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Optimal periodic gain scheduling for bipedal walking with hybrid dynamics

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SUMMARY
We present an optimal gain scheduling control design for bipedal walking with minimum tracking error. We obtained a linear approximation by linearizing the nonlinear hybrid dynamic model about a nominal periodic trajectory. This linearization allows us to identify the linear model as a linear periodic system. An optimal feedback was designed using Bellman's dynamic programming. The linear periodic system allows us to determine a linear quadratic regulator (LQR) for a single period and to set the Hamilton-Jacobi-Bellman (HJB) function in a linear quadratic form. In this way, the dynamic programming yielded an admissible continuous gain scheduling that was designed with regard to the hybrid dynamics of the system. We tuned the optimization parameters such that the tracking error and the average energy consumption are minimized. Due to linearization, we were able to examine the stability of the approximated periodic system achieved by the periodic gain according to Floquet's theory, by calculating the monodromy matrix of the closed-loop hybrid system. In addition to determining stability, the eigenvalues of this approximated monodromy matrix allowed us to evaluate the settling time of the system. This approach presents a direct method for optimal solution of locomotion control according to a given reference trajectory.

KEYWORDS: Bipedal locomotion; Linearization; Floquet theory; Dynamic programming; LQR; Periodic linear system.

1. Introduction

Human bipedal locomotion is a highly efficient process as demonstrated by McGeer's famous *passive walker*.[1] A substantial part of the power required for locomotion is generated from inertia and gravity and only a small amount of actuation power is needed to propel the body forward. Although walking is natural, bipedal locomotion is a complicated problem for control design. This is mainly due to the biped system being a nonlinear system characterized by an equilibrium trajectory, meaning that biped locomotion stability is achieved dynamically. These phenomena make the common control design methods of *LTI systems* irrelevant and force us to look for more advanced solutions. The known approaches can be divided into several types: *quasi-static* approaches and the *dynamic* approaches. The *quasi-static* approach is such that a set of equilibrium equations is solved for every time-step to find appropriate control inputs for a specific configuration. This method ensures stability since it relies on an equilibrium. However, it also results in an unnatural and energy wasteful gait. A more advanced solution, which also relies on an equilibrium principle, is Vukobratović's *zero moment point* (*ZMP*)[2]that also takes the inertia forces into account to stabilize the robot.

The *dynamic* approach obtains a more natural gait, because like human locomotion, it relies on a dynamic stability. Using this approach for gaining stable locomotion, it is important to design a controller that yields an accurate control signal. When an appropriate controller is obtained, it creates a more natural locomotion. This approach's main obstacle is the non-linearity of the model describing the system's dynamics, thus requiring a *nonlinear control* solution. Various solutions were suggested for obtaining an appropriate control for dynamic nonlinear biped systems. The classic control approach

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is to force the biped to track on a desired trajectory. Chevallereau *et al.*[3]introduced a control for a point-feet biped that tracks over an optimal trajectory[4] for a particular biped. According to their suggestion, the tracking is not on the reference motion but only on the geometric evolution, while the time scaling parameter, defined as the sequence of the biped configurations, is controlled. An advanced solution for a similar point-feet biped was suggested by Plestan *et al.*[5] A *linearization via feedback* associated with a set of *virtual constraints* enables an *asymptotically stable* locomotion for a point-feet under-actuated biped. Stabilization is achieved using a definition of angular momentum about the stance leg tip as a *Lyapunov function*, and it was numerically examined with the use of *Poincaré map*. This approach yields a fine independent stable walking, due to the virtual constraints defining the stability and the desired functionality of the robot. Another approach was presented by Braun and Goldfarb.[6] The desired *generalized forces*' vector is considered as a *linear PD feedback* that operates according to the functionality expected from the system (e.g. torso angle, maximum knee extension, etc.). Locomotion of a seven-link biped was divided into four different states. Constant 'stiffness' and 'damping' coefficients of the feedback, and constant attraction points for the proportional control, were chosen for every state. In this way, a natural and efficient locomotion of a biped with feet is obtained.

Our approach is based on the classic *linearization* method. The advantage of using the linearization approach, compared to the approaches mentioned above, is that it simplifies the control design. Linearization is the immediate choice when handling nonlinear systems since it allows us to handle nonlinear systems with *linear systems* tools. While in most cases, linearization is performed about an isolated operating point, in the current development, linearization was performed about a nominal stable trajectory. If the trajectory depends explicitly on time then the new system can be recognized as a *linear time-variant system*. Moreover, if the trajectory is periodic, the linearized system can be recognized as a *linear periodic system*. This recognition eases the development of the feedback control and the stability analysis of the closed loop system. The control design, however, is not simple since the dynamic model is a hybrid, i.e. a combination of continuous and instantaneous phases. Furthermore, part of the walking cycle (the impact phase) is not directly controlled. Therefore, the feedback of the controlled phase needs to be designed in a manner that the uncontrolled phase can be successfully managed by the close-loop system and that the system will stay periodically stable. To accomplish this goal, we used methods from the *optimal control* field.

An optimal feedback, based on *Bellman's dynamic programming,* was designed to obtain minimum tracking error according to a human reference trajectory. One major problem with dynamic programming is that the *Hamilton-Jacobi-Bellman (HJB) function* is unknown for the majority of the systems. In order to manage this problem, Jacobson[7] designed an algorithm established on *differential dynamic programming* that can numerically find the neighboring optimality. Using Jacobson's algorithm, Liu *et al.*[8] designed a periodic optimal control and applied it on a bipedal robot system. Todorov *et al.*[9] suggested a local dynamic programming that iteratively finds the neighboring optimality. In contrast, our linearization allows us to define the HJB function in a linear quadratic form. The dynamic programming is associated with a *linear quadratic regulator (LQR)* as a minimization criterion which enables to analytically obtain a continuous optimal *gain scheduling*. With the recognition of the system as a linear periodic one, the optimization can also be preformed for a single walking cycle, such that the optimal control will be an appropriate periodic control with no dependence on the number of steps desired. However, the major problem is that continuous LQR is not designed for hybrid systems, and therefore, cannot handle the instantaneous changes that occur when the swing leg hits the ground. This problem does not enable periodic stability. To obtain the desired periodic stability, we have suggested here a simple fix for the LQR's terminal condition so that the system will not diverge after impact, but will be maintained in proximity to the desired trajectory. This fix is possible due to the linearized hybrid model, and yields a stable *periodic gain scheduling*, designed with regard to the hybrid dynamics of the system.

Another advantage of using linearization and the linear-periodic recognition is that the stability of the periodic gain can be analytically examined off-line using *Floquet's theory*. Unlike examining the discrete Poincaré map used by Plestan *et al.*[5] and others[3,10] during simulation, with our method we calculate the *monodromy matrix* of the close-loop linear hybrid system for a given feedback, due to the linearized hybrid model. This allows us to verify the stability of the system off-line. Moreover, this method also allows for the evaluation of the system's settling time, according to the dominant eigenvalue of the monodromy matrix.

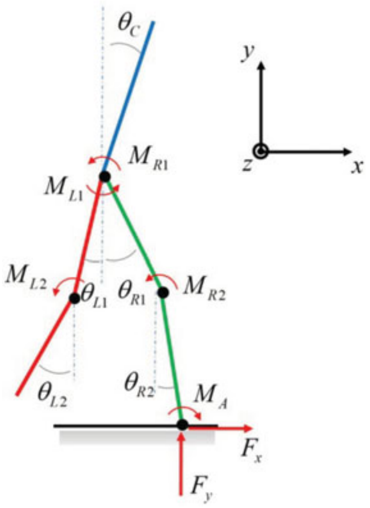

Fig. 1. Planar bipedal model. Angles θ_C, θ_{R1}, θ_{R2}, θ_{L1} and θ_{L2} are the angles of the torso, right thigh, right shank, left thigh and left shank, respectively. M_{R1}, M_{R2}, M_{L1}, M_{L2} and M_A are the torques at the right hip, right knee, left hip, left knee and right ankle, respectively. Forces F_x and F_y are the friction force and the normal reaction exerted on the biped's body by the walking surface, respectively.

2. Bipedal Model

The biped's structure is a five-link planar structure (Fig. 1) which includes a torso, two thighs and two shanks. All links are attached by revolute joints, and actuators are placed at all joints. Two supplementary actuators are placed at the legs tips, and act when the leg touches the ground. These two actuators simulate an actuated ankle during the contact with the ground, where the mass-less foot stays horizontal ('flat-foot' state) throughout the entire walking cycle. Neglecting the feet's mass and tilt, yields a point feet model that is fully-actuated. These assumptions were used to simplify the model and examination of the suggested control law, avoiding the problem of under-actuation.

We assume that at the moment the swing leg touches the ground the stance leg lifts off, so there is no double support phase. Thus, the biped's locomotion is a combination of two phases: a single support phase and an impact phase. This division of the point-feet locomotion is suggested in previous works.[3,5]

2.1. *Single support model*

Single support phase is where one leg touches the ground and supports the body, while the other leg swings forward. Assuming that during single support phase, the stance leg's tip does not rebound or slip, we can say that the contact point acts as a pivot. This assumption yields another revolute constraint and reduces the system by two Cartesian DOF. Hence, the generalized coordinates' vector in this case is $q = [\theta_C\ \theta_{R1}\ \theta_{L1}\ \theta_{R2}\ \theta_{L2}]^T$. According to this generalized coordinate vector and by using *Lagrange dynamics*,[11] we obtain the nonlinear model

$$A(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu \tag{1}$$

where $A(q)$ is the inertia matrix, $C(q,\dot{q})$ is the Coriolis matrix and $G(q)$ is the vector of gravity forces. B is a constant matrix that maps the actuators torques' vector $u(t)$ to the generalized forces space. This set of differential equations can be presented in a state space presentation

$$\dot{x}(t) = f(x(t)) + g(x(t)) \cdot u = \begin{bmatrix} \dot{q} \\ -A^{-1}(q)\,[C(q,\dot{q})\dot{q} + G(q)] \end{bmatrix} + \begin{bmatrix} 0 \\ A^{-1}(q)B \end{bmatrix} u \tag{2}$$

where $x(t)$ is the state vector $x^T(t) = \left[\, q^T(t)\ \dot{q}^T(t) \,\right] \in R^n$.

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2.2. Impact model
Impact occurs when the swing leg strikes the ground. We assume that the impact is instantaneous and there is no rebounding or slipping of the colliding leg during the impact. An additional assumption is that since impact is instantaneous, the configuration does not change during impact, but the velocities can change instantaneously. Using these assumptions, the change in the *conjugate angular momentum* may be used to describe the dynamics of the impact.[5,12] Using the unconstrained model (i.e. without the pivot constrain) while assuming that actuators cannot generate impulse torques, the angular momentum is given by

$$A_u(q_u^+)\dot{q}_u^+ - A_u(q_u^-)\dot{q}_u^- = B_F(q_u^-)F_G^{\text{imp}} \tag{3}$$

where q_u^- and q_u^+ denotes the configurations before and after impact, respectively. A_u is the inertia matrix of the unconstrained model, and B_F is a matrix that maps the impact forces vector F_G^{imp} to the unconstrained generalized coordinates space. Since the configuration does not change during impact, we can substitute $q_u^+ = q_u^-$ in Eq. (3). Another equation can be obtained from the no rebounding and slipping assumption

$$B_F^T(q_u^-)\dot{q}_u^+ = 0 \tag{4}$$

From this assumption we may also express the generalized coordinates q_u in terms of the generalized coordinates q of the single support model $q_u^T = \left[\, Z^T(q)\ q^T \,\right]$, where $Z(q)$ is the geometric reference between the impact point and the hip.

$$Z(q) = \begin{bmatrix} x_G - l_2\sin\theta_{L2} - l_1\sin\theta_{L1} \\ y_G + l_2\cos\theta_{L2} + l_1\cos\theta_{L1} \end{bmatrix} \tag{5}$$

where l_1 and l_2 are, the thigh's and shank's lengths respectively, and (x_G, y_G) are the coordinates of the impact point. By combining Eqs. (3) and (4) we can obtain the impact model in state space and in terms of $x(t)$:

$$x(t_k^+) = h(x(t_k^-)) = \begin{bmatrix} q(t_k^-) \\ [0_{5\times 2} \quad I_{5\times 5}]\alpha \cdot \dot{q}(t_k^-) \end{bmatrix} \tag{6}$$

where

$$\alpha = \left[-A_u^{-1}B_F\left(B_F^T A_u^{-1} B_F\right)^{-1} B_F^T + I_{7\times 7}\right]\begin{bmatrix} \frac{\partial}{\partial q} Z(q^-) \\ I_{5\times 5} \end{bmatrix} \tag{7}$$

2.3. Hybrid model
Both models, single support and impact, constitute the *hybrid dynamic model*

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), & t_k < t < t_{k+1} \\ x(t_k^+) = h(x(t_k^-)), & k = 1, 2, \ldots \end{cases} \tag{8}$$

The *hybrid model* is a combination of a continuous model and a discrete-instantaneous model that describe the dynamics of the locomotion in its two phases.

3. Linearization

The hybrid model (7) is nonlinear and therefore we performed a linearization of the single support and impact models about a nominal equilibrium trajectory $x_e(t)$. By deriving the single support model (2) in deviations of $\Delta x(t) = x(t) - x_e(t)$, a linear model is obtained. Since $x_e(t)$ is a function of time and changes uniformly with respect to it, the linearized model can be classified as a *linear time-variant system*

$$\Delta\dot{x}(t) = \tilde{\mathbf{A}}(t)\Delta x(t) + \tilde{\mathbf{B}}(t)\Delta u(t) \tag{9}$$

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where $\tilde{\mathbf{A}}(t) = \partial f(x(t))/\partial x(t)|_{x(t)=x_e(t)}$ and $\tilde{\mathbf{B}}^T(t) = \left[0_{5\times 5}\ -B^T A^T(q_e)\right]$. The matrices of the linearized model are in bold and are additionally denoted by a tilde to distinguish them from the nonlinear model matrices. Furthermore, $x_e(t)$ is also periodic, such that $x_e(t) = x_e(t+T)$. We can therefore conclude that (8) is a *linear periodic system*, satisfying $\tilde{\mathbf{A}}(t) = \tilde{\mathbf{A}}(t+T)$, $\tilde{\mathbf{B}}(t) = \tilde{\mathbf{B}}(t+T)$.

In a similar fashion, linearization for the impact model can be obtained

$$\Delta x(t_k^+) = \tilde{\mathbf{H}}\Delta x(t_k^-) \tag{10}$$

where $\tilde{\mathbf{H}} = \partial h(x)/\partial x|_{x=x_e^-}$. For a symmetric constant velocity walking, the system (9) is a *linear time-invariant system*, such that $\tilde{\mathbf{H}}(kT) = \text{const}$, for all natural k.

As a result of the linearization of these two subsystems, the hybrid nonlinear model (7) is transformed to the following *linear hybrid model*

$$\begin{cases} \Delta\dot{x}(t) = \tilde{\mathbf{A}}(t)\Delta x(t) + \tilde{\mathbf{B}}(t)\Delta u(t), & t_k < t < t_{k+1} \\ \Delta x(t_k^+) = \tilde{\mathbf{H}}\Delta x(t_k^-), & k = 1, 2, \ldots \end{cases} \tag{11}$$

This approximated linear model is the model that will be used in the following development. Matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{H}}$ are nonsingular. There is a possibility that for certain postures of the robot singularity could happen, however, we avoided these possibilities for the reference trajectory.

4. Optimal Control Design

Defining the desired outputs of the system, the connection between the output vector $y(t)$ and the state vector $x(t)$ is given by $y(t) = \mathbf{C}x(t)$, where $\mathbf{C}$ is a constant coefficients matrix. The state vector contains the angular configuration and the angular velocities of the generalized coordinates. Generally, the demand is for a correct geometric evolution according to the prescribed reference trajectory $r(t)$, while the velocities are less important. However, due to the transfer at the impact moment, it is important that the velocities after impact will more or less correspond to the referenced post-impact initial conditions. Therefore, the velocities should also be considered as the desired outputs. Hence, matrix $\mathbf{C}$ is 10×10 identity matrix, and we can identify that $y(t) \equiv x(t)$.

Through optimization, we will try to minimize the tracking error $e(t)$ between the system's output $y(t)$ and the reference signal $r(t)$. According to the identity between $y(t)$ and $x(t)$, $e(t) = x(t) - r(t)$.

4.1. Linear quadratic regulator (LQR)

Kalman's Linear Quadratic Regulator (LQR)[13] enables us to regulate between values inside the vectors that should be minimized. The advantages of the linear quadratic form will be appreciated in the continuation when the optimization strategy will be developed. The optimization goal is to achieve minimum tracking error throughout the walking cycle. As previously stated, agreement between the pre-impact state and the post-impact state is necessary. Therefore, the LQR for a single step in the time interval $[t_0, t_f]$ is defined in a *finite-horizon* form

$$L = \frac{1}{2}e^T(t_f)P_f e(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[e^T(\tau)Qe(\tau) + u^T(\tau)Ru(\tau)\right]d\tau \tag{12}$$

where $P_f, Q \in R^{10\times 10}$ and $R \in R^{5\times 5}$ are diagonal constant coefficients matrices, defining the minimization 'weights' of the minimized variables. Minimization of the magnitude of the actuators' torques $u(t)$ is done to obtain minimum energy consumption and also to prevent an infinity solution.

However, the minimization criterion (11) in its current form does not ensure a stable periodic control for the hybrid system and therefore we suggesting the following fix. It is clear that t_f refers to the pre-impact moment t_k^-. Moreover, it can be said that the reference trajectory's and the nominal equilibrium trajectory's terminal entries, $r(t_f)$ and $x_e(t_f)$, are identical. Hence, according to this identity and by using the linearized impact model (9), it can be defined that $e(t_f) \equiv \Delta x(t_k^-) = \tilde{\mathbf{H}}^{-1}\Delta x(t_k^+)$. According to periodicity, the post-impact equilibrium state is the equilibrium's initial value $x_e(t_k^+) = x_e(t_0)$. We

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then can write the finite horizon criterion as

$$\frac{1}{2}e^T(t_f)P_f e(t_f) = \frac{1}{2}\left(x(t_k^+) - x_e(t_0)\right)^T \tilde{\mathbf{H}}^{-T} P_f \tilde{\mathbf{H}}^{-1}\left(x(t_k^+) - x_e(t_0)\right) \tag{13}$$

This expression will be used in the following *Riccati equation* development.

4.2. Dynamic programming

Bellman's dynamic programming method enables us to obtain a *periodic gain scheduling* such that it will generate the appropriate locomotion according to the reference trajectory. One problem in using dynamic programming is the *HJB function* which is mostly unknown. However, we can overcome this difficulty by using the linearized model.

According to Bellman's *principle of optimality*,[14] the HJB function is defined as the minimal *cost-to-go* with regard to the initial conditions. Derivations of the HJB function with respect to time yield the well known partial differential equation:

$$v_t^*(x^*(t), t) = -\min_{u(t)}\left[\mathcal{L}(x^*(t), u(t), t) + v_x^*(x^*(t), t)F(x^*(t), u(t), t)\right] \tag{14}$$

where $(^*)$ denotes the optimal value. As previously mentioned, the problem is that the HJB function $v(x(t), t)$ is unknown for most of the systems. However, due to Kalman's LQR, we may consider the HJB function in a *linear quadratic form*. Therefore, we can use the linearized single-support model (8). Consequently, the HJB function for the tracking problem is

$$v(x(t), t) = \frac{1}{2}x^T S(t)x + V^T(t)x \tag{15}$$

where $S(t) \in R^{n\times n}$ is a symmetric matrix, and $V(t) \in R^n$, both derived from the following *Riccati equation*. Performing derivations of Eq. (14) with respect to time and the state vector x and substituting them with the linear single support model in Eq. (13) we obtain:

$$\begin{aligned}\frac{1}{2}x^T\dot{S}(t)x + \dot{V}^T(t)x &= -\min_{u(t)}\left[\frac{1}{2}(x - r(t))^T Q(x - r(t))\right.\\ &\left.+\frac{1}{2}u^T(t)Ru(t) + \left[x^T S(t) + V^T(t)\right]\left[\tilde{\mathbf{A}}(t)\Delta x + \tilde{\mathbf{B}}(t)\Delta u(t)\right]\right]\end{aligned} \tag{16}$$

A minimum value is obtained by a derivation of Eq. (15) with respect to the control signal $u(t)$ and equalizing it to zero. Isolating $u(t)$ from the obtained expression we obtain the optimal control expression

$$u(t) = K(t)x(t) - \tilde{r}(t) \tag{17}$$

where $K(t) = -R^{-1}\tilde{\mathbf{B}}^T(t)S(t)$, represents the optimal scheduled gain, and $\tilde{r}(t) = R^{-1}\tilde{\mathbf{B}}^T(t)V(t)$, represents the optimal reference vector. By substituting Eq. (16) in Eq. (15) and by comparing coefficients x and $x^T x$, we obtain a set of *Riccati equations*:

$$\begin{aligned}\dot{S}(t) &= S(t)\tilde{\mathbf{B}}(t)R^{-1}\tilde{\mathbf{B}}^T(t)S(t) - S(t)\tilde{\mathbf{A}}(t) - \tilde{\mathbf{A}}^T(t)S(t) - Q,\ S(t_f) = \tilde{\mathbf{H}}^{-T}P_f\tilde{\mathbf{H}}^{-1}\\ \dot{V}(t) &= \left(\tilde{\mathbf{B}}(t)R^{-1}\tilde{\mathbf{B}}^T(t)S(t) - \tilde{\mathbf{A}}(t)\right)^T V(t) + Qx_e(t),\ V(t_f) = \tilde{\mathbf{H}}^{-T}P_f x_e(t_0)\end{aligned} \tag{18}$$

Solving these equations yields the solution for the optimal control signal $u(t)$.

Although $\tilde{\mathbf{A}}(t)$ and $\tilde{\mathbf{B}}(t)$ are periodic, the Riccati equation's variables, $S(t)$ and $V(t)$, are not necessarily periodic. However we assume that the solution of the Riccati equation is periodic (or close to being periodic), such that the feedback control (16) is an appropriate optimal periodic control. This assumption will be more accurate when the settling time will be smaller than the period terminal time t_f. Simulations verified the reliability of this assumption.

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Table I. Biped's parameters. m_C, m_1 and m_2 are the masses of the torso and the thigh and shank limbs, respectively. The moments of inertia, I_C, I_1 and I_2, are calculated about the COMs of the torso, thigh and shank respectively, that are distant by r_C, r_1 and r_2, respectively, from a distal joint. l_1 and l_2 are the lengths of the thigh and shank limbs, respectively.

$m_C\ (kg)$	$m_1\ (kg)$	$m_2\ (kg)$	$I_C\ (kg \cdot m^2)$	$I_1\ (kg \cdot m^2)$	$I_2\ (kg \cdot m^2)$
50.4	8.85	3.45	1.596	0.199	0.047
$r_C\ (m)$	$r_1\ (m)$		$r_2\ (m)$	$l_1\ (m)$	$l_2\ (m)$
0.285	0.176		0.188	0.406	0.432

Table II. Optimization parameters.

Q^q	diag (1.5	7	13	5	2)					
R	diag ($3e-5$	$8e-6$	$5e-6$	$8e-5$	$1e-6$					
P_f	diag ($5e-4$	$5e-4$	$5e-4$	$9e-4$	$1e-5$	0	$6e-4$	$5e-6$	$6e-5$	$3e-5$)

5. Simulation and Results

In order to simulate walking, human locomotion measurements[15] were used as an equilibrium trajectory for the linearization and as the tracked reference trajectory. Only the evolutions of the thigh and shank limbs were considered since the feet are assumed to be not tilting. The torso was constrained to bend forward at an angle of 10 degrees. Although this is not the optimal trajectory for the considered model, it is appropriate enough to use for examining the suggested control law. Since it is a stable human trajectory with similar characteristics, we assumed that the control law is robust enough to cope with the small deviations from the optimal trajectory. The robot's parameters are such that they have the same characteristics as a human which was taken from one of the reference trajectories.[16] The parameters taken are presented in Table I. We used human measurements as a reference trajectory under the assumption that the right parameter approximations will achieve a stable trajectory although, not an optimal one.

To obtain appropriate results, it was important to choose the values of the optimization parameters properly. Since we use a regulator, any value that will give the desired performance in one aspect may violate it in another. For example, a high preference for a minimal tracking error will cause a higher power demand to maintain the error in the minimum value. However, it is not necessarily true in the multibody system where sometimes reducing or increasing a coefficient will achieve something opposite than expected, for better or for worse. Therefore, searching for optimal coefficients is done not only intuitively but also by trial and error. The results presented are for a single set of parameters that after several attempts yielded the desired results. However, these suggested parameters are not necessarily the optimal ones. It is possible that other values in the large space of the optimization parameters may result in better results.

Since the velocity tracking throughout the single support phase is not desired, the parameters in Q that multiply the velocities in $\Delta x(t)$ are zeroed. Hence, only the diagonal sub-matrix (denoted by Q^q) of the first five components on the diagonal of Q are tuned, while the last five components are zeroed. The orders of magnitude of the optimization parameters were determined according to the minimization preferences. In general, all tracking error needed to be zeroed while the torques should have some magnitude. Therefore, the R elements get a lower order of magnitude than the Q^q and P_f elements. Inside Q^q, the parameter of the torso's angle gets a lower value since the torso needs to maintain its position only in the neighborhood of the constrained angle. After this first determination, the values were tuned to improve the results.

While simulating the walking, it was important to ensure that the preliminary assumptions are not violated. Therefore, during the simulation, the program verified that the terms $F_y > 0$ and $\left|F_x/F_y\right| < \mu$ were fulfilled. If any of these terms were violated, the simulation was ordered to stop, and the parameters were then tuned.

The set of parameters determined after several attempts is presented in Table II. Running simulations with these values yielded the fine walking process, described in Fig. 2 by the configuration flow.

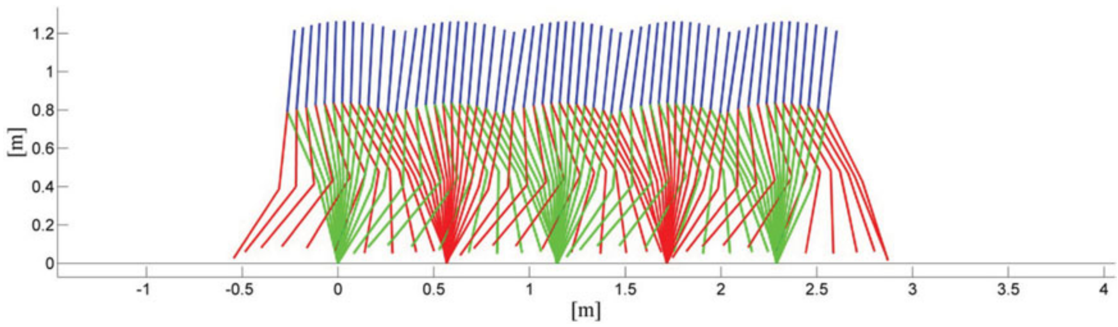

Fig. 2. Configuration flow of five steps.

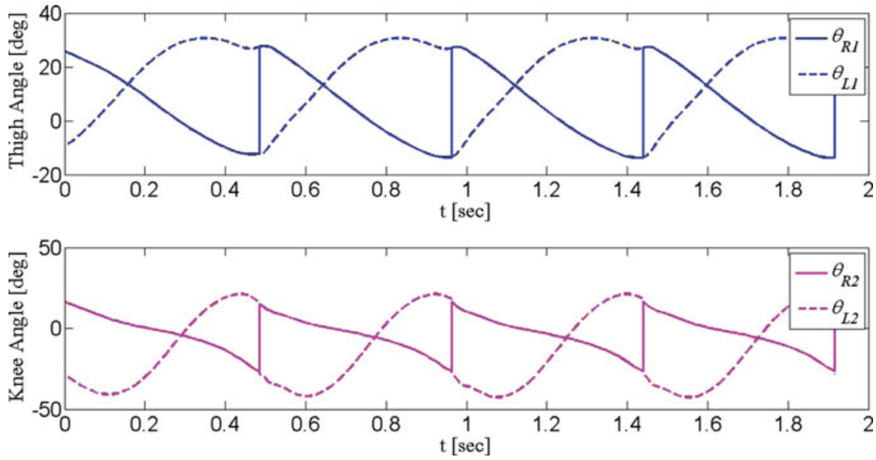

Fig. 3. Thigh and shank angles for the first four steps. Solid lines represent the swing leg's elements while dashed lines represent the stance leg elements. The assumption of symmetric locomotion allowed using the same model for both legs while switching initial conditions between legs after impact.

Figure 3 represents the angles of the thigh and the shank throughout the first four steps. The solid lines represent the stance leg's elements while the dashed lines represent the swing leg's elements. Symmetric walking assumption enables us to use the same model for every step only by replacing the initial conditions between the legs after impact. The continuity between the solid and dashed lines at the impact moments can be clearly seen. The results of the tracking square error can be used to observe the behavior of the system at steady state. It will be more convenient to examine the system's global behavior by presenting the sums of square errors for every cycle - $\bar{e}_k^2 = \int_{t_k}^{t_{k+1}} e^2(\tau)d\tau$. Figure 4 shows the summed square errors over 10 steps, where it can be seen that after five steps the summed square errors of all elements are constant, and the system reaches a steady state. In the sense of *nonlinear systems*, this is the stage where the close-loop system gets its *limit cycle*.

Figures 5–7 show the torques generated by the actuators. The graph shown in Fig. 7 represents only the stance ankle torque. The swing ankle torque is zero since the actuator is placed on the open edge of the swing leg, and therefore does not work in this stage.

The high initial value of the ankle torque is caused by the initial conditions which were taken from the reference trajectory.[15]

6. Stability

By using an approximated linear model, stability analysis can be done using the *Floquet's principle*[17] for *linear periodic systems*. According to this principle, a linear periodic system is considered to be *asymptotically stable* if all the *monodromy matrix's* eigenvalues (also called *characteristic multipliers*), are inside the unit circle. The monodromy matrix for our particular hybrid linear system

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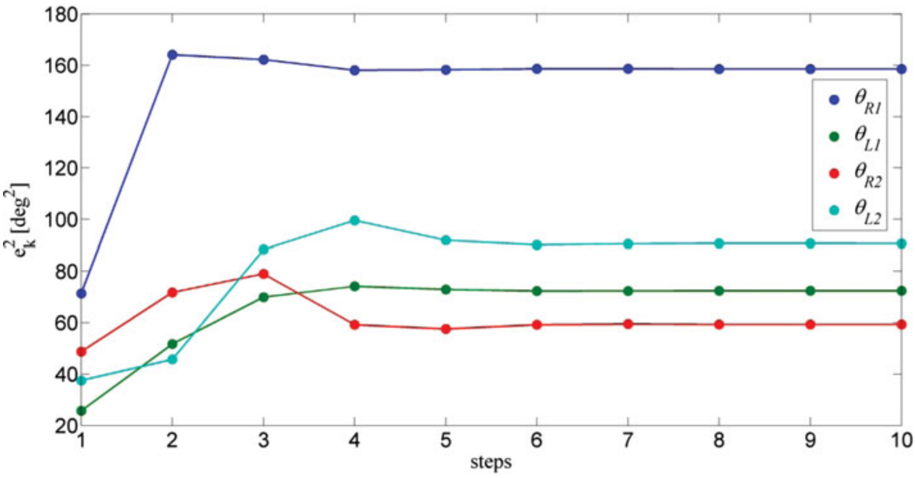

Fig. 4. Summed square errors over 10 steps.

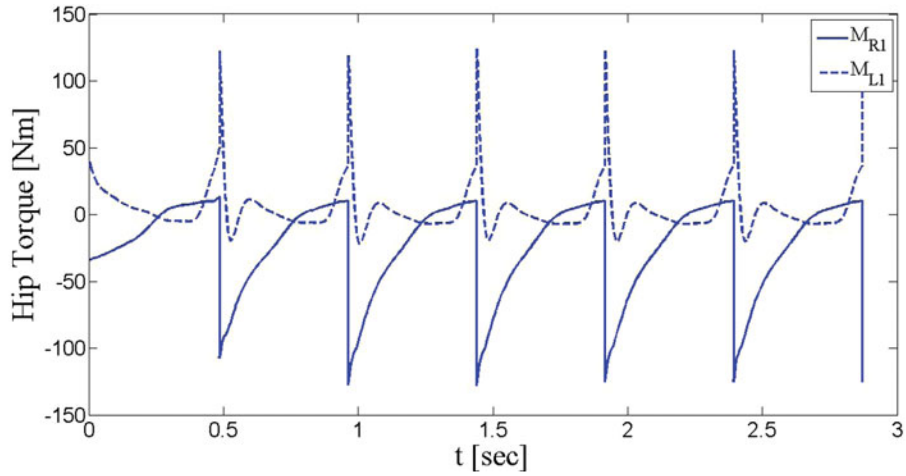

Fig. 5. Torques at the hips for the first six steps.

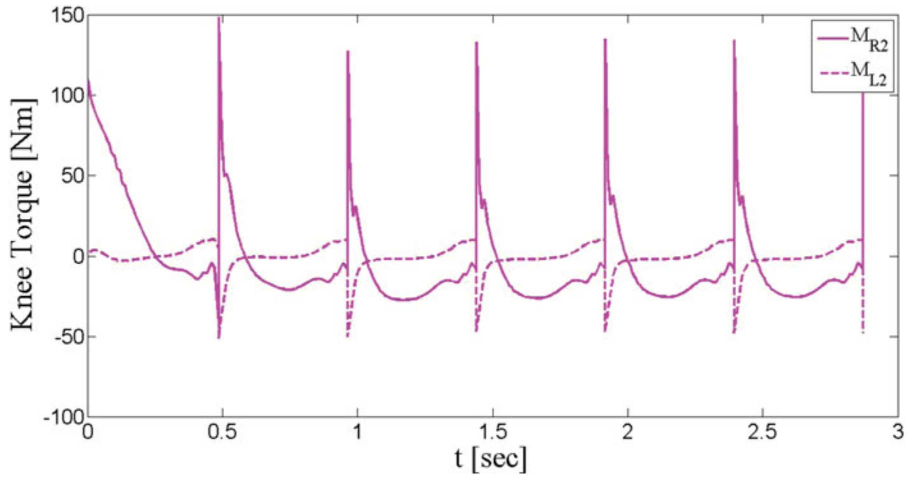

Fig. 6. Torques at the knees for the first six steps.

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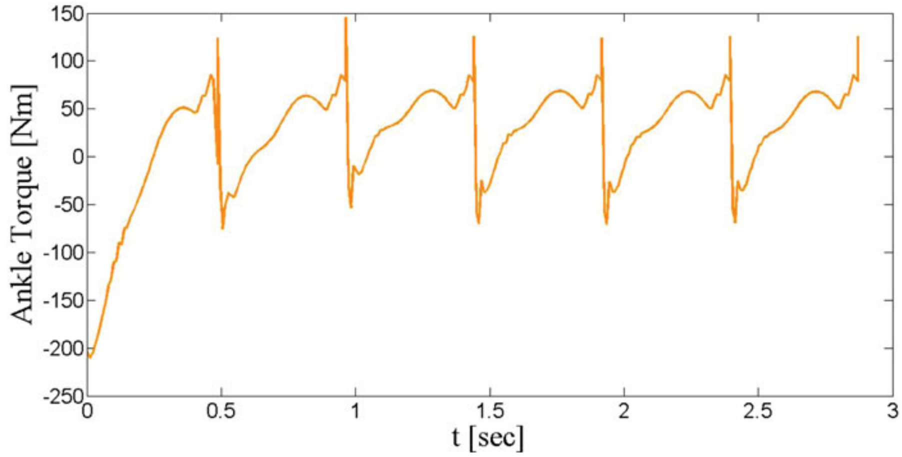

Fig. 7. Torques at the stance ankle for the first six steps.

is given by[18,19]

$$\Psi = \Phi_s(T, 0) \cdot \tilde{\mathbf{H}} \tag{19}$$

where $\Phi_s(T, 0)$ is the transition matrix of the close-loop system in its single support phase.

Calculation of the eigenvalues of this matrix can also be used to evaluate the settling time of the periodic system,[20] according to Ψ's dominant eigenvalue: $t_s \cong -5T/\ln(\max\{|\lambda_i|\})$; $i = 1, \ldots, n$.

For the parameters presented in Table II, the evaluated settling time was 3.1 seconds, whereas in Fig. 4 it seems that steady state is achieved after five cycles that correspond to about 2.5 seconds. However, the evaluation refers only to the approximated linear system. Moreover, because Fig. 1 refers only to the discrete times, it is very likely that the steady state is achieved somewhere between 2.5 and 3 seconds. Therefore, the inaccuracy of the settling time evaluation is reasonable and it can be considered to be in good agreement with the results.[21]

7. Discussion

We have shown that a linearization approach can be useful for the control design of dynamical hybrid systems, and a periodic feedback can be analytically obtained according to a periodic linearized hybrid model. In addition, it enabled us to examine the stability of the hybrid dynamic system before conducting a simulation, by only examining the monodromy's eigenvalues that are calculated according to the designed output feedback. This stability analysis also allows us to predict the settling time. By choosing the LQR's optimization parameters carefully, we can obtain an admissible gain scheduling according to the reference trajectory. Moreover, except for the instantaneous changes at the cycles' edges, the control signals are continuous and do not suffer from 'bang-bang' effects as sometimes happens with nonlinear systems.

However, the torques' magnitudes are slightly elevated as compared to those of a human with the same parameters. There are some possible explanations for these results. It is clear that since the reference trajectory is taken from human measurements it does not constitute an optimal trajectory for a point feet biped with the parameters taken. Hence, there are some disagreements between the natural dynamics of the model and its constrained trajectory. Another explanation is that the optimal feedback was obtained according to a linear approximation. Since the feedback was eventually activated on a nonlinear system, its response may be different than that of a linear system's response. Considering these circumstances, the results are very reasonable.

More satisfying results may be obtained by planning the optimal trajectory for the current model and using it as a reference trajectory. Alternatively, a complete dynamic model that includes feet and refers to all of the characteristics of the human's locomotion may be determined. This kind of model will be more in tune with the human reference trajectory and will reduce the torques generated by

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the actuators. By applying the methods used in this paper, more optimal results should be obtained. These suggestions will be considered in future studies.

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