Canadian Mathematical Bulletin
http://dx.doi.org/10.4153/CMB-2016-031-7

Canonical Systems of Basic Invariants for Unitary Reflection Groups

Norihiro Nakashima, Hiroaki Terao, and Shuhei Tsujie

Abstract. It is known that there exists a canonical system for every finite real reflection group. In a previous paper, the first and the third authors obtained an explicit formula for a canonical system. In this article, we first define canonical systems for the finite unitary reflection groups, and then prove their existence. Our proof does not depend on the classification of unitary reflection groups. Furthermore, we give an explicit formula for a canonical system for every unitary reflection group.

1 Introduction

Let V be an n-dimensional unitary space and let $W \subseteq U(V)$ be a finite unitary reflection group. Each reflection fixes a hyperplane in V pointwise. Let S denote the symmetric algebra $S(V^*)$ of the dual space V^*, and S_k the vector space consisting of homogeneous polynomials of degree k together with the zero polynomial. Then W acts contravariantly on S by $(w \cdot f)(v) = f(w^{-1} \cdot v)$ for $v \in V$, $w \in W$, and $f \in S$. The action of W on S preserves the degree of homogeneous polynomials, and W also acts on S_k. The subalgebra $R = S^W$ of W-invariant polynomials of S is generated by n algebraically independent homogeneous polynomials by Chevalley [2]. A system of such generators is called a system of *basic invariants* of R.

Let $x_1, \dots, x_n$ be an orthonormal basis for V^* and let $\partial_1, \dots, \partial_n$ be the basis for V^{**} dual to $x_1, \dots, x_n$. The symmetric algebra of V^{**} acts naturally on S as differential operators (*e.g.,* Kane [8, §25-2]). Let $\overline{c}$ denote the complex conjugate of $c \in \mathbb{C}$. For $f = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} x^{\boldsymbol{a}} \in S$, a differential operator f^* is defined by

$$f^* := \overline{f}(\partial) := \sum_{\boldsymbol{a}} \overline{c_{\boldsymbol{a}}} \partial^{\boldsymbol{a}},$$

where $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$, $c_{\boldsymbol{a}} \in \mathbb{C}$, $x^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n}$, and $\partial^{\boldsymbol{a}} = \partial_1^{a_1} \cdots \partial_n^{a_n}$. Note that $(cf)^* = \overline{c} f^*$, $w \cdot (f^*) = (w \cdot f)^*$ and $w \cdot (f^* g) = (w \cdot f)^* (w \cdot g)$ for $c \in \mathbb{C}$, $w \in W$, $f, g \in S$.

Flatto and Wiener introduced canonical systems to solve a mean value problem related to vertices for polytopes in [3–5]. They proved that there exists a canonical system for every finite real reflection group. Later, Iwasaki [7] gave a new definition of the canonical system as well as explicit formulas for canonical systems for some types of reflection groups. The first and the third authors, in their previous work [9], obtained an explicit formula for a canonical system for every reflection group. In this

Received by the editors September 22, 2014; revised May 8, 2016.
Published electronically June 1, 2016.
AMS subject classification: **13A50**, 20F55.
Keywords: basic invariant, invariant theory, finite unitary reflection group.

article, we extend the definition of canonical system to the finite *unitary* reflection groups as follows.

Definition 1.1 A system $\{f_1,\dots,f_n\}$ of basic invariants is said to be *canonical* if it satisfies the following system of partial differential equations:

$$f_i^* f_j = \delta_{ij}$$

for $i, j = 1,\dots,n$, where δ_{ij} is the Kronecker delta.

Our main result is the following existence theorem.

Theorem 1.2 *There exists a canonical system for every finite unitary reflection group.*

Our proof of Theorem 1.2 is classification free. Furthermore, we give an explicit formula (Theorem 4.3) for a canonical system that is also classification free. This formula is the same as one obtained in [9] for the real case, and we improve the proof.

The organization of this article is as follows. In Section 2, we introduce Lemma 2.2, which will play an important role in Section 3 when we prove Theorem 1.2. In Section 4, we give an explicit formula for a canonical system.

2 Basic Invariants

Let R_+ be the ideal of R generated by homogeneous elements of positive degrees and let $I = SR_+$ be the ideal of S generated by R_+. We define a unitary inner product $\langle\cdot,\cdot\rangle\colon S\times S\to\mathbb{C}$ by

$$\langle f,g\rangle = f^* g|_{x=0} = \overline{f}(\partial)g|_{x=0} \qquad (f,g\in S), \tag{2.1}$$

where $x=(x_1,\dots,x_n)$ and $\partial=(\partial_1,\dots,\partial_n)$. Let e_H be the order of the cyclic group generated by the reflection $s_H\in W$ corresponding to a reflecting hyperplane H. Fix $L_H\in V^*$ satisfying $\ker L_H = H$. Let Δ denote the product of $L_H^{e_H-1}$ as H runs over the set of all reflecting hyperplanes. Then Δ is skew-invariant, *i.e.*, $w\cdot\Delta = (\det w)\Delta$ for any $w\in W$. Set $\mathcal{H} := \{f^*\Delta \mid f\in S\}$. The following lemma was obtained by Steinberg [12].

Lemma 2.1 *Let $f\in S$ be a homogeneous polynomial. Then we have the following:*

(i) *$f\in I$ if and only if $f^*\Delta = 0$;*
(ii) *$g^* f = 0$ for all $g\in I$ if and only if $f\in\mathcal{H}$.*

It follows from Lemma 2.1(ii) that I is the orthogonal complement of $\mathcal{H}$ with respect to the inner product (2.1) degreewise.

In the sequel, we assume that W acts on V irreducibly. We fix a W-stable graded subspace U of S such that $S = I\oplus U$. It is known that the U is isomorphic to the regular representation of W (see Bourbaki [1, Chap. 5 §5 Theorem 2]). Hence the multiplicity of V in U is equal to $\dim_{\mathbb{C}} V = n$. Let $\pi\colon S\to U$ be the second projection with respect to the decomposition $S = I\oplus U$. Then π is a W-homomorphism. Let $\{h_1,\dots,h_n\}$ be a system of basic invariants with $\deg h_1\le\cdots\le\deg h_n$. The multiset of degrees $m_i := \deg h_i$ $(i=1,\dots,n)$ does not depend on a choice of basic invariants.

There exists a unique linear map $d\colon S \to S \otimes_{\mathbb{C}} V^*$ satisfying $d(fg) = fd(g) + gd(f)$ for $f, g \in S$ and $dL := 1 \otimes L \in \mathbb{C} \otimes_{\mathbb{C}} V^*$ for $L \in V^*$. The map d is called the *differential map*. The differential 1-form dh is expressed as

$$dh = \sum_{j=1}^{n} \partial_j h \otimes x_j = \sum_{j=1}^{n} (\partial_j h) dx_j$$

for $h \in S$. Note that dh is invariant if h is invariant. Define a W-homomorphism

$$\varepsilon\colon (S \otimes_{\mathbb{C}} V^*)^W \longrightarrow R_+$$

by

$$\varepsilon\Big(\sum_{j=1}^{n} h_j dx_j\Big) = \sum_{j=1}^{n} x_j h_j.$$

Then $\varepsilon \circ d(h) = (\deg h)h$ for any homogeneous polynomial h. The projection $\pi\colon S \to U$ is extended to a W-homomorphism $\widetilde{\pi}\colon (S \otimes V^*)^W \to (U \otimes V^*)^W$ defined by $\widetilde{\pi}(\sum_{j=1}^{n} g_j \otimes x_j) := \sum_{j=1}^{n} \pi(g_j) \otimes x_j$.

Lemma 2.2 *Let $\{h_1, \dots, h_n\}$ be a system of basic invariants. Put $f_i := (\varepsilon \circ \widetilde{\pi})(dh_i)$, and $\{f_1, \dots, f_n\}$ is a system of basic invariants.*

Proof Since $h_1, \dots, h_n$ are invariants, so are the 1-forms $dh_1, \dots, dh_n$. Thus, each f_i is invariant, because both ε and $\widetilde{\pi}$ are W-homomorphisms.

Next we prove that $\{f_1, \dots, f_n\}$ is a system of basic invariants. Define $f_{ij} := \pi(\partial_j f_i)$. Then $f_i = \sum_{j=1}^{n} x_j f_{ij}$. For $j = 1, \dots, n$, we express $\partial_j h_i = f_{ij} + r_{ij}$ for some $r_{ij} \in I$. Then $r_{ij} = \sum_{k=1}^{\ell} h_k g_{ijk}$ for some $g_{ijk} \in S$. Put $r_i := \sum_{j=1}^{n} x_j r_{ij}$ for $i = 1, \dots, n$. Then we have

$$m_i h_i = \sum_{j=1}^{n} x_j \partial_j h_i = f_i + r_i.$$

Since f_i is invariant, the polynomial $r_i = m_i h_i - f_i$ is also invariant. This implies

$$r_i = r_i^\sharp = \sum_{k=1}^{\ell} \Big(\sum_{j=1}^{n} x_j g_{i,j,k}\Big)^\sharp h_k \in I^2 \cap R,$$

where $\sharp$ denotes the averaging operator, *i.e.*,

$$f^\sharp = \frac{1}{|W|} \sum_{w \in W} w \cdot f$$

for $f \in S$. Thus, we have $\partial_j r_i \in I$ for $i, j \in \{1, \dots, n\}$. Let $J(g_1, \dots, g_n)$ denote the Jacobian for $g_1, \dots, g_n \in S$. Then

$$\begin{aligned} J(m_1 h_1, \dots, m_n h_n) &= \det[\partial_j h_i]_{i,j} = \det[\partial_j f_i + \partial_j r_i]_{i,j} \\ &\equiv \det[\partial_j f_i]_{i,j} = J(f_1, \dots, f_n) \pmod{I}. \end{aligned}$$

It immediately follows that $\Delta \notin I$ by Lemma 2.1(i). Since $J(h_1, \dots, h_n)$ is a nonzero constant multiple of Δ, we obtain $J(f_1, \dots, f_n) \notin I$, and thus $J(f_1, \dots, f_n) \neq 0$. By the Jacobian criterion (*e.g.*, [6, Proposition 3.10]), $\{f_1, \dots, f_n\}$ is algebraically independent. Therefore, $\{f_1, \dots, f_n\}$ is a system of basic invariants because $\deg f_i = \deg h_i$ for $i = 1, \dots, n$. ∎

Remark There exists a W-stable subspace U' of S such that $S = I \oplus U'$ and $df_1, \dots, df_n \in (U' \otimes_{\mathbb{C}} V^*)^W$. However, we do not know whether U' coincides with U or not. In Section 3, we see that U and U' coincide when $U = \mathcal{H}$.

3 Existence of a Canonical System

In this section, we prove Theorem 1.2, which is the existence theorem of a canonical system. The following lemma is widely known.

Lemma 3.1 *Let $g \in S$ be a homogeneous polynomial, and put $g_j := \partial_j g$ for $j = 1, \dots, n$. Then, for any $h \in S$, we have $g^*(x_j h) = x_j g^* h + g_j^* h$.*

Proof We only need to verify the assertion when g is a monomial. We verify it by induction on $\deg g$. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a multi-index with $|\boldsymbol{a}| = \deg g$. Then

$$\begin{aligned}\partial^{\boldsymbol{a}}(x_j h) &= \partial^{\boldsymbol{a}-\boldsymbol{e}_j}\partial_j(x_j h) = \partial^{\boldsymbol{a}-\boldsymbol{e}_j} h + \partial^{\boldsymbol{a}-\boldsymbol{e}_j}(x_j \partial_j h)\\ &= \partial^{\boldsymbol{a}-\boldsymbol{e}_j} h + \left(x_j \partial^{\boldsymbol{a}-\boldsymbol{e}_j}\partial_j h + (a_j - 1)\partial^{\boldsymbol{a}-2\boldsymbol{e}_j}\partial_j h\right)\\ &= x_j \partial^{\boldsymbol{a}} h + a_j \partial^{\boldsymbol{a}-\boldsymbol{e}_j} h. \qquad \blacksquare\end{aligned}$$

By Lemma 2.1, I is the orthogonal complement of $\mathcal{H}$ with respect to the inner product (2.1), and the W-stable graded space S is decomposed into the direct sum of the W-stable graded subspaces I and $\mathcal{H}$, *i.e.*, $S = I \oplus \mathcal{H}$. Let $\pi: S \to \mathcal{H}$ be the second projection with respect to the decomposition $S = I \oplus \mathcal{H}$. Let $h_1, \dots, h_n$ be an arbitrary system of basic invariants. Put $f_{ij} := \pi(\partial_j h_i)$ for $i, j = 1, \dots, n$, and $f_i := (\varepsilon \circ \widetilde{\pi})(dh_i) = \sum_{j=1}^n x_j f_{ij}$ for $i = 1, \dots, n$. Then $\{f_1, \dots, f_n\}$ is a system of basic invariants by Lemma 2.2. We are now ready to give a proof of Theorem 1.2.

Let $g \in R_+$ be a homogeneous invariant polynomial with $\deg g < m_i$. By using Lemmas 2.1 and 3.1 we obtain

$$g^* f_i = \sum_{j=1}^n g^*(x_j f_{ij}) = \sum_{j=1}^n (x_j g^* f_{ij} + g_j^* f_{ij}) = \sum_{j=1}^n g_j^* f_{ij} \in \mathcal{H}.$$

Meanwhile, $g^* f_i$ is an invariant polynomial of positive degree, since g and f_i are invariant and $\deg g < m_i$. Therefore, we have $g^* f_i \in \mathcal{H} \cap I = \{0\}$. In particular, when $g = f_j$ with $\deg f_j < m_i$, we have $f_j^* f_i = 0$. It immediately follows that $f_j^* f_i = 0$ if $\deg f_j > \deg f_i$. Applying the Gram–Schmidt orthogonalization with respect to the inner product (2.1), we obtain a canonical system of basic invariants. This completes our proof of Theorem 1.2.

The subspace spanned by a canonical system can be characterized as follows.

Proposition 3.2 *Let $\{f_1, \dots, f_n\}$ be a canonical system and $\mathcal{F} := \langle f_1, \dots, f_n\rangle_{\mathbb{C}}$. Then*

(i) $\mathcal{F} = \bigoplus_{k=1}^{\infty}\{f \in R_k \mid g^* f = 0 \text{ for } g \in R_\ell \text{ with } 0 < \ell < k\}$, *where* $R_k := R \cap S_k$,

(ii) $\langle df_1, \dots, df_n\rangle_{\mathbb{C}} = (\mathcal{H} \otimes_{\mathbb{C}} V^*)^W$,

(iii) $\mathcal{F} = \varepsilon\left((\mathcal{H} \otimes_{\mathbb{C}} V^*)^W\right)$.

https://doi.org/10.4153/CMB-2016-031-7 Published online by Cambridge University Press

Proof Define

$$\mathcal{G} := \bigoplus_{k=1}^{\infty} \{ f \in R_k \mid g^* f = 0 \text{ for } g \in R_\ell \ \text{ with } 0 < \ell < k \}.$$

Let $f \in \mathcal{G}$ be a homogeneous polynomial. For any $g \in I$, it is not hard to see that $g^*(\partial_j f) = \partial_j(g^* f) = 0$. Thus we have $\partial_j f \in \mathcal{H}$ for $j = 1, \dots, n$ by Lemma 2.1. This implies $d(\mathcal{G}) \subseteq (\mathcal{H} \otimes_{\mathbb{C}} V^*)^W$. The inclusion $\mathcal{F} \subseteq \mathcal{G}$ follows immediately, because $\{f_1, \dots, f_n\}$ is a canonical system. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{d} & (\mathcal{H} \otimes_{\mathbb{C}} V^*)^W \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathcal{F} & \xrightarrow{d} & \langle df_1, \dots, df_n \rangle_{\mathbb{C}}. \end{array}$$

Since $\ker(d) = \mathbb{C}$ and $\mathcal{G}$ does not contain any nonzero constant, the horizontal maps d are both injective. Recall that V is an irreducible representation and that $\mathcal{H}$ affords the regular representation of W. Hence, we have $\dim(\mathcal{H} \otimes_{\mathbb{C}} V^*)^W = \dim V = n$, because $(\mathcal{H} \otimes_{\mathbb{C}} V^*)^W \simeq \operatorname{Hom}_W(V, \mathcal{H})$; this isomorphism can be found in [10] or the proof of [11, Lemma 6.45]. By comparing the dimensions, we have $\mathcal{F} = \mathcal{G}$, which is (i). Sending both sides of this equality by d, we obtain

$$\langle df_1, \dots, df_n \rangle_{\mathbb{C}} = d(\mathcal{F}) = d(\mathcal{G}) = (\mathcal{H} \otimes_{\mathbb{C}} V^*)^W, \tag{3.1}$$

so equality (ii) is proved. Moreover, we have

$$\langle f_1, \dots, f_n \rangle_{\mathbb{C}} = \mathcal{F} = \mathcal{G} = \varepsilon\big((\mathcal{H} \otimes_{\mathbb{C}} V^*)^W \big)$$

by applying ε to (3.1). This verifies (iii). ∎

4 An Explicit Construction of a Canonical System

The following is a key to our explicit formula for a canonical system.

Definition 4.1 (*cf.* [9]) Define a linear map $\phi\colon S \longrightarrow \mathcal{H}$ by $\phi(f) := (f^* \Delta)^* \Delta$ for $f \in S$. The map ϕ induces a W-homomorphism

$$\widetilde{\phi}\colon (S \otimes V^*)^W \longrightarrow (\mathcal{H} \otimes V^*)^W$$

defined by $\widetilde{\phi}(\sum f \otimes x) := \sum \phi(f) \otimes x$.

One has

$$\begin{aligned} w \cdot \phi(f) &= \big((w \cdot f)^* (w \cdot \Delta) \big)^* (w \cdot \Delta) = \big((w \cdot f)^* (\det(w) \Delta) \big)^* \big(\det(w) \Delta \big) \\ &= \overline{\det(w)} \det(w) \big((w \cdot f)^* \Delta \big)^* (\Delta) = \phi(w \cdot f) \end{aligned}$$

for $w \in W$ and $f \in S$. Therefore, ϕ is a W-homomorphism, and so is $\widetilde{\phi}$.

Let $\{h_1, \dots, h_n\}$ be an arbitrary system of basic invariants, and assume $\deg h_i = m_i$ for $i = 1, \dots, n$. Let $\{f_1, \dots, f_n\}$ be a canonical system with $\deg f_i = m_i$. We have already shown in Proposition 3.2(i) that $(\mathcal{H} \otimes V^*)^W = \langle df_1, \dots, df_n \rangle_{\mathbb{C}}$.

https://doi.org/10.4153/CMB-2016-031-7 Published online by Cambridge University Press

Lemma 4.2 *The restriction*

$$\widetilde{\phi}|_{\langle dh_1,\dots,dh_n\rangle_{\mathbb{C}}} : \langle dh_1,\dots,dh_n\rangle_{\mathbb{C}} \longrightarrow (\mathcal{H}\otimes V^*)^W = \langle df_1,\dots,df_n\rangle_{\mathbb{C}}$$

is isomorphic.

Proof It is enough to prove the injectivity. Fix $h \in \langle h_1,\dots,h_n\rangle_{\mathbb{C}}$ with $\widetilde{\phi}(dh) = 0$. It follows from Lemma 2.1 that $\ker\widetilde{\phi} = (I\otimes_{\mathbb{C}} V^*)^W$. Then we have $dh \in (I\otimes_{\mathbb{C}} V^*)^W$. At the same time, since $\{f_1,\dots,f_n\}$ is a system of basic invariants, we can write

$$h = \sum_{k=1}^{n} \lambda_k f_k + P,$$

where $\lambda_k \in \mathbb{C}$ and $P \in I^2 \cap R$. Then the 1-form dP lies in $(I\otimes_{\mathbb{C}} V^*)^W$, and $df_k \in (\mathcal{H}\otimes_{\mathbb{C}} V^*)^W$ for $k = 1,\dots,n$ by Proposition 3.2. Hence, we have

$$\sum_{k=1}^{n} \lambda_k df_k = dh - dP \in (\mathcal{H}\otimes_{\mathbb{C}} V^*)^W \cap (I\otimes_{\mathbb{C}} V^*)^W = \{0\}.$$

This implies $\lambda_k = 0$ for all $k = 1,\dots,n$, since $\{df_1,\dots,df_n\}$ is linearly independent over $\mathbb{C}$. Thus, we have $h = P \in I^2 \cap R$. The algebraic independence of $h_1,\dots,h_n$ implies $\langle h_1,\dots,h_n\rangle \cap I^2 = \{0\}$. Therefore, $h = 0$. ∎

The image of $\widetilde{\phi}|_{\langle dh_1,\dots,dh_n\rangle_{\mathbb{C}}}$ coincides with $\langle df_1,\dots,df_n\rangle_{\mathbb{C}}$ by Lemma 4.2. Therefore, we have a chain of the linear maps

$$\langle h_1,\dots,h_n\rangle_{\mathbb{C}} \xrightarrow{d} \langle dh_1,\dots,dh_n\rangle_{\mathbb{C}} \xrightarrow{\widetilde{\phi}} \langle df_1,\dots,df_n\rangle_{\mathbb{C}} \xrightarrow{\varepsilon} \langle f_1,\dots,f_n\rangle_{\mathbb{C}}. \tag{4.1}$$

The image of $\{h_1,\dots,h_n\}$ by the composition of all the maps in (4.1) forms a basis for $\langle f_1,\dots,f_n\rangle_{\mathbb{C}}$. Thus, we have the following explicit formula for a canonical system of basic invariants.

Theorem 4.3 (*cf.* [9]) *Let $h_1,\dots,h_n$ be an arbitrary system of basic invariants. Applying the Gram–Schmidt orthogonalization to*

$$\left\{ \varepsilon\circ\widetilde{\phi}(dh_i) = \sum_{j=1}^{n} x_j \phi(\partial_j h_i) \;\middle|\; i = 1,\dots,n \right\}$$

with respect to the inner product (2.1)*, we obtain a canonical system of basic invariants.*

Remark Theorem 4.3 asserts the same formula as [9, Theorem 3.4] for the real case. In [9], to prove the theorem, we showed the symmetricity of ϕ with respect to the inner product (2.1) and considered eigenvectors of $\widetilde{\phi}$. In contrast, the proof of this paper does not require these arguments.

References

[1] N. Bourbaki, *Groupes et Algèbres de Lie.* Chapitres 4, 5, et 6, Hermann, Paris, 1968.
[2] C. Chevalley, *Invariants of finite groups generated by reflections.* Amer. J. Math, 77(1955), no. 4, 778–782. http://dx.doi.org/10.2307/2372597
[3] L. Flatto, *Basic sets of invariants for finite reflection groups.* Bull. Amer. Math. Soc. 74(1968), no. 4, 730–734. http://dx.doi.org/10.1090/S0002-9904-1968-12017-8

[4] ———, *Invariants of finite reflection groups and mean value problems. II.* Amer. J. Math. **92**(1970), 552–561. http://dx.doi.org/10.2307/2373360
[5] L. Flatto and M. M. Wiener, *Invariants of finite reflection groups and mean value problems.* Amer. J. Math. **91**(1969), no. 3, 591–598. http://dx.doi.org/10.2307/2373340
[6] J. E. Humphreys, *Reflection groups and Coxeter groups.* Cambridge Studies in Mathematics, 29, Cambridge University Press, Cambridge, 1990.
[7] K. Iwasaki, *Basic invariants of finite reflection groups.* J. Algebra. **195**(1997), no. 2, 538–547. http://dx.doi.org/10.1006/jabr.1997.7066
[8] R. Kane, *Reflection groups and invariant theory.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 5, Springer-Verlag, New York, 2001. http://dx.doi.org/=10.1007/978-1-4757-3542-0
[9] N. Nakashima and S. Tsujie, *A canonical system of basic invariants of a finite reflection group.* J. Algebra **406**(2014), 143–153. http://dx.doi.org/10.1016/j.jalgebra.2014.02.012
[10] P. Orlik and L. Solomon, *Unitary reflection groups and cohomology.* Invent. Math. **59**(1980), no. 1, 77–94. http://dx.doi.org/10.1007/BF01390316
[11] P. Orlik and H. Terao, *Arrangements of hyperplanes.* Grundlehren dermatematischen Wissenschaften, 300, Springer-Verlag, Berlin, 1992.
[12] R. Steinberg, *Differential equations invariant under finite reflection groups.* Trans. Amer. Math. Soc. **112**(1964), no. 3, 392–400. http://dx.doi.org/10.1090/S0002-9947-1964-0167535-3

School of Information Environment, Tokyo Denki University, Inzai, 270-1382, Japan
e-mail: nakashima@mail.dendai.ac.jp

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
e-mail: hterao00@za3.so-net.ne.jp tsujie@math.sci.hokudai.ac.jp