}$, its conjugate vector space $M(x)^{\text{op}}$ is, by definition, the fibre of $\mathscr{M}^{\text{op}} = (M^{\text{op}} \to X^{\text{op}})$ over x^{op}. Clearly, $\mathscr{M}^{\text{op}}$ is also an upper semi-continuous Banach bundle if M^{op} and X^{op} are given the same topologies as M and X, respectively. Moreover, the right $\mathscr{B}$-action on $\mathscr{M}$ induces a left $\mathscr{B}$-action on $\mathscr{M}^{\text{op}}$ if we define $b \vDash m^{\text{op}} \coloneqq (m \dashv b^*)^{\text{op}}$. The reader should compare all of this with [22, Example 6.7] in the setting where $\mathscr{M}$ is an equivalence.

Note that $\mathscr{K}$ is the bundle-analog of the space $X {}_{\sigma}{*}_{\sigma} X^{\text{op}}$ in the world of groupoids and of the unbalanced tensor product $\mathbf{X} \otimes \mathbf{X}^{\text{op}}$ of Hilbert C*-modules in the world of C*-algebras. What we are actually after, though, is a bundle-analog of their *quotients*, that is, $X \times_{\mathcal{H}} X^{\text{op}}$ respectively $\mathbf{X} \otimes_A \mathbf{X}^{\text{op}}$: To get $\mathbf{X} \otimes_A \mathbf{X}^{\text{op}}$ from $\mathbf{X} \otimes \mathbf{X}^{\text{op}}$, we quotient out by elements of the form

$$(\mathbf{x} \cdot a) \otimes \mathbf{y} - \mathbf{x} \otimes (a \cdot \mathbf{y})$$

where $\mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{X}^{\text{op}}$, and $a \in A$. For bundles, such a difference does not make sense, since the two elementary tensors may live in different fibres: if $m \in M(x)$, $n \in M(y)$, and $b \in B(h)$ are such that $\sigma(x) = r_{\mathcal{H}}(h)$ and $s_{\mathcal{H}}(h) = \sigma(y)$, then

$$\begin{aligned} &(m \dashv b) \otimes n^{\text{op}} \in M(x \triangleleft h) \otimes_{s(h)} M(y)^{\text{op}}, \text{ while} \\ &m \otimes (b \vDash n^{\text{op}}) \in M(x) \otimes_{r(h)} M(y \triangleleft h^{-1})^{\text{op}}. \end{aligned}$$

In order to identify $(m \dashv b) \otimes n^{\text{op}}$ with $m \otimes (b \vDash n^{\text{op}}) = m \otimes (n \dashv b^*)^{\text{op}}$, we therefore need a way to identify the above two fibres of $\mathscr{K}$.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Lemma 3.3 (cf. [4, Theorem 5.20]) *For any $u, v \in \mathcal{H}^{(0)}$ and $h \in u\mathcal{H}v$, there exists a map*

$$\Psi_h\colon \quad \bigsqcup_{\substack{x\in Xu\\ y\in Xv}} M(x \triangleleft h) \otimes_v M(y)^{op} \to \bigsqcup_{\substack{x\in Xu\\ y\in Xv}} M(x) \otimes_u M(y \triangleleft h^{-1})^{op}$$

determined on elementary tensors by

$$\Psi_h((m \blacktriangleleft b) \otimes n^{op}) = m \otimes (n \blacktriangleleft b^*)^{op}, \tag{3.3}$$

where $\sigma_{\mathscr{M}}(m) = u$, $\sigma_{\mathscr{M}}(n) = v$, and $b \in B(h)$. These maps have the following properties:

(Ψ1) *When restricted to a single fibre $K(x \triangleleft h, y^{op}) \to K(x, (y \triangleleft h^{-1})^{op})$, each Ψ_h is linear.*
(Ψ2) *Each Ψ_h is isometric, i.e., $\|\Psi_h(\xi)\| = \|\xi\|$.*
(Ψ3) *$\Psi_{h'} \circ \Psi_h = \Psi_{h'h}$ for $(h', h) \in \mathcal{H}^{(2)}$.*
(Ψ4) *$\Psi_{h^{-1}}$ is inverse to Ψ_h.*
(Ψ5) *Ψ_u for $u \in \mathcal{H}^{(0)}$ is the identity map.*

Proof We construct Ψ_h on each of the fibres $M(x \triangleleft h) \otimes_v M(y)^{\mathrm{op}}$. Consider the map

$$\begin{gathered} M(x) \times B(h) \times M(y)^{\mathrm{op}} \to M(x) \otimes_u M(y \triangleleft h^{-1})^{\mathrm{op}}, \\ (m, b, n^{\mathrm{op}}) \mapsto m \otimes (n \blacktriangleleft b^*)^{\mathrm{op}}. \end{gathered}$$

Since it is multilinear, it descends to a linear map with domain $[M(x) \odot B(h)] \odot M(y)^{\mathrm{op}}$, where $\odot$ denotes the algebraic tensor product. Because of the $B(u)$-balancing in the codomain, the map descends to a map with domain $[M(x) \odot_{B(u)} B(h)] \odot M(y)^{\mathrm{op}}$. Since $[n \blacktriangleleft b_v^*] \blacktriangleleft b^* = n \blacktriangleleft [bb_v]^*$ for any $b_v \in B(v)$, it further descends to a map with domain $[M(x) \odot_{B(u)} B(h)] \odot_{B(v)} M(y)^{\mathrm{op}}$. It is easy to check that that map is isometric (when $M(x) \odot_{B(u)} B(h)$ is given the norm of the balanced tensor product of Hilbert modules), and therefore extends to $[M(x) \otimes_u B(h)] \otimes_v M(y)^{\mathrm{op}}$. Now, Assumption (DE8) implies that every element of $M(x \triangleleft h)$ is the limit of sums of elements of the form $m \blacktriangleleft b$; in other words, the linear map $M(x) \odot_{B(u)} B(h) \to M(x \triangleleft h)$ determined by $m \odot b \mapsto m \blacktriangleleft b$ has dense range. We conclude the existence of Ψ_h. The remaining properties are easy to check. ∎

The collection of maps Ψ_h stitch together to give an isomorphism of upper semi-continuous Banach bundles as follows. Let $t\colon X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}} \to \mathcal{H}^{(0)}$ be given by $t(x, y^{\mathrm{op}}) := \sigma(x) = \sigma^{\mathrm{op}}(y^{\mathrm{op}})$, and consider the continuous projection map

$$f\colon \quad \mathcal{H}\,{}_{s}{*}_{t}\,(X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}}) \to X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}}, \quad (h, x, y^{\mathrm{op}}) \mapsto (x, y^{\mathrm{op}}).$$

The pull-back bundle of the upper semi-continuous Banach bundle $\mathscr{K}$ via f is the bundle over the domain of f defined by

$$\begin{aligned} &\{(h, x, y^{\mathrm{op}}, \xi) \in [\mathcal{H}\,{}_{s}{*}_{t}\,(X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}})] \times K : f(h, x, y^{\mathrm{op}}) = q_{\mathscr{K}}(\xi)\} \\ &\quad \cong \{(h, \xi) \in \mathcal{H} \times K : s_{\mathcal{H}}(h) = t(q_{\mathscr{K}}(\xi))\}. \end{aligned}$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

While this bundle is often denoted by $f^*(\mathscr{K})$, we will denote it by $\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}$ instead, so that the letter f does not have to be introduced. We make an analogous definition for $\mathscr{K}\,{}_{t}{*}_{r}\,\mathcal{H}$.

It was shown in [4, Lemma 2.8] that basic open sets of this upper semi-continuous Banach bundle are of the form $U\,{}_{s}{*}_{t}\,V$ for $U \subseteq \mathcal{H}$, $V \subseteq K$ open, and the sections

$$\mathcal{H}\,{}_{s}{*}_{t}\,(X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}}) \to \mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}, \quad (h, x, y^{\mathrm{op}}) \mapsto (h, \tau(x, y^{\mathrm{op}})),$$

for τ a continuous section of $\mathscr{K}$, are the continuous sections that uniquely determine the topology on $\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}$.

Lemma 3.4 (cf. [4, Theorem 5.20]) *The map*

$$\Psi\colon \quad \mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K} \to \mathscr{K}\,{}_{t}{*}_{r}\,\mathcal{H}, \quad (h, \xi) \mapsto (\Psi_h(\xi), h),$$

is an isomorphism of upper semi-continuous Banach bundles covering the homeomorphism

$$\psi\colon \quad \mathcal{H}\,{}_{s}{*}_{t}\,(X\,{}_{\sigma}{*}_{\sigma}\,X^{op}) \to (X\,{}_{\sigma}{*}_{\sigma}\,X^{op})\,{}_{t}{*}_{r}\,\mathcal{H}, \quad (h, x \triangleleft h, y^{op}) \mapsto (x, (y \triangleleft h^{-1})^{op}, h),$$

In particular, Ψ is jointly continuous.

We remind the reader that 'Ψ *covers* ψ' means that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K} \arrow[r, "\Psi"] \arrow[d, "q"] & \mathscr{K}\,{}_{t}{*}_{r}\,\mathcal{H} \arrow[d, "q"] \\
\mathcal{H}\,{}_{s}{*}_{t}\,(X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}}) \arrow[r, "\psi"] & (X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}})\,{}_{t}{*}_{r}\,\mathcal{H}.
\end{tikzcd}$$

So given $\xi \in K$ and $h \in \mathcal{H}$ with $s_{\mathcal{H}}(h) = t(q_{\mathscr{K}}(\xi))$, this means that

$$q_{\mathscr{K}}(\xi) = (x \triangleleft h, y^{\mathrm{op}}) \iff q_{\mathscr{K}}(\Psi_h(\xi)) = (x, (y \triangleleft h^{-1})^{\mathrm{op}}). \tag{3.4}$$

Proof To see that Ψ is an isomorphism, we will invoke Lemma B.2 and [4, Proposition A.8]. Because of Lemma 3.1 and our comment preceding Lemma 3.4, the set

$$\Gamma = \{(h, x, y^{\mathrm{op}}) \mapsto (h, \mu(x) \otimes \nu(y)^{\mathrm{op}}) : \mu, \nu \in \Gamma_0(X; \mathscr{M})\}$$

is a collection of continuous sections of $\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}$. As the fibre of $\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}$ over a given (h, x, y^{op}) is just $K(x, y^{\mathrm{op}})$ and since the linear span of $\{\mu(x) \otimes \nu(y)^{\mathrm{op}} : \mu, \nu \in \Gamma_0(X; \mathscr{M})\}$ is dense in $K(x, y^{\mathrm{op}})$, we conclude that the linear span of $\{\gamma(h, x, y^{\mathrm{op}}) : \gamma \in \Gamma\}$ is likewise dense in the fibre of $\mathcal{H}\,{}_{s}{*}_{t}\,\mathscr{K}$. We claim that, given any element $\gamma\colon (h, x, y^{\mathrm{op}}) \mapsto (h, \mu(x) \otimes \nu(y)^{\mathrm{op}})$ of Γ, the section $\Psi \circ \gamma \circ \psi^{-1}$ of $\mathscr{K}\,{}_{t}{*}_{r}\,\mathcal{H}$ is continuous. Indeed, it suffices to check that, if

$$(x_\lambda, (y_\lambda \triangleleft h_\lambda^{-1})^{\mathrm{op}}, h_\lambda) \to (x, (y \triangleleft h)^{\mathrm{op}}, h) \text{ in } (X\,{}_{\sigma}{*}_{\sigma}\,X^{\mathrm{op}})\,{}_{t}{*}_{r}\,\mathcal{H},$$

then

$$\Psi_{h_\lambda}\big(\mu(x_\lambda \triangleleft h_\lambda) \otimes \nu(y_\lambda)^{\mathrm{op}}\big) \to \Psi_h\big(\mu(x \triangleleft h) \otimes \nu(y)^{\mathrm{op}}\big) \text{ in } K\,{}_{t}{*}_{r}\,\mathcal{H}.$$

Fix $\varepsilon > 0$. As mentioned earlier, Assumption (DE8) implies that $M(x \triangleleft h) = M(x) \prec B(h)$, so we may take finitely many elements $m_i \in M(x)$ and $b_i \in B(h)$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

such that

$$\left\| \mu(x \triangleleft h) - \sum_{i=1}^{k} m_i \triangleleft b_i \right\| < \varepsilon. \tag{3.5}$$

For each $1 \leq i \leq k$, fix a section μ_i of $\mathscr{M}$ and τ_i of $\mathscr{B}$ with $\mu_i(x) = m_i$ respectively $\tau_i(h) = b_i$. Since addition in $\mathscr{M}$ and the right $\mathscr{B}$-action are continuous, the net $\zeta_\lambda := \sum_i \mu_i(x_\lambda) \triangleleft \tau_i(h_\lambda)$ converges to $\zeta := \sum_i \mu_i(x) \triangleleft \tau_i(h)$ in M. By [4, Lemma A.3, $(i) \Longrightarrow (iii)$], we therefore have

$$\limsup_\lambda \|\mu(x_\lambda \triangleleft h_\lambda) - \zeta_\lambda\| \leq \|\mu(x \triangleleft h) - \zeta\| < \varepsilon. \tag{3.6}$$

Note that

$$\Psi_{h_\lambda}\big(\zeta_\lambda \otimes \nu(y_\lambda)^{\mathrm{op}}\big) = \sum_{i=1}^{k} \mu_i(x_\lambda) \otimes \big(\nu(y_\lambda) \triangleleft \tau_i(h_\lambda)^*\big)^{\mathrm{op}},$$

and likewise without subscript-λ's. Thus, by continuity of μ_i, ν, τ_i, of the involution in $\mathscr{B}$, of the right $\mathscr{B}$-action, and of addition in $\mathscr{K}$, combined with Remark 3.2, this implies that

$$\Psi_{h_\lambda}(\zeta_\lambda \otimes \nu(y_\lambda)^{\mathrm{op}}) \to \Psi_h(\zeta \otimes \nu(y)^{\mathrm{op}}). \tag{3.7}$$

Since Ψ_h is isometric (Ψ2), we then have for large enough λ

$$\begin{aligned}
&\|\Psi_{h_\lambda}(\zeta_\lambda \otimes \nu(y_\lambda)^{\mathrm{op}}) - \Psi_{h_\lambda}(\mu(x_\lambda \triangleleft h_\lambda) \otimes \nu(y_\lambda)^{\mathrm{op}})\| \\
&\quad = \|(\zeta_\lambda - \mu(x_\lambda \triangleleft h_\lambda)) \otimes \nu(y_\lambda)^{\mathrm{op}}\| \\
&\quad \leq \|\zeta_\lambda - \mu(x_\lambda \triangleleft h_\lambda)\| \, \|\nu(y_\lambda)\| \overset{(*)}{\leq} \varepsilon(\varepsilon + \|\nu(y)\|),
\end{aligned}$$

where $(*)$ follows from (3.6) and continuity of ν. The same computation without subscripts yields

$$\|\Psi_h(\zeta \otimes \nu(y)^{\mathrm{op}}) - \Psi_h(\mu(x \triangleleft h) \otimes \nu(y)^{\mathrm{op}})\| \leq \varepsilon(\varepsilon + \|\nu(y)\|).$$

As ε was arbitrary, Lemma B.1 states that this combined with (3.7) implies

$$\Psi_{h_\lambda}\big(\mu(x_\lambda \triangleleft h_\lambda) \otimes \nu(y_\lambda)^{\mathrm{op}}\big) \to \Psi_h\big(\mu(x \triangleleft h) \otimes \nu(y)^{\mathrm{op}}\big).$$

Because each Ψ_h is linear, isometric, and surjective, the claim now follows from Lemma B.2 and [4, Proposition A.8]. ∎

We are now able to proceed with the balancing by defining a relation R on the set K as follows; the results are adapted from [4, Section 6].

Lemma 3.5 *If $\xi_1, \xi_2 \in K$ with $q_{\mathscr{K}}(\xi_i) = (x_i, y_i^{op})$, let*

$$\xi_1 \,\mathrm{R}\, \xi_2 :\Longleftrightarrow \exists h \in \sigma(x_1)\mathcal{H}\sigma(x_2) \text{ such that } x_2 = x_1 \triangleleft h,\, y_2 = y_1 \triangleleft h,\, \Psi_h(\xi_2) = \xi_1.$$

This defines a closed equivalence relation on the total space K of $\mathscr{K}$ whose quotient map $Q\colon K \to K/\mathrm{R}$ is open.

We often write $[\xi]$ for $Q(\xi)$.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Proof The proof that R is closed is verbatim that given for [4, Lemma 6.2], where it is also explained why R is an equivalence relation.

To show that the quotient map Q is open, the proof of [4, Proposition 6.8] goes through for $\mathscr{N} := \mathscr{M}^{\mathrm{op}}$, even though $\mathscr{M}$ was assumed to be an equivalence and $\mathcal{H}$ was assumed to be étale in [4]. In fact, étaleness was only needed to be allowed to invoke [4, Lemma 6.10], but that lemma's proof holds as long as the range map of $\mathcal{H}$ is open (which we assumed here). ∎

Lemma 3.6 *On the quotient of K by R, the map $p: K/\mathrm{R} \to \mathcal{G}, [\xi] \mapsto [q_{\mathscr{K}}(\xi)]$, is well defined, surjective, continuous, and open.*

Proof The proof of [4, Lemma 6.5] for $\mathscr{N} := \mathscr{M}^{\mathrm{op}}$ goes through, even though $\mathscr{M}$ was assumed to be an equivalence in that lemma. ∎

▸ **For the moment**, we will write $\mathscr{A} = (p_{\mathscr{A}} = p: A \to \mathcal{G})$ for the quotient bundle of $\mathscr{K}$ by R. Once we have shown that $\mathscr{A}$ is a Fell bundle that is equivalent to $\mathscr{B}$ via $\mathscr{M}$, we will change notation.

Lemma 3.7 (cf. [4, Proposition 6.6. and Lemma 6.7]) *Fix $g = {}^{x}_{\mathcal{G}}\{x \mid y^{op}\} \in \mathcal{G}$. With respect to the subspace topology of $A = K/\mathrm{R}$ on the fibre $A(g)$ of $\mathscr{A}$, the restriction of the quotient map $Q: K \to A$ to the fibre over (x, y^{op}) is a homeomorphism:*

$$Q_{(x,y^{op})}: \quad K(x, y^{op}) = M(x) \otimes_{\sigma(x)} M(y)^{op} \xrightarrow{\approx} A(g). \tag{3.8}$$

The above lemma means that we can give $A(g)$ the Banach space structure that makes all maps $Q_{(x,y^{\mathrm{op}})}$ *isomorphisms*: for $[\xi_1], [\xi_2] \in A(g)$, there exists a unique $h \in \mathcal{H}$ such that $q_{\mathscr{K}}(\xi_1) = q_{\mathscr{K}}(\Psi_h(\xi_2))$, and we may let

$$\lambda_1[\xi_1] + \lambda_2[\xi_2] := [\lambda_1\xi_1 + \lambda_2\Psi_h(\xi_2)] \quad \text{and} \quad \|[\xi]\| := \|\xi\|_{K(q(\xi))}.$$

Note that, in light of the Hilbert module isomorphism in Equation (3.1), we can also think of the fibres of $\mathscr{A}$ as generalized compact operators:

$$\mathbb{K}_{B(\sigma(x))}\big(M(y), M(x)\big) \xrightarrow{\approx} A([x, y^{\mathrm{op}}]). \tag{3.9}$$

Proof If $\xi, \eta \in K(x, y^{\mathrm{op}})$, then since the $\mathcal{H}$-action on X is free, $h = \sigma(x) \in \mathcal{H}^{(0)}$ is the only possible element that allows $\xi \,\mathrm{R}\, \eta$, in which case $\Psi_h(\eta) = \eta$ (since h is a unit) and hence $\xi = \eta$. Thus, $Q_{(x,y^{\mathrm{op}})}$ is injective. For surjectivity, note that Q is surjective, so any element a of $A(g)$ can be written as $a = Q(\xi)$ for $\xi \in K$ with $[q_{\mathscr{K}}(\xi)] = g = {}^{x}_{\mathcal{G}}\{x \mid y^{\mathrm{op}}\}$. In particular, there exists $h \in \mathcal{H}$ with $q_{\mathscr{K}}(\xi) = (x \triangleleft h, (y \triangleleft h)^{\mathrm{op}})$, so that $\Psi_h(\xi) \in K(x, y^{\mathrm{op}})$ and $a = Q(\xi) = Q_{(x,y^{\mathrm{op}})}(\Psi_h(\xi))$.

Since $p = p_{\mathscr{A}}$ is continuous, $A(g) = p_{\mathscr{A}}^{-1}(\{g\})$ is closed in A, and so continuity of Q implies continuity of $Q_{(x,y^{\mathrm{op}})}$. To see that $Q_{(x,y^{\mathrm{op}})}$ is closed (and hence a homeomorphism), we follow the idea in the proof of [4, Lemma 6.7]: Let $F \subseteq K(x, y^{\mathrm{op}})$ be a closed set. To show that $Q(F) \subseteq A(g)$ is closed, it suffices to show that it is closed in A, i.e., that $Q^{-1}(Q(F))$ is closed in K. So assume $\xi_\lambda \to \xi$ in K with $Q(\xi_\lambda) \in Q(F)$; we must prove that $Q(\xi) \in Q(F)$. Since $Q(\xi_\lambda) \in F$, there exist $\eta_\lambda \in F$ with $\eta_\lambda \,\mathrm{R}\, \xi_\lambda$. Since $F \subseteq K(x, y^{\mathrm{op}})$, this means there exist $h_\lambda \in \sigma(x)\mathcal{H}$ with $q_{\mathscr{K}}(\xi_\lambda) = (x \triangleleft h_\lambda, (y \triangleleft h_\lambda)^{\mathrm{op}})$ and $\Psi_{h_\lambda}(\xi_\lambda) = \eta_\lambda$. Since $\xi_\lambda \to \xi$, we have

$$(x \triangleleft h_\lambda, (y \triangleleft h_\lambda)^{\mathrm{op}}) = q_{\mathscr{K}}(\xi_\lambda) \to q_{\mathscr{K}}(\xi) =: (x_0, y_0^{\mathrm{op}}).$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Since the $\mathcal{H}$-action on X is proper, it follows from [28, Proposition 2.17] that (a subnet of) $\{h_\lambda\}_\lambda$ converges to, say, h; since X is Hausdorff, it follows that $(x_0, y_0^{\mathrm{op}}) = (x \triangleleft h, (y \triangleleft h)^{\mathrm{op}})$. Since Ψ is jointly continuous, we conclude that (a subnet of) $\eta_\lambda = \Psi_{h_\lambda}(\xi_\lambda)$ converges to $\Psi_h(\xi)$ in K. Since $\eta_\lambda \in F$ and F is closed in $K(x, y^{\mathrm{op}})$ and hence also in K, we have $\Psi_h(\xi) \in F$. Since $\xi \, \mathrm{R} \, \Psi_h(\xi)$, this means that $\xi \in Q^{-1}(Q(F))$, as needed. ∎

Lemma 3.8 *The bundle $\mathscr{A}$—equipped with the quotient topology on $A = K/\mathrm{R}$ and with the fibrewise linear structure inherited from $\mathscr{K}$ via the maps given in (3.8)—is an upper semi-continuous Banach bundle. With respect to this topology, all sections of the form*

$$[x, y^{op}] \mapsto [\mu(x) \otimes \nu(y)^{op}]$$

are continuous, where μ, ν are continuous sections of $\mathscr{M}$.

Proof That $\mathscr{A}$ is an upper semi-continuous Banach bundle follows from an application of [4, Proposition 6.13]; here, we need that both ${}^{X}_{\mathcal{G}}\{\_ \mid \_\} : X \,{}_\sigma{*}_\sigma\, X^{\mathrm{op}} \to \mathcal{G}$ and $Q: K \to A$ are open quotient maps (Lemma 3.5) and that $Q_{(x,y^{\mathrm{op}})}$ is surjective (Lemma 3.7) and a linear isometry (by definition of the linear structure on each fibre of $\mathscr{A}$).

That the given section is continuous follows since ${}^{X}_{\mathcal{G}}\{\_ \mid \_\}$ is open, $\mu \otimes \nu^{\mathrm{op}}$ is a continuous section of $\mathscr{K}$ (Lemma 3.1), and Q is continuous. ∎

At this point, we will diverge from what was done in [4] and prove that $\mathscr{A} = (p_{\mathscr{A}}: A \to \mathcal{G})$ is not just an upper semi-continuous Banach bundle but actually a Fell bundle; see Theorem 3.17. In particular, we need to construct two maps, namely a multiplication $\_ \cdot \_ : \mathscr{A}^{(2)} \to A$ and an involution $\_^* : A \to A$. Conceptually, the involution is easier, so this is where we will start.

Lemma 3.9 *There exists a unique bijective, fibrewise isometric and conjugate-linear map $\mathtt{Flip}: \mathscr{K} \to \mathscr{K}$ that covers the homeomorphism*

$$\mathtt{flip}: X \,{}_\sigma{*}_\sigma\, X^{op} \to X \,{}_\sigma{*}_\sigma\, X^{op}, \quad (x, y^{op}) \mapsto (y, x^{op}),$$

and that is fibrewise given on dense spanning elements by

$$\begin{aligned} K(x, y^{op}) &\to K(y, x^{op}) \\ m \otimes n^{op} &\mapsto n \otimes m^{op}. \end{aligned} \tag{3.10}$$

Note that the following diagram commutes:

$$\begin{array}{ccc} K(x \triangleleft h, y^{\mathrm{op}}) & \xrightarrow{\mathtt{Flip}} & K(y, (x \triangleleft h)^{\mathrm{op}}) \\ \Big\downarrow{\scriptstyle \Psi_h} & & \Big\downarrow{\scriptstyle \Psi_h} \\ K(x, (y \triangleleft h^{-1})^{\mathrm{op}}) & \xrightarrow{\mathtt{Flip}} & K(y \triangleleft h^{-1}, x^{\mathrm{op}}). \end{array} \tag{3.11}$$

Proof To see that the map $\mathtt{Flip}$ exists on each fibre, fix $(x, y^{\mathrm{op}}) \in X \,{}_\sigma{*}_\sigma\, X^{\mathrm{op}}$ and let $u = \sigma(x) = \sigma(y)$. Recall that $K(x, y^{\mathrm{op}})$ is isomorphic to $\mathbb{K}_{B(u)}(M(y), M(x))$ as bi-Hilbert $\mathbb{K}_{B(u)}(M(x)) - \mathbb{K}_{B(u)}(M(y))$-bimodules. With this identification, the restriction of $\mathtt{Flip}$ to this fibre is given by $|m\rangle\langle n| \mapsto |n\rangle\langle m|$, or in other words, $\mathtt{Flip}$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

is the adjoint map $T \mapsto T^*$. In particular, `Flip` is fibrewise antilinear and isometric. Its continuity is built into the topology on $\mathscr{K}$: If μ, ν are continuous sections of $\mathscr{M}$, then $\mu \otimes \nu^{\mathrm{op}}$ and $\nu \otimes \mu^{\mathrm{op}}$ are continuous sections of $\mathscr{K}$. Since `Flip` transforms the former into the latter, it is continuous and open by an application of [4, Propositions A.7 and A.8].[5] ∎

Continuity of `Flip` and continuity and openness of the quotient map Q now immediately imply the following.

Corollary 3.10 *On the quotient bundle $\mathscr{A}$, there exists a unique continuous, fibrewise conjugate-linear map $\_^*\colon A \to A$ given by $[\xi]^* = [\mathtt{Flip}(\xi)]$.*

Now that we have (a candidate for) the involution on $\mathscr{A}$, we proceed to construct the multiplication. Recall that our end goal is not only to show that $\mathscr{A}$ is a Fell bundle but also that there is a left action $\vDash$ of $\mathscr{A}$ on $\mathscr{M}$. We want this action to behave nicely with respect to the multiplication in the sense that $a_1 \vDash (a_2 \vDash m) = (a_1 \cdot a_2) \vDash m$; it is therefore easier to *first* construct a candidate for the left-action and then use it to construct the multiplication. Like with the involution, we first do everything on the level of $\mathscr{K}$ before we move to its quotient $\mathscr{A}$. For clarity, we remind the reader that the fibre of $\mathscr{K}$ over $(x, y^{\mathrm{op}}) \in X\,{}_\sigma{*}_\sigma\, X^{\mathrm{op}}$ is given by $K(x, y^{\mathrm{op}}) = M(x) \otimes_{\sigma(x)} M(y)^{\mathrm{op}}$.

Lemma 3.11 *For $(x, y^{op}) \in X\,{}_\sigma{*}_\sigma\, X^{op}$, there exists a continuous bilinear map*

$$\Phi_{x,y}\colon \quad K(x, y^{op}) \times M(y) \to M(x)$$

determined on elementary tensors by

$$\Phi_{x,y}(m \otimes n^{op}, k) = m \dashv \langle n \mid k \rangle_{\mathscr{B}}. \tag{3.12}$$

For all $\xi \in K(x, y^{op})$ and $k \in M(y)$, it satisfies

$$\left\| \Phi_{x,y}(\xi, k) \right\| \leq \|\xi\| \, \|k\| \,. \tag{3.13}$$

Proof The existence of $\Phi_{x,y}$ follows from the following, well-studied isomorphisms of bi-Hilbert $A-A$-modules for any full right-Hilbert A-module $\mathbf{Y}$: Firstly, the map $\mathbf{Y}^{\mathrm{op}} \otimes_{\mathbb{K}} \mathbf{Y} \to A$ determined by $\mathbf{y}_1^{\mathrm{op}} \otimes \mathbf{y}_2 \mapsto \langle \mathbf{y}_1 \mid \mathbf{y}_1 \rangle_A^{\mathbf{Y}}$, and secondly, $\mathbf{Y} \otimes_A A \to \mathbf{Y}$ determined by $\mathbf{y} \otimes a \mapsto \mathbf{y} \cdot a$. In our situation, $A = B(u)$ where $u = \sigma(x) = \sigma(y)$, and we use first $M(y)$ and then $M(x)$ to take the rôle of $\mathbf{Y}$. To be precise, if we write $\mathbb{K}$ for $\mathbb{K}_{B(u)}(M(y))$, then

$$K(x, y^{\mathrm{op}}) \times M(y) \twoheadrightarrow \big(M(x) \otimes_u M(y)^{\mathrm{op}}\big) \otimes_{\mathbb{K}} M(y) \cong M(x) \otimes_u B(u) \cong M(x).$$

The claim about the norm is also well known, but we add it here for completion. Since $\Phi_{x,y}$ is continuous, it suffices to prove the claim for $\xi = \sum_{i=1}^{\ell} m_i \otimes n_i^{\mathrm{op}}$, a sum of elementary tensors. We have

$$\left\| \Phi_{x,y}(\xi, k) \right\|^2 = \left\| \sum_i m_i \dashv \langle n_i \mid k \rangle_{\mathscr{B}} \right\|^2 = \left\| \sum_{i,j} | m_i \dashv \langle n_i \mid k \rangle_{\mathscr{B}} \rangle \langle m_j \dashv \langle n_j \mid k \rangle_{\mathscr{B}} | \right\|,$$

[5] To be pedantic, [4] only deals with fibrewise linear rather than antilinear maps, so the cited results give an isomorphism $\mathscr{K} \to \mathscr{K}^{\mathrm{op}}$ of Banach bundles determined by $m \otimes n^{\mathrm{op}} \mapsto (n \otimes m^{\mathrm{op}})^{\mathrm{op}}$ covering the map $(x, y^{\mathrm{op}}) \mapsto (y, x^{\mathrm{op}})^{\mathrm{op}}$. But we can then compose that map with the homeomorphism $\mathscr{K}^{\mathrm{op}} \to \mathscr{K}$, $\xi^{\mathrm{op}} \mapsto \xi$.

where the norm on the right-hand side is the operator norm on $\mathbb{K}_{B(u)}(M(x))$. Note that $\langle n_i \mid k\rangle_{\mathscr{B}} \in B(u)$ acts by $\mathbb{K}_{B(u)}(M(x))$-adjointable operators, so that

$$\left\|\Phi_{x,y}(\xi,k)\right\|^2 = \left\|\sum_{i,j} |m_i\rangle\langle (m_j \triangleleft \langle n_j \mid k\rangle_{\mathscr{B}}) \triangleleft \langle k \mid n_i\rangle_{\mathscr{B}}|\right\|.$$

Now, $\langle n_j \mid k\rangle_{\mathscr{B}}\langle k \mid n_i\rangle_{\mathscr{B}} = \langle n_j \mid k \triangleleft \langle k \mid n_i\rangle_{\mathscr{B}}\rangle_{\mathscr{B}}$ by (DE4). Since $k, n_i \in M(y)$, we further know that $k \triangleleft \langle k \mid n_i\rangle_{\mathscr{B}} = |k\rangle\langle k|(n_i)$. Recall that $|k\rangle\langle k|$ is a positive element of $\mathbb{K}_{B(u)}(M(y))$, so we can write it as T^*T. In particular,

$$\left\|\Phi_{x,y}(\xi,k)\right\|^2 = \left\|\sum_{i,j} |m_i\rangle\langle m_j \triangleleft \langle Tn_j \mid Tn_i\rangle_{\mathscr{B}}|\right\| \overset{(3.2)}{=} \left\|\sum_i m_i \otimes (Tn_i)^{\mathrm{op}}\right\|^2.$$

Since $\|\sum_i m_i \otimes (Tn_i)^{\mathrm{op}}\| \le \|1 \otimes T\| \, \|\sum_i m_i \otimes n_i^{\mathrm{op}}\|$ and $\|1 \otimes T\| = \|T\| = \|k\|$, we conclude that

$$\left\|\Phi_{x,y}(\xi,k)\right\| \le \|\xi\| \, \|k\|,$$

as claimed. ∎

Bilinearity of $\Phi_{x,y}$ helps us prove the following result.

Lemma 3.12 *For any $u \in \mathcal{H}^{(0)}$ and any $y \in Xu$, there exists a map*

$$U_y\colon \bigsqcup_{x,z \in Xu} K(x, y^{op}) \times K(y, z^{op}) \to \bigsqcup_{x,z \in Xu} K(x, z^{op})$$

determined by

$$U_y(m \otimes n_1^{op}, n_2 \otimes k^{op}) = (m \triangleleft \langle n_1 \mid n_2\rangle_{\mathscr{B}}) \otimes k^{op}. \tag{3.14}$$

These maps have the following properties:

(U1) *When restricted to a fibre $K(x, y^{op}) \times K(y, z^{op}) \to K(x, z^{op})$, U_y is bilinear.*
(U2) *Each U_y satisfies $\|U_y(\xi,\eta)\| \le \|\xi\| \, \|\eta\|$ and $\|U_y(\xi, \mathtt{Flip}(\xi))\| = \|\xi\|^2$.*
(U3) *If $h \in u\mathcal{H}$, then $\Psi_h \circ U_{y \triangleleft h} = U_y \circ (\Psi_h \times \Psi_h)$.*
(U4) *We have $\mathtt{Flip}(U_y(\xi,\eta)) = U_y(\mathtt{Flip}(\eta), \mathtt{Flip}(\xi))$.*

Note that "equivariance" of U (Condition (U3)) is equivalent to commutativity of the following diagram for all x, z:

$$\begin{tikzcd} K(x \triangleleft h, h^{-1} \triangleright y^{\mathrm{op}}) \times K(y \triangleleft h, h^{-1} \triangleright z^{\mathrm{op}}) \arrow[r, "U_{y \triangleleft h}"] \arrow[d, "\Psi_h \times \Psi_h"'] & K(x \triangleleft h, h^{-1} \triangleright z^{\mathrm{op}}) \arrow[d, "\Psi_h"] \\ K(x, y^{\mathrm{op}}) \times K(y, z^{\mathrm{op}}) \arrow[r, "U_y"] & K(x, z^{\mathrm{op}}). \end{tikzcd} \tag{3.15}$$

Proof of Lemma 3.12 The existence of U_y follows from observations of general Hilbert C*-modules similar to those following Lemma 1.5: If $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are full right-Hilbert C*-modules over a C*-algebra A, then the maps

$$(\mathbf{X} \otimes_A \mathbf{Y}^{\mathrm{op}}) \otimes_{\mathbb{K}_A(\mathbf{Y})} (\mathbf{Y} \otimes_A \mathbf{Z}^{\mathrm{op}}) \longrightarrow \mathbf{X} \otimes_A \mathbf{Z}^{\mathrm{op}} \longrightarrow \mathbb{K}_A(\mathbf{Z}, \mathbf{X}) \tag{3.16}$$

determined by

$$(\mathbf{x} \otimes \mathbf{y}_1^{\mathrm{op}}) \otimes (\mathbf{y}_2 \otimes \mathbf{z}^{\mathrm{op}}) \quad \longmapsto \quad (\mathbf{x} \cdot \langle \mathbf{y}_1 \mid \mathbf{y}_2 \rangle_A^{\mathbf{Y}}) \otimes \mathbf{z}^{\mathrm{op}} \quad \longmapsto \quad |\mathbf{x} \cdot \langle \mathbf{y}_1 \mid \mathbf{y}_2 \rangle_A^{\mathbf{Y}} \rangle \langle \mathbf{z}|$$

are isomorphisms of bi-Hilbert $\mathbb{K}_A(\mathbf{X}) - \mathbb{K}_A(\mathbf{Z})$-modules. The above isomorphism can be precomposed with the universal bilinear map $\mathbf{X} \otimes_A \mathbf{Y}^{\mathrm{op}} \times \mathbf{Y} \otimes_A \mathbf{Z}^{\mathrm{op}} \to (\mathbf{X} \otimes_A \mathbf{Y}^{\mathrm{op}}) \otimes_{\mathbb{K}_A(\mathbf{Y})} (\mathbf{Y} \otimes_A \mathbf{Z}^{\mathrm{op}})$. We apply this to the case where $\mathbf{Y} = M(y)$ and $A = B(u)$. Condition (U1) is therefore by construction.

(U2) Since the maps in (3.16) are isomorphisms of modules (in particular, they are isometric), we have $\|U_y(\xi, \eta)\| = \|\xi \otimes \eta\|$ for $\xi \in K(x, y^{\mathrm{op}})$ and $\eta \in K(y, z^{\mathrm{op}})$, which is known (or easily shown) to be bounded by $\|\xi\| \, \|\eta\|$. In particular, U_y is not just linear but also continuous on each fibre. For the second claim, it therefore suffices to consider one of the dense elements $\xi = \sum_i m_i \otimes n_i^{\mathrm{op}}$ in which case $\mathtt{Flip}(\xi) = \sum_j n_j \otimes m_j^{\mathrm{op}}$ and

$$U(\xi, \mathtt{Flip}(\xi)) = \sum_{i,j} (m_i \dashv \langle n_i \mid n_j \rangle_{\mathscr{B}}) \otimes m_j^{\mathrm{op}}. \tag{3.17}$$

Again, since the maps in (3.16) are isometric, we see that

$$\|U(\xi, \mathtt{Flip}(\xi))\| = \left\| \sum_{i,j} |m_i \dashv \langle n_i \mid n_j \rangle_{\mathscr{B}} \rangle \langle m_j| \right\|,$$

which equals $\|\xi\|^2$ by Equation (3.2).

(U3) The properties of $\dashv$ and of $\langle \_ \mid \_ \rangle_{\mathscr{B}}$ imply that

$$(m \dashv b) \dashv \langle n_1 \mid n_2 \rangle_{\mathscr{B}} = m \dashv (b \langle n_1 \mid n_2 \rangle_{\mathscr{B}}) = m \dashv \langle n_1 \dashv b^* \mid n_2 \rangle_{\mathscr{B}}$$

whenever $m, n_1, n_2 \in M$ and $b \in B$ are chosen such that the left (and hence each) side of the above equations makes sense. Likewise, we have $\langle n_1 \mid n_2 \dashv b \rangle_{\mathscr{B}} = \langle n_1 \mid n_2 \rangle_{\mathscr{B}} b$. For $h \in u\mathcal{H}$, suppose we are given elements $b_i \in B(h), m \in M(x), n_1 \in M(y \triangleleft h), n_2 \in M(y), k \in M(z \triangleleft h)$. Then

$$\begin{aligned}
&(\Psi_h \circ U_{y \triangleleft h})\big((m \dashv b_1) \otimes n_1^{\mathrm{op}}, (n_2 \dashv b_2) \otimes k^{\mathrm{op}}\big) \\
&= \Psi_h\big((m \dashv b_1) \dashv \langle n_1 \mid n_2 \dashv b_2 \rangle_{\mathscr{B}} \otimes k^{\mathrm{op}}\big) = \Psi_h\big([m \dashv (b_1 \langle n_1 \mid n_2 \rangle_{\mathscr{B}})] \dashv b_2 \otimes k^{\mathrm{op}}\big) \\
&= \big(m \dashv \langle n_1 \dashv b_1^* \mid n_2 \rangle_{\mathscr{B}}\big) \otimes (b_2 \vdash k^{\mathrm{op}}) \\
&= U_y\big(m \otimes (b_1 \vdash n_1^{\mathrm{op}}), n_2 \otimes (b_2 \vdash k^{\mathrm{op}})\big) \\
&= (U_y \circ \Psi_h \times \Psi_h)\big((m \dashv b_1) \otimes n_1^{\mathrm{op}}, (n_2 \dashv b_2) \otimes k^{\mathrm{op}}\big).
\end{aligned}$$

Because of continuity and (bi)linearity of Ψ_h and of $U_y, U_{y \triangleleft h}$, we conclude that $\Psi_h \circ U_{y \triangleleft h} = U_y \circ \Psi_h \times \Psi_h$, as claimed. One likewise checks that (U4) holds for for elementary tensors, and again uses (bi)linearity and continuity of U_y and of $\mathtt{Flip}$ to deduce the claim. ∎

Similarly to how we constructed Ψ in Lemma 3.4 out of the maps Ψ_h from Lemma 3.3, we now want to "stich together" the maps $\Phi_{x,y}$ on the one hand and the maps U_y on the other hand, to bundle maps. To do so, we first need a definition.

Definition 3.1 For two upper semi-continuous Banach bundles $\mathscr{M} = (M \to X)$ and $\mathscr{N} = (N \to Y)$, consider the product bundle $\mathscr{M} \times \mathscr{N} = (M \times N \to X \times Y)$; the norm on its fibre over (x, y) can be chosen as the maximum of the norms of $M(x)$ and $N(y)$, and it is given the component-wise vector space structure. The global topology of the total space $M \times N$ of $\mathscr{M} \times \mathscr{N}$ is induced by the $\mathbb{C}$-linear span of sections of the form $(x, y) \mapsto (\mu(x), \nu(y))$, where μ and ν are continuous sections of $\mathscr{M}$ respectively $\mathscr{N}$.

If $f\colon X \to Z$ and $g\colon Y \to Z$ are continuous functions into some other topological space, then we write $\mathscr{M}\,{}_f{*}_g\,\mathscr{N}$ for the restriction of $\mathscr{M} \times \mathscr{N}$ to the closed subset $X\,{}_f{*}_g\,Y$ of the base $X \times Y$.

We now let $\mathtt{fir}, \mathtt{sec}\colon X\,{}_\sigma{*}_\sigma\,X^{\mathrm{op}} \to X$ be given by $\mathtt{fir}(x, y^{\mathrm{op}}) = x$ respectively $\mathtt{sec}(x, y^{\mathrm{op}}) = y$ ("$\mathtt{fir}$" for "first", "$\mathtt{sec}$" for "second").

Lemma 3.13 *Writing $\Phi_{(x,y^{op})}$ for the map $\Phi_{x,y}$ of Lemma 3.11, the map*

$$\Phi\colon \quad \mathscr{K}\,{}_{\mathtt{sec}}{*}_q\,\mathscr{M} \to \mathscr{M}, \qquad\qquad (\xi, m) \mapsto \Phi_{q(\xi)}(\xi, m),$$

is bilinear, jointly continuous, and covers the continuous surjection

$$(X\,{}_\sigma{*}_\sigma\,X^{op})\,{}_{\mathtt{sec}}{*}_{\mathbf{id}}\,X \to X, \qquad\qquad (x, y^{op}, y) \mapsto x.$$

Note that Φ is not a homomorphism of Banach bundles, since Φ is not fibrewise linear.

Proof Since $\Phi_{x,y}$ lands in $M(x)$ by construction, Φ covers $(x, y^{\mathrm{op}}, y) \mapsto x$. To see that Φ is jointly continuous, assume that we are given a convergent net $(\xi_\lambda, k_\lambda) \to (\xi, k)$ in $K\,{}_{\mathtt{sec}}{*}_q\,M$, and let $q(\xi_\lambda, k_\lambda) = (x_\lambda, y_\lambda^{\mathrm{op}}, y_\lambda)$ and $q(\xi, k) = (x, y^{\mathrm{op}}, y)$; note that $x_\lambda \to x$ and $y_\lambda \to y$ in X. To prove that $\Phi(\xi_\lambda, k_\lambda) \to \Phi(\xi, k)$, fix $\delta > 0$; we must find $\kappa \in \Gamma_0(X; \mathscr{M})$ with $\|\Phi(\xi, k) - \kappa(x)\| < \delta$ and $\|\Phi(\xi_\lambda, k_\lambda) - \kappa(x_\lambda)\| < \delta$ for large λ [4, Lemma A.3].

Because of how the topology on $\mathscr{K}$ is defined and because $\xi_\lambda \to \xi$ in $\mathscr{K}$, we can find finitely many nets $m_{j,\lambda} \to m_j$ and $n_{j,\lambda} \to n_j$ in M such that

$$\left\| \xi - \sum_{j=1}^{\ell} m_j \otimes n_j^{\mathrm{op}} \right\| < \frac{\delta}{2(\|k\| + 1)}, \qquad \left\| \xi_\lambda - \sum_{j=1}^{\ell} m_{j,\lambda} \otimes n_{j,\lambda}^{\mathrm{op}} \right\| < \frac{\delta}{2(\|k\| + 1)}. \tag{3.18}$$

Since $\langle \llcorner \mid \llcorner \rangle_{\mathscr{B}}$ is jointly continuous, this implies that for each j, $\langle n_{j,\lambda} \mid k_\lambda \rangle_{\mathscr{B}} \to \langle n_j \mid k \rangle_{\mathscr{B}}$ in B. Since $\llcorner \triangleleft \llcorner$ is jointly continuous, this in turn implies that

$$\sum_{j=1}^{\ell} m_{j,\lambda} \triangleleft \langle n_{j,\lambda} \mid k_\lambda \rangle_{\mathscr{B}} \to \sum_{j=1}^{\ell} m_j \triangleleft \langle n_j \mid k \rangle_{\mathscr{B}} \text{ in } M.$$

That means that we can find a section κ of $\mathscr{M}$ with

$$\left\| \sum_{j=1}^{\ell} m_j \triangleleft \langle n_j \mid k \rangle_{\mathscr{B}} - \kappa(x) \right\| < \frac{\delta}{2}, \qquad \left\| \sum_{j=1}^{\ell} m_{j,\lambda} \triangleleft \langle n_{j,\lambda} \mid k_\lambda \rangle_{\mathscr{B}} - \kappa(x_\lambda) \right\| < \frac{\delta}{2} \tag{3.19}$$

for all $\lambda \geq \lambda_0$. Let λ be large enough such that we also have $|\,\|k_\lambda\| - \|k\|\,| < 1$. Linearity of Φ in the first component yields:

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

$$\begin{aligned}\|\Phi(\xi,k)-\kappa(x)\| &\leq \Big\|\Phi(\xi,k)-\sum_{j=1}^{\ell} m_j \triangleleft \langle n_j \mid k\rangle_{\mathscr{B}}\Big\| + \Big\|\sum_{j=1}^{\ell} m_j \triangleleft \langle n_j \mid k\rangle_{\mathscr{B}} - \kappa(x)\Big\| \\ &\overset{(3.19)}{\leq} \Big\|\Phi\Big(\xi-\sum_{j=1}^{\ell} m_j\otimes n_j^{\mathrm{op}},k\Big)\Big\| + \frac{\delta}{2} \\ &\overset{(3.13)}{\leq} \Big\|\xi-\sum_{j=1}^{\ell} m_j\otimes n_j^{\mathrm{op}}\Big\|\,\|k\| + \frac{\delta}{2} \overset{(3.18)}{<} \delta,\end{aligned}$$

and the exact same computation with subscript-λ's yields $\|\Phi(\xi_\lambda,k_\lambda)-\kappa(x_\lambda)\|<\delta$ for large λ, which proves the claim. ∎

We proceed with stitching together the U_y's.

Lemma 3.14 *The map*

$$U\colon\ \mathscr{K}\,{}_{\mathtt{sec}}{*}_{\mathtt{fir}}\,\mathscr{K}\to\mathscr{K},\qquad (\xi,\eta)\mapsto U_{\mathtt{sec}(\xi)}(\xi,\eta),$$

is bilinear, jointly continuous, and covers the continuous surjection

$$(X\times X^{op})\,{}_{\mathtt{sec}}{*}_{\mathtt{fir}}\,(X\times X^{op})\to X\times X^{op},\qquad (x,y^{op},y,z^{op})\mapsto(x,z^{op}).$$

Moreover, given $\mu,\xi,\eta\in K$ and $m\in M$ with $(\mu,\xi),(\xi,\eta)\in\mathscr{K}\,{}_{\mathtt{sec}}{*}_{\mathtt{fir}}\,\mathscr{K}$ and $(\xi,m)\in\mathscr{K}\,{}_{\mathtt{sec}}{*}_{q}\,\mathscr{M}$, we have

$$\Phi(U(\mu,\xi),m)=\Phi(\mu,\Phi(\xi,m))\ \textit{and thus}\ U\big(U(\mu,\xi),\eta\big)=U\big(\mu,U(\xi,\eta)\big). \tag{3.20}$$

Proof To see that U is jointly continuous, assume that $\{(\xi_\lambda,\eta_\lambda)\}_\lambda$ is a net in $\mathscr{K}\,{}_{\mathtt{sec}}{*}_{\mathtt{fir}}\,\mathscr{K}$ that converges to (ξ,η). Let $(x_\lambda,y_\lambda^{\mathrm{op}}):=q_{\mathscr{K}}(\xi_\lambda)$, which converges to $(x,y^{\mathrm{op}}):=q_{\mathscr{K}}(\xi)$, and let $(y_\lambda,z_\lambda^{\mathrm{op}}):=q_{\mathscr{K}}(\eta_\lambda)$, which converges to $(y,z^{\mathrm{op}}):=q_{\mathscr{K}}(\eta)$. Fix an arbitrary $\delta\in(0,1)$; by [4, Lemma A.3], we should find a section $\kappa\in\Gamma_0(X\,{}_\sigma{*}_\sigma\,X^{\mathrm{op}};\mathscr{K})$ with

$$\big\|U\big(\xi,\eta\big)-\kappa(x,z^{\mathrm{op}})\big\|<4\delta \text{ and } \big\|U\big(\xi_\lambda,\eta_\lambda\big)-\kappa(x_\lambda,z_\lambda^{\mathrm{op}})\big\|<4\delta$$

for large λ. Because of how the topology on $\mathscr{K}$ is defined and because $\xi_\lambda\to\xi,\eta_\lambda\to\eta$ in $\mathscr{K}$, there exist elements

$$\zeta_\lambda\in M(x_\lambda)\odot M(y_\lambda)^{\mathrm{op}} \text{ converging to } \zeta\in M(x)\odot M(y)^{\mathrm{op}} \text{ in } K$$

and elements

$$\chi_\lambda\in M(y_\lambda)\odot M(z_\lambda)^{\mathrm{op}} \text{ converging to } \chi\in M(y)\odot M(z)^{\mathrm{op}} \text{ in } K$$

such that

$$\|\xi-\zeta\|,\|\xi_\lambda-\zeta_\lambda\|<\frac{\delta}{\|\eta\|+1},\qquad \text{and } \|\eta-\chi\|,\|\eta_\lambda-\chi_\lambda\|<\frac{\delta}{\|\xi\|+1} \tag{3.21}$$

for large λ. Here, it is important to point out that $\odot$ refers to the *algebraic* tensor product, so each of the elements considered above is a finite sum of elementary tensors. Note that

$$U(m\otimes n^{\mathrm{op}},k\otimes\ell^{\mathrm{op}})=\Phi(m\otimes n^{\mathrm{op}},k)\otimes\ell^{\mathrm{op}}, \tag{3.22}$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

so using bilinearity of U and continuity of Φ, the fact that $\zeta_\lambda \to \zeta$ and $\chi_\lambda \to \chi$ in K and that addition and scalar multiplication is continuous on $\mathscr{K}$, imply that

$$U(\zeta_\lambda, \chi_\lambda) \to U(\zeta, \chi)$$

for these particular sums of elementary tensors $\zeta_\lambda, \chi_\lambda$. This convergence means that we can find a section κ of $\mathscr{K}$ with

$$\|U(\zeta_\lambda, \chi_\lambda) - \kappa(x_\lambda, z_\lambda^{\mathrm{op}})\| < \delta \text{ and } \|U(\zeta, \chi) - \kappa(x, z^{\mathrm{op}})\| < \delta. \tag{3.23}$$

Combining this with Condition (U2), we conclude for large λ:

$$\begin{aligned}\|U(\xi,\eta) - \kappa(x, z^{\mathrm{op}})\| &\le \|U(\xi, [\eta-\chi])\| + \|U([\xi-\zeta],\chi)\| + \|U(\zeta,\chi) - \kappa(x, z^{\mathrm{op}})\| \\ &< \|\xi\|\ \|\eta - \chi\| + \|\xi - \zeta\|\ \|\chi\| + \delta \qquad \text{by (3.23), (U2)} \\ &< 3\delta \qquad \text{by (3.21).}\end{aligned}$$

We likewise get for λ large enough such that $|\,\|\xi\| - \|\xi_\lambda\|\,|$ and $|\,\|\eta\| - \|\eta_\lambda\|\,|$ are bounded by 1,

$$\begin{aligned}\|U(\xi_\lambda, \eta_\lambda) - \kappa(x_\lambda, z_\lambda^{\mathrm{op}})\| &< \|\xi_\lambda\|\ \|\eta_\lambda - \chi_\lambda\| + \|\xi_\lambda - \zeta_\lambda\|\ \|\chi_\lambda\| + \delta \\ &\overset{(3.21)}{<} (\|\xi\|+1)\frac{\delta}{\|\xi\|+1} + \frac{\delta}{\|\eta\|+1}\ \|\chi_\lambda\| + \delta.\end{aligned}$$

Note that

$$\|\chi_\lambda\| \le \|\chi_\lambda - \eta_\lambda\| + \|\eta_\lambda\| \le \|\chi_\lambda - \eta_\lambda\| + \|\eta\| + 1 \le \frac{\delta}{\|\xi\|+1} + \|\eta\| + 1,$$

so that

$$\frac{\delta}{\|\eta\|+1}\ \|\chi_\lambda\| \le \frac{\delta}{\|\eta\|+1}\left[\frac{\delta}{\|\xi\|+1} + \|\eta\| + 1\right] \le 2\delta.$$

We conclude that

$$\|U(\xi_\lambda, \eta_\lambda) - \kappa(x_\lambda, z_\lambda^{\mathrm{op}})\| < (\|\xi\|+1)\frac{\delta}{\|\xi\|+1} + 2\delta + \delta < 4\delta,$$

as needed.

For (3.20), it likewise suffices to prove the claims for elementary tensors $\mu = \ell \otimes m_1^{\mathrm{op}}$, $\xi = m_2 \otimes n_1^{\mathrm{op}}$, and $\eta = n_2 \otimes k^{\mathrm{op}}$. The computations make use of $\mathscr{B}$-linearity of $\langle \_ \mid \_ \rangle_{\mathscr{B}}$ and are left to the reader. ■

For the next result, we remind the reader that we denote the groupoid $X \times_{\mathcal{H}} X^{\mathrm{op}} = (X\, {}_\sigma{*}_\sigma X^{\mathrm{op}})/\mathcal{H}$ by $\mathcal{G}$ and that we write $r_{\mathscr{A}} := r_{\mathcal{G}} \circ p_{\mathscr{A}}$ and $s_{\mathscr{A}} := s_{\mathcal{G}} \circ p_{\mathscr{A}}$.

Lemma 3.15 *On the quotient bundle $\mathscr{A} = (p_{\mathscr{A}}: A \to \mathcal{G})$, the following map is well defined, continuous, and fibrewise bilinear:*

$$\_ \cdot \_ : \qquad \mathscr{A}^{(2)} := \{([\xi],[\eta]) \in A \times A : s_{\mathscr{A}}([\xi]) = r_{\mathscr{A}}([\eta])\} \to A,$$

given by

$$[\xi]\cdot[\eta] := \left[U(\xi, \Psi_h(\eta))\right],$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

where U is as defined in Lemma 3.14 and $h \in \mathcal{H}$ is the unique element such that $\mathtt{sec}_{\mathscr{K}}(\xi) = \mathtt{fir}_{\mathscr{K}}\big(\Psi_h(\eta)\big)$.

Remark 3.16 If $(g_1, g_2) \in \mathcal{G}^{(2)}$, we may choose representatives (x, y^{op}) and (y^{op}, z) of g_1 respectively g_2. If we then identify the fibres $A(g_1), A(g_2)$ with $K(x, y^{\mathrm{op}}) = M(x) \otimes_{\sigma(x)} M(y)^{\mathrm{op}}$ and $K(y, z^{\mathrm{op}})$ via (3.8), then the map $\_\cdot\_$ is fibrewise on dense spanning sets given by

$$\begin{aligned} \_\cdot\_ : \quad & A(g_1) \times A(g_2) \to K(x, z^{\mathrm{op}}) \cong A(g_1 g_2) \\ & \big(m_1 \otimes n_1^{\mathrm{op}}, m_2 \otimes n_2^{\mathrm{op}}\big) \mapsto \big(m_1 \triangleleft \langle n_1 \mid m_2 \rangle_{\mathscr{B}}^{\mathscr{M}}\big) \otimes n_2^{\mathrm{op}}. \end{aligned} \tag{3.24}$$

Proof of Lemma 3.15 Since $U_h \circ (\Psi_h \times \Psi_h) = \Psi_h \circ U_{y \triangleleft h}$, the map $\mathfrak{m} := \_\cdot\_$ is well defined. Since U_y and Ψ_h are linear, linearity of $\mathfrak{m}$ follows directly from the definition of the fibrewise Banach space structure on $\mathscr{A}$.

To see that it is continuous, we will invoke openness of the quotient map Q: Suppose $\{\chi_\lambda\}_\lambda$ is a net in $\mathscr{A}^{(2)}$ which converges to χ. Since it suffices to show that a *subnet* of $\{\mathfrak{m}(\chi_\lambda)\}_\lambda$ converges to $\mathfrak{m}(\chi)$, we can without loss of generality assume that the *entire* net $\{\chi_\lambda\}_\lambda$ lifts to a net $\{(\xi_\lambda, \eta_\lambda)\}_\lambda$ in $K \times K$ that converges to a lift (ξ, η) of χ (here, we have made use of [28, Proposition 1.1 (Fell's criterion)] for the open map Q). Denote $q_{\mathscr{K}}(\xi_\lambda) = (x_\lambda, y_\lambda^{\mathrm{op}})$. Since $\chi_\lambda \in \mathscr{A}^{(2)}$, we may let $h_\lambda, h \in \mathcal{H}$ be the unique elements such that $q_{\mathscr{K}}\big(\Psi_{h_\lambda}(\eta_\lambda)\big) = (y_\lambda, z_\lambda^{\mathrm{op}})$ and $q_{\mathscr{K}}\big(\Psi_h(\eta)\big) = (y, z^{\mathrm{op}})$ for some $z_\lambda, z \in X$. Since $\xi_\lambda \to \xi$, it follows that $y_\lambda \to y$. Since $\eta_\lambda \to \eta$ and $q_{\mathscr{K}}\big(\Psi_{h_\lambda}(\eta_\lambda)\big) = (y_\lambda, z_\lambda^{\mathrm{op}})$, we further have that $(y_\lambda \triangleleft h_\lambda, h_\lambda^{-1} \triangleright z_\lambda^{\mathrm{op}}) \to q_{\mathscr{K}}(\eta) = (y \triangleleft h, h^{-1} \triangleright z^{\mathrm{op}})$. Since both y_λ and $y_\lambda \triangleleft h_\lambda$ converge, it follows from properness and freeness of the $\mathcal{H}$-action on X that a subnet of $\{h_\lambda\}_\lambda$ converges to h; again, without loss of generality we can assume that the entire net converges. By continuity of Ψ, we conclude that $\Psi_{h_\lambda}(\eta_\lambda) \to \Psi_h(\eta)$. Since U is jointly continuous by Lemma 3.14, we therefore have

$$U\big(\xi_\lambda, \Psi_{h_\lambda}(\eta_\lambda)\big) \longrightarrow U\big(\xi, \Psi_h(\eta)\big)$$

in K, which suffices since Q is continuous. ∎

We arrive at our first main result.

Theorem 3.17 *With respect to the multiplication in Lemma 3.15 and the involution in Corollary 3.10, the upper semi-continuous Banach bundle $\mathscr{A}$ described in Lemma 3.8 is a saturated Fell bundle over the groupoid $\mathcal{G} = X \times_{\mathcal{H}} X^{op}$, called* the imprimitivity Fell bundle of $\mathscr{M}$.

Proof We will check the conditions as stated in [4, Definition 2.9]. It is clear that multiplication $\_\cdot\_$ is bilinear and that $\_^*$ is conjugate linear and self-inverse; this takes care of Conditions (F2), (F6), and (F8). For the following, fix $([\xi], [\eta]) \in \mathscr{A}^{(2)}$, and let $h \in \mathcal{H}$ be the unique element such that $q_{\mathscr{K}}(\xi) = (x, y^{\mathrm{op}})$ and $q_{\mathscr{K}}(\Psi_h(\eta)) = (y, z^{\mathrm{op}})$, meaning that $q_{\mathscr{K}}(\eta) = (y \triangleleft h, h^{-1} \triangleright z^{\mathrm{op}})$.

(F1) Since $U_y\big(\xi, \Psi_h(\eta)\big) \in K(x, z^{\mathrm{op}})$ and since $p_{\mathscr{A}}([\xi]) = [q_{\mathscr{K}}(\xi)]$, we get the first and last equality in the following computation:

$$p_{\mathscr{A}}([\xi] \cdot [\eta]) = [x, z^{\mathrm{op}}] = [x, y^{\mathrm{op}}] \cdot [y, z^{\mathrm{op}}] = p_{\mathscr{A}}([\xi]) \cdot p_{\mathscr{A}}([\eta]).$$

(F3) Associativity of the multiplication can be shown using uniqueness of h, the second identity in (3.20), commutativity of Diagram (3.15), and the fact that $\Psi_{hh'} = \Psi_h \circ \Psi_{h'}$; the details are left as an exercise.

(F4) The definition of the norm on the fibres of $\mathscr{A}$ implies that

$$\|[\xi]\cdot[\eta]\| = \left\|U_y\big(\xi, \Psi_h(\eta)\big)\right\| \overset{(\mathrm{U2})}{\leq} \|\xi\|\,\|\Psi_h(\eta)\| \overset{(\Psi 2)}{=} \|\xi\|\,\|\eta\| = \|[\xi]\|\,\|[\eta]\|,$$

proving that multiplication is norm-decreasing.

(F5) Since $p_{\mathscr{A}}([\xi]^*) = [q_{\mathscr{K}}(\mathtt{Flip}(\xi))]$ and since $[x, y^{\mathrm{op}}]^{-1} = [y, x^{\mathrm{op}}]$ in $\mathcal{G}$, we have that $\_^*$ maps $A(g)$ to $A(g^{-1})$.

(F7) We must show that $([\xi]\cdot[\eta])^* = [\eta]^*\cdot[\xi]^*$. We have

$$\begin{aligned}
([\xi]\cdot[\eta])^* &= \left[\mathtt{Flip}\left(U_y\big(\xi, \Psi_h(\eta)\big)\right)\right] \\
&= \left[U_y\Big(\mathtt{Flip}(\Psi_h(\eta)), \mathtt{Flip}(\xi)\Big)\right] && \text{by (U4)} \\
&= \left[U_y\Big(\Psi_h(\mathtt{Flip}(\eta)), \mathtt{Flip}(\xi)\Big)\right] && \text{by commutativity of Diag. (3.11)} \\
&= \left[U_y\Big(\mathtt{Flip}(\eta), \Psi_{h^{-1}}(\mathtt{Flip}(\xi))\Big)\right] && \text{by (U3) and definition of } \mathrm{R},
\end{aligned}$$

which is exactly $[\eta]^*\cdot[\xi]^*$.

(F9) It is clear that $\|[\xi]\| = \|[\xi]^*\|$. To show $\|[\xi]\cdot[\xi]^*\| = \|[\xi]\|^2$, note that $\eta \coloneqq \mathtt{Flip}(\xi)$ has $q_{\mathscr{K}}(\eta) = (y, x^{\mathrm{op}})$; in particular, $u \coloneqq \sigma(x)$ is the unique element of $\mathcal{H}$ with $\mathtt{sec}_{\mathscr{K}}(\xi) = \mathtt{fir}_{\mathscr{K}}\big(\Psi_u(\eta)\big)$. Since Ψ_u is the identity, we conclude that

$$[\xi]\cdot[\xi]^* = \left[U_y\big(\xi, \mathtt{Flip}(\xi)\big)\right],$$

which has norm equal to $\|\xi\|^2 = \|[\xi]\|^2$ by (U2), as needed.

(F10) To see that $[\xi]\cdot[\xi]^* \geq 0$, we again invoke [24, Lemma 2.65]: The matrix with i,j-entry $\langle n_i \mid n_j\rangle_{\mathscr{B}}$ is positive, so there exists a matrix $(b_{i,j})_{i,j}$ over $B(u)$ such that $\langle n_i \mid n_j\rangle_{\mathscr{B}} = \sum_{l=1} b_{il} b_{jl}^*$. In particular, the $B(u)$-balancing in $M(x)\otimes_u M(y)^{\mathrm{op}}$ allows us to write

$$\begin{aligned}
\sum_{i,j}(m_i \dashv \langle n_i \mid n_j\rangle_{\mathscr{B}}) \otimes m_j^{\mathrm{op}} &= \sum_{i,j,l}(m_i \dashv (b_{il} b_{jl}^*)) \otimes m_j^{\mathrm{op}} \\
&= \sum_l \big(\sum_i m_i \dashv b_{il}\big) \otimes \big(\sum_j m_j \dashv b_{jl}\big)^{\mathrm{op}}.
\end{aligned}$$

If we let $\mathbf{x}_l \coloneqq \sum_i m_i \dashv b_{il}$, then this combined with (3.17) shows that

$$[\xi]\cdot[\xi]^* = \left[U_y\big(\xi, \mathtt{Flip}(\xi)\big)\right] = \sum_l \left[\mathbf{x}_l \otimes \mathbf{x}_l^{\mathrm{op}}\right].$$

Since the element $\mathbf{x}_l \otimes \mathbf{x}_l^{\mathrm{op}}$ of $K(x, x^{\mathrm{op}})$ is positive (it corresponds to the positive operator $|\mathbf{x}_l\rangle\langle\mathbf{x}_l|$ in $\mathbb{K}_{B(u)}(M(x))$), and since the algebra $A([x, x^{\mathrm{op}}])$ is *-isomorphic to $K(x, x^{\mathrm{op}})$ (see Remark 3.16), the claim follows.

Lastly, to see that $\mathscr{A}$ is saturated, we must show that the linear span of elements of the form $[\xi]\cdot[\eta]$ for $[\xi] \in A([x, y^{\mathrm{op}}])$ and $[\eta] \in A([y, z^{\mathrm{op}}])$ is dense in $A([x, z^{\mathrm{op}}])$. When we consider that $A([x, y^{\mathrm{op}}]) \cong M(x)\otimes_u M(y)^{\mathrm{op}}$ and that these isomorphisms respect the multiplicative and linear structure we defined on $\mathscr{A}$, then the claim follows directly from the first isomorphism in Equation (3.16). ∎

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

4 Equivalence from the imprimitivity Fell bundle to the "coefficient bundle"

We now proceed to equip the right $\mathscr{B}$-demi-equivalence $\mathscr{M}$ with the structure of a left $\mathscr{A}$-demi-equivalence, where $\mathscr{A} = (p_{\mathscr{A}} : A \to \mathcal{G})$ is the quotient of $\mathscr{K} = (K \to X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}})$ by the equivalence relation R defined in Lemma 3.5; we have shown in Theorem 3.17 that $\mathscr{A}$ is a Fell bundle.

Proposition 4.1 *There is a map* $\_ \vDash \_ : A\,{}_{s}{*}_{\rho}\, M \to M$ *given by*

$$[\xi] \vDash m := \Phi_{x,y}\big(\Psi_h(\xi), m\big) \tag{4.1}$$

where $y = q_{\mathscr{M}}(m)$ *and where* $(x, h) \in X\,{}_{\sigma}{*}_{r}\,\mathcal{H}$ *is the unique element such that* $q_{\mathscr{K}}(\Psi_h(\xi)) = (x, y^{\mathrm{op}})$. *This map furthermore has the following properties.*[6]

(LA1) *It covers the map* $\_ \triangleright \_ : \mathcal{G}\,{}_{s}{*}_{\rho}\, X \to X$; *and*

(LA2) $[\xi] \vDash (m \dashv b) = ([\xi] \vDash m) \dashv b$ *for all appropriate* $b \in B$.

Of course, $\_ \vDash \_$ will turn out to be an action in the sense of [4, Definition 2.10]; in particular, it is continuous, which one can indeed show by hand. But thanks to Proposition 2.3, we can make do with proving fewer properties.

Proof of Proposition 4.1 Write $p_{\mathscr{A}}([\xi]) = [x_1, x_2^{\mathrm{op}}]$. Since $([\xi], m) \in A\,{}_{s}{*}_{\rho}\, M$, we have $s_{\mathcal{G}}([x_1, x_2^{\mathrm{op}}]) = \rho_{\mathscr{M}}(m) = [y, y^{\mathrm{op}}]$, which means that there exists a unique $h \in \mathcal{H}$ such that $[x_1, x_2^{\mathrm{op}}] = [x_1 \triangleleft h, y^{\mathrm{op}}]$. Now that the second component is fixed as y^{op}, the first component $x := x_1 \triangleleft h$ is also uniquely determined, and satisfies $\sigma(x) = s_{\mathcal{H}}(h)$. In other words, we have argued that the equality $s_{\mathscr{A}}([\xi]) = \rho_{\mathscr{M}}(m)$ implies that there exists a *unique* representative $\Psi_h(\xi) \in K$ of the R-equivalence class $[\xi]$ for which $q_{\mathscr{K}}(\Psi_h(\xi))$ has $q_{\mathscr{M}}(m)^{\mathrm{op}}$ as its second component: $\Psi_h(\xi) \in M(x) \otimes_{\sigma(x)} M(y^{\mathrm{op}})$. We can now invoke Lemma 3.11 to conclude that $[\xi] \vDash m$ is a well-defined element of $M(x)$.

(LA1) Follows from $p_{\mathscr{A}}([\xi]) \triangleright q_{\mathscr{M}}(m) = [x_1, x_2^{\mathrm{op}}] \triangleright y = [x, y^{\mathrm{op}}] \triangleright y = x = q_{\mathscr{M}}([\xi] \vDash m)$.

(LA2) We compute

$$\begin{aligned}\Phi_{x,y}\big(m \otimes n^{\mathrm{op}}, k\big) \dashv b &= (m \dashv \langle n \mid k\rangle_{\mathscr{B}}) \dashv b = m \dashv (\langle n \mid k\rangle_{\mathscr{B}} b)\\ &= m \dashv \langle n \mid k \dashv b\rangle_{\mathscr{B}} = \Phi_{x,y}\big(m \otimes n^{\mathrm{op}}, k \dashv b\big),\end{aligned}$$

so that bilinearity and continuity of Φ implies

$$\Phi_{x,y}\big(\eta, k\big) \dashv b = \Phi_{x,y}\big(\eta, k \dashv b\big)$$

for any appropriate $\eta \in K$. The claim follows. ∎

Proposition 4.2 *The map* ${}_{\mathscr{A}}\langle \_ \mid \_ \rangle : M\,{}_{\sigma}{*}_{\sigma}\, M \to A$ *given by*

$${}_{\mathscr{A}}\langle m \mid n\rangle := [m \otimes n^{op}]$$

has the following properties.[7]

[6] The LA in "(LAn)" stands for "left action."

[7] The LIP in "(LIPn)" stands for "left inner product."

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

(LIP1) *It covers the map* ${}_{\mathcal{G}}^{X}\{\_ \mid \_\}: X\,{}_{\sigma}{*}_{\sigma} X \to \mathcal{G}$, *meaning that* $p_{\mathscr{A}}({}_{\mathscr{A}}\langle m \mid n\rangle) \triangleright q_{\mathscr{M}}(n) = q_{\mathscr{M}}(m)$;
(LIP2) *it is continuous and fibrewise linear in the first and antilinear in the second component;*
(LIP3) ${}_{\mathscr{A}}\langle m \mid n\rangle^* = {}_{\mathscr{A}}\langle n \mid m\rangle$;
(LIP4) $[\xi] \cdot {}_{\mathscr{A}}\langle m \mid n\rangle = {}_{\mathscr{A}}\langle [\xi] \vDash m \mid n\rangle$ *for all* $[\xi] \in A$ *with* $s_{\mathscr{A}}([\xi]) = \rho_{\mathscr{M}}(m)$; *and*
(LIP5) ${}_{\mathscr{A}}\langle m_1 \mid m_2\rangle \vDash m_3 = m_1 \dashv \langle m_2 \mid m_3\rangle_{\mathscr{B}}$.

Proof The map is clearly well defined. Property (LIP1) follows immediately from Equation (1.3).

(LIP2) Sesquilinearity is obvious. For continuity, it suffices to check that the map $M\,{}_{\sigma}{*}_{\sigma} M \to K$, $(m, n) \mapsto m \otimes n^{\mathrm{op}}$, is continuous since the quotient map $\mathscr{K} \to \mathscr{A}$ is continuous. But said continuity is built into the definition of $\mathscr{K}$'s topology; see Lemma 3.1.

(LIP3) Follows from the definition of the involution * on $\mathscr{A}$.

(LIP4) Recall from Proposition 4.1 that

$$[\xi] \vDash m_1 = \Phi_{x,y}\big(\Psi_h(\xi), m_1\big)$$

where $y = q_{\mathscr{M}}(m_1)$ and where $(x, h) \in X\,{}_{\sigma}{*}_{r} \mathcal{H}$ is the unique element such that $q_{\mathscr{K}}(\Psi_h(\xi)) = (x, y^{\mathrm{op}})$, so that

$$\begin{aligned}
{}_{\mathscr{A}}\langle [\xi] \vDash m_1 \mid m_2\rangle &= \big[\Phi_{x,y}\big(\Psi_h(\xi), m_1\big) \otimes m_2^{\mathrm{op}}\big] \\
&= \big[U\big(\Psi_h(\xi), m_1 \otimes m_2^{\mathrm{op}}\big)\big] && \text{by Equation (3.22)} \\
&= [\xi] \cdot [m_1 \otimes m_2^{\mathrm{op}}] = [\xi] \cdot {}_{\mathscr{A}}\langle m_1 \mid m_2\rangle,
\end{aligned}$$

so (LIP4) holds.

(LIP5) Let $y = q_{\mathscr{M}}(m_3)$ and let $h \in \mathcal{H}$ and $x \in X$ be the unique elements such that $q_{\mathscr{K}}(\Psi_h(m_1 \otimes m_2^{\mathrm{op}})) = (x, y^{\mathrm{op}})$. Because of (3.4), we know that $q_{\mathscr{M}}(m_1) = x \triangleleft h$ and $q_{\mathscr{M}}(m_2) = y \triangleleft h$. In particular, we can approximate m_1 by $\sum_i n_i \dashv b_i$ for finitely many $n_i \in M(x)$ and $b_i \in B(h)$. In the following final computation, the first instance of $\approx$ is valid because $\Phi_{x,y}$ and Ψ_h are continuous and linear (see (Ψ2)), and the second because the right $\mathscr{B}$-action on $\mathscr{M}$ is continuous by (DE16).

$$\begin{aligned}
{}_{\mathscr{A}}\langle m_1 \mid m_2\rangle \vDash m_3 = [m_1 \otimes m_2^{\mathrm{op}}] \vDash m_3 &\overset{(4.1)}{=} \Phi_{x,y}\big(\Psi_h(m_1 \otimes m_2^{\mathrm{op}}), m_3\big) \\
&\approx \textstyle\sum_i \Phi_{x,y}\big(\Psi_h([n_i \dashv b_i] \otimes m_2^{\mathrm{op}}), m_3\big) \\
&\overset{(3.3)}{=} \textstyle\sum_i \Phi_{x,y}\big(n_i \otimes (m_2 \dashv b_i^*)^{\mathrm{op}}, m_3\big) \\
&\overset{(3.12)}{=} \textstyle\sum_i n_i \dashv \langle m_2 \dashv b_i^* \mid m_3\rangle_{\mathscr{B}} \\
&= \textstyle\sum_i n_i \dashv (b_i \langle m_2 \mid m_3\rangle_{\mathscr{B}}) \approx m_1 \dashv \langle m_2 \mid m_3\rangle_{\mathscr{B}}.
\end{aligned}$$

■

Corollary 4.3 *The saturated Fell bundle* $\mathscr{A}$ *is equivalent to* $\mathscr{B}$ *via* $\mathscr{M}$ *with the* $\mathscr{A}$*-action and* $\mathscr{A}$*-inner product specified in Propositions 4.1 and 4.2, respectively. In particular, the left action is continuous.*

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

▸ **For the remainder of the paper**, we will frequently denote the imprimitivity Fell bundle $\mathscr{A}$ of $\mathscr{M}$ by $\mathscr{A} = \mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ in analogy with $X \times_{\mathcal{H}} X^{\mathrm{op}}$ and with $\mathbf{X} \otimes_A \mathbf{X}^{\mathrm{op}}$. However, the notation $\mathscr{A} = \mathbb{K}(\mathscr{M}_{\mathscr{B}})$ also lends itself well and has the advantage that it makes explicit reference to the underlying bundle $\mathscr{B}$ (and the fact that it is acting on the right-hand side).

Proof of Corollary 4.3 We first check that $\mathscr{M}$ is a left $\mathscr{A}$-demi-equivalence. In the following lists, all references are either to Proposition 4.1 or to Proposition 4.2.

(DE1) was shown in (LA1);
(DE2) was shown in (LIP1);
(DE3) was shown in (LIP2);
(DE4) was shown in (LIP4); and
(DE5) was shown in (LIP3).

For the remaining conditions necessary for a demi-equivalence that we have not checked previously, note that each $M(x)$ is a full right-Hilbert C*-module by (DE9), and so it is an imprimitivity bimodule between $M(x) \otimes_u M(x)^{\mathrm{op}}$ and $B(u)$ for $u := \sigma(x)$. All structure with which we equipped the bundle $\mathscr{A}$ is compatible with the homeomorphism $A(\rho(x)) \cong M(x) \otimes_u M(x)^{\mathrm{op}}$ in Lemma 3.7; in other words, with the restriction to $M(x)$ of the given $\mathscr{A}$-action and $\mathscr{A}$-inner product on $\mathscr{M}$, we recover exactly the structure that $M(x)$ naturally carries. In particular:

(DE6): We have ${}_{\mathscr{A}}\langle m \mid m \rangle \geq 0$ in the C*-algebra $A(\rho_{\mathscr{M}}(m))$, and ${}_{\mathscr{A}}\langle m \mid m \rangle = 0$ only if $m = 0$.
(DE7): By [24, Lemma 2.30], the norm of $\langle m \mid m \rangle_{\mathscr{B}}$ in $B(\sigma(x))$ coincides with the norm of $m \otimes m^{\mathrm{op}}$ in $M(x) \otimes_u M(x)^{\mathrm{op}}$ which, by definition, is exactly the norm of $[m \otimes m^{\mathrm{op}}] = {}_{\mathscr{A}}\langle m \mid m \rangle$ in $A(\rho(x))$. Since $\mathscr{M}$ is a right $\mathscr{B}$-demi-equivalence, the norm $m \mapsto \|\langle m \mid m \rangle_{\mathscr{B}}\|^{1/2}$ agrees with the norm that the upper semi-continuous Banach bundle $\mathscr{M}$ already carries, which is therefore in turn identical to the norm $m \mapsto \|{}_{\mathscr{A}}\langle m \mid m \rangle\|^{1/2}$.
(DE8): For each $x \in X$, the linear span of $\{{}_{\mathscr{A}}\langle m_1 \mid m_2 \rangle : m_i \in M(x)\}$ is dense in $A(\rho(x))$, since $M(x)$ is an imprimitivity bimodule.

That $A(\rho(x)) \cong M(x) \otimes_u M(x)^{\mathrm{op}}$ further implies that Assumption (2) of Proposition 2.3 is satisfied. Since Assumptions (1) and (3) were shown in (LA2) and (LIP5), respectively, we can conclude that $\mathscr{M}$ is an equivalence by Proposition 2.3. ■

5 The imprimitivity Fell bundle: Uniqueness

For the next result, we remind the reader of Remark 1.1: given a $(\mathcal{G}, \mathcal{H})$-groupoid equivalence X and two elements $x, y \in X$ with the same $\mathcal{H}$-anchor, we write ${}^{X}_{\mathcal{G}}\{x \mid y^{\mathrm{op}}\}$ for the unique element of $\mathcal{G}$ that satisfies ${}^{X}_{\mathcal{G}}\{x \mid y^{\mathrm{op}}\} \triangleright y = x$.

Proposition 5.1 (cf. [1, Corollary 3.11]) *Suppose $\mathscr{A} = (A \to \mathcal{G})$, $\mathscr{A}' = (A' \to \mathcal{G}')$, and $\mathscr{B} = (B \to \mathcal{H})$ are Fell bundles over groupoids $\mathcal{G}, \mathcal{G}', \mathcal{H}$ respectively, and that $\mathscr{M} = (q_{\mathscr{M}}\colon M \to X)$ is both an $(\mathscr{A}, \mathscr{B})$- and an $(\mathscr{A}', \mathscr{B})$-Fell bundle equivalence. Then* *there exists a unique Fell bundle isomorphism $\Omega\colon \mathscr{A} \to \mathscr{A}'$ determined by the following commutative diagram.*

$$\begin{tikzcd}
\mathscr{A} \arrow[rrr, "\Omega"] \arrow[ddd, "p_{\mathscr{A}}"'] & & & \mathscr{A}' \arrow[ddd, "p_{\mathscr{A}'}"] \\
& {}_{\mathscr{A}}^{\mathscr{M}}\langle m \mid n\rangle \arrow[r, mapsto] \arrow[d, mapsto] & {}_{\mathscr{A}'}^{\mathscr{M}}\langle m \mid n\rangle \arrow[d, mapsto] & \\
& {}_{\mathcal{G}}^{X}\{q_{\mathscr{M}}(m) \mid q_{\mathscr{M}}(n)^{\mathrm{op}}\} \arrow[r, mapsto] & {}_{\mathcal{G}'}^{X}\{q_{\mathscr{M}}(m) \mid q_{\mathscr{M}}(n)^{\mathrm{op}}\} & \\
\mathcal{G} \arrow[rrr, "\omega"'] & & & \mathcal{G}'
\end{tikzcd}$$

Proof of Proposition 5.1 By Corollary 4.3, we may without loss of generality assume that $\mathscr{A} = \mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ and $\mathcal{G} = X \times_{\mathcal{H}} X^{\mathrm{op}}$. In that setting, ${}_{\mathcal{G}}^{X}\{q_{\mathscr{M}}(m) \mid q_{\mathscr{M}}(n)^{\mathrm{op}}\}$ is exactly $[q_{\mathscr{M}}(m), q_{\mathscr{M}}(n)] \in \mathcal{G}$; see Lemma 1.3. We also remind the reader that $\omega\colon X \times_{\mathcal{H}} X \to \mathcal{G}', [x, y^{\mathrm{op}}] \mapsto {}_{\mathcal{G}'}^{X}\{x \mid y^{\mathrm{op}}\}$, is known to be an isomorphism of topological groupoids. It is therefore clear that $\mathscr{A}' \cong \omega^*(\mathscr{A}')$, and so we may without loss of generality assume that $\mathcal{G}' = \mathcal{G}$ (and hence $\omega = \mathrm{id}$ and $\mathscr{A}' = \omega^*(\mathscr{A}')$). As before, we will denote the anchor maps of X by $\sigma\colon X \to \mathcal{H}^{(0)}$ and $\rho\colon X \to \mathcal{G}^{(0)}, x \mapsto [x, x^{\mathrm{op}}]$. Recall that $\mathcal{G}^{(0)} \cong X/\mathcal{H}$ via $[x, x^{\mathrm{op}}] \mapsto x \triangleleft \mathcal{H}$.

To construct Ω, we will first construct a surjective bundle map $\tilde{\Omega}\colon \mathscr{K} \to \mathscr{A}'$ covering the quotient map $X\,{}_{\sigma}{*}_{\sigma}\, X^{\mathrm{op}} \to \mathcal{G}, (x, y^{\mathrm{op}}) \mapsto [x, y^{\mathrm{op}}]$. We will then show that $\tilde{\Omega}$ is constant on $\mathbb{R}$-equivalence classes (where $\mathbb{R}$ is as in Lemma 3.5), so that $\tilde{\Omega}$ induces a bundle map Ω from the quotient $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}} = Q(\mathscr{K})$ to $\mathscr{A}'$ covering the identity map $\mathcal{G} \to \mathcal{G}$. After we have shown that Ω is injective, we will then prove the conditions in [4, Propositions A.7 and A.8] to deduce that Ω is open and continuous and hence the claimed isomorphism of Fell bundles.

We define $\tilde{\Omega}$ fibrewise. Fix $u \in \mathcal{H}^{(0)}$ and $x, y \in Xu$, let $g := [x, y^{\mathrm{op}}]$, and consider

$$M(x) \times M(y)^{\mathrm{op}} \to A'(g), \quad (m, n^{\mathrm{op}}) \mapsto {}_{\mathscr{A}'}^{\mathscr{M}}\langle m \mid n\rangle. \tag{5.1}$$

By the assumption on the left $\mathscr{A}'$-valued inner product on $\mathscr{M}$, this map is bilinear. Moreover, by [4, Corollary 4.6][8], we have ${}_{\mathscr{A}'}^{\mathscr{M}}\langle m \prec b \mid n\rangle = {}_{\mathscr{A}'}^{\mathscr{M}}\langle m \mid n \prec b^*\rangle$ for all $b \in B(u)$, meaning that the map in (5.1) descends to a linear map

$$\tilde{\Omega}_{(x,y^{\mathrm{op}})}\colon M(x) \odot_{B(u)} M(y)^{\mathrm{op}} \to A'(g),$$

determined by

$$\tilde{\Omega}_{(x,y^{\mathrm{op}})}(m \odot n^{\mathrm{op}}) = {}_{\mathscr{A}'}^{\mathscr{M}}\langle m \mid n\rangle,$$

where $M(x) \odot_{B(u)} M(y)^{\mathrm{op}}$ denotes the balanced algebraic tensor product.

We claim that $\tilde{\Omega}_{(x,y^{\mathrm{op}})}$ has dense range. Since $\mathscr{A}'$ is a Fell bundle, $A'(g)$ is an $A'(x \triangleleft \mathcal{H}) - A'(y \triangleleft \mathcal{H})$-imprimitivity bimodule, so by [24, Proposition 2.33 (Hewitt-Cohen)], any element of $A'(g)$ can be written as $a_g a_{y\triangleleft\mathcal{H}}$ for some $a_g \in A'(g)$ and some $a_{y\triangleleft\mathcal{H}} \in A'(y \triangleleft \mathcal{H})$, where the product is the multiplication in the Fell bundle

[8] Note that we cannot simply invoke the assumption that each $M(x)$ is an imprimitivity bimodule ([22, Definition 6.1(c)]), because we are in the setting where x might not equal y.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

$\mathscr{A}'$. By assumption, $M(y)$ is a $A'(y \triangleleft \mathcal{H}) - B(u)$-imprimitivity bimodule. In particular, ${}^{\mathscr{M}}_{\mathscr{A}'}\langle M(y) \mid M(y)\rangle$ is dense in $A'(y \triangleleft \mathcal{H})$, meaning we can approximate $a_{y\triangleleft\mathcal{H}}$ by linear combinations of elements of the form ${}^{\mathscr{M}}_{\mathscr{A}'}\langle n' \mid n\rangle$ for $n', n \in M(y)$. Since $a_g\, {}^{\mathscr{M}}_{\mathscr{A}'}\langle n' \mid n\rangle = {}^{\mathscr{M}}_{\mathscr{A}'}\langle a_g \vDash n' \mid n\rangle$ and $a_g \vDash n' \in M(g \triangleright y) = M(x)$, we conclude that the arbitrary element $a_g a_{y\triangleleft\mathcal{H}}$ of $A'(g)$ can be approximated by linear combinations of elements of the form ${}^{\mathscr{M}}_{\mathscr{A}'}\langle m \mid n\rangle$ for $m \in M(x), n \in M(y)$, as claimed.

Since $\mathscr{M}$ is an equivalence, $M(x)$ is an imprimitivity bimodule between $A'(\rho(x)) = A'(x \triangleleft \mathcal{H})$ and $B(\sigma(x)) = B(u)$, so that there exists a canonical isomorphism $A'(x \triangleleft \mathcal{H}) \cong \mathbb{K}_{B(u)}(M(x))$ of C*-algebras. In particular, the norm with which we equip $M(x) \odot_{B(u)} M(y)^{\mathrm{op}}$, is unambiguously given by Equation (3.2) and, using the $*$-isomorphism, can be rewritten to

$$\left\| \sum_{i=1}^{k} m_i \otimes n_i^{\mathrm{op}} \right\| = \left\| \sum_{i,j=1}^{k} {}^{\mathscr{M}}_{\mathscr{A}'}\langle n_i \prec \langle m_i \mid m_j\rangle^{\mathscr{M}}_{\mathscr{B}} \mid n_j\rangle \right\|^{1/2}.$$

Using the properties [4, Definition 2.9, (F9)] and [4, Definition 2.11, (FE2.b), (FE2.c), (FE2.d)], we therefore see that each $\tilde{\Omega}_{(x,y^{\mathrm{op}})}$ is isometric and hence extends to a map on the completion $K(x, y^{\mathrm{op}})$ of $M(x) \odot_{B(u)} M(y)^{\mathrm{op}}$. All in all, we get a fibrewise linear map

$$\tilde{\Omega}\colon \quad K = \bigsqcup_{(x,y^{\mathrm{op}})\in X_\sigma *_\sigma X^{\mathrm{op}}} K(x, y^{\mathrm{op}}) \to \bigsqcup_{g\in\mathcal{G}} A'(g),$$

determined by

$$\tilde{\Omega}(m \otimes n^{\mathrm{op}}) = {}^{\mathscr{M}}_{\mathscr{A}'}\langle m \mid n\rangle,$$

covering the quotient map $X_\sigma *_\sigma X^{\mathrm{op}} \mapsto \mathcal{G}$.

Next, fix $u, v \in \mathcal{H}^{(0)}$, $h \in u\mathcal{H}v$, and $x \in Xu$, $y \in Xv$. For an arbitrary but fixed element $\xi = \sum_i (m_i \prec b_i) \odot n_i^{\mathrm{op}}$ of $M(x \triangleleft h) \odot_{B(v)} M(y)^{\mathrm{op}}$, we have

$$\begin{aligned} \tilde{\Omega}_{(x,y^{\mathrm{op}})}(\Psi_h(\xi)) &= \sum_i {}^{\mathscr{M}}_{\mathscr{A}'}\langle m_i \mid n_i \prec b_i^*\rangle && \text{by definition of } \Psi\text{, see (3.3)} \\ &= \sum_i {}^{\mathscr{M}}_{\mathscr{A}'}\langle m_i \prec b_i \mid n_i\rangle = \tilde{\Omega}_{(x,y^{\mathrm{op}})}(\xi) && \text{by [4, Corollary 4.6].} \end{aligned}$$

Since $\tilde{\Omega}_{(x,y^{\mathrm{op}})}(\Psi_h(\xi)) = \tilde{\Omega}_{(x,y^{\mathrm{op}})}(\xi)$ for all ξ in the algebraic tensor product, we conclude that $\tilde{\Omega}$ descends to a bundle map

$$\Omega\colon \quad \mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}} \to \mathscr{A}', \quad \text{determined by} \quad \Omega([m \otimes n^{\mathrm{op}}]) = {}^{\mathscr{M}}_{\mathscr{A}'}\langle m \mid n\rangle,$$

covering the identity map $\mathcal{G} \to \mathcal{G}$. We must now show that this map is a continuous open bijection.

Since Ω covers the identity map, surjectivity of Ω follows from the fact that each $\tilde{\Omega}_{(x,y^{\mathrm{op}})}$ has dense range, and injectivity of Ω follows since each $\tilde{\Omega}_{(x,y^{\mathrm{op}})}$ is isometric. To see that Ω is continuous, suppose that we have a convergent net in $\mathscr{A} = \mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$, say $[\xi_\lambda] \to [\xi]$; by (2) $\implies$ (1) in [23, Theorem 18.1], it suffices to show that a subnet of $\{\Omega([\xi_\lambda])\}_\lambda$ converges to $\Omega([\xi])$. Since the quotient map $Q\colon \mathscr{K} \to \mathscr{A}$ is open (Lemma 3.5), we can without loss of generality assume that $\xi_\lambda \to \xi$ in $\mathscr{K}$. Let $q_{\mathscr{K}}(\xi_\lambda) = (x_\lambda, y_\lambda^{\mathrm{op}})$ with limit $q_{\mathscr{K}}(\xi) = (x, y)$ in $X_\sigma *_\sigma X^{\mathrm{op}}$. We

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

now invoke the definition of the topology on $\mathscr{K}$ (Lemma 3.1; see also [4, Lemma A.3]): for any given $\varepsilon > 0$, we can find finitely many continuous sections μ_i, ν_i of $\mathscr{M}$ and a λ_0 such that for all $\lambda \geq \lambda_0$, we have

$$\left\| \xi_\lambda - \sum_{i=1}^{k} \mu_i(x_\lambda) \otimes \nu_i(y_\lambda)^{\mathrm{op}} \right\| < \varepsilon \text{ and } \left\| \xi - \sum_{i=1}^{k} \mu_i(x) \otimes \nu_i(y)^{\mathrm{op}} \right\| < \varepsilon. \tag{5.2}$$

The norm on each fibre $A([x, y^{\mathrm{op}}])$ of $\mathscr{A}$ is defined as the norm on one of its lifts under Q, meaning that Q restricted to $M(x) \otimes_u M(y)^{\mathrm{op}}$ is isometric. Since Ω is likewise isometric, (5.2) implies

$$\begin{aligned} \left\| \Omega([\xi_\lambda]) - \Omega\left(\left[\textstyle\sum_i \mu_i(x_\lambda) \otimes \nu_i(y_\lambda)^{\mathrm{op}}\right]\right) \right\| < \varepsilon \text{ and} \\ \left\| \Omega([\xi]) - \Omega\left(\left[\textstyle\sum_i \mu_i(x) \otimes \nu_i(y)^{\mathrm{op}}\right]\right) \right\| < \varepsilon. \end{aligned} \tag{5.3}$$

By definition of Ω,

$$\Omega\left(\left[\textstyle\sum_i \mu_i(x) \otimes \nu_i(y)^{\mathrm{op}}\right]\right) = \textstyle\sum_i {}_{\mathscr{A}'}^{\mathscr{M}}\langle \mu_i(x) \mid \nu_i(y) \rangle.$$

By assumption on $\mathscr{M}$, ${}_{\mathscr{A}'}^{\mathscr{M}}\langle \_ \mid \_ \rangle$ is continuous, so that continuity of μ_i, ν_i and of addition in $\mathscr{A}$ implies

$$\textstyle\sum_i {}_{\mathscr{A}'}^{\mathscr{M}}\langle \mu_i(x_\lambda) \mid \nu_i(y_\lambda) \rangle \to \sum_i {}_{\mathscr{A}'}^{\mathscr{M}}\langle \mu_i(x) \mid \nu_i(y) \rangle.$$

Combining (5.3) with Lemma B.1, we conclude that $\Omega([\xi_\lambda]) \to \Omega([\xi])$, so Ω is continuous. Lastly, since Ω is isometric, it follows from [4, Proposition A.8] that Ω is open as well. ■

This also finishes the proof of the main theorem, Theorem 1.6, since it is a combination of Corollary 4.3 and Proposition 5.1.

Remark 5.2 A brief comment on another property related to equivalences that has a version not only in the realm of groupoids and C*-algebras but also of Fell bundles: linking objects.

By [24, Theorem 3.19], two C*-algebras are strongly Morita equivalent if and only if they are complementary full corners of another algebra, called the *linking algebra*. In [27, Theorem 4.1], this result was moved into groupoid-land: an equivalence of groupoids gives rise to a *linking groupoid* whose C*-algebra is the linking algebra that witnesses the strong Morita equivalence. Likewise, a Fell bundle equivalence has a *linking (Fell) bundle* (see [26, Section 3], [1, Theorem 3.2]) whose C*-algebra is the linking algebra of the Fell bundle C*-algebras [26, Theorem 14].

It would be interesting to study whether "one-sided" versions of these linking objects exist. In the paper at hand, we considered principal groupoid-spaces X, full right-Hilbert C*-modules $\mathbf{X}$, and Fell bundle-demi-equivalences $\mathscr{M}$, and turned them into equivalences between the coefficient object and the "generalized compacts", that is, the imprimitivity groupoid $X \times_{\mathcal{H}} X^{\mathrm{op}}$, the C*-algebra of compact operators $\mathbb{K}(\mathbf{X}_A)$, and the imprimitivity Fell bundle $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}} = \mathbb{K}(\mathscr{M}_{\mathscr{B}})$, respectively. Do these results have analogs in the language of linking objects?

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

6 Applications

The original impetus for the paper at hand was Proposition A.2.3 in the appendix of [19]; it can easily seen to be a corollary of our main theorem in the special case that $\mathcal{H}$ is a group that acts freely and properly on a space X. Before we study other interesting applications, let us first do some sanity checks. In analogy to the isomorphism of imprimitivity groupoids in Example 1.2, we have:

Corollary 6.1 *Let $\mathscr{B}$ be a Fell bundle, considered as a self-Fell bundle equivalence [22, Example 6.6]. Then there is an isomorphism of Fell bundles $\mathbb{K}(\mathscr{B}_{\mathscr{B}}) = \mathscr{B} \otimes_{\mathscr{B}} \mathscr{B}^{\mathrm{op}} \cong \mathscr{B}$ determined by $[b_1 \otimes b_2^{op}] \mapsto b_1 \cdot b_2^*$.*

In analogy to the isomorphism of Hilbert C*-bimodules in (3.16), we have:

Corollary 6.2 *Suppose that $\mathscr{M}, \mathscr{N}, \mathscr{K}$ are right $\mathscr{B}$-demi-equivalences, and let $\mathscr{A} = \mathbb{K}(\mathscr{N}_{\mathscr{B}})$ be the imprimitivity Fell bundle of $\mathscr{N}$. Then $(\mathscr{M} \otimes_{\mathscr{B}} \mathscr{N}^{\mathrm{op}}) \otimes_{\mathscr{A}} (\mathscr{N} \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}}) \cong \mathscr{M} \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}}$ as upper semi-continuous Banach bundles.*

Proof sketch First, let us make sure that the objects we have written down are not nonsense: By Theorem 1.6, all three demi-equivalences are equivalences between their respective imprimitivity Fell bundles and $\mathscr{B}$. As explained in [22, Example 6.7], their opposites are then likewise equivalences, just in the other direction. It was shown in [4] that equivalences can be concatenated: $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{N}^{\mathrm{op}}$ is an equivalence from the imprimitivity Fell bundle of $\mathscr{M}$ to $\mathscr{A}$, and so forth. In particular, both the left- and the right-hand side of the alleged isomorphism is an equivalence between the imprimitivity Fell bundle of $\mathscr{M}$ and that of $\mathscr{K}$.

Since the balanced tensor product of imprimitivity bimodule is associative, it is easy to see that the same is true for $\otimes_{\mathscr{B}}$ of Fell bundle equivalences. In particular,

$$(\mathscr{M} \otimes_{\mathscr{B}} \mathscr{N}^{\mathrm{op}}) \otimes_{\mathscr{A}} (\mathscr{N} \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}}) \cong \big(\mathscr{M} \otimes_{\mathscr{B}} (\mathscr{N}^{\mathrm{op}} \otimes_{\mathscr{A}} \mathscr{N})\big) \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}}.$$

By Proposition 5.1 (applied to $\mathscr{N}^{\mathrm{op}}$), the Fell bundle $\mathbb{K}(\mathscr{N}^{\mathrm{op}}_{\mathscr{A}}) = \mathscr{N}^{\mathrm{op}} \otimes_{\mathscr{A}} \mathscr{N}$ is isomorphic to $\mathscr{B}$. Since the balanced tensor product of imprimitivity bimodule absorbs the coefficient algebra (meaning that $\mathbf{X} \otimes_A A \cong \mathbf{X}$), it is easy to see that $\otimes_{\mathscr{B}}$ absorbs the coefficient Fell bundle. Thus,

$$(\mathscr{M} \otimes_{\mathscr{B}} \mathscr{N}^{\mathrm{op}}) \otimes_{\mathscr{A}} (\mathscr{N} \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}}) \cong (\mathscr{M} \otimes_{\mathscr{B}} \mathscr{B}) \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}} \cong \mathscr{M} \otimes_{\mathscr{B}} \mathscr{K}^{\mathrm{op}},$$

as claimed. ■

We can also conclude some other, well-known results.

Corollary 6.3 *Suppose $\mathcal{H}$ is a locally compact Hausdorff groupoid with Haar system $\{\lambda_u\}_{u \in \mathcal{H}^{(0)}}$ and $\mathscr{B}$ is a Fell bundle over $\mathcal{H}$. Then $C^*(\mathscr{B})$ is strongly Morita equivalent to $M_n(C^*(\mathscr{B}))$.*

Note that the right choice of $\mathscr{B}$ allows C$^*(\mathscr{B})$ to model the full C*-algebras of groupoids (with or without a twist á la Kumjian) and full crossed product C*-algebras.

Proof Take $X = \mathcal{H}$, with $\sigma\colon X \to \mathcal{H}^{(0)}$ the source map of $\mathcal{H}$. The existence of a Haar system implies that the source map of $\mathcal{H}$ (and hence the anchor map of X) is open [28, Proposition 1.23], so we are in good shape to use our main theorem, provided we

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

can find the right $\mathscr{B}$-demi-equivalence. Let $\mathscr{M}$ to be the bundle over X with fibres $\mathbb{C}^n \times \mathscr{B}(h)$ for each $h \in \mathcal{H}$. We define

$$\llcorner\!\lrcorner \triangleleft \llcorner\!\lrcorner \colon \qquad \mathscr{M}\, {}_\sigma{*}_r\, \mathscr{B} \to \mathscr{M}, \quad (\vec{x}, b) \triangleleft b' \coloneqq (\vec{x}, b \cdot b'),$$

and

$$\langle \llcorner\!\lrcorner \mid \llcorner\!\lrcorner \rangle^{\mathscr{M}}_{\mathscr{B}} \colon \qquad \mathscr{M}\, {}_\sigma{*}_\sigma\, \mathscr{M} \to \mathscr{B}, \quad \langle (\vec{x}, b) \mid (\vec{y}, c) \rangle^{\mathscr{M}}_{\mathscr{B}} \coloneqq \langle \vec{x} \mid \vec{y} \rangle^{\mathbb{C}^n}_{\mathbb{C}}\, b^* \cdot c,$$

where we choose the inner product on $\mathbb{C}^n$ to be conjugate linear in the first coordinate to match up with our definitions of Hilbert C*-modules. Then, using Corollary 6.1, it is easy to see that

$$\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}} \cong \mathrm{M}_n(\mathbb{C}) \times \mathscr{B} \text{ via } [(\vec{e}_i, b_1) \otimes (\vec{e}_j, b_2)^{\mathrm{op}}] \mapsto (E_{i,j}, b_1 \cdot b_2^*),$$

where $\vec{e}_i$ is the standard basis vector of $\mathbb{C}^n$ and $E_{i,j}$ is the matrix unit with a 1 in the i^{th} row and j^{th} column and 0s everywhere else. By Theorem 1.6, the bundles $\mathscr{B}$ and $\mathrm{M}_n(\mathbb{C}) \times \mathscr{B}$ are equivalent. By [22, Theorem 6.4], it follows that $\mathrm{C}^*(\mathscr{B})$ is strongly Morita equivalent to $\mathrm{C}^*(\mathrm{M}_n(\mathbb{C}) \times \mathscr{B}) \cong \mathrm{M}_n(\mathrm{C}^*(\mathscr{B}))$. ∎

6.1 Kumjian's Stabilization trick

In this subsection, we will see that our main theorem implies [17, Corollary 4.5], which in its original form was only shown for principal r-discrete groupoids. The approach here is inspired by that preceding [20, Theorem 15]. We will go through it in great detail to help the reader understand our technical results.

We start with a (saturated) Fell bundle $\mathscr{B} = (B \to \mathcal{H})$ over a locally compact Hausdorff groupoid $\mathcal{H}$ with Haar system $\{\lambda_v\}_{v \in \mathcal{H}^{(0)}}$. Fix $h \in \mathcal{H}$, and let $v \coloneqq r(h)$ and $u \coloneqq s(h)$. We define $M_0(h) \coloneqq \Gamma_c(\mathcal{H}u; \mathscr{B})$, sections of the restriction of $\mathscr{B}$ to $\mathcal{H}u = s^{-1}(s(h))$. If $g \in \mathcal{H}$ is another element with $r(g) = v$, then we define a sesquilinear form

$$\langle \llcorner\!\lrcorner \mid \llcorner\!\lrcorner \rangle_{B(g^{-1}h)} \colon M_0(g) \times M_0(h) \to B(g^{-1}h)$$

by

$$\langle \mu \mid \xi \rangle_{B(g^{-1}h)} = \int_{\mathcal{H}s(g)} \mu(\ell)^* \xi(\ell g^{-1} h)\, \mathrm{d}\lambda_{s(g)}(\ell). \tag{6.1}$$

Note that this indeed makes sense: if $\ell \in \mathcal{H}s(g)$, then ℓ is in the domain of μ and ℓg^{-1} is defined. Moreover, $s(\ell g^{-1} h) = s(h)$, so $\ell g^{-1} h$ is in the domain of ξ. Since $\mu(\ell)^* \in B(\ell^{-1})$ and $\xi(\ell g^{-1} h) \in B(\ell g^{-1} h)$, their product is indeed an element of $B(\ell^{-1} \ell g^{-1} h) = B(g^{-1} h)$. And since μ and ξ are compactly supported, the integral exists.

For $h = g$, the form is valued in the C*-algebra $B(u)$, so we may take the completion $M(h)$ of $M_0(h)$ with respect to the induced norm

$$\|\xi\|_{M(h)} \coloneqq \left\| \langle \xi \mid \xi \rangle_{B(u)} \right\|^{1/2}.$$

If $k \in u\mathcal{H}w$, so that $(h, k) \in \mathcal{H}^{(2)}$, then the fibre $B(k)$ can act on an element ξ of $M_0(h)$ and deliver an element $\xi \triangleleft b$ of $M_0(hk)$: if ℓ is in $\mathcal{H}w$ (the domain of any element of

$M_0(hk)$), then $\ell k^{-1} \in \mathcal{H}u$ is in the domain of ξ and so

$$(\xi \prec b)(\ell) := \xi(\ell k^{-1}) \cdot b \text{ is defined and an element of } B(\ell k^{-1}) \cdot B(k) \subseteq B(\ell).$$

This extends to a map $M(h) \times B(k) \to M(hk)$. In fact, it induces an isomorphism $M(h) \otimes_u B(k) \cong M(hk)$ of right-Hilbert C^*-$B(w)$-modules.

We let $\mathscr{M} = (q_{\mathscr{M}}: M \to \mathcal{H})$ be the bundle with fibres $M(h)$, which we will now topologize. If $\tau \in \Gamma_c(\mathcal{H}; \mathscr{B})$ is a section of $\mathscr{B}$ and $h \in \mathcal{H}u$ is arbitrary, then $\tilde{\tau}(h) := \tau|_{\mathcal{H}u}$ is a continuous, compactly supported section $\mathcal{H}u \to \mathscr{B}$; i.e., $\tilde{\tau}(h)$ is an element of $M(h)$, so $\tilde{\tau}$ is a section of the bundle $\mathscr{M}$. The family $\{\tilde{\tau}: \tau \in \Gamma_c(\mathcal{H}; \mathscr{B})\}$ uniquely induces a topology on $\mathscr{M}$ making it upper semi-continuous. The attentive reader will have expected what comes next: we will show that $\mathscr{M}$ is a right $\mathscr{B}$-demi-equivalence over the principal $\mathcal{H}$-space $X = \mathcal{H}$ (Example 1.2).

Remark 6.4 Let us compare what we have done so far with what was done in [20, 17]; this will also give us an indicator as to what will happen next. Muhly constructs first an upper semi-continuous Banach bundle $\mathscr{V} = (V \to \mathcal{H}^{(0)})$, which is exactly $\mathscr{V} = \mathscr{M}|_{\mathcal{H}^{(0)}}$ and which Kumjian calls the "Hilbert $\mathscr{B}^{(0)}$-module bundle over $\mathcal{H}^{(0)}$" in [17, Section 4.2]; he points out that $\mathscr{V}$ is full.

From $\mathscr{V}$, they then construct a bundle $\mathscr{E} = (E \to \mathcal{H})$; in the notation of the paper at hand, $\mathscr{E}$ can be constructed as the pullback bundle of

$$\mathscr{V} \otimes_{\mathscr{B}^{(0)}} \mathscr{B} = \left(\bigsqcup_{(v,h)} V(v) \otimes_v B(h) \to \mathcal{H}^{(0)}{}_{\mathrm{id}}{*}_r \mathcal{H} \right) \tag{6.2}$$

along the isomorphism $\mathcal{H} \cong \mathcal{H}^{(0)}{}_{\mathrm{id}}{*}_r \mathcal{H}, h \mapsto (r(h), h)$.[9]

It is then shown that $\mathscr{E}$ is an equivalence between $\mathscr{B}$ and the semi-direct product Fell bundle of a certain $\mathcal{H}$-action on another bundle which Kumjian and Muhly denote by $\mathcal{K}(\mathscr{V})$. Following [17, Section 1.7], $\mathcal{K}(\mathscr{V})$ is the C^*-algebraic bundle with fibre $\mathbb{K}(V(u)_{B(u)})$ over $u \in \mathcal{H}^{(0)}$. It is apparent that $\mathcal{K}(\mathscr{V})$ coincides with our $\mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}}) = \mathscr{V} \otimes_{\mathscr{B}^{(0)}} \mathscr{V}^{\mathrm{op}}$, the imprimitivity Fell bundle of the right $\mathscr{B}^{(0)}$-demi-equivalence $\mathscr{V}$.[10] By construction $\mathscr{E} \cong \mathscr{M}$ via the map $\xi \otimes b \mapsto \xi \prec b$, and so in light of our main theorem, we should expect that the imprimitivity Fell bundle $\mathbb{K}(\mathscr{M}_{\mathscr{B}})$ of $\mathscr{M}$ is (isomorphic to) a semi-direct product of $\mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})$ by $\mathcal{H}$.

By Example 1.2, the anchor maps of $X = \mathcal{H}$ are exactly the range and the source map once we identify its imprimitivity groupoid $\mathcal{G} = X \times_{\mathcal{H}} X^{\mathrm{op}}$ with $\mathcal{H}$. This means that the $\mathscr{B}$-valued inner product

$$\langle \_ \mid \_ \rangle_{\mathscr{B}}: M {}_r{*}_r M \to B$$

covers the map $\{\_ \mid \_\}^X_{\mathcal{H}}$ (Condition (DE2)). Moreover, the map

$$\_ \prec \_: M {}_s{*}_r B \to M$$

[9] The bundle $\mathscr{V} \otimes_{\mathscr{B}^{(0)}} \mathscr{B}$ is constructed in the same fashion as the bundle $\mathscr{K} = \mathscr{M} \otimes_{\mathscr{B}^{(0)}} \mathscr{M}^{\mathrm{op}}$ in Lemma 3.1; a more general construction of bundles of the form $\mathscr{M} \otimes_{\mathscr{B}^{(0)}} \mathscr{N}$ can be found in [4].

[10] To nitpick, Kumjian's $\mathcal{K}(\mathscr{V})$ is the pullback of $\mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})$ along the homeomorphism $\mathcal{H}^{(0)} \to \mathcal{H}^{(0)} \times_{\mathcal{H}^{(0)}} (\mathcal{H}^{(0)})^{\mathrm{op}}, u \mapsto [u, u^{\mathrm{op}}]$.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

covers the multiplication map of $\mathcal{H}$ (or, in other words, the map that describes the right $\mathcal{H}$-action on $X = \mathcal{H}$; Condition (DE1)). Let us indicate how to check all other conditions for $\mathscr{M}$ to be a demi-equivalence.

(DE3) Follows from the properties of Haar systems (Property (HS2) in [28, Definition 1.19]) and the definition of the topology on $\mathscr{M}$ as given above.

(DE4) Straight-forward computation.

(DE5) Follows from the properties of Haar systems (Property (HS3) in [28, Definition 1.19]).

(DE6) Since the integrand of $\langle \xi \mid \xi \rangle_{\mathscr{B}}$ takes positive values in $B(s(h))$, so does the inner product itself, and $\langle \xi \mid \xi \rangle_{\mathscr{B}} = 0$ implies $\xi(\ell) = 0$ for all $\ell \in \mathcal{H}s(h)$, which is the entire domain of ξ.

(DE7) Holds by definition of the Banach space structure on $M(h)$.

(DE8) Note that $M(h) = M(u)$ as right-Hilbert C*-$B(u)$-modules, so it suffices to point out that $M(u) = V(u)$ is known to be full since $\mathscr{B}$ is a saturated Fell bundle.

It follows from Theorem 1.6 that $\mathscr{B}$ is equivalent to the imprimitivity Fell bundle $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ of $\mathscr{M}$. Let us describe $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ as a bundle $\mathscr{A} = (A \to \mathcal{H})$ over $\mathcal{H}$. There are multiple variants here, depending on our choice of section $\mathcal{H} \to X\,{}_s{*}_s\, X^{\mathrm{op}}$ of the quotient map $X\,{}_s{*}_s\, X^{\mathrm{op}} \to X \times_{\mathcal{H}} X^{\mathrm{op}} \cong \mathcal{H}$.

In view of $\mathscr{E}$ as described in Equation (6.2) and in view of the fact that $B(h) \cong B(h^{-1})^{\mathrm{op}}$ by the Fell bundle properties, let us declare[11]

$$A(h) := M(v) \otimes_v M(h^{-1})^{\mathrm{op}} \text{ for } h \in \mathcal{H}_u^v.$$

To describe the multiplication map, we must be a bit careful: if $k \in \mathcal{H}_w^u$, then our choice of section turns the composable pair (h, k) into $([v, (h^{-1})^{\mathrm{op}}], [u, (k^{-1})^{\mathrm{op}}])$ in the imprimitivity groupoid. But h^{-1} and u do not coincide, and so neither will the "inner" parts of $A(h)$ and $A(k)$:

$$A(h) \times A(k) = \left(M(v) \otimes_v M(h^{-1})^{\mathrm{op}}\right) \times \left(M(u) \otimes_u M(k^{-1})^{\mathrm{op}}\right).$$

On the level of groupoids, we must therefore replace (for example) the representative $(u, (k^{-1})^{\mathrm{op}})$ by $(h^{-1}, (k^{-1}h^{-1})^{\mathrm{op}})$.[12] On the level of $\mathscr{A}$, this is being done by the Ψ-map from Lemma 3.4:

$$\begin{tikzcd}
M(u) \otimes_u M(k^{-1})^{\mathrm{op}} \arrow[d, "\cong"] \arrow[ddd, bend right] & {[\xi \triangleleft b] \otimes \eta^{\mathrm{op}}} \\
{[M(h^{-1}) \otimes_v B(h)] \otimes_u M(k^{-1})^{\mathrm{op}}} \arrow[d, "\cong"] & {[\xi \otimes b] \otimes \eta^{\mathrm{op}}} \arrow[u, mapsto] \arrow[d, mapsto] \\
M(h^{-1}) \otimes_v [M(k^{-1}) \otimes_u B(h^{-1})]^{\mathrm{op}} \arrow[d, "\cong"] & \xi \otimes [\eta \otimes b^*]^{\mathrm{op}} \arrow[d, mapsto] \\
M(h^{-1}) \otimes_v M(k^{-1}h^{-1})^{\mathrm{op}} & \xi \otimes [\eta \triangleleft b^*]^{\mathrm{op}}
\end{tikzcd}$$

[11]Notationally, it would have been nicer to choose $A(h) = M(h) \otimes_u M(u)^{\mathrm{op}}$, but then our description would drift farther away from those in [17, 20].

[12]Again, this choice is not very pleasant, but it is indeed the best in this scenario.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Having found compatible representatives, we can then use the U-map from Lemma 3.14:

$$\begin{aligned} A(h)\times A(k) &\xrightarrow{\mathrm{id}\times\Psi_h} (M(v)\otimes_v M(h^{-1})^{\mathrm{op}})\times(M(h^{-1})\otimes_v M(k^{-1}h^{-1})^{\mathrm{op}}) \\ &\xrightarrow{U_{h^{-1}}} M(v)\otimes_v M(k^{-1}h^{-1})^{\mathrm{op}} = A(hk), \end{aligned}$$

where

$$U_{h^{-1}}(\xi\otimes\eta_1^{\mathrm{op}},\eta_2\otimes\zeta^{\mathrm{op}}) = (\xi \triangleleft \langle \eta_1 \mid \eta_2\rangle_{\mathscr{B}})\otimes\zeta^{\mathrm{op}}.$$

We therefore see that, with the choice of $A(h)$ that we have made, the multiplication $A(h)\times A(k)\to A(hk)$ is given by $U_{h^{-1}}\circ(\mathrm{id}\times\Psi_h)$. The involution on A is easier to decipher: it is given by

$$A(h) = M(v)\otimes_v M(h^{-1})^{\mathrm{op}} \xrightarrow{\texttt{Flip}} M(h^{-1})\otimes_v M(v)^{\mathrm{op}} \xrightarrow{\Psi_{h^{-1}}} M(u)\otimes_u M(h)^{\mathrm{op}} = A(h^{-1}).$$

Now, $M(h)$ and $M(s(h))$ are *equal* as right-Hilbert C*-modules; the letter h is merely needed to keep track of the "mixed" inner product in Equation (6.1). This means that we have an isomorphism

$$\Psi_h\colon \qquad V(u)\otimes_u V(u)^{\mathrm{op}} \longrightarrow V(v)\otimes_u V(v)^{\mathrm{op}}.$$

If we make the canonical identification of $V(u)\otimes_u V(u)^{\mathrm{op}}$ with $\mathbb{K}\big(V(u)_{B(u)}\big) = \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(s(h))$, then the above can be written as

$$h \triangleright \llcorner\!\lrcorner\colon \qquad \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(s(h)) \longrightarrow \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(r(h)).$$

In other words, Ψ encodes an action of $\mathcal{H}$ on the bundle $\mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})$ which covers the map $\llcorner\!\lrcorner \triangleright \llcorner\!\lrcorner\colon h\triangleright s(h) = r(h)$. In this picture, the map `Flip` is just the adjoint:

$$\begin{array}{ccccc} \xi\otimes\eta^{\mathrm{op}} & \longleftarrow\!\!\shortmid & |\xi\rangle\langle\eta| & \in & \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(v) \\ {\scriptstyle\texttt{Flip}}\big\downarrow & & & & \big\downarrow{\scriptstyle *} \\ \eta\otimes\xi^{\mathrm{op}} & \longmapsto & |\eta\rangle\langle\xi| = |\xi\rangle\langle\eta|^* & \in & \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(v) \end{array}$$

and $U_{h^{-1}}$ is just juxtaposition of compact operators:

$$\begin{array}{ccccc} (\xi\otimes\eta_1^{\mathrm{op}},\eta_2\otimes\zeta^{\mathrm{op}}) & \longleftarrow\!\!\shortmid & \big(|\xi\rangle\langle\eta_1|,|\eta_2\rangle\langle\zeta|\big) & \in & \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(v)\times\mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(v) \\ {\scriptstyle U_{h^{-1}}}\big\downarrow & & & & \big\downarrow{\scriptstyle \llcorner\!\lrcorner\circ\llcorner\!\lrcorner} \\ (\xi\triangleleft\langle\eta_1\mid\eta_2\rangle_{\mathscr{B}})\otimes\zeta^{\mathrm{op}} & \longmapsto & |\xi\triangleleft\langle\eta_1\mid\eta_2\rangle_{\mathscr{B}}\rangle\langle\zeta| & & \\ & & = |\xi\rangle\langle\eta_1|\circ|\eta_2\rangle\langle\zeta| & \in & \mathbb{K}(\mathscr{V}_{\mathscr{B}^{(0)}})(v) \end{array}$$

If we write

$$\begin{aligned} A(h) &= [V(v)\times\{v\}]\otimes_v [V(v)\times\{h^{-1}\}]^{\mathrm{op}} = [V(v)\otimes_v V(v)^{\mathrm{op}}]\times\{h\} \\ &\cong \mathbb{K}\big(V(v)_{B(v)}\big)\times\{h\}, \end{aligned}$$

then as sets, $\mathscr{A} = \mathbb{K}\big(\mathscr{V}_{\mathscr{B}^{(0)}}\big)\rtimes\mathcal{H}$. Moreover, we found that the multiplication $A(h)\times A(k)\to A(hk)$ is given by $U_{h^{-1}}\circ(\mathrm{id}\times\Psi_h)$ and the involution $A(h)\to A(h^{-1})$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

by $\Psi_{h^{-1}} \circ \mathtt{Flip}$, which we have shown translates to

$$(T_1, h) \cdot (T_2, k) = \big(T_1 \circ (h \triangleright T_2), hk\big) \quad \text{and} \quad (T, h)^* = (T^*, h^{-1}),$$

which are exactly the formulas in the semi-direct product. We deduce that the imprimitivity Fell bundle $\mathscr{A}$ of the $\mathscr{B}$-action on $\mathscr{M}$ is exactly the Fell bundle $\mathbb{K}\big(\mathscr{V}_{\mathscr{B}^{(0)}}\big) \rtimes \mathcal{H}$, showing that Theorem 1.6 recovers[13] Kumjian's Stabilization trick.

6.2 Higher order operators *à la* Abadie–Ferraro

As alluded to earlier, a theorem similar to our main theorem has appeared in [1] in the setting that $\mathcal{H} = X = \mathcal{G}$ is a group. Before we can cite it, let us first establish the following bridge between their terminology and ours.

Lemma 6.5 *Suppose $\mathscr{B}$ is a saturated Fell bundle over a locally compact Hausdorff group $\mathcal{H}$. Then $\mathscr{B}$-demi-equivalences over the same group are exactly right Hilbert $\mathscr{B}$-bundles in the sense of [1, Definition 2.1] that are fibrewise full.*

Proof *A priori*, the norm of a demi-equivalence $\mathscr{M} = (M \to \mathcal{H})$ is only upper semi-continuous. However, since $\mathscr{B}$ is a Fell bundle over a group, it follows from [2, Lemma 3.30] that $b \mapsto \|b\|$ is continuous (not only upper semi-continuous) on $\mathscr{B}$, and since the norm on $\mathscr{M}$ is given by $\|\langle m \mid m \rangle_{\mathscr{B}}\|^{1/2}$ by Assumption (DE7), it is the concatenation of continuous maps and hence itself continuous. Therefore, the upper semi-continuous Banach bundle $\mathscr{M}$ is actually a *continuous* Banach bundle in the sense of [9, Definition II.13.4], as needed for [1, Definition 2.1].

As in [1], the inner product and the right-action are continuous by (DE3) and (DE16), respectively. The assumption that the Hilbert bundle be fibrewise full corresponds to (DE8) (and is stronger than the assumption (7R) in [1]). The remaining items of [1, Definition 2.1] correspond to our assumptions as follows.

(1R) corresponds to (DE1) and (DE2), using that $\{h_1^{\mathrm{op}} \mid h_2\}_{\mathcal{H}}^{\mathcal{H}} = h_1^{-1}h_2$ as explained in Example 1.2;
(2R) corresponds to (DE13);
(3R) corresponds to (DE3);
(4R) corresponds to (DE4) and (DE5);
(5R) corresponds to (DE6); and
(6R) corresponds to (DE7).

■

We can now restate the result of Abadie and Ferraro to which we want to compare ours, in our terminology. We will artificially add the assumption that the fibres are all full, to align with our situation. We will further use different letters for elements of the group $\mathcal{H}$: we use x if we think of $\mathcal{H}$ as a principal $\mathcal{H}$-space; we use h if we think of $\mathcal{H}$ as the group acting on the right, and g if it is acting on the left; in particular, $x \triangleleft h$ and $g \triangleright x$ will just be the products xh respectively gx in $\mathcal{H}$.

[13] Of course, there is still the difficulty of having to come up with a suitable $\mathscr{M}$ in the first place; we have taken it for granted here.

Theorem 6.6 ([1, Definition 3.6, Theorem 3.9, Corollary 3.10]) *Let $\mathcal{H}$ be a locally compact Hausdorff group, $\mathscr{B} = (B \to \mathcal{H})$ a Fell bundle, and $\mathscr{M} = (M \to \mathcal{H})$ a right $\mathscr{B}$-demi-equivalence, also over $\mathcal{H}$. For $g \in \mathcal{H}$, let $\mathbb{B}_g(\mathscr{M})$ be the collection of continuous maps $S\colon M \to M$ such that*

- *there exists $c \in \mathbb{R}$ such that $\|Sm\| \leq c\,\|m\|$ for all $m \in M$;*
- *$S(M(x)) \subseteq M(g \triangleright x)$ for all $x \in \mathcal{H}$; and*
- *there exists $S^*\colon M \to M$ such that $\langle Sm_1 \mid m_2\rangle_{\mathscr{B}} = \langle m_1 \mid S^* m_2\rangle_{\mathscr{B}}$ for all $m_i \in M$.*

Then there exists a unique Fell bundle $\mathbb{K}(\mathscr{M})$ over $\mathcal{H}$ such that:

(i) *for all $g \in \mathcal{H}$, the fibre $\mathbb{K}(\mathscr{M})_g$ is, as a Banach space, the closure in $\mathbb{B}_g(\mathscr{M})$ of*

$$\mathit{span}\{|m\rangle\langle n| : m \in M(g \triangleright x), n \in M(x), x \in \mathcal{H}\}$$

where $|m\rangle\langle n|\colon M \to M, k \mapsto m \triangleleft \langle n \mid k\rangle_{\mathscr{B}}$;

(ii) *given $\psi, \varphi \in C_c(M)$ and $x \in \mathcal{H}$, the function $[\psi, \varphi, x]\colon \mathcal{H} \to \mathbb{K}(\mathscr{M})$ given by $[\psi, \varphi, x](g) = [\psi(g \triangleright x), \varphi(x)]$, is a continuous section of $\mathbb{K}(\mathscr{M})$.*

Moreover, $\mathscr{M}$ is a $\mathbb{K}(\mathscr{M})$ – $\mathscr{B}$-equivalence bundle with the action $\mathbb{K}(\mathscr{M}) \times M \to M$ given by $(S, m) \to S(m)$ and the left inner product $M \times M \to \mathbb{K}(\mathscr{M})$ given by $(m, n) \to |m\rangle\langle n|$.

Abadie and Ferraro call the elements of $\mathbb{B}_g(\mathscr{M})$ *adjointable operators of order g*. We conclude:

Corollary 6.7 *Suppose $\mathcal{H}$, $\mathscr{B}$, and $\mathscr{M}$ are as in Theorem 6.6. The map*

$$\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}} \to \mathbb{K}(\mathscr{M})$$

which maps $[m \otimes n^{op}]$ in the fibre over $[x, y^{op}]$ of $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ to the operator

$$|m\rangle\langle n|: \quad M \ni k \mapsto m \triangleleft \langle n \mid k\rangle_{\mathscr{B}}^{\mathscr{M}} \in M$$

in the fibre over xy^{-1} of $\mathbb{K}(\mathscr{M})$, is an isomorphism of Fell bundles, covering the map f from Example 1.2.

Proof By Lemma 6.5, $\mathscr{M}$ is a right Hilbert $\mathscr{B}$-bundle in the sense of [1, Definition 2.1]. It is proven in [1, Corollary 3.10] that $\mathscr{M}$ is an equivalence between $\mathscr{B}$ and $\mathbb{K}(\mathscr{M})$. Since $\mathscr{M}$ is also an equivalence between $\mathscr{B}$ and $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$, it thus follows from [1, Corollary 3.11] (or equivalently, from Proposition 5.1), that the displayed map is the claimed isomorphism. ∎

Remark 6.8 As alluded to in Remark 2.4, the description *à la* Abadie and Ferraro of the imprimitivity Fell bundle $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ as "adjointable operators with a shift" is also built into our construction, albeit less visibly so.

Even when we consider Fell bundles over groups, Theorem 1.6 can cover some examples that are not covered by [1], since there, $X = \mathcal{H}$ always. One example is [19, Proposition A.2.3]; another one is Example 6.9.

6.3 Imprimitivity theorems

Earlier, we alluded to a relationship between Theorem 1.6 and imprimitivity theorems; let us give one example.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Example 6.9 ([20, Example 14]) If X is a locally compact Hausdorff group with closed subgroup H, then H acts on X by right translation ($x \triangleleft h := xh$) and X is a principal H-space. Its imprimitivity groupoid $X \times_H X^{\mathrm{op}}$ is isomorphic to the transformation group groupoid $X \ltimes X/H$ of X acting on the quotient space X/H via

$$X \times_H X^{\mathrm{op}} \to X \ltimes X/H, \quad [x, y^{\mathrm{op}}] \mapsto (xy^{-1}, yH). \tag{6.3}$$

In terms of $G := X \ltimes X/H$, the anchor map described in (1.2) becomes the quotient map,

$$\rho\colon X \to G^{(0)} \cong X/H, \quad x \mapsto (1_X, xH) \triangleq xH,$$

and the left action on X as described in (1.3) becomes the action $(x, yH) \triangleright y := xy$. We conclude that G is equivalent to the group H via X. To sum up,

$$X^{\mathrm{op}}\,{}_{\rho}{*}_{\rho}\, X \to H,\ \{x^{\mathrm{op}} \mid y\}_H^X = x^{-1}y, \quad \text{and} \quad X \times X^{\mathrm{op}} \to G,\ {}_G^X\{x \mid y^{\mathrm{op}}\} = (xy^{-1}, yH).$$

Note that $\{\llcorner\!\lrcorner \mid \llcorner\!\lrcorner\}_H^X$ indeed lands in H since $\rho(x) = \rho(y)$.

Now suppose that A is a C^*-algebra with action $\alpha\colon X \to \mathrm{Aut}(A)$. Let $\mathscr{B} = A \rtimes H$ be the Fell bundle over H that encodes the restriction $\alpha|_H$, meaning that, for each $h \in H$, the fibre $B(h)$ is A and the structure maps of $\mathscr{B}$ are given by

$$(a_1, h_1) \cdot (a_2, h_2) := (a_1\alpha_{h_1}(a_2), h_1h_2) \quad \text{and} \quad (a, h)^* := (\alpha_{h^{-1}}(a)^*, h^{-1}).$$

Let $\mathscr{M} = A \times X$ be the "trivial A-bundle" over X. Then $\mathscr{B}$ acts on $\mathscr{M}$ via

$$M \times B \to M, \quad (m, x) \blacktriangleleft (b, h) := (m\alpha_x(b), xh),$$

and we have a $\mathscr{B}$-valued inner product given by

$$M\,{}_{\rho}{*}_{\rho}\, M \to B, \quad \langle (m, x) \mid (n, y)\rangle_{\mathscr{B}} := \big(\alpha_{x^{-1}}(m^*n), x^{-1}y\big).$$

One quickly checks that $\mathscr{M}$ is a $\mathscr{B}$-demi-equivalence, so $\mathscr{B}$ is equivalent via $\mathscr{M}$ to $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$. Let us identify the fibre of the latter Fell bundle over an element $(xy^{-1}, yH) = {}_G^X\{x \mid y^{\mathrm{op}}\}$ of G: As explained in Theorem 3.17, we use (3.8) to identify

$$(\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}})(xy^{-1}, yH) \cong M(x) \otimes_e M(y)^{\mathrm{op}}$$

as $\mathbb{K}_{B(e)}(M(x)) - \mathbb{K}_{B(e)}(M(y))$-imprimitivity bimodules. Over a unit $(1_G, yH)$ of G, the fibre is (canonically isomorphic to) the C^*-algebra $\mathbb{K}_{B(e)}(M(y))$ itself, where y is any representative of yH. These $B(e)$-compact operators on $M(y)$ are generated by $|n_1\rangle\langle n_2|$ for $n_i \in M(y)$, and one quickly computes that such a rank-one operator just multiplies $n \in M(y)$ on the left by $n_1n_2^*$.

The balancing in $M(x) \otimes_e M(y)^{\mathrm{op}}$ identifies

$$(m\alpha_x(b), x) \otimes (n, y)^{\mathrm{op}} = (m, x) \otimes (n\alpha_y(b^*), y)^{\mathrm{op}}$$

for all $b \in B(e) = A$. If we let $A(xy^{-1}, yH) := A$ as a Banach space, then we can therefore show that the map

$$\begin{aligned} M(x) \otimes_e M(y)^{\mathrm{op}} &\to A(xy^{-1}, yH) \\ (m, x) \otimes (n, y)^{\mathrm{op}} &\mapsto m\alpha_{xy^{-1}}(n)^* \end{aligned}$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

is a Banach space isomorphism. These isomorphisms create a new Fell bundle $\mathscr{A} = (p_{\mathscr{A}}\colon A \to G)$ out of $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$: $\mathscr{A}$ has the fibre $A(xy^{-1}, yH)$ over (xy^{-1}, yH), and the structure maps of $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ described in (3.24) and Corollary 3.10 translate to

$$\llcorner\!\lrcorner \cdot \llcorner\!\lrcorner\colon \qquad A(x, yzH) \times A(y, zH) \to A(xy, zH), \quad (a, b) \mapsto a\alpha_x(b),$$

and

$$\llcorner\!\lrcorner^*\colon \qquad A(x, yH) \to A(x^{-1}, xyH), \quad a \mapsto \alpha_{x^{-1}}(a^*).$$

The Fell bundle $\mathscr{A}$ is therefore exactly the semi-direct product Fell bundle $X \ltimes (A \times X/H)$ which encodes the left-X action $x \cdot (a, yH) = (\alpha_x(a), xyH)$ on the constant C*-algebraic bundle $A \times X/H$ over X/H. We have therefore recovered the same Fell bundle that was mentioned as being equivalent to $\mathscr{B}$ in [20, Example 14].

Example 6.9 is the extreme case where X is a group; the other extreme case is when X is just a principal H-space: We can likewise define a free and proper action of the transformation group groupoid $X \rtimes H$ on X by letting (x, h) transform x into $x \triangleleft h$. The associated imprimitivity groupoid $X \times_{X \rtimes H} X^{\mathrm{op}}$ is exactly the quotient X/H of the original action on X, and so an application of [21, Theorem 2.8] recovers Green's theorem that $C_0(X/H)$ and $C_0(X) \rtimes_r H$ are strongly Morita equivalent.

There is a plethora of generalizations of Green's result. For example, we can allow the object X that is being acted on to be not just a space (as in Green's theorem) or a group (as in Example 6.9), but a groupoid $\mathcal{X}$, and we allow the underlying bundles to be more than line-bundles: if $\mathscr{M} = (M \to \mathcal{X})$ is a *Fell bundle* that carries a free and proper action of a group K, then $\mathscr{M}$ is an equivalence between the semi-direct product Fell bundle $\mathscr{B} = \mathscr{M} \rtimes K$ and the quotient Fell bundle $K\backslash\mathscr{M}$.[14] Theorem 1.6 can recover such results, in that $\mathscr{M}$ is a $\mathscr{B}$-demi-equivalence, and $\mathscr{M} \otimes_{\mathscr{B}} \mathscr{M}^{\mathrm{op}}$ can be computed to be isomorphic to the quotient bundle.

A Hilbert C*-Modules

Lemma A.1 *If $\mathbf{X}$ is a full right Hilbert C*-module over a C*-algebra B and $\mathbf{x}_i \in \mathbf{X}$, then the operator-norm of the B-compact, positive operator $\sum_{i=1}^k |\mathbf{x}_i\rangle\langle\mathbf{x}_i|$ is exactly the norm of $\sum_{i=1}^k \langle \mathbf{x}_i \mid \mathbf{x}_i\rangle_B^{\mathbf{X}}$ in B.*

Proof Denote the compact operator in question by T. For any $\mathbf{z} \in \mathbf{X}$, we have

$$\langle T\mathbf{z} \mid \mathbf{z}\rangle_B^{\mathbf{X}} = \sum_i \langle \mathbf{x}_i \cdot \langle \mathbf{x}_i \mid \mathbf{z}\rangle_B^{\mathbf{X}} \mid \mathbf{z}\rangle_B^{\mathbf{X}} = \sum_i \langle \mathbf{z} \mid \mathbf{x}_i\rangle_B^{\mathbf{X}} \langle \mathbf{x}_i \mid \mathbf{z}\rangle_B^{\mathbf{X}}.$$

Note that the right-hand side is a sum of positive elements and hence positive, so T is indeed a positive operator by [24, Lemma 2.28]. By the Cauchy–Schwarz inequality for Hilbert modules [24, Lemma 2.5], we have for each i

$$\langle \mathbf{z} \mid \mathbf{x}_i\rangle_B^{\mathbf{X}} \langle \mathbf{x}_i \mid \mathbf{z}\rangle_B^{\mathbf{X}} \leq \|\mathbf{z}\|^2 \langle \mathbf{x}_i \mid \mathbf{x}_i\rangle_B^{\mathbf{X}}$$

[14] This is a special case of [15, Theorem 3.1], where they consider *two* group actions on a groupoid $\mathcal{X}$. [5, Theorem 6.1] goes two steps further, by firstly allowing groupoid actions, and secondly by allowing the actions to be only *self-similar* rather than by homomorphisms; this yields a Fell bundle equivalence between the Zappa–Szép product Fell bundles $(\mathscr{M}/\mathcal{K}_1) \bowtie \mathcal{K}_2$ and $\mathcal{K}_1 \bowtie (\mathcal{K}_2\backslash\mathscr{M})$ as constructed in [6].

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

in the C*-algebra B, and hence we conclude that

$$\langle T\mathbf{z} \mid \mathbf{z}\rangle_B^{\mathbf{X}} \leq \|\mathbf{z}\|^2 \sum_i \langle \mathbf{x}_i \mid \mathbf{x}_i\rangle_B^{\mathbf{X}}.$$

Using [24, Remark 2.29] and positivity of T in the first equality of the following, we deduce

$$\|T\| = \sup\{\|\langle T\mathbf{z} \mid \mathbf{z}\rangle_B^{\mathbf{X}}\| : \|\mathbf{z}\| \leq 1\} \leq \left\|\sum_i \langle \mathbf{x}_i \mid \mathbf{x}_i\rangle_B^{\mathbf{X}}\right\|.$$

For the reverse inequality, note that $\mathbf{X}$ is an imprimitivity bimodule between $\mathbb{K}_B(\mathbf{X})$ and B, meaning that B is exactly the algebra of $\mathbb{K}_B(\mathbf{X})$-compact operators on the *left*-Hilbert C*-module $\mathbf{X}$. In particular, the same proof for T replaced by $\sum_i \langle \mathbf{x}_i \mid \mathbf{x}_i\rangle_B^{\mathbf{X}}$ yields the other estimate. ∎

The following result is well known.

Lemma A.2 *Suppose $\mathbf{X}$ is an $A-B$-imprimitivity bimodule and $\mathbf{Y}$ is a $C-B$-imprimitivity bimodule. Then the balanced tensor product $\mathbf{X} \otimes_B \mathbf{Y}^{op}$ is isomorphic to the B-compact operators $\mathbb{K}_B(\mathbf{Y}, \mathbf{X})$ as $A-C$-imprimitivity bimodule s.*

Proof First recall that $\mathbb{K}_B(\mathbf{Y}, \mathbf{X})$ is densely spanned by the maps

$$|\mathbf{x}\rangle\langle\mathbf{y}|: \quad \mathbf{y}' \mapsto \mathbf{x} \cdot \langle \mathbf{y} \mid \mathbf{y}'\rangle_B$$

for $\mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{Y}$. Its bimodule structure is given by

$$a \cdot |\mathbf{x}\rangle\langle\mathbf{y}| := |a \cdot \mathbf{x}\rangle\langle\mathbf{y}| \quad \text{and} \quad |\mathbf{x}\rangle\langle\mathbf{y}| \cdot c := |\mathbf{x}\rangle\langle c^* \cdot \mathbf{y}|.$$

The map

$$\mathbf{X} \times \mathbf{Y}^{\mathrm{op}} \to \mathbb{K}_B(\mathbf{Y}, \mathbf{X}), \quad (\mathbf{x}, \mathbf{y}^{\mathrm{op}}) \mapsto |\mathbf{x}\rangle\langle\mathbf{y}|,$$

is bilinear, and so it descends to a linear map with domain $\mathbf{X} \odot \mathbf{Y}^{\mathrm{op}}$. For any $b \in B$, we have

$$(\mathbf{x} \cdot b) \cdot \langle \mathbf{y} \mid \mathbf{y}'\rangle_B = \mathbf{x} \cdot \langle \mathbf{y} \cdot b^* \mid \mathbf{y}'\rangle_B, \text{ so that } |\mathbf{x} \cdot b\rangle\langle\mathbf{y}| = |\mathbf{x}\rangle\langle\mathbf{y} \cdot b^*|.$$

Since $(\mathbf{y} \cdot b^*)^{\mathrm{op}} = b \cdot \mathbf{y}^{\mathrm{op}}$, we conclude that we have a linear map

$$\mathbf{X} \odot_B \mathbf{Y}^{\mathrm{op}} \to \mathbb{K}_B(\mathbf{Y}, \mathbf{X}) \text{ determined by } \mathbf{x} \odot \mathbf{y}^{\mathrm{op}} \mapsto |\mathbf{x}\rangle\langle\mathbf{y}|.$$

Clearly, this map is an $A-C$-bimodule map and, by definition of $\mathbb{K}_B(\mathbf{Y}, \mathbf{X})$, it has dense range. Thus, if we can show that the map is isometric, then it extends to an isomorphism of Hilbert bimodules. For $\mathbf{x}_i \in \mathbf{X}$ and $\mathbf{y}_i, \mathbf{y} \in \mathbf{Y}$, we have

$$\begin{aligned}\left\|\sum_i |\mathbf{x}_i\rangle\langle\mathbf{y}_i|(\mathbf{y})\right\|_{\mathbf{X}}^2 &= \left\|\sum_{i,j} \langle \mathbf{x}_i \cdot \langle \mathbf{y}_i \mid \mathbf{y}\rangle_B \mid \mathbf{x}_j \cdot \langle \mathbf{y}_j \mid \mathbf{y}\rangle_B\rangle_B\right\|_B \\ &= \left\|\sum_{i,j} \langle \mathbf{y} \mid \mathbf{y}_i\rangle_B \langle \mathbf{x}_i \mid \mathbf{x}_j\rangle_B \langle \mathbf{y}_j \mid \mathbf{y}\rangle_B\right\|_B\end{aligned}$$

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

As in [24, Lemma 2.65], we can write $\langle \mathbf{x}_j \mid \mathbf{x}_i \rangle_B^{\mathbf{X}} = \sum_l b_{j,l} b_{i,l}^*$ for some $b_{i,j} \in B$, so that

$$= \left\| \sum_{i,j,l} \langle \mathbf{y} \mid \mathbf{y}_i \rangle_B b_{i,l} b_{j,l}^* \langle \mathbf{y}_j \mid \mathbf{y} \rangle_B \right\|_B$$
$$= \left\| \sum_{i,j,l} \langle \mathbf{y} \mid \mathbf{y}_i \cdot b_{i,l} \rangle_B \langle \mathbf{y}_j \cdot b_{j,l} \mid \mathbf{y} \rangle_B \right\|_B .$$

Write $\mathbf{z}_l := \sum_i \mathbf{y}_i \cdot b_{i,l}$. Since

$$\langle \mathbf{y} \mid \mathbf{z}_l \rangle_B^{\mathbf{Y}} \langle \mathbf{z}_l \mid \mathbf{y} \rangle_B^{\mathbf{Y}} = \langle \mathbf{y} \mid \mathbf{z}_l \cdot \langle \mathbf{z}_l \mid \mathbf{y} \rangle_B^{\mathbf{Y}} \rangle_B^{\mathbf{Y}} = \langle \mathbf{y} \mid {}_C^{\mathbf{Y}}\langle \mathbf{z}_l \mid \mathbf{z}_l \rangle \cdot \mathbf{y} \rangle_B^{\mathbf{Y}},$$

it follows that

$$\left\| \sum_i |\mathbf{x}_i\rangle\langle \mathbf{y}_i| \right\|_{\mathbb{K}}^2 = \sup_{\|\mathbf{y}\| \le 1} \left\| \sum_i |\mathbf{x}_i\rangle\langle \mathbf{y}_i|(\mathbf{y}) \right\|_{\mathbf{X}}^2$$
$$= \sup_{\|\mathbf{y}\| \le 1} \left\| \langle \mathbf{y} \mid \sum_l {}_C^{\mathbf{Y}}\langle \mathbf{z}_l \mid \mathbf{z}_l \rangle \cdot \mathbf{y} \rangle_B^{\mathbf{Y}} \right\|_B \overset{(\dagger)}{=} \left\| \sum_l {}_C^{\mathbf{Y}}\langle \mathbf{z}_l \mid \mathbf{z}_l \rangle \right\|,$$

where $(\dagger)$ follows from [24, Remark 2.29] applied to the positive operator $\sum_l {}_C^{\mathbf{Y}}\langle \mathbf{z}_l \mid \mathbf{z}_l \rangle$. Since B acts by C-adjointable operators on $\mathbf{Y}$, we have

$$\left\| \sum_l {}_C^{\mathbf{Y}}\langle \mathbf{z}_l \mid \mathbf{z}_l \rangle \right\|_C = \left\| \sum_{i,j,l} {}_C^{\mathbf{Y}}\langle \mathbf{y}_i \mid \mathbf{y}_j \cdot (b_{j,l} b_{i,l}^*) \rangle \right\|_C = \left\| \sum_{i,j} {}_C^{\mathbf{Y}}\langle \mathbf{y}_i \mid \mathbf{y}_j \cdot \langle \mathbf{x}_j \mid \mathbf{x}_i \rangle_B^{\mathbf{X}} \rangle \right\|_C$$
$$= \left\| \sum_{i,j} \langle \mathbf{y}_i^{\mathrm{op}} \mid \langle \mathbf{x}_i \mid \mathbf{x}_j \rangle_B^{\mathbf{X}} \cdot \mathbf{y}_j^{\mathrm{op}} \rangle_C^{\mathbf{Y}^{\mathrm{op}}} \right\|_C \overset{(3.2)}{=} \left\| \sum_i \mathbf{x}_i \otimes \mathbf{y}_i^{\mathrm{op}} \right\|_{\mathbf{X} \otimes_B \mathbf{Y}^{\mathrm{op}}}^2,$$

as claimed. ■

B Upper semi-continuous Banach bundles

The following lemma explains that, in an upper semi-continuous Banach bundle, being arbitrarily close to convergent nets implies convergence. It is, in essence, just a restatement of [29, Proposition C.20], where we drop the assumption that the bundle be a C*-bundle.

Lemma B.1 *Let $\mathscr{M} = (q_{\mathscr{M}} : M \to X)$ be an upper semi-continuous Banach bundle over a locally compact Hausdorff space X. Suppose we are given a net $(m_\lambda)_\lambda$ in $\mathscr{M}$ and a point $m \in \mathscr{M}$ such that $q_{\mathscr{M}}(m_\lambda) \to q_{\mathscr{M}}(m)$. Then $m_\lambda \to m$ if and only if, for every $\varepsilon > 0$, there exists a convergent net $(n_\lambda^\varepsilon)_\lambda$ in $\mathscr{M}$ with $q_{\mathscr{M}}(n_\lambda^\varepsilon) = q_{\mathscr{M}}(m_\lambda)$ such that for all large λ, we have*

$$\|m_\lambda - n_\lambda^\varepsilon\| < \varepsilon \text{ and } \left\| m - \lim_\lambda n_\lambda^\varepsilon \right\| < \varepsilon.$$

Proof The forwards implication is trivial. The backwards implication follows from an application of [4, Lemma A.3, (i) ⇒ (ii)] (applied to the convergent net $(n_\lambda^\varepsilon)_\lambda$), the triangle inequality, and an application of [4, Lemma A.3, (i) ⇐ (ii)]. ■

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Lemma B.2 (cf. [4, Proposition A.7]) *Let $\mathscr{M} = (q_{\mathscr{M}}: M \to X)$ and $\mathscr{N} = (q_{\mathscr{N}}: N \to Y)$ be two upper semi-continuous Banach bundles. Suppose we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\Omega} & \mathscr{N} \\ {\scriptstyle q_{\mathscr{M}}}\downarrow & & \downarrow{\scriptstyle q_{\mathscr{N}}} \\ X & \xrightarrow{\omega} & Y \end{array} \tag{B.1}$$

and that the maps satisfy the following conditions.

(a) *for each $x \in X$, $\Omega|_{M_x}$ is linear;*
(b) *there exists a constant $K > 0$ such that $\|\Omega(m)\| \leq K\|m\|$ for all $m \in M$;*
(c) *ω is a continuous map; and*
(d) *there exists a collection Γ of continuous sections of $\mathscr{M}$ such that the $\mathbb{C}$-linear span of $\{\gamma(x) : \gamma \in \Gamma\}$ is dense in each M_x and, for each $\gamma \in \Gamma$, the section $\Omega \circ \gamma: x \mapsto \big(x, \Omega(\gamma(x))\big)$ of the pull-back bundle $\omega^*(\mathscr{N})$ is continuous.*

Then Ω is continuous.

Proof First we point out that the proof of [4, Proposition A.7] does not require Γ to be a vector space but only that $\mathrm{span}_{\mathbb{C}}\{\gamma(x) : \gamma \in \Gamma\}$ is dense in $M(x)$; in other words, we already know that Lemma B.2 holds in the case that ω is a homeomorphism.

Consider the commutative diagram

$$\begin{array}{ccccc} m & \longmapsto & \big(q_{\mathscr{M}}(m), \Omega(m)\big) & \longmapsto & \Omega(m) \\ \pitchfork & & \pitchfork & & \pitchfork \\ \mathscr{M} & \xrightarrow{\Psi} & \omega^*(\mathscr{N}) & \longrightarrow & \mathscr{N} \\ {\scriptstyle q_{\mathscr{M}}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle q_{\mathscr{N}}} \\ X & = & X & \xrightarrow{\omega} & Y \end{array} \tag{B.2}$$

We claim that the map Ψ is continuous, which will imply that Ω is continuous since Diagram B.2 commutes. Since the identity map on X is a homeomorphism, we are in good shape to invoke [4, Proposition A.7] for the square on the left-hand side; because of Assumption (d), it only remains to verify that Ψ is fibrewise linear and bounded. But clearly, Assumption (a) implies linearity of $\Psi|_{M_x}: M_x \to \omega^*(\mathscr{N})_x = \{x\} \times \mathscr{N}_{\omega(x)}$ by the definition of the linear structure on $\omega^*(\mathscr{N})$. And likewise, by definition of the norm on $\omega^*(\mathscr{N})_x$, we have $\|\Psi(m)\| = \|\Omega(m)\|$, and hence $\|\Psi(m)\| \leq K\|m\|$ for all $m \in M$ by (b). Using [4, Proposition A.7], we conclude that Ψ and hence Ω is continuous. ∎

Lemma B.3 (cf. [4, Proposition A.8]) *Let $\mathscr{M} = (q_{\mathscr{M}}: M \to X)$ and $\mathscr{N} = (q_{\mathscr{N}}: N \to Y)$ be two upper semi-continuous Banach bundles, and suppose we have the commutative Diagram (B.1) and that the maps Ω, ω satisfy the following conditions.*

(a) *for each $x \in X$, $\Omega|_{M_x}$ is linear;*
(b) *there exists a constant $k > 0$ such that $\|\Omega(m)\| \geq k\|m\|$ for all $m \in M$;*
(c) *ω is an embedding (i.e., a homeomorphism onto its image); and*
(d) *Ω is continuous.*

Then $\Omega^{-1}: \Omega(M) \to M$ is continuous.

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press

Proof As in the proof of Lemma B.2, consider the map Ψ in Diagram (B.2). Again, since $\Omega|_{M_x}$ is linear, so is $\Psi|_{M_x}$, and since Ω is bounded below, so is Ψ. This time, continuity of Ω implies continuity of Ψ. We can therefore apply [4, Proposition A.8] to the square on the left-hand side of Diagram (B.2) to conclude that the map $\Psi^{-1}\colon \Psi(M) \to M$ is continuous.

Since $q_{\mathscr{N}}(\Omega(m)) = \omega(q_{\mathscr{M}}(m))$ by commutativity of Diagram (B.1), it follows from Assumption (c) that the map $f = q_{\mathscr{M}} \circ \omega^{-1} \circ q_{\mathscr{N}}\colon \Omega(M) \to X, \Omega(m) \mapsto q_{\mathscr{M}}(m)$, is well defined and continuous. By definition of Ψ, we have

$$\big(f(\Omega(m)), \Omega(m)\big) = \big(q_{\mathscr{M}}(m), \Omega(m)\big) = \Psi(m),$$

so that for $n \in \Omega(M)$, $\Psi^{-1}(f(n), n) = \Omega^{-1}(n)$. It follows that Ω^{-1} is continuous as concatenation of continuous maps. ∎

Acknowledgements I would like to thank Bram Mesland for asking me a question that the main theorem answers, and I would further like to thank him and Arthur Pander Maat for multiple helpful discussions during the production of this paper. Lastly, I would like to thank the anonymous referee for their helpful report.

References

[1] F. Abadie and D. Ferraro, *Equivalence of Fell bundles over groups*. J. Operator Theory **81**(2019), no. 2, 273–319. https://doi.org/10.7900/jot.

[2] A. Buss, R. Meyer, and C. Zhu, *A higher category approach to twisted actions on C^*-algebras*. Proc. Edinb. Math. Soc. (2) **56**(2013), 387–426. https://doi.org/10.1017/S0013091512000259.

[3] A. Duwenig and H. Emerson, *Transversals, duality, and irrational rotation*. Trans. Amer. Math. Soc. Ser. B. 7(2020), 254–289. https://doi.org/10.1090/btran/54.

[4] A. Duwenig and B. Li, *Equivalence of Fell bundles is an equivalence relation*. Münster J. Math. **16**(2023), no. 1, 95–145. https://doi.org/10.17879/51009605127.

[5] A. Duwenig and B. Li, *Imprimitivity theorems and self-similar actions on Fell bundles*. Journal of Functional Analysis **288**(2025), no. 2, 110699. https://doi.org/10.1016/j.jfa.2024.110699.

[6] A. Duwenig and B. Li, *The Zappa-Szép product of a Fell bundle and a groupoid*. J. Funct. Anal. **282**(2022), no. 1, 109268. https://doi.org/10.1016/j.jfa.2021.109268.

[7] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*. Mem. Amer. Math. Soc. **180**(2006), pp. viii+ 169. https://doi.org/10.1090/memo/0850.

[8] R. Exel, *Morita-Rieffel equivalence and spectral theory for integrable automorphism groups of $C*$-algebras*. J. Funct. Anal. **172**(2000), no. 2, 404–465. https://doi.org/10.1006/jfan.1999.3537.

[9] J. M. G. Fell and R. S. Doran, *Representations of *-algebras, locally compact groups, and Banach* -algebraic bundles: Basic representation theory of groups and algebras. Vol.* 1, Pure and Applied Mathematics, 125, Academic Press, Inc., Boston, MA, 1988, pp. xviii+ 746.

[10] D. Ferraro, *Fixed-point algebras for weakly proper Fell bundles*. New York J. Math. **27**(2021), 943–980.

[11] P. Green, *C^* -algebras of transformation groups with smooth orbit space*. Pacific J. Math. **72**(1977), 71–97. http://projecteuclid.org/euclid.pjm/1102811272.

[12] K. H. Hofmann, *Bundles and sheaves are equivalent in the category of Banach spaces*. In: K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, GA, 1975), Lecture Notes in Math, 575, Springer, Berlin-New York, 1977, pp. 53–69.

[13] A. Huef, S. Kaliszewski, I. Raeburn, and D. P. Williams, *Naturality of symmetric imprimitivity theorems*. Proc. Amer. Math. Soc. **141**(2013), no. 7, 2319–2327. https://doi.org/10.1090/S0002-9939-2013-11712-0.

[14] M. Ionescu and D. P. Williams, *A classic Morita equivalence result for Fell bundle C^*-algebras*. Math. Scand. **108**(2011), no. 2, 251–263. https://doi.org/10.7146/math.scand.a-15170.

[15] S. Kaliszewski, P. S. Muhly, J. Quigg, and D. P. Williams, *Fell bundles and imprimitivity theorems*. Münster J. Math. **6**(2013), no. 1, 53–83. https://doi.org/10.1017/s1446788713000153.

[16] S. Kaliszewski, P. S. Muhly, J. Quigg, and D. P. Williams, *Fell bundles and imprimitivity theorems: towards a universal generalized fixed point algebra*. Indiana Univ. Math. J. **62**(2013), no. 6, 1691–1716. https://doi.org/10.1512/iumj.2013.62.5107.

[17] A. Kumjian, *Fell bundles over groupoids*. Proc. Amer. Math. Soc. **126**(1998), no. 4, 1115–1125. https://doi.org/10.1090/S0002-9939-98-04240-3 .

[18] A. J. Lazar, *A selection theorem for Banach bundles and applications*. J. Math. Anal. Appl. **462**(2018), no. 1, 448–470. https://doi.org/10.1016/j.jmaa.2018.02.008.

[19] B. Mesland and M. H. Sengun, *Stable Range Local Theta Correspondence as a Strong Morita Equivalence*, in preparation. 2023.

[20] P. S. Muhly, *Bundles over groupoids*. In: *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, Contemporary Mathematics, 282, American Mathematical Society, Providence, RI, 2001, pp. 67–82. https://doi.org/10.1090/conm/282/04679.

[21] P. S. Muhly, J. N. Renault, and D. P. Williams, *Equivalence and isomorphism for groupoid $C*$-algebras*. J. Operator Theory **17**(1987), no. 1, 3–22.

[22] P. S. Muhly and D. P. Williams, *Equivalence and disintegration theorems for Fell bundles and their C^*-algebras*. Diss. Math. **456**(2008), 1–57. https://doi.org/10.4064/dm456-0-1.

[23] J. R. Munkres, *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000, pp. xvi+ 537.

[24] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*- algebras*, Mathematical Surveys and Monographs, 60, American Mathematical Society, Providence, RI, 1998, pp. xiv+ 327. https://doi.org/10.1090/surv/060.

[25] M. A. Rieffel, *Proper actions of groups on C^*-algebras*. In: Mappings of operator algebras (Philadelphia, PA, 1988), Progress in Mathematics, 84, Birkhäuser, Boston, MA, 1990, pp. 141–182.

[26] A. Sims and D. P. Williams, *An equivalence theorem for reduced Fell bundle C^*-algebras*. New York J. Math. **19**(2013), 159–178. http://nyjm.albany.edu:8000/j/2013/19_159.html.

[27] A. Sims and D. P. Williams, *Renault's equivalence theorem for reduced groupoid C^*-algebras*. J. Operator Theory **68**(2012), no. 1, 223–239.

[28] D. P. Williams, *A tool kit for groupoid C^*- algebras*, Mathematical Surveys and Monographs, 241, American Mathematical Society, Providence, RI, 2019, pp. xv+ 398. https://doi.org/10.1016/j.physletb.2019.06.021.

[29] D. P. Williams, *Crossed products of C^*- algebras*, Mathematical Surveys and Monographs, 134, American Mathematical Society, Providence, RI, 2007, pp. xvi+ 528. https://doi.org/10.1090/surv/134.

[30] D. P. Williams, *Haar systems on equivalent groupoids*. Proc. Amer. Math. Soc. Ser. B. **3**(2016), 1–8. https://doi.org/10.1090/bproc/22.

KU Leuven, Department of Mathematics, Leuven, Belgium
e-mail: anna.duwenig@kuleuven.be alduwenig@gmail.com

https://doi.org/10.4153/S0008414X24000907 Published online by Cambridge University Press