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Pontrjagin duality and full completeness for multiplicative linear logic (without Mix)

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Dedicated to Jim Lambek on the occasion of his 75th birthday

We prove a full completeness theorem for MLL without the Mix rule. This is done by interpreting a proof as a dinatural transformation in a $*$-autonomous category of reflexive topological abelian groups first studied by Barr, denoted $\mathscr{RTA}$. In Section 2, we prove the *unique interpretation theorem* for a binary provable MLL-sequent. In Section 3, we prove a completeness theorem for binary sequents in MLL without the Mix rule, where we interpret formulas in the category $\mathscr{RTA}$. The theorem is proved by investigating the concrete structure of $\mathscr{RTA}$, especially that arising from Pontrjagin's work on duality.

1. Introduction

E. S. Bainbridge, P. J. Freyd, A. Scedrov and P. J. Scott (Bainbridge *et al.* 1990) first proposed the idea that dinatural transformations could provide a semantics for proofs in a logical system: given a category $\mathscr{C}$ in which there exist functors corresponding to the logical connectives, one must interpret a formula by a multivariant functor, since an atom can occur both positively and negatively in the syntax. In this setting, they discovered that a dinatural transformation between multivariant functors provides an interpretation of a proof. In spite of the peculiarity that dinatural transformations do not allow composition in general, dinatural transformations provide a powerful semantics for proofs. J-Y. Girard, A. Scedrov, and P. J. Scott (Girard *et al.* 1992) have shown that a dinatural interpretation in the framework of *cartesian closed categories* is sound with respect to cut-free proofs in Intuitionistic Logic. Developing the above results, R. F. Blute and P. J. Scott (Blute and Scott 1996) have proved a full completeness theorem for $MLL + Mix$ using a dinatural interpretation in the framework of a concrete $*$-autonomous category of reflexive topological vector spaces. The notion of a full completeness theorem was introduced by S. Abramsky and R. Jagadeesan (Abramsky and Jagadeesan 1994) in the framework of categorical logic. Blute and Scott call their fully complete semantics *Linear Läuchli Semantics*, since their methods are reminiscent of Läuchli's semantics (Läuchli 1970) for Intuitionistic Logic in the sense that the notion of 'invariance' plays an essential role in both semantics. Developing the methods of Blute and Scott, the author in Hamano (1998) proved a full completeness theorem using only the condition of *invariance* while dropping the requirement of dinaturality.

Recently, there have been several other full completeness results based on dinaturality. Pratt (Pratt 1997) obtained a full completeness theorem using the Chu construction. Tan (Tan 1997) obtained several such results using the method of *double gluing*, a generalization of Loader's linear logical predicates (Loader 1994). However, we would suggest that their methods reveal properties intrinsic to the Chu construction and to the gluing construction, respectively, rather than to dinaturality, given that these results rely on full completeness in the category of sets or in a compact closed category. Hence, natural questions arise as to whether dinaturality can provide a direct uniform method for obtaining the full completeness theorems in mathematically concrete and natural categories such as those in Barr (1976) and Barr (1977), which led Barr to establish the general theory of $*$-autonomous categories in Barr (1979). Of course the methods should be simpler and more abstract than those in Blute and Scott (1996), and the author's methods in Hamano (1998). In Section 2 of this paper we shall partly address this issue in the case of full completeness for a provable sequent. This is the *unique interpretation theorem.* We prove, using only category theoretic methods, that the dinatural interpretation of a proof for a binary sequent is unique provided that the tensor unit for the given $*$-autonomous category $\mathscr{C}$ is a generator (Theorem 2):

Unique interpretation theorem:

$$\text{A binary sequent } \vdash \Gamma \text{ is provable } \Rightarrow Dinat_{\mathscr{C}}(\ \vdash \Gamma) = < \pi^* >,$$

where $< \pi^* >$ denotes the module spanned by the interpretation π^* of the unique (modulo permutations of inferences) cut-free proof π of $M \vdash N$.

Our method does not depend on any concrete property of a given category. Our procedure for the proof follows the Hyland–Ong method (Hyland and Ong 1993), where the full completeness theorem for MLL without the Mix rule was proved in terms of a game semantics. First we shall prove the theorem for a semisimple binary sequent. Then, in the presence of the canonical natural transformations, called *weak distributivities*, in a $*$-autonomous category, the theorem for a binary sequent will be derived. When a given category is a Mix category (*cf.* Cockett and Seely (1997)), our proof can be considered as an alternative proof of Blute and Scott's unique interpretation theorem (*cf.* Lemma 10.2 of Blute and Scott (1996)).

Although the *Mix* rule is not contained in the original syntax rules for MLL, the category $\mathscr{RTVEC}$, which is the framework of the interpretation in Blute and Scott (1996) and Hamano (1998), satisfies the *Mix* rule. This means that this category does not provide a pure semantics for MLL-proofs although this category has very nice mathematical properties. Hence there arises a natural question as to whether there exists a naturally occurring $*$-autonomous category in which the dinatural interpretation provides the full completeness theorem for MLL without the Mix-rule. In Section 3, we shall answer this question by showing that the completeness theorem holds for Barr's $*$-autonomous category $\mathscr{RTA}$ (Barr 1977) of reflexive topological abelian groups. Combining this completeness theorem with the unique interpretation theorem in Section 2, we shall obtain the full completeness theorem (without the Mix rule) via a dinatural interpretation in $\mathscr{RTA}$. In Section 3, we shall first of all observe that Barr's $*$-autonomous category $\mathscr{RTA}$ of reflexive topological abelian groups does not satisfy the Mix rule, hence $\mathscr{RTA}$ could be a candidate for a complete model of linear logic without the Mix rule. M. Barr (Barr 1976) formulates this category by investigating the Pontrjagin duality theorem, which is one of the most well-known duality theorems:

For a locally compact abelian group G, let $G^{\perp}$ denote the character group of G: that is, $G^{\perp} := Hom(G, \mathbf{T})$ with $\mathbf{T} := \mathbf{R}/\mathbf{Z}$. Then the canonical map $G \to G^{\perp\perp}$ is an isomorphism.

Although Barr constructs $\mathscr{RTA}$ analogously to $\mathscr{RTVEC}$, there are differences between the two. This is due to the fact that the objects of $\mathscr{RTA}$ have more complicated structure than those of $\mathscr{RTVEC}$, and this fact makes the construction of $\mathscr{RTA}$ more difficult. And the most basic difference for our purpose, that is, as a model of linear logic, is the following:

In $\mathscr{RTA}$, the tensor unit $\mathbf{Z}$ (the additive group of integers), which is the interpretation of the MLL-constant $\mathbf{1}$, and the dualizing object $\mathbf{T}$ (the torus group), which is the interpretation of the MLL-constant $\perp$, do not coincide.

Moreover, there exists no morphism except 0 from $\mathbf{T}$ to $\mathbf{Z}$. Thus, in linear logic terminology, $\otimes$ does not imply $\invamp$, or equivalently $\mathscr{RTA}$ does not satisfy the Mix rule.

In contrast with the unique interpretation theorem of Section 2, which is derived by only using a category theoretical argument, the classical completeness theorem requires more mathematically concrete notions. This is because for the completeness theorem we have to construct a concrete counter-model that does not validate the unprovable sequent. This is also the case in Blute and Scott (1996). The crucial lemma for their result was that $V \otimes V^{\perp}$ has no fixed point, except 0, under arbitrary automorphisms of V (*cf.* Lemma 9.2 of Blute and Scott (1996)).

The fundamental structure used in Section 3 is the duality theorem between the category $\mathscr{CA}$ of compact abelian groups and the category $\mathscr{DA}$ of discrete abelian groups: $(\ )^{\perp}$ determines a contravariant functor between $\mathscr{CA}$ and $\mathscr{DA}$ that $(\ )^{\perp\perp}$ becomes naturally equivalent to the identity functor on each of $\mathscr{CA}$ and $\mathscr{DA}$. We also have that for $\prod_I \mathbf{T} \in \mathscr{CA}$ and $\sum_I \mathbf{Z} \in \mathscr{DA}$, which are dual to each other, $\prod_I \mathbf{T} \otimes (\prod_I \mathbf{T})^{\perp} \cong (\sum_I \mathbf{Z})^{\perp} \otimes \sum_I \mathbf{Z}$ has no fixed point except 0 under the permutations of the index set I, assuming I is infinite (Proposition 5). By using this, we can prove the completeness theorem for a binary sequent (Theorem 4) of the following form:

Completeness theorem:

$$\text{A binary sequent } \vdash \Gamma \text{ is not provable } \Rightarrow Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\ \vdash \Gamma) = \mathbf{0}.$$

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See Definition 2 for the definition of $Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\ )$ with $\mathscr{F}$ denoting the subcategory consisting of the countable, compact, connected and locally connected abelian groups and of their character groups.

As was shown in Pontrjagin (1986), the Pontrjagin duality theorem on $\mathscr{F}$ is fundamental to obtaining the theorem on the category of the locally compact abelian groups. Combining the unique interpretation theorem of Section 2 and the completeness theorem of Section 3, we shall achieve the full completeness theorem for a sequent of MLL (without the Mix rule).

2. Dinatural interpretation of provable sequents

In this section, we consider a dinatural interpretation for MLL-sequent in a $*$-autonomous category $\mathscr{C}$ that satisfies the following (a), (b) and (c):

(a) The tensor unit I is a generator for $\mathscr{C}$. (Equivalently, the dualizing object $\perp$ is a cogenerator.)

(b) $\mathscr{C}$ has a preadditive structure (that is, each hom-set $\mathscr{C}(A, B)$ has an abelian group structure, which is denoted by $f + g$).

(c) The abelian group $\mathscr{C}(A, B)$ (of (b)) is an R-module over a principal ideal domain R, and $\mathtt{dim}_R(\mathscr{C}(I, I)) = 1$ holds.

In this framework, we shall prove (in Theorem 1) that for a binary semisimple sequent $\vdash \Gamma$ (see Notation 5)

$$\mathtt{dim}_R(Dinat_{\mathscr{C}}(\ \vdash \Gamma)) = 1.$$

We shall begin this section by reminding the reader of the definitions of generator and cogenerator.

Definition 1 (Generator, cogenerator). An object G of $\mathscr{C}$ is called a generator if for arbitrary parallel morphisms $h_1, h_2 : A \overset{\to}{\to} B$, we have $\forall f : G \to A \quad h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Dually, an object G of $\mathscr{C}$ is called a cogenerator if for arbitrary parallel morphisms $h_1, h_2 : A \overset{\to}{\to} B$, we have $\forall f : B \to G \quad f \circ h_1 = f \circ h_2$ implies $h_1 = h_2$.

The following are examples of $*$-autonomous categories that satisfy the Condition (a) above (see Barr (1976), Blute and Scott (1996) and Hamano (1998) for the definitions of $\mathscr{RTVEC}$ and $\mathscr{RTMOD}(G)$).

Example 1.

— The (discrete) field $\mathbf{k}$, which is the tensor unit for $\mathscr{RTVEC}$, becomes a generator for $\mathscr{RTVEC}$.

— The trivial action $(\mathbf{1}, \mathbf{k})$, which is the tensor unit for $\mathscr{RTMOD}(G)$, becomes a generator for $\mathscr{RTMOD}(G)$.

— The additive group $\mathbf{Z}$ of integers, which is the tensor unit for $\mathscr{RTA}$, becomes a generator for $\mathscr{RTA}$ (see Section 3 of this paper for the definition of $\mathscr{RTA}$).

Proposition 1. If a $*$-autonomous category satisfies Condition (b) above and the following (c′), then it satisfies (c).

(c′) The compositions of morphisms satisfy the distributive law: that is, $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ and $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$) and the endomorphism ring $\mathscr{C}(I, I)$ of the tensor unit I becomes a principal ideal domain.

(The endomorphism ring of an object A is a ring $\mathscr{C}(A, A)$ with its addition and its multiplication defined by (b) and by the composition, respectively.)

Proof. The endomorphism ring $\mathscr{C}(I, I)$ of I has a natural action on $\mathscr{C}(I, A \multimap B)$ defined by

$$\mathscr{C}(I, I) \times \mathscr{C}(I, A \multimap B) \longrightarrow \mathscr{C}(I, A \multimap B) \qquad [\ (h, f) \mapsto f \circ h].$$

By virtue of the following natural isomorphism, the above defines a $\mathscr{C}(I, I)$-action on $\mathscr{C}(A, B)$:

$$\mathscr{C}(I, A \multimap B) \cong \mathscr{C}(A, B) \tag{1}$$

Then, automatically, $dim_R(\mathscr{C}(I, I)) = 1$ with $R := \mathscr{C}(I, I)$, since $dim_R(R) = 1$ for an arbitrary ring R. □

Since the above Conditions (b) and the distributive law of (c′) are part of the definition of an additive category, the following corollary holds.

Corollary 1. If a $*$-autonomous category is an additive category such that the endomorphism ring $\mathscr{C}(I, I)$ is a principal ideal domain, then it satisfies the above (b) and (c). In particular, the categories $\mathscr{RTVEC}$, $\mathscr{RTMOD}(G)$ (and $\mathscr{RTA}$ in the next section) satisfy the above Conditions (b) and (c).

In the following, we shall remind the reader of the definition of dinatural interpretations of proofs of MLL in the framework of a $*$-autonomous category, which is due to Girard *et al.* (1992), Blute (1993), and Blute and Scott (1996). In the following, unless otherwise mentioned, $\mathscr{C}$ is an arbitrary $*$-autonomous category (which does not necessarily satisfy the above Conditions (a), (b) and (c)). $\mathscr{D}$ denotes an arbitrary fixed subcategory (not necessarily $*$-autonomous) containing I and $\perp$ of $\mathscr{C}$.

We begin with the definition of dinatural transformations between multivariant functors whose variables range only over a subcategory $\mathscr{D}$ of a $*$-autonomous category $\mathscr{C}$.

Definition 2 (Di(agonalizable)natural transformation). We assume that $\mathscr{C}$ denotes a $*$-autonomous category and $\mathscr{D}$ denotes a subcategory containing the tensor unit I and the dualizing object $\perp$. For multivariant functors $E(\vec{X}; \vec{Y}), F(\vec{X}; \vec{Y}) : (\mathscr{D}^{op})^n \times \mathscr{D}^n \to \mathscr{C}$,

$$\theta \in Dinat^{\mathscr{D}}_{\mathscr{C}}(E(\vec{X}; \vec{Y}), F(\vec{X}; \vec{Y}))$$

is a family of morphisms $\theta := \{\theta_{\vec{A}} : E(\vec{A}; \vec{A}) \to F(\vec{A}; \vec{A}) \mid \vec{A} \in \mathscr{D}^n\}$ such that for all

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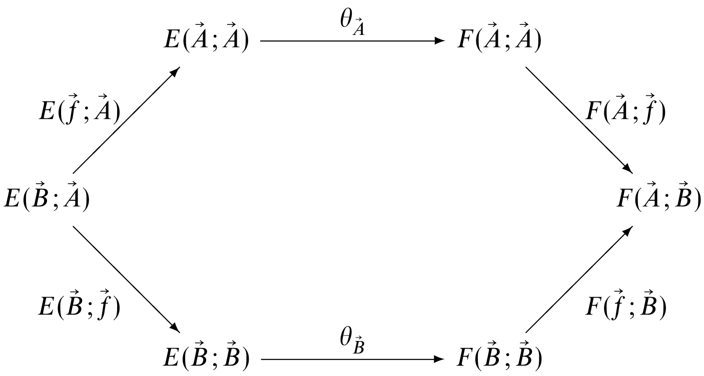
$\vec{f} : \vec{A} \to \vec{B}$ in $\mathscr{D}^n$ the following hexagonal diagram commutes:



If $\theta \in Dinat^{\mathscr{D}}_{\mathscr{C}}(E,F)$, then a family $\theta^{\mathscr{D}}$ of morphisms, which is defined by $\theta^{\mathscr{D}} := \{\theta_{\vec{D}} : E(\vec{D},\vec{D}) \to F(\vec{D},\vec{D}) \mid \vec{D} \in \mathscr{D}^n\}$, becomes an element of $Dinat^{\mathscr{D}}_{\mathscr{C}}(E,F)$, that is, $\{\theta^{\mathscr{D}}_{\mathscr{C}} \mid \theta \in Dinat_{\mathscr{C}}(E,F)\} \subset Dinat^{\mathscr{D}}_{\mathscr{C}}(E,F)$.

Throughout this paper, we write $Dinat_{\mathscr{C}}$ for $Dinat^{\mathscr{D}}_{\mathscr{C}}$ where $\mathscr{D}$ is an arbitrary subcategory containing I and $\perp$.

In the following we define an interpretation of the MLL-syntax. We will also define a formula $E(\vec{p})$ by the diagonalization $E^*(\vec{X};\vec{X})$ of a multivariant functor

$$E^*(\vec{X};\vec{Y}) : (\mathscr{D}^{op})^n \times \mathscr{D}^n \to \mathscr{C}$$

and define a proof of $E(\vec{p}) \vdash F(\vec{p})$ by the dinatural transformation in

$$Dinat^{\mathscr{D}}_{\mathscr{C}}(E^*(\vec{X};\vec{Y}), F^*(\vec{X};\vec{Y})).$$

Definition 3 (Syntax interpretation (Girard *et al.* 1992; Blute 1993; Blute and Scott 1996)).

A formula as a multivariant functor: An MLL-formula $E(p_1,\dots,p_n)$ built up from atoms $p_1,\dots,p_n$ and the constants $\mathbf{1}$ and $\perp$ is interpreted by the diagonalization $E^*(\vec{X};\vec{X})$ of the multivariant (contravariant and covariant) functor

$$E^*(\vec{X};\vec{Y}) : (\mathscr{D}^{op})^n \times \mathscr{D}^n \to \mathscr{C},$$

with $\vec{X} := X_1,\dots,X_n$ and $\vec{Y} := Y_1,\dots,Y_n$, which is defined inductively as follows:

— If $E(\vec{p}) := \mathbf{1}$, then $E^* = I$ (a constant functor).
— If $E(\vec{p}) := \perp$, then $E^* = \perp$ (a constant functor).
— If $E(p_1,\dots,p_n) := p_i$ with $1 \leqslant i \leqslant n$, then $E^*(\vec{X};\vec{Y}) := Y_i$.
— If $E(\vec{p}) := E_1(\vec{p}) \otimes E_2(\vec{p})$, then $E^*(\vec{X};\vec{Y}) := E_1^*(\vec{X};\vec{Y}) \otimes E_2^*(\vec{X};\vec{Y})$.
— If $E(\vec{p}) := E_1(\vec{p}) \multimap E_2(\vec{p})$, then $E^*(\vec{X};\vec{Y}) := E_1^*(\vec{Y};\vec{X}) \multimap E_2^*(\vec{X};\vec{Y})$.
— If $E(\vec{p}) := E_1(\vec{p})^{\perp}$, then $E^*(\vec{X};\vec{Y}) := E_1^*(\vec{Y};\vec{X})^{\perp}$.

A cut-free proof as a dinatural transformation: A proof π of a sequent $\Gamma(\vec{p}) \vdash \Delta(\vec{p})$ is interpreted by the dinatural transformation

$$|\pi| \quad \in \quad Dinat^{\mathscr{D}}_{\mathscr{C}}((\otimes\Gamma)^*(\vec{X};\vec{Y}),\ (⅋\ \Delta)^*(\vec{X};\vec{Y}))$$

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as follows:

— If π consists of only an initial sequent $p \vdash p$, $\quad \vdash \mathbf{1}$, or $\bot \vdash$, then $|\pi|$ is $\{id_A : A \to A \mid A \in \mathscr{D}\}$, $\{id_I : I \to I\}$, or $\{id_\bot : \bot \to \bot\}$, respectively.

— If π is derived from π_1 (and π_2) by an inference rule with $|\pi_i| = \{\theta^i_{\vec{A}} : (\mathscr{D}^{op})^n \times \mathscr{D}^n \to \mathscr{C} \mid \vec{A} \in \mathscr{D}^n\}$, then for each $\vec{A}$, we have $\theta^i_{\vec{A}}$ yields the morphism, say $\theta_{\vec{A}} : (\otimes\Gamma)^*(\vec{A};\vec{A}) \to (⅋\ \Delta)^*(\vec{A};\vec{A})$ by means of the instantiation of $\vec{A}$ to the corresponding functor of the inference rule. We define $|\pi|$ by the family $\{\theta_{\vec{A}} \mid \vec{A} \in \mathscr{D}^n\}$. Analogously to Section 4 of Girard *et al.* (1992), it can be verified that $|\pi|$ becomes a dinatural transformation.

The set $Dinat^{\mathscr{D}}_{\mathscr{C}}((\otimes\Gamma)^*(\vec{X};\vec{Y}),\ (⅋\ \Delta)^*(\vec{X};\vec{Y}))$ is denoted by $Dinat^{\mathscr{D}}_{\mathscr{C}}(\Gamma \vdash \Delta)$ and is called the interpretation of a sequent $\Gamma \vdash \Delta$.

Let $E(p_0,\vec{p})$ denote an MLL-formula built from atoms p_0 and $\vec{p}$. For an object $\mathbf{d}$ of $\mathscr{D}$, $E(\mathbf{d},\vec{X};\mathbf{d},\vec{X})$ is defined by

$$\begin{array}{ccc} (\mathscr{D}^{op})^n \times \mathscr{D}^n & \xrightarrow{\ \mathbf{d}\ } & (\mathscr{D}^{op})^{n+1} \times \mathscr{D}^{n+1} \\ & \searrow^{E(\mathbf{d},\vec{X};\mathbf{d},\vec{X})} \quad \swarrow_{E(X_0,\vec{X};X_0,\vec{X})} & \\ & \mathscr{C} & \end{array}$$

where $\mathbf{d}$ assigns each object $(\vec{A},\vec{B})$ to the object $(\mathbf{d},\vec{A},\mathbf{d},\vec{B})$ and assigns each morphism $(\vec{f},\vec{g})$ to the morphism $(id_{\mathbf{d}},\vec{f},id_{\mathbf{d}},\vec{g})$. Note that if $\mathbf{d}$ is the tensor unit I or the dualizing object $\bot$, then $E(\mathbf{d},\vec{X};\mathbf{d},\vec{X})$ becomes a functor-interpretation of the formula $E(\mathbf{1},\vec{p})$ or $E(\bot,\vec{p})$, respectively.

Let $E(p_0,\vec{p})$ and $F(p_0,\vec{p})$ be MLL-formulas and $\theta_{X_0,\vec{X}} \in Dinat^{\mathscr{D}}_{\mathscr{C}}(E(p_0,\vec{p}) \vdash F(p_0,\vec{p}))$. We define the family of morphisms $\theta_{\mathbf{d},\vec{X}} := \{\theta_{\mathbf{d},\vec{A}} : E^*(\mathbf{d},\vec{A};\mathbf{d},\vec{A}) \to F^*(\mathbf{d},\vec{A};\mathbf{d},\vec{A})\}$. Then it is easily checked that

$$\theta_{\mathbf{d},\vec{X}} \in Dinat^{\mathscr{D}}_{\mathscr{C}}(E^*(\mathbf{d},\vec{X};\mathbf{d},\vec{Y}), F^*(\mathbf{d},\vec{X};\mathbf{d},\vec{Y})).$$

Hence, if $\mathbf{d}$ is the tensor unit I or the dualizing object $\bot$, then $\theta_{\mathbf{d},\vec{X}}$ belongs to

$$Dinat^{\mathscr{D}}_{\mathscr{C}}(E(\mathbf{1},\vec{p}) \vdash F(\mathbf{1},\vec{p}))$$

or

$$Dinat^{\mathscr{D}}_{\mathscr{C}}(E(\bot,\vec{p}) \vdash F(\bot,\vec{p})),$$

respectively.

Before the following soundness theorem, we note that if $\mathscr{C}$ satisfies Condition (b), then every $Dinat^{\mathscr{D}}_{\mathscr{C}}(E(\vec{X};\vec{Y}),F(\vec{X};\vec{Y}))$ has an abelian group structure; that is, for dinatural transformations θ and σ, a dinatural transformation $\theta+\sigma$ is defined by $(\theta+\sigma)_{\vec{A}} := \theta_{\vec{A}}+\sigma_{\vec{A}}$.

Proposition 2 (Soundness (*cf.* Girard *et al.* (1992), Blute (1993) and Blute and Scott (1996)). If $\vdash \Gamma$ is provable in MLL, then for an arbitrary $*$-autonomous category $\mathscr{C}$ satisfying Condition (b),

$$Dinat_{\mathscr{C}}(\ \vdash \Gamma) \neq \mathbf{0}.$$

Proof. Since $\vdash \Gamma$ is provable, there exists a cut free proof π of Γ. It is easily proved that $|\pi| \neq 0$ by induction on the length of π. □

Notation 1. A *tensor context* $\vdash \Gamma(\ )$ is a context such that the hole appears exactly once, and all connectives that bind the hole are tensor connectives. For example, $\vdash (\ ), p \otimes q$ and $\vdash p \otimes (\ ) \otimes q$ are tensor contexts, but $\vdash p \otimes (\ )^{\perp} \otimes q$ is not a tensor context. In particular, in the tensor context $\vdash \Gamma(\ )$, the hole occurs positively.

The following proposition is the crucial proposition in this section.

Proposition 3. Let $\Gamma(\ )$ denote a tensor context and p denote an atom that does not occur in the context. For $\mathscr{C}$ satisfying Conditions (a) and (b),

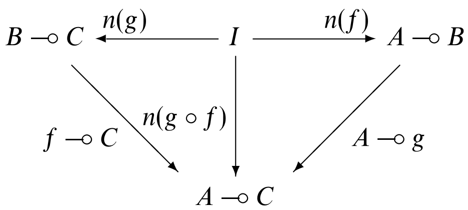
$$\forall\ \ \theta_{X_0,\vec{X}} \in Dinat_{\mathscr{C}}(\ \ \vdash \Gamma(p^{\zeta_1}), p^{\zeta_2})\ \ [\ \theta_{\zeta_1,\vec{X}} = 0 \quad \Rightarrow \quad \theta_{X_0,\vec{X}} = 0\ ],$$

where $\{\zeta_1, \zeta_2\} = \{I, \perp\}$ with $p^I := p$. Note that in the above, X_0 denotes the corresponding variable to the atom p.

For the proof of Proposition 3, we require the following lemma. We begin by reminding the reader of the *name* $n(f)$ of a morphism f, which is defined using the following canonical morphism:

$$\mathscr{C}(A,B) \ \stackrel{\sim}{\rightarrow}\ \mathscr{C}(I, A \multimap B) \qquad\qquad [\ f \mapsto n(f)\ ] \tag{2}$$

Then it is easily checked that the following diagram commutes for every $f : A \to B$ and every $g : B \to C$:



$$That\ is, \quad n(g \circ f) = (A \multimap g) \circ n(f) = (f \multimap C) \circ n(g). \tag{3}$$

The canonical isomorphism (2) with the help of (3) makes the diagram

$$I \xrightarrow{f} A \ \underset{k_2}{\overset{k_1}{\rightrightarrows}}\ B$$

correspond to the diagram

$$I \ \underset{n(k_2)}{\overset{n(k_1)}{\rightrightarrows}}\ (A \multimap B) \xrightarrow{f \multimap B} (I \multimap B).$$

Thus we have obtained the following lemma.

Lemma 1. Let $\mathscr{C}$ denote a $*$-autonomous category satisfying Condition (a). Then, given any parallel morphisms $h_1, h_2 : I \rightrightarrows (A \multimap B)$, the following holds:

$$\forall f : I \to A\ [\ (f \multimap B) \circ h_1 = (f \multimap B) \circ h_2\ ] \quad \Rightarrow \quad h_1 = h_2.$$

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Dually, the following holds:

$$\forall f : B \to \bot \; [\, (A \multimap f) \circ h_1 = (A \multimap f) \circ h_2 \,] \quad \Rightarrow \quad h_1 = h_2.$$

Now we can prove Proposition 3 using the above lemma.

Proof of Proposition 3. First we remind the reader of the definition of a dinatural transformation $\theta : I \stackrel{..}{\to} E(\vec{X};\vec{X})$: θ as a family of morphisms $\{\theta_{\vec{A}} : I \to E(\vec{A};\vec{A}) \mid \vec{A} \in \mathscr{C}^n\}$ such that the following diagram commutes for all $\vec{A}$, $\vec{B}$ and for all $\vec{f} : \vec{A} \to \vec{B}$:

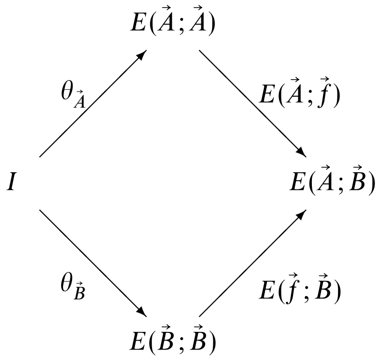


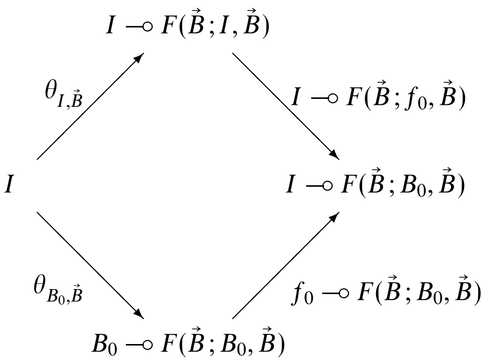
Diagram 1

Since the hole in $\Gamma(\ )$ occurs positively and p does not occur in $\Gamma(\ )$, a multivariant functor interpretation of $\Gamma(p^{\zeta_1})$ is written in the form:

$$\begin{cases} F(\vec{X};X_0,\vec{X}) & \text{if } \zeta_1 = I \\ F(X_0,\vec{X};\vec{X}) & \text{if } \zeta_1 = \bot, \end{cases}$$

where X_0 is the variable corresponding to the atom p. Note that the notation $F(\vec{X};X_0,\vec{X})$ and $F(X_0,\vec{X};\vec{X})$ is used to stress that X_0 does not occur contravariantly and covariantly, respectively. Let $E(X_0,\vec{X};X_0,\vec{X})$ denote a multivariant functor interpreting $\Gamma(P^{\zeta_1}) \parr P^{\zeta_2}$.

Case $\zeta_1 = \mathbf{I}$
$E(X_0,\vec{X};X_0,\vec{X}) := X_0 \multimap F(\vec{X};X_0,\vec{X})$.
Diagram 1 with $A_0 := I$, $\vec{A} := \vec{B}$ and $\vec{f} := \vec{B}$ implies that the following diagram commutes for all $f_0 : I \to B_0$:



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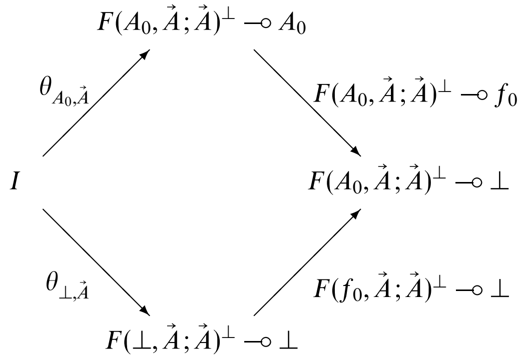
$\theta_{I,\vec{B}} = 0$ implies that the upper leg of the diagram is 0. By the commutativity of the diagram, the lower leg is 0. That is,

$$\forall f_0 : I \to B_0\ [\ (f_0 \multimap F(\vec{B}; B_0, \vec{B})) \circ \theta_{B_0,\vec{B}} = 0\ ].$$

From Lemma 1

$$\theta_{B_0,\vec{B}} = 0.$$

Case $\zeta_1 = \perp$
$E(X_0, \vec{X}; X_0, \vec{X}) := F(X_0, \vec{X}; \vec{X})^{\perp} \multimap X_0$.
In this case the proof is exactly the dual to that of the previous case: Diagram 1 with $B_0 := \perp$, $\vec{B} := \vec{A}$ and $\vec{f} := \vec{A}$ implies that the following diagram commutes for all $f_0 : A_0 \to \perp$:



$\theta_{\perp,\vec{A}} = 0$ implies that the lower leg of the diagram is 0. By the commutativity of the diagram, the upper leg is 0. That is,

$$\forall f_0 : A_0 \to \perp\ [\ (F(A_0, \vec{A}; \vec{A})^{\perp} \multimap f_0) \circ \theta_{A_0,\vec{A}} = 0\ ].$$

From Lemma 1

$$\theta_{A_0,\vec{A}} = 0.$$

$\square$

Notation 2. For a tensor context $\vdash \Gamma(\ )$, the sequent $\vdash \Gamma(*)$ is defined as follows: if $\vdash \Gamma(\ )$ is of the form $\vdash \Delta_1, (\ ), \Delta_2$, then $\vdash \Gamma(*)$ is defined to be $\vdash \Delta_1, \mathbf{1}, \Delta_2$; otherwise $\vdash \Gamma(*)$ denotes the sequent resulting from $\vdash \Gamma(\ )$ by deleting the hole together with the innermost tensor that binds the hole. For example, if $\vdash \Gamma(\ )$ is $\vdash (\ ), p \otimes q$, then $\vdash \Gamma(*)$ is $\vdash \mathbf{1}, p \otimes q$, and if $\vdash \Gamma(\ )$ is $\vdash p \otimes (\ ) \otimes q$, then $\vdash \Gamma(*)$ is $\vdash p \otimes q$.

If each hom-set of $\mathscr{C}$ has a module structure in addition to an abelian group structure (that is, $\mathscr{C}$ satisfies Condition (c)), the following holds as a corollary of Proposition 3.

Corollary 2. For a tensor MLL-context $\Gamma(\ )$ and an atom p that does not occur in the context and for $\mathscr{C}$ satisfying (a), (b) and (c),

$$\mathtt{dim}(Dinat_{\mathscr{C}}(\ \ \vdash \Gamma(*))) \geqslant \mathtt{dim}(Dinat_{\mathscr{C}}(\ \ \vdash \Gamma(p^{\zeta_1}), p^{\zeta_2})), \tag{4}$$

where $\{\zeta_1, \zeta_2\} = \{I, \perp\}$ with $p^I := p$.

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Proof. It suffices to prove the following: for a linearly independent $\{\theta^{\,i}_{X_0,\vec{X}}\}_{i=1}^{n}$ in $Dinat(\ \vdash \Gamma(p^{\zeta_1}), p^{\zeta_2})$, the family $\{\theta^{\,i}_{\zeta_1,\vec{X}}\}_{i=1}^{n}$ becomes linearly independent in $Dinat(\ \vdash \Gamma(*))$. The assertion is verified as follows:

$$0 = \textstyle\sum_{i=1}^{n} m^i \theta_{\zeta_1,\vec{X}} := (\sum_{i=1}^{n} m^i \theta)_{\zeta_1,\vec{X}} \overset{Prop\ 3}{\Rightarrow} 0 = (\sum_{i=1}^{n} m^i \theta)_{X_0,\vec{X}} := \sum_{i=1}^{n} m^i \theta_{X_0,\vec{X}}$$

$$\Rightarrow \quad \forall i (m_i = 0) \qquad \square$$

Corollary 2 is the crucial proposition leading to the unique interpretation theorem for a semisimple binary sequent (Theorem 1) stating that the interpretation of a provable sequent coincides with the module spanned by the interpretation of the unique (modulo permutations of inferences) cut free proof of the sequent. To obtain the theorem, we introduce the syntactic notation $\rhd$ and restate Corollary 2 in terms of the notation.

Notation 3 (The relation $\equiv$ between sequents). The relation $\vdash A_1, \ldots, A_n \equiv \vdash B_1, \ldots, B_n$ for two sequents is defined if $A_i \sim^* B_{\sigma(i)}$ $(i = 1, \ldots, n)$ for some permutation σ on $\{1, \ldots, n\}$, where $\sim^*$ denotes the reflexive, symmetric, and transitive closure of the relation $\sim$ between formulas such that $A \sim \mathbf{1} \otimes A$ and $A \sim A \otimes \mathbf{1}$.

It is obvious that if $\vdash \Gamma \equiv \vdash \Delta$, then $Dinat_{\mathscr{C}}(\ \vdash \Gamma) \cong Dinat_{\mathscr{C}}(\ \vdash \Delta)$.

Notation 4 (the reduction relation $\rhd$ between sequents).
The reduction $\vdash \Gamma \rhd\ \vdash \Gamma'$ is defined if there exists a tensor context $\tilde{\Gamma}(\ )$ such that the following hold:

(i) $\vdash \Gamma' = \vdash \tilde{\Gamma}(*)$,
(ii) $\vdash \Gamma \equiv \vdash \tilde{\Gamma}(p^{\zeta_1}), p^{\zeta_2}$ for some atom p with $\{\zeta_1, \zeta_2\} = \{I, \perp\}$ and $p^I := p$.

For example, $\Gamma \rhd \tilde{\Gamma}(*)$ holds in the following (i) and (ii):

(*i*)					(*ii*)		
Γ	:=	$p_1^{\perp}$, p_0,	$p_1 \otimes p_0^{\perp} \otimes p_2$,	$p_2^{\perp}$	Γ	:=	p_0, $p_0^{\perp}$
$\tilde{\Gamma}(\ )$	:=	$p_1^{\perp}$,	$p_1 \otimes (\ ) \otimes p_2$,	$p_2^{\perp}$	$\tilde{\Gamma}(\ )$	:=	$(\ )$
$\tilde{\Gamma}(*)$	:=	$p_1^{\perp}$,	$p_1 \otimes p_2$,	$p_2^{\perp}$	$\tilde{\Gamma}(*)$	:=	$\mathbf{1}$

Now that we have introduced the reduction relation, Corollary 2 can be restated as follows.

Corollary 2′. (Reduction Lemma) If $\vdash \Gamma$ is a binary sequent and $\vdash \Gamma \rhd\ \vdash \Delta$, then for $\mathscr{C}$ satisfying (a), (b) and (c), we have

$$\mathtt{dim}(Dinat_{\mathscr{C}}(\ \vdash \Gamma)) \leqslant \mathtt{dim}(Dinat_{\mathscr{C}}(\ \vdash \Delta)).$$

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By means of the reduction relation $\rhd$, every provable semisimple sequent is reduced to a sequent of an elementary form (Syntactic Lemma 1 below). We shall begin by reminding the reader of some terminology.

Notation 5 (Semisimple sequent (Hyland and Ong 1993) and simple sequent (Abramsky and Jagadeesan 1994)). A sequent of the form $\vdash A_1, \ldots, A_n$ is called semisimple if each A_i is built up from the constants and literals by using tensor connectives only. A semisimple sequent is called *simple* if each A_i is either a literal or the tensor product of two literals.

Syntactic Lemma 1. Every MLL-provable semisimple sequent $\vdash \Gamma$ is reducible by means of $\rhd$ to a semisimple provable sequent of the form $\vdash \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \bot, \ldots, \bot$.

Proof. We use induction on the number of linear negations in Γ.

Base case: Every MLL-provable sequent where no linear negations occur is of the form $\vdash \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \bot, \ldots, \bot$.

Induction case: In this case, there exists a sequent $\vdash \Gamma_1 \equiv \vdash \Gamma$ and a literal p^{ζ_1} that is bounded by no logical connectives in $\vdash \Gamma_1$. Then $\vdash \Gamma \equiv \vdash \tilde{\Gamma}(p^{\zeta_2}), p^{\zeta_1}$ holds. Hence $\vdash \Gamma \rhd \vdash \tilde{\Gamma}(*)$. □

Now, by virtue of Syntactic Lemma 1 and Corollary 2′, we reach the unique interpretation theorem for binary semisimple sequent as follows.

Theorem 1 (Unique interpretation theorem for binary semisimple sequent). For a binary semisimple sequent $\vdash \Gamma$ that is provable in MLL and $\mathscr{C}$ satisfying Conditions (a), (b) and (c),

$$\mathtt{dim}_{\mathscr{C}(I,I)}(Dinat_{\mathscr{C}}(\ \vdash \Gamma)) = 1.$$

Proof. By virtue of Syntactic Lemma 1 and Corollary 2′, it suffices to prove the theorem when $\vdash \Gamma$ is of the form $\vdash \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \bot, \ldots, \bot$. This is obvious from $Dinat(\ \vdash \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \bot, \ldots, \bot) = \mathscr{C}(I, I)$ and from (c). □

From Theorem 1, we obtain the uniqueness of the interpretation of an arbitrary binary provable sequent. Our approach from Theorem 1 to Theorem 2 follows that taken by Hyland and Ong (Hyland and Ong 1993): they reduce a provable binary sequent to provable semisimple sequents by using a lemma, which is explained in our framework below. Key tools for the reduction to semisimple sequents in our framework are the following canonical natural transformations of a $*$-autonomous category $\mathscr{C}$ between functors $\mathscr{C}^3 \to \mathscr{C}$, called 'weak distributivities' (*cf.* Cockett and Seely (1997)):

$$X \otimes (Y \parr Z) \to (X \otimes Y) \parr Z \quad \text{and} \quad X \otimes (Y \parr Z) \to (X \otimes Z) \parr Y.$$

Now the following lemma of our framework corresponds to the Hyland–Ong lemma (*cf.* Proposition 4.5 of Hyland and Ong (1993)).

Lemma 2. For an arbitrary binary provable sequent $\vdash \Gamma$, there exist binary semisimple provable sequents $\vdash \Gamma_1, \ldots, \vdash \Gamma_n$ such that every dinatural transformation in $Dinat(\ \vdash \Gamma)$ comes from a dinatural transformation in $Dinat(\ \vdash \Gamma_i)$ for some i by left composing

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weak distributivities. Note that dinaturality is preserved under compositions with natural transformations.

By virtue of Theorem 1 with the help of Lemma 2, the following theorem can be derived, provided that:

(d) The weak distributivities of $\mathscr{C}$ are monic.

Theorem 2 (Unique interpretation theorem for a binary sequent).
For a binary sequent $\vdash \Gamma$ that is provable in MLL and $\mathscr{C}$ satisfying (a), (b), (c) and (d)

$$\mathtt{dim}_{\mathscr{C}(I,I)}(Dinat_{\mathscr{C}}(\ \vdash \Gamma)) = 1.$$

Proof. From Lemma 2, $\mathtt{dim}(Dinat_{\mathscr{C}}(\Gamma)) \leqslant \mathtt{dim}(Dinat_{\mathscr{C}}(\Gamma_i))$ holds for all i. Hence the assertion holds from Proposition 2 and Theorem 1. □

As a corollary of Theorem 2, we obtain the unique interpretation theorem of a binary provable sequent in $MLL{+}Mix$. Since the Mix rule corresponds to the following canonical natural transformation, called the 'Mix transformation'

$$X \otimes Y \quad \rightarrow \quad X \parr Y.$$

The following Lemma 3 corresponds to the Abramsky–Jagadeesan lemma (Proposition 9 of Abramsky and Jagadeesan (1994)):

Lemma 3. For an arbitrary binary $MLL + Mix$ provable sequent $\vdash \Gamma$, there exist binary $MLL + Mix$ provable sequents $\vdash \Gamma_1, \ldots, \vdash \Gamma_n$ such that every dinatural transformation in $Dinat_{\mathscr{C}}(\ \vdash \Gamma)$ comes from a dinatural transformation in $Dinat_{\mathscr{C}}(\ \vdash \Gamma_i)$ for some i by left composing weak distributivities and Mix transformations.

With the help of the above lemma, the corresponding theorem to Theorem 1 in $MLL + Mix$ is obtained as a corollary provided that:

(e) The Mix morphisms of $\mathscr{C}$ are monic.

Corollary 3 (Unique interpretation theorem, binary case). For a binary sequent $\vdash \Gamma$ that is provable in $MLL + Mix$ and $\mathscr{C}$ satisfying (a), (b), (c), (d) and (e),

$$\mathtt{dim}_{\mathscr{C}(I,I)}(Dinat_{\mathscr{C}}(\ \vdash \Gamma)) = 1.$$

Proof. From Lemma 3, we have $\mathtt{dim}(Dinat_{\mathscr{C}}(\Gamma)) \leqslant \mathtt{dim}(Dinat_{\mathscr{C}}(\Gamma_i))$ holds for all i. Hence the assertion holds from Proposition 2 and Theorem 1. □

Note that the category $\mathscr{RTVEC}$ of reflexive topological vector spaces satisfies the conditions of Corollary 3: our proof in this section when applied to $\mathscr{RTVEC}$ is considered as an alternative proof of the unique interpretation theorem of Blute and Scott (*cf.* Lemma 10.2 of Blute and Scott (1996)).

3. Pontrjagin duality and completeness for MLL (without Mix)

In this section, we shall prove a completeness theorem for MLL without Mix via the dinatural interpretation in Barr's $*$-autonomous category $\mathscr{RTA}$ of reflexive topological

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abelian groups (Barr 1977). Combined with the results of Section 2, we shall obtain the full completeness theorem for MLL in the category $\mathscr{RTA}$.

Barr (Barr 1977) formulated $\mathscr{RTA}$, which is a subcategory of the category $\mathscr{TG}$ of topological abelian groups and continuous homomorphisms, based on the famous Pontrjagin Duality theorem for locally compact abelian groups. Let $\mathscr{LCA}$ denote the full subcategory of locally compact abelian groups. The torus group $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ (with $\mathbf{R}$ and $\mathbf{Z}$ denoting the additive groups of reals and integers, respectively) is an object of $\mathscr{LCA}$, indeed $\mathbf{T}$ is isomorphic to the compact circle group consisting of complex numbers of absolute value 1. For $G \in \mathscr{LCA}$, $G^{\perp}$ denotes the character group of G; that is, $G^{\perp} := \mathscr{LCA}(G, \mathbf{T})$. Then $G^{\perp}$ becomes an object of $\mathscr{LCA}$, taken with point-wise multiplication and the compact open topology. The Pontrjagin duality theorem states that the canonical map $G \to G^{\perp\perp}$ becomes an isomorphism of $\mathscr{LCA}$. (See Dikranjan *et al.* (1989) for the precise properties of the structure of locally compact abelian groups.) But the category $\mathscr{LCA}$ of locally compact abelian groups has some problematic properties, for example, for locally compact groups G and H, neither $\mathscr{LCA}(G, H)$ nor its tensor product $G \otimes H$ is necessarily locally compact. Inspired by work of Kaplan (Kaplan 1948; Kaplan 1950), Barr (Barr 1977) constructed a $*$-autonomous category $\mathscr{RTA}$, which is still self dual in the sense of the Pontrjagin duality theorem and which is, furthermore, complete and cocomplete.

We begin this Section with Barr's definition of the category $\mathscr{RTA}$. For more details, see Barr (1996). Recently Barr (Barr 1999) presented an elegant Chu construction that works uniformly to obtain almost all the $*$-autonomous categories (including $\mathscr{RTA}$) of his monograph Barr (1979).

Definition 4. If $L \in \mathscr{LCA}$, let L^* denote its character group with the compact open topology. Let $\mathscr{SPLC}$ denote the full subcategory of topological abelian groups that are subgroups of products of locally compact abelian groups. Now, if $A \in \mathscr{SPLC}$, then let A^* denote its character group. We endow A^* with the weak topology with respect to those homomorphisms $f^* : A^* \to L^*$ where $f : L \to A$ is a continuous homomorphism and $L \in \mathscr{LCA}$. For $A, B \in \mathscr{SPLC}$, we use $A \multimap\!\!\bullet\, B$ to denote the group of continuous homomorphisms from A to B topologized as a subspace of $B^{|A|} \times A^{*|B^*|}$. It follows that $(A \multimap\!\!\bullet\, \mathbf{T}) \cong A^*$. Then $\mathscr{RTA}$ denotes the full subcategory of reflexive groups of $\mathscr{SPLC}$, (that is, $A^{**} \to A$ becomes an isomorphism). Then $\mathscr{RTA}$ becomes a $*$-autonomous category with $A \multimap B$ defined by $(A \multimap\!\!\bullet\, B)^{**}$, $A \otimes B$ defined by $(A \multimap B^{\perp})^{\perp}$, where $(\ )^{\perp} := (\ ) \multimap \mathbf{T}$. The ring $\mathbf{Z}$ of integers becomes the unit for the tensor product, and the torus group $\mathbf{T}$ becomes the dualizing object.

Note that, in the above definition, if $A, B \in \mathscr{LCA}$, then $A \multimap B = A \multimap\!\!\bullet\, B$ from the Pontrjagin duality theorem. And for $A, B \in \mathscr{SPLC}$, there is a natural surjective map

$$|A| \otimes |B| \quad \twoheadrightarrow \quad |(A \multimap\!\!\bullet\, B^*)^*| \qquad [a \otimes b \mapsto (f \mapsto b(f(a)))]. \tag{5}$$

Barr's construction is analogous to his construction in Barr (1976) of the $*$-autonomous category $\mathscr{RTVEC}$. But there is a crucial difference between the two categories when considered as models of linear logic. In $\mathscr{RTA}$, the tensor unit $\mathbf{Z}$ and the dualizing object $\mathbf{T}$ for $\mathscr{RTA}$ do not coincide. Since $\mathbf{T}$ and $\mathbf{Z}$ are the interpretations of the MLL-constants $\perp$ and $\mathbf{1}$, respectively, one can see that the Mix rule is not satisfied in $\mathscr{RTA}$ (Corollary 4).

Lemma 4 (*cf.* Lemma 6.1 of Cockett and Seely (1997)). The following is equivalent in *MLL*:

1 $\bot \vdash \mathbf{1}$

2 $A \otimes B \vdash A \parr B$

3 $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ \mathit{MixRule}$

Proof. (1) ⇒ (2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, \bot} \qquad \dfrac{\bot \vdash \mathbf{1} \qquad \dfrac{B \vdash B}{\mathbf{1}, B \vdash B}}{\bot, B \vdash B}}{A, B \vdash A, B}}{A \otimes B \vdash A, B}}{A \otimes B \vdash A \parr B}}{}$$

(2) ⇒ (1)

$$\dfrac{\dfrac{\dfrac{\bot \vdash \bot \quad \vdash \mathbf{1}}{\bot \vdash \bot \otimes \mathbf{1}} \qquad \bot \otimes \mathbf{1} \vdash \bot \parr \mathbf{1}}{\bot \vdash \bot \parr \mathbf{1}} \qquad \dfrac{\bot \vdash \quad \mathbf{1} \vdash \mathbf{1}}{\bot \parr \mathbf{1} \vdash \mathbf{1}}}{\bot \vdash \mathbf{1}}$$

(2) ⇔ (3) This is obvious. □

We remind the reader of some basic terminology in group theory, which will be used in this paper frequently: a group G is called *divisible* if $\{x^n \mid x \in G\} = G$ holds for every $n = 1, 2, \ldots$.

Proposition 4. $\mathscr{TG}(\mathbf{T}, \mathbf{Z}) = \mathbf{0}$.

Proof. **T** is divisible, while no subgroup except 0 of **Z** is divisible. □

Since **T** and **Z** are the interpretations of *MLL*-constants $\bot$ and **1**, respectively, the following is a corollary of Lemma 4.

Corollary 4. The Mix rule is not satisfied in the $*$-autonomous category $\mathscr{RTA}$.

For a family $\{A_i\}_{i \in I}$ of abelian groups, *the direct sum* $\sum_{i \in I} A_i$ denotes the subgroup of the direct product $\prod_{i \in I} A_i$ consisting of all (a_i) such that $a_i = 0$ for all but a finite number of indices $i \in I$. It is well known that the following canonical morphisms become isomorphisms of $\mathscr{TG}$ for an arbitrary object $B \in \mathscr{TG}$, provided that the direct products are equipped with their product topologies, and the direct sums are equipped with their sum topologies:

$$\mathscr{TG}\left(B, \prod_{i \in I} A_i\right) \xrightarrow{\sim} \prod_{i \in I} \mathscr{TG}(B, A_i). \tag{6}$$

$$\mathscr{TG}\left(\sum_{i \in I} A_i, B\right) \xrightarrow{\sim} \prod_{i \in I} \mathscr{TG}(A_i, B). \tag{7}$$

Especially (7) with $B := \mathbf{T}$ is

$$\left(\sum_{i\in I} A_i\right)^{\perp} \xrightarrow{\sim} \prod_{i\in I} A_i^{\perp}.$$

The following is a key property for our completeness theorem.

Fact 1 (*cf.* Proposition 3.1.2 of Dikranjan *et al.* (1989)). For a family $\{A_i\}_{i\in I}$ of topological abelian groups, if $\Pi_{i\in I} A_i$ is compact or I is countable, then the following canonical morphisms are isomorphisms (sums and products are given their usual topologies):

$$\sum_{i\in I} \mathscr{T}\mathscr{G}(A_i, \mathbf{T}) \xrightarrow{\sim} \mathscr{T}\mathscr{G}\left(\prod_{i\in I} A_i, \mathbf{T}\right) \tag{8}$$

That is,

$$\sum_{i\in I} A_i^{\perp} \xrightarrow{\sim} \left(\prod_{i\in I} A_i\right)^{\perp}.$$

Suppose that $\{A_i\}_{i\in I}$ is a family of abelian groups such that each A_i is topologically isomorphic to a fixed A. Then, given a permutation $\sigma : I \to I$ on the index set, there is an induced isomorphism both on $\prod_{i\in I} A_i$ and on $\sum_{i\in I} A_i$. For $\prod_{i\in I} A_i$, the set of points fixed by all such isomorphisms is of the form $(a_i)_{i\in I}$ where $a_i = a$ for all i. But $\sum_{i\in I} A_i$ has no fixed points except 0, provided that the index set I is infinite. More generally, if each A_i depends on the index set I, say A_i^I, a permutation σ determines the composition of homomorphisms $A_i^I \to A_{\sigma(i)}^I \to A_{\sigma(i)}^{\sigma(I)}$, where the first homomorphism is for changing a component, and the second one is the endomorphism induced by σ on $A_{\sigma(i)}^I$. Thus a permutation induces the endomorphisms both on $\prod_{i\in I} A_i^I$ and on $\sum_{i\in I} A_i^I$. Again, $\sum_{i\in I} A_i^I$ has no fixed points except 0 under all permutations, provided that the index set I is infinite.

The following proposition is crucial for our proof of the completeness theorem for *MLL* (*cf.* Proposition 6 and Theorem 3 below).

Proposition 5. Let $\{A_i\}_{i\in I}$ be a family of topological abelian groups such that I is infinite and each A_i is isomorphic to a fixed A. If A is compact or I is countable, then both

$$\left(\prod_{i\in I} A_i\right) \otimes \left(\prod_{j\in I} A_j\right)^{\perp} \text{ and } \left(\sum_{i\in I} A_i\right) \otimes \left(\sum_{j\in I} A_j\right)^{\perp}$$

have no fixed points, except 0, under the isomorphisms induced by the permutations of indices I.

Proof.

$$\begin{aligned}
\textstyle\prod_{i\in I} A_i \otimes (\prod_{j\in I} A_j)^{\perp} &:= (\textstyle\prod_{i\in I} A_i \multimap \prod_{j\in I} A_j)^{\perp} \\
&\cong \{\textstyle\prod_{j\in I}(\prod_{i\in I} A_i \multimap A_j)\}^{\perp} \quad \text{from (6)} \\
&\cong \textstyle\sum_{j\in I}(\prod_{i\in I} A_i \multimap A_j)^{\perp} \quad \text{from (8).}
\end{aligned}$$

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$$
\begin{aligned}
\textstyle\sum_{i\in I} A_i \otimes (\sum_{j\in I} A_j)^\perp &:= (\textstyle\sum_{i\in I} A_i \multimap \sum_{j\in I} A_j)^\perp \\
&\cong \{\textstyle\prod_{i\in I}(A_i \multimap \sum_{j\in I} A_j)\}^\perp \quad \text{from (7)} \\
&\cong \textstyle\sum_{i\in I}(A_i \multimap \sum_{j\in I} A_j)^\perp \quad \text{from (8).}
\end{aligned}
$$

The above indicates that both groups are topologically isomorphic to direct sums of fixed groups, hence neither group has any points, except 0, fixed under all permutations of indices I (*cf.* the paragraph after the above Fact 1). □

Note that with the help of (7) and (8),

$$\left(\sum_{i\in I} A_i\right)^\perp \otimes \sum_{j\in J} A_j^{\perp\perp} \cong \left(\prod_{i\in I} A_i^\perp\right) \otimes \left(\prod_{j\in J} A_j^\perp\right)^\perp$$

holds. In particular, taking into account that $\mathbf{T} \multimap \mathbf{T} \cong \mathbf{Z}$ and $\mathbf{Z} \multimap \mathbf{T} \cong \mathbf{T}$, the following holds:

$$
\begin{aligned}
(\textstyle\sum_{i\in I} \mathbf{Z}_i)^\perp \otimes (\sum_{j\in I} \mathbf{Z}_j) &\cong (\textstyle\prod_{i\in I} \mathbf{T}_i) \otimes (\prod_{j\in I} \mathbf{T}_j)^\perp \\
&\cong \textstyle\sum_{j\in I}(\prod_{i\in I} \mathbf{T}_i \multimap \mathbf{T}_j)^\perp \quad \text{from Proposition 5} \\
&\cong \textstyle\sum_{j\in I}(\sum_{i\in I}(\mathbf{T}_i \multimap \mathbf{T}_j))^\perp \quad \text{from (8)} \\
&\cong \textstyle\sum_{j\in I}(\sum_{i\in I} \mathbf{Z}_{i,j})^\perp \\
&\cong \textstyle\sum_{j\in I} \prod_{i\in I} \mathbf{Z}_{i,j}^\perp \quad \text{from (7)} \\
&\cong \textstyle\sum_{j\in I} \prod_{i\in I} \mathbf{T}_{i,j} \qquad (9)
\end{aligned}
$$

Our aim in this section is to obtain a classical completeness theorem using a dinatural interpretation in the $*$-autonomous category $\mathscr{RTA}$. To obtain the theorem, we shall investigate a specific subcategory $\mathscr{D}$ of $\mathscr{RTA}$ in the framework $Dinat^{\mathscr{D}}_{\mathscr{RTA}}$ (*cf.* Definition 2 of Section 2), which we will show is sufficient for the completeness theorem. Remember that $Dinat^{\mathscr{D}}_{\mathscr{RTA}}$ is described as follows: we restrict the interpretation of formulas in such a way that the variables of multivariant functors (whose diagonalizations are interpretations of formulas) range over a fixed subcategory $\mathscr{D}$ of $\mathscr{RTA}$. Note that $\mathscr{D}$ is not necessarily $*$-autonomous. Then the resulting interpretations of the cut-free MLL-proofs of Γ belong to $Dinat^{\mathscr{D}}_{\mathscr{RTA}}(\ \vdash \Gamma)$.

Our techniques are based on the following well-known facts due to Pontrjagin (Pontrjagin 1986):

(i) There is Pontrjagin correspondence between the closed subgroups of a locally compact abelian group and those of its character group.

(ii) There is a duality between compact abelian groups and discrete abelian groups.

The first was established by Pontrjagin (Pontrjagin 1986) through his proof of the duality theorem. The second was the most fundamental but important restriction of his duality

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theorem of the locally compact abelian groups. (i) and (ii) are stated more precisely as follows (Fact 2 and Fact 3).

Fact 2 (Pontrjagin correspondence). For a subgroup H of a locally compact abelian group $G \in \mathscr{LCA}$, we define $A_G(H) := \{\chi \in G^\perp \mid \chi(h) = 1 \quad \forall h \in H\}$, which is a subgroup of $G^\perp$, called the *annihilator* of H in G. The annihilator gives a one-to-one correspondence between the closed subgroups of G and those of $G^\perp$, as the following hold for any closed subgroup H:

1 $H = A_{G^\perp}(A_G(H))$,
2 $(G/H)^\perp \cong A_G(H)$,
3 $G^\perp/A_G(H) \cong H^\perp$.

Let $\mathscr{CA}$ denote the subcategory of compact groups of $\mathscr{LCA}$. Then the Pontrjagin duality theorem restricted to the category $\mathscr{CA}$ gives a duality between $\mathscr{CA}$ and the category $\mathscr{DA}$ of the discrete abelian groups. Obviously, we can treat objects of $\mathscr{DA}$ algebraically, as there is no topology to consider. We remind the reader of several key algebraic properties of abelian groups. The reader may refer to Fuchs (1970) for the general theory of abelian groups: an abelian group is said to be *free* if it is a direct sum of a finite or infinite number of infinite cyclic groups. For example, $\sum_I \mathbf{Z}$ is free for an arbitrary index I, but $\prod_I \mathbf{Z}$ is not free for an infinite index I. Interestingly, it is known that any countable subgroup of $\prod_I \mathbf{Z}$ is free. For a ring $\mathbf{Z}$ of integers, an abelian group G has a natural $\mathbf{Z}$-module structure defined by $n \cdot g := sgn(n)(\overbrace{g + \cdots + g}^{|n|-times})$ for $n \in \mathbf{Z}$ and $g \in G$, where $sgn(n)$ denotes the sign of an integer n. A subset of an abelian group is said to be *linearly independent* if for every finite subset it is linearly independent over $\mathbf{Z}$. Every abelian group has a maximal independent subset and it turns out that any two such subsets have the same cardinality, which we call the *rank* of G. An abelian group is said to be finitely generated if it is spanned by a finite subset (not necessarily linearly independent) over $\mathbf{Z}$. An elementary but important result is the structure theorem for finitely generated abelian groups: that is, every finitely generated abelian group is a direct sum of a finite number of cyclic groups; that is, has a finite basis over $\mathbf{Z}$ (*cf.* Theorem 15.5 of Fuchs (1970)). An abelian group G is called *torsion free* if $n \cdot g \neq e$ for all $g \neq e$ in G and for all $n \neq 0 \in \mathbf{Z}$. An important notion, *property L* (*cf.* (A) page 260 of Pontrjagin (1986)), is defined as follows: a group $H \in \mathscr{DA}$ is said to have *property L* if every finite subset of H is contained in a finitely generated subgroup S such that G/S is torsion free.

The Pontrjagin duality theorem between compact abelian groups and discrete abelian groups can now be described as follows: $(\ )^\perp$ is a contravariant functor between $\mathscr{CA}$ and $\mathscr{DA}$ such that $(\ )^{\perp\perp}$ is naturally equivalent to the identity functor on each of $\mathscr{CA}$ and $\mathscr{DA}$. The following fact is a well-known consequence of the theorem.

Fact 3 (Pontrjagin duality between $\mathscr{CA}$ and $\mathscr{DA}$). Let $\{\zeta_1, \zeta_2\} = \{1, \perp\}$ with $G^1 := G$.

1 Every $C \in \mathscr{CA}$ is topologically isomorphic to a closed subgroup of a product $\prod_{i \in I} \mathbf{T}_i$

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with $\mathbf{T}_i \cong \mathbf{T}$ for all $i \in I$: that is, there exists a monomorphism of $\mathscr{C}\mathscr{A}$

$$C \hookrightarrow \prod_{i\in I} \mathbf{T}_i.$$

Dually, taking into account that $(\prod_I \mathbf{T})^\perp \cong \sum_I \mathbf{T}^\perp \cong \sum_I \mathbf{Z}$ and by the Pontrjagin correspondence (3), every $D \in \mathscr{D}\mathscr{A}$ is isomorphic to a homomorphic image of $\sum_{i\in I} \mathbf{Z}_i$ with $\mathbf{Z}_i \cong \mathbf{Z}$ for all $i \in I$: that is, there exists an epimorphism of $\mathscr{D}\mathscr{A}$

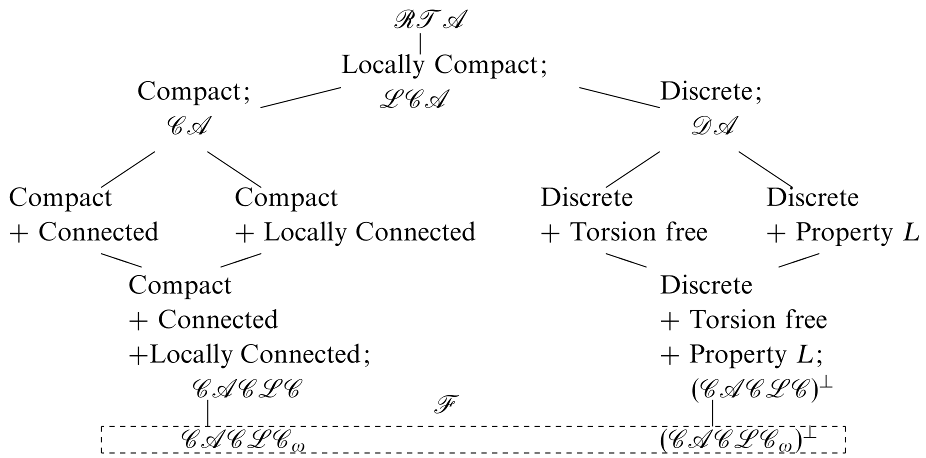
$$\sum_{i\in I} \mathbf{Z}_i \twoheadrightarrow D.$$

2 The following are equivalent (*cf.* Corollary 3.3.8 of Dikranjan *et al.* (1989)):
— $G^{\zeta_1} \in \mathscr{C}\mathscr{A}$ is connected.
— $G^{\zeta_2} \in \mathscr{D}\mathscr{A}$ is torsion free.

3 The following are equivalent (*cf.* Theorem 48 of Pontrjagin (1986)):
— $G^{\zeta_1} \in \mathscr{C}\mathscr{A}$ is locally connected.
— $G^{\zeta_2} \in \mathscr{D}\mathscr{A}$ has property L (*cf.* the paragraph below Fact 2).

4 As a special case of the above (3) where G^{ζ_2} is torsion free with finite rank, the following is equivalent (*cf.* (B) of page 260 of Pontrjagin (1986)):
— $G^{\zeta_1} \in \mathscr{C}\mathscr{A}$ is locally connected.
— $G^{\zeta_2} \in \mathscr{D}\mathscr{A}$ is finitely generated.

$\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}$ denotes the full subcategory of connected, locally connected groups of $\mathscr{C}\mathscr{A}$. $\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}_\omega$ denotes the subcategory of countable objects of $\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}$. Then our completeness theorem will be obtained with respect to a dinatural interpretation $Dinat^{\mathscr{F}}_{\mathscr{R}\mathscr{T}\mathscr{A}}$ where $\mathscr{F}$ denotes the subcategory of $\mathscr{L}\mathscr{C}\mathscr{A}$ consisting of the objects both from $\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}_\omega$ and from $(\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}_\omega)^\perp$, where $(\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}_\omega)^\perp$ denotes the category consisting of the character groups of the objects of $\mathscr{C}\mathscr{A}\mathscr{C}\mathscr{L}\mathscr{C}_\omega$. See the figure below, where the above subcategories of $\mathscr{L}\mathscr{C}\mathscr{A}$ are illustrated:



We prove the following lemma on an object A of $\mathscr{F}$. Proposition 5 together with the lemma becomes crucial in proving the completeness theorem.

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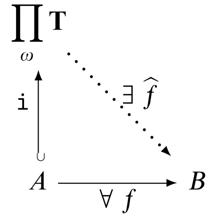
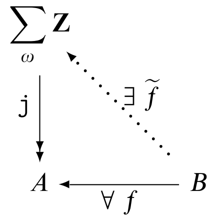
Lemma 5. Let $A \in \mathscr{CACLC}_\omega$, and assume that we have a monomorphism $\mathtt{i} : A \hookrightarrow \prod_\omega \mathbf{T}$. Then every morphism $f : A \to B$ of $\mathscr{LCA}$ can be extended to a morphism $\widehat{f} : \prod_\omega \mathbf{T} \to B$ of $\mathscr{LCA}$ in such a way that the following diagram commutes:



Dually (by virtue of Fact 3 (1)), if $A \in (\mathscr{CACLC}_\omega)^\perp$ and we have an epimorphism $\mathtt{j} : \sum_\omega \mathbf{Z} \twoheadrightarrow A$. Then every morphism $f : B \to A$ of $\mathscr{LCA}$ can be coextended to $\widetilde{f} : B \to \sum_\omega \mathbf{Z}$ in such a way that the following diagram commutes:



Proof. We prove the latter assertion. It suffices to prove that $ker(\mathtt{j})$ is a direct summand of $\sum_\omega \mathbf{Z}$ (note that the proof is purely algebraic since A and $\sum_\omega \mathbf{Z}$ are discrete). If we prove this, then $\sum_\omega \mathbf{Z} \cong ker(\mathtt{j}) \oplus A$ holds since $\mathtt{j}(\sum_\omega \mathbf{Z}) = A$, and we may define $\widetilde{f}$ by the composition of f and the canonical inclusion of A into the summand of $\sum_\omega \mathbf{Z}$. It is well known that if B is a subgroup of A such that A/B is free, then B is a direct summand of A (*cf.* Theorem 14.4 of Fuchs (1970)), thus it is sufficient to prove in the following that $(\sum_\omega \mathbf{Z})/ker(\mathtt{j})$ is free.

By the structure theorem for finitely generated abelian groups (*cf.* the paragraph after Fact 2), every finitely generated torsion free abelian group is free. In the case of countable abelian groups, the following criterion due to Pontrjagin is well known.

Fact 4 (Pontrjagin's criterion (*cf.* Theorem 19.1 of Fuchs (1970))). A countable torsion free group is free if and only if every subgroup of finite rank is free.

We must first show the following lemma.

Lemma 6. Let D be a discrete abelian group that is torsion free. If D has property L (*cf.* the paragraph below Fact 2), then every subgroup of D with finite rank is free.

Proof. From Fact 3 (2) the compact group $D^\perp$ becomes locally connected. Let H be such a subgroup of D that has a finite rank. Then $H^\perp$ is locally connected, since $H^\perp$ is a quotient group of locally connected $D^\perp$ by Pontrjagin correspondence (3). By Fact 3 (4), H is finitely generated. This, together with the fact that H is torsion free, implies that H is free by the structure theorem on finitely generated abelian groups. □

We will now use the above Lemma 6 and Fact 4 to prove that $(\sum_\omega \mathbf{Z})/ker(\mathtt{j})$ is free: since

A is connected and locally connected, A is discrete torsion free and has the property L. By Lemma 6 every subgroup of A with a finite rank is free. This implies by Fact 4 that A is free. Hence $(\sum_\omega \mathbf{Z})/ker(\mathtt{j})$ is free because $(\sum_\omega \mathbf{Z})/ker(\mathtt{j}) \cong \mathtt{j}(\sum_\omega \mathbf{Z}) = A$. □

The following is a corollary of Lemma 5.

Corollary 5. For arbitrary $C \in \mathscr{CACLC}_\omega$ and $D \in (\mathscr{CACLC}_\omega)^\perp$,

$$\mathscr{TG}(C, D) = \mathbf{0}.$$

Proof. From Fact 3 (1) and the duality between $\mathscr{CA}$ and $\mathscr{DA}$, fix a monomorphism $i : C \hookrightarrow \prod_\omega \mathbf{T}$ and an epimorphism $j : \sum_\omega \mathbf{Z} \twoheadrightarrow D$. Lemma 5 implies that for an arbitrary $f : C \to D$ there exists $\widetilde{f} : C \to \sum_\omega \mathbf{Z}$ such that $f = j \circ \widetilde{f}$. Again Lemma 5 tells us that for the $\widetilde{f}$ there exists $\widehat{\widetilde{f}} : \prod_\omega \mathbf{T} \to \sum_\omega \mathbf{Z}$ such that $\widetilde{f} = \widehat{\widetilde{f}} \circ i$. Hence $f = j \circ \widehat{\widetilde{f}} \circ i$ – see the diagram below:

$$\begin{array}{ccc} \prod_\omega \mathbf{T} & \xrightarrow{\;\widehat{\widetilde{f}}\;} & \sum_\omega \mathbf{Z} \\ {\scriptstyle i}\uparrow & \overset{\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle j} \\ C & \xrightarrow[\;f\;]{} & D \end{array}$$

Now $\widehat{\widetilde{f}} \in \mathscr{TG}(\prod_\omega \mathbf{T}, \sum_\omega \mathbf{Z}) = \mathbf{0}$ because $\prod_\omega \mathbf{T}$ is divisible, while 0 is the only divisible subgroup of $\sum_\omega \mathbf{Z}$. This yields $f = 0$. □

Before going to the main result, we first remind the reader of Plotkin's observation, first mentioned in Blute and Scott (1999), which we shall use later.

Lemma 7 (Plotkin's observation (*cf.* Proposition 2.5 of Blute and Scott (1999))). For an arbitrary isomorphism $f : \vec{A} \to \vec{B} \in \mathscr{D}^n$, if

$$\theta \in Dinat^{\mathscr{D}}_{\mathscr{C}}(E(\vec{X};\vec{Y}), F(\vec{X};\vec{Y})),$$

the following diagram commutes:

$$\begin{array}{ccc} E(\vec{A};\vec{A}) & \xrightarrow{\;\theta_{\vec{A}}\;} & F(\vec{A};\vec{A}) \\ {\scriptstyle E((\vec{f})^{-1};\vec{f})}\downarrow & & \downarrow{\scriptstyle F((\vec{f})^{-1};\vec{f})} \\ E(\vec{B};\vec{B}) & \xrightarrow{\;\theta_{\vec{B}}\;} & F(\vec{B};\vec{B}) \end{array}$$

For the remainder of this section, all sequents considered are binary and semisimple (*cf.* Notation 5 in Section 2). We shall remind the reader of the *proof structure* representation of a binary sequent: A binary MLL-sequent $\vdash \Gamma$ corresponds to a unique proof structure without cut links. Moreover, if a sequent is semisimple then the Danos–Regnier condition is simplified, since there exists only one switching in the proof structure. Hence, in the following we often identify a binary semisimple sequent with its uniquely corresponding

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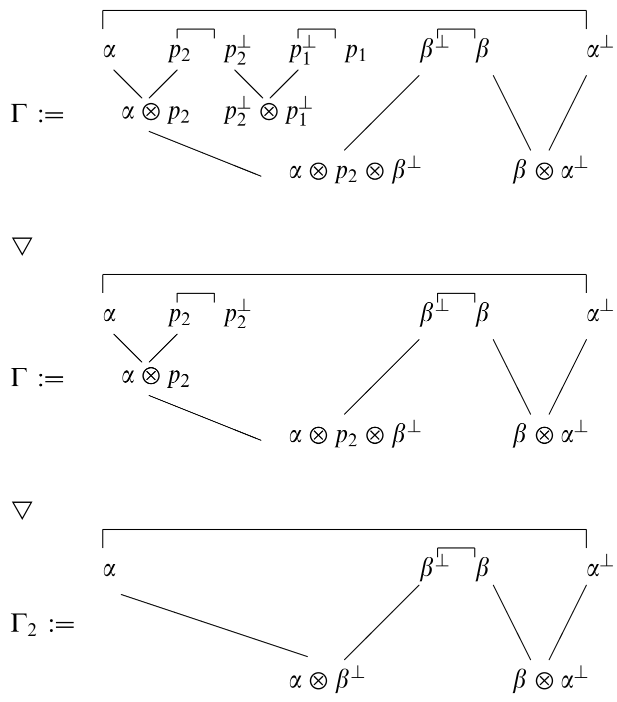
proof structure, which we call *the associated proof structure*, and we often say that a sequent is connected (respectively, has a cycle) when the corresponding proof structure is connected (respectively, has a cycle) with respect to its uniquely determined switching.

The following is a basic syntactic lemma in this section for a semisimple sequent that has a cycle. We shall recall for the reader Notation 4 in Section 2 for the definition of the reduction relation $\rhd$.

Syntactic Lemma 2. Every semisimple sequent $\vdash \Gamma$ that has a cycle is reducible by means of $\rhd$ to a simple sequent $\vdash \Delta$ that has a cycle.

Proof. A *border* vertex in a proof structure is a vertex to which only one link is attached. If the associated proof structure of $\vdash \Gamma$ has a border vertex, then there exists an atom p and a context $\tilde{\Gamma}(\ )$ such that $\vdash \Gamma \equiv \vdash \tilde{\Gamma}(p^{\zeta_2}), p^{\zeta_1}$; hence $\vdash \Gamma \rhd \vdash \tilde{\Gamma}(*)$ and $\vdash \tilde{\Gamma}(*)$ is semisimple and has a cycle. If the associated proof structure of $\vdash \Gamma$ has no border vertices, $\vdash \Gamma$ must be simple. □

For example, the following is a reduction sequence $\Gamma \rhd \Gamma_1 \rhd \Gamma_2$, starting from a semisimple sequent Γ that has a cycle and terminating in a simple sequent Γ_2 that still has a cycle:



Proposition 6. Let $\vdash \Gamma$ be a a semisimple connected binary sequent that has a cycle,

$$Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\ \vdash \Gamma) = \mathbf{0}.$$

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Proof of Proposition 6. By virtue of Syntactic Lemma 2 and Corollary 2′, it suffices to prove the lemma for a simple sequent that has a cycle: that is, we shall prove

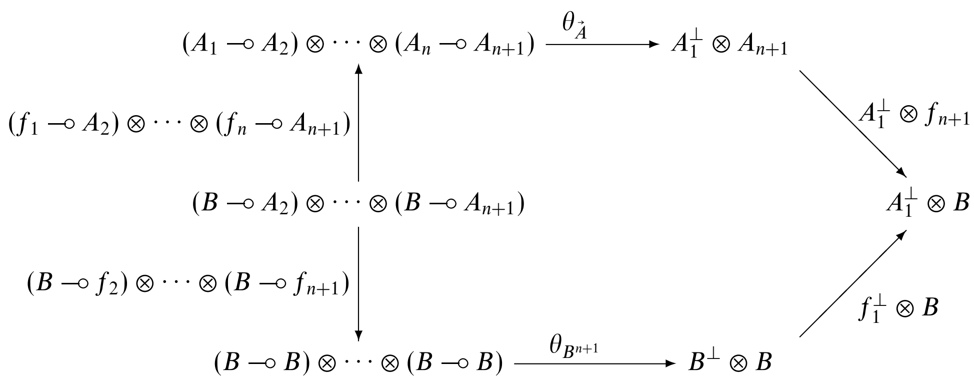
$$Dinat^{\mathscr{F}}_{\mathscr{RTA}}((p_1 \multimap p_2) \otimes \cdots \otimes (p_n \multimap p_{n+1}) \vdash \quad p_1^{\perp} \otimes p_{n+1}) \quad = \quad \mathbf{0}. \tag{10}$$

We first need the following lemma.

Lemma 8 (*cf.* Lemma 9.3 of Blute and Scott (1996)). For an arbitrary θ from the left-hand side of (10) and an arbitrary object X, if $\theta_{X^{n+1}}(id_X, \ldots, id_X) = 0$, then $\forall f_1 \cdots f_n \theta_{X^{n+1}}(f_1, \ldots, f_n) = 0$ with f_i $(i = 1, \ldots, n)$ an endomorphism of X.

Now take θ from the left-hand side of (10). We show that $\theta_{\vec{A}} = 0$ for an arbitrary $\vec{A} \in \mathscr{F}^{n+1}$. We consider three cases:

Case 1: $\vec{A} \in (\mathscr{CACLC}_\omega)^{n+1}$.
From Fact 3 (1), for each k there exists a monomorphism $f_k : A_k \hookrightarrow \prod_{i\in I} \mathbf{T}_i$. Then consider the hexagonal diagram with $B_k := B := \prod_I \mathbf{T}_i$ and f_k described in the above:



By virtue of the natural surjective map of (5), we denote every element in $X \otimes Y$ by $x \otimes y$ with $x \in X$ and $y \in Y$, although it is not uniquely denoted in general. By Proposition 5, $B^{\perp} \otimes B := (\prod_{j\in I} \mathbf{T}_j)^{\perp} \otimes (\prod_{i\in I} \mathbf{T}_i) \overset{(9)}{\cong} \sum_{j\in I} \prod_{i\in I} \mathbf{T}_{ij}$ has no fixed point except 0 under the permutations of I. This implies by Lemma 7 (Plotkin's observation) that $\theta_{B^{n+1}}(id_B, \ldots, id_B) = 0$. By Lemma 8, we obtain that $\theta_{B^{n+1}} = 0$. Then the lower leg of the diagram is 0. Now given an arbitrary $h_i \in A_i \multimap A_{i+1}$, from Lemma 5 there exists an extension $\widehat{h_i} \in B_i \multimap A_{i+1}$ of h_i. Thus $h_i = (f_i \multimap A_{i+1}) \circ \widehat{h_i}$. This implies that $\forall h_i \in A_i \multimap A_{i+1}((A_1^{\perp} \otimes f_{n+1}) \circ \theta_{\vec{A}}(h_1, \ldots, h_n)) = 0$. Since f_{n+1} is monic, $\theta_{\vec{A}} = 0$.

Case 2: $\vec{A} \in ((\mathscr{CACLC}_\omega)^{\perp})^{n+1}$.
In this case, the proof is exactly the dual to that of Case 1. From Fact 3 (1), for each k there exists an epimorphism $f_k : \sum \mathbf{Z} \twoheadrightarrow A_k$. Then consider the hexagonal diagram with

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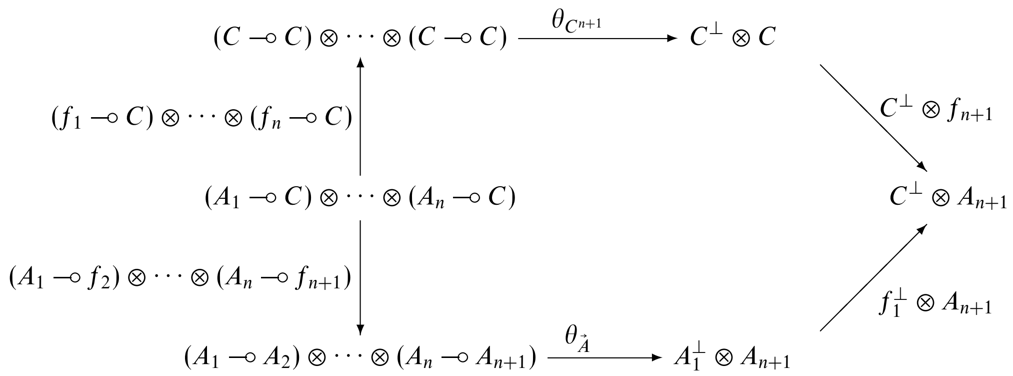
$A_k := \sum_I \mathbf{Z} := C$, $B_k := A_k$ and f_k described in the above:



First consider the uppermost morphism $\theta_{C^{n+1}}$. Since $C^\perp \otimes C := (\sum_{i\in I} \mathbf{Z}_i)^\perp \otimes \sum_{j\in I} \mathbf{Z}_j \overset{(9)}{\cong} \sum_{j\in I} \prod_{i\in I} \mathbf{T}_{ij}$ has no fixed point except 0 under the permutations of I, exactly the same proof as that of Case 1 yields $\theta_{C^{n+1}} = 0$. Then the upper leg of the diagram is 0. Now consider the lowest morphism $\theta_{\vec{A}}$. Given an arbitrary $h_i \in A_i \multimap A_{i+1}$, from Lemma 5 there exists a coextension $\widetilde{h_i} \in A_i \multimap C$ of h_i. Thus $h_i = (A_i \multimap f_{i+1}) \circ \widetilde{h_i}$. This implies $\forall h_i \in A_i \multimap A_{i+1} ((f_1^\perp \otimes A_{n+1}) \circ \theta_{\vec{A}}(h_1, \ldots, h_n) = 0)$. Since f_1 is epic, $\theta_{\vec{A}} = 0$.

Case 3: Remaining Case.

Since the sequent is MLL-equivalent to $(p_1 \multimap p_2) \otimes \cdots \otimes (p_n \multimap p_{n+1}) \otimes (p_{n+1} \multimap p_1) \vdash$, there exists $k+1 \pmod{n+1}$ such that $A_k \in \mathscr{CACLC}_\omega$ and $A_{k+1} \in (\mathscr{CACLC}_\omega)^\perp$. Thus $A_k \multimap A_{k+1} = 0$ from Corollary 5. Hence $\theta_{\vec{A}} = 0$. □

Using Proposition 6, we shall now prove the following completeness theorem for a binary semisimple sequent.

Theorem 3 (Completeness theorem for a binary semisimple sequent). If a binary semisimple sequent $\vdash \Gamma$ is not provable in MLL, then

$$Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\;\vdash \Gamma) = \mathbf{0}.$$

Proof.

The case where $\vdash \Gamma$ is connected: From Danos and Regnier's condition, the corresponding proof structure of $\vdash \Gamma$ must have a cycle. Hence the assertion holds by Proposition 6.

The case where $\vdash \Gamma$ is disconnected: In this case $\vdash \Gamma$ is of the form $\vdash \Gamma_1, \ldots, \Gamma_{n+1}$ with Γ_i $(i = 1, \ldots, n+1)$ being connected and $n \geqslant 1$. From the two Syntactic Lemmas (in Section 2 and Section 3), each $\vdash \Gamma_i$ is reduced to a connected $\vdash \Delta_i$ that is either of the form $\vdash \otimes\mathbf{1}$ or an unprovable simple sequent, depending on whether $\vdash \Gamma_i$ is provable or not. Then $\vdash \Gamma_1, \ldots, \Gamma_{n+1}$ is reduced to $\vdash \Delta_1, \ldots, \Delta_{n+1}$. On the other hand, from Corollary 2′, $\mathtt{dim}(Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\;\vdash \Gamma_1, \ldots, \Gamma_{n+1}\;)) \leqslant \mathtt{dim}(Dinat^{\mathscr{F}}_{\mathscr{RTA}}(\;\vdash \Delta_1, \ldots, \Delta_{n+1}\;))$. Hence it suffices to prove the theorem for $\vdash \Delta_1, \ldots, \Delta_{n+1}$. Without loss of generality we may assume, there exists m with $0 \leqslant m \leqslant n$ such that if $1 \leqslant j \leqslant m$, then $\vdash \Delta_j$ is of the form $\vdash \otimes\mathbf{1}$ and that if $m < j \leqslant n+1$, then $\vdash \Delta_j$ is an unprovable simple sequent.

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If $m \geqslant 2$, by virtue of Proposition 4, the theorem for $\vdash \Delta_1, \ldots, \Delta_{n+1}$ is proved by observing: $Dinat_{\mathscr{RTA}}(\ \vdash \Delta_1, \ldots, \Delta_n, \Delta_{n+1}) := \otimes(\Delta_{n+1}^{\perp}) \multimap \cdots \multimap \otimes(\Delta_2{}^{\perp}) \multimap (⅋\ \ \Delta_1) = \otimes(\Delta_{n+1}^{\perp}) \multimap \cdots \multimap I^{\perp} \multimap I \cong \otimes(\Delta_{n+1}^{\perp}) \multimap \cdots \multimap \perp \multimap I \cong \otimes(\Delta_{n+1}^{\perp}) \multimap \cdots \multimap \mathbf{0} \cong \mathbf{0}$.

We now assume $m \leqslant 1$ in the following. Because $\vdash \Delta_i$ with $m < i \leqslant n+1$ is an unprovable simple sequent, it suffices to prove the following assertion with $k \geqslant 1$:

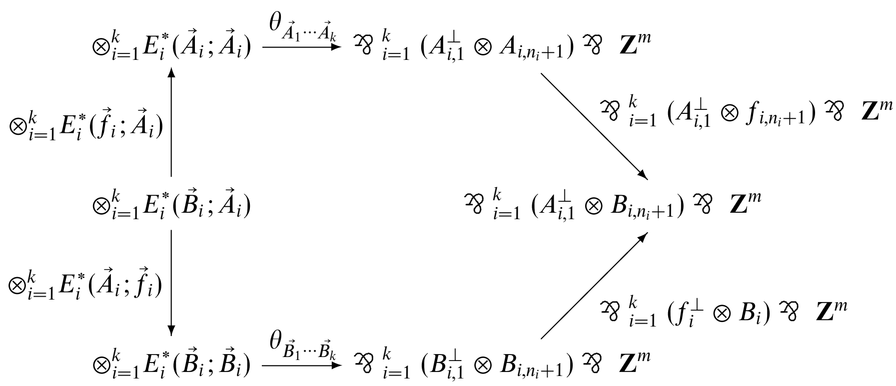
$$Dinat^{\mathscr{T}}_{\mathscr{RTA}}(\ \otimes_{i=1}^{k} [(p_{i,1} \multimap p_{i,2}) \otimes \cdots \otimes (p_{i,n_i} \multimap p_{i,n_i+1})]\ \vdash ⅋_{i=1}^{k} (p_{i,1}^{\perp} \otimes p_{i,n_i+1}),\ (\otimes\mathbf{1})^m) = \mathbf{0}.$$

In the above, $(\otimes\mathbf{1})^m$ denotes $\otimes\mathbf{1}$ if $m = 1$ and does not exist if $m = 0$. We prove in the following that $\theta_{\vec{A}_1\cdots\vec{A}_k} = 0$ holds for an arbitrary dinatural transformation θ from the left-hand side of the above equation. The proof is divided into the following three cases according to objects $\vec{A}_1$. The proof of each case is just a straightforward generalization of the corresponding case of the proof of Proposition 6:

Case 1: $\vec{A}_1 \in (\mathscr{CACLC}_{\omega})^{n_1+1}$.
Let $E_i(X_1, \ldots, X_{n_i+1}) := (X_1 \multimap X_2) \otimes \cdots \otimes (X_{n_i} \multimap X_{n_i+1})$. Consider the following hexagonal diagram with

$$\begin{cases} B_{1,j} := B := \prod_{I_1} \mathbf{T} \text{ and } f_{1,j} : A_{1,j} \hookrightarrow \prod_{I_1} \mathbf{T} & \text{if } i = 1 \\ B_{i,j} := A_{i,j} \text{ and } f_{i,j} := id_{A_{i,j}} & \text{if } i \neq 1. \end{cases}$$

In the above the existence of the above $f_{1,j}$ is guaranteed by Fact 3 (1).



Note that $\mathbf{Z}^m$ denotes $\mathbf{Z}$ if $m = 1$ and denotes $\perp$ if $m = 0$.

First consider the lowest morphism $\theta_{\vec{B}_1\vec{B}_2\cdots\vec{B}_k} := \theta_{(B)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}$. We remind the reader that every action ρ both on X and on Y defines the conjugate action on $X \multimap\!\bullet\, Y$: that is, $f^{\rho}(x) := \rho(f(\rho^{-1}(x)))$ for $f \in X \multimap\!\bullet\, Y$ and for $x \in X$. Since $C\ ⅋\ \ D \cong (D^{\perp} \multimap\!\bullet\, C)^{**}$, we have the following provided that actions considered are trivial on the interpretation $\mathbf{T}$ of $\perp$: if an arbitrary non-zero element of C has an action that does not fix the element but fixes every element of D, then an arbitrary non-zero element of $C\ ⅋\ \ D$ has an action that does not fix the element. Now, since the permutations of I_1 fix every element of $(⅋_{i=2}^{k} (B_{i,1}^{\perp} \otimes B_{i,n_i+1}))\ ⅋\ \ \mathbf{Z}^m$, Proposition 5 implies that the codomain $((\prod_{I_1} \mathbf{T})^{\perp} \otimes \prod_{I_1} \mathbf{T})\ ⅋\ \ ((⅋_{i=2}^{k} (B_{i,1}^{\perp} \otimes B_{i,n_i+1}))\ ⅋\ \ \mathbf{Z}^m)$ of $\theta_{(B)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}$ has no fixed point except 0 under the permutations of I_1.

On the other hand, the element $(id)^{n_1+1} \otimes \vec{h}_2 \otimes \cdots \otimes \vec{h}_k$ in the domain of $\theta_{(B)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}$ is

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fixed under the permutations of I_1. This implies by Lemma 7 (Plotkin's observation) that for all $\vec{h}_2 \cdots \vec{h}_k$,

$$\theta_{(B)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}((id)^{n_1+1}, \vec{h}_2, \ldots, \vec{h}_k) = 0.$$

On the other hand, we can show by the same proof of Lemma 8 that

$$\theta_{(X)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}((id)^{n_1+1}, \vec{h}_2, \ldots, \vec{h}_k) = 0 \Rightarrow \forall \vec{h}_1 \ \{\theta_{(X)^{n_1+1}\vec{A}_2\cdots\vec{A}_k}(\vec{h}_1, \vec{h}_2, \ldots, \vec{h}_k) = 0\}.$$

We have thus obtained that $\theta_{(B)^{n_1+1}\vec{A}_2\cdots\vec{A}_k} = 0$, and hence that the lower leg of the diagram is 0.

Secondly, consider the uppermost morphism $\theta_{\vec{A}_1\cdots\vec{A}_k}$ of the diagram and an arbitrary $\vec{h}_1 \otimes \vec{h}_2 \otimes \cdots \otimes \vec{h}_k$ in its domain. By virtue of Lemma 5, there exist extensions $\widehat{h_{1,1}}, \ldots, \widehat{h_{1,n_1+1}}$, say $\hat{\vec{h}}_1$, of $\vec{h}_1 := h_{1,1}, \ldots, h_{1,n_1+1}$ to $B_{1,1} := \cdots = B_{1,n_1+1} := \prod_\omega \mathbf{T}$. Then $(\otimes_{i=1}^k E_i^*(\vec{f}_i; \vec{A}_i))(\hat{\vec{h}}_1 \otimes \vec{h}_2 \otimes \cdots \otimes \vec{h}_k) = \vec{h}_1 \otimes \vec{h}_2 \otimes \cdots \otimes \vec{h}_k$. This yields $\theta_{\vec{A}_1\cdots\vec{A}_k} \circ (\otimes_{i=1}^k E_i^*(\vec{f}_i; \vec{A}_i)) = 0$. Since f_{1,n_1+1} is monic, $\theta_{\vec{A}_1\cdots\vec{A}_k} = 0$.

Case 2: $\vec{A}_1 \in ((\mathscr{CACLC}_\omega)^\perp)^{n_1+1}$.
In this case the proof is exactly the dual of that of Case 1. Consider the hexagonal diagram of Case 1 with $\vec{A}_i$ and $\vec{B}_i$ exchanged and the following setting:

$$\begin{cases} B_{1,j} := C := \sum_{I_1} \mathbf{Z} \text{ and } f_{1,j} : \sum_{I_1} \mathbf{Z} \twoheadrightarrow A_{1,j} & \text{if } i = 1 \\ B_{i,j} := A_{i,j} \text{ and } f_{i,j} := id_{A_{i,j}} & \text{if } i \neq 1. \end{cases}$$

Since $((\sum_{I_1} \mathbf{Z})^\perp \otimes \sum_{I_1} \mathbf{Z}) ⅋ (⅋_{i=2}^k (B_{i,1}^\perp \otimes B_{i,n_i+1}) ⅋ \mathbf{Z}^m)$ of $\theta_{C^{n_1+1}\vec{A}_2\cdots\vec{A}_k}$ has no fixed point except 0 under the permutations of I_1 and f_{1,n_1+1} is epic, $\theta_{\vec{A}_1\cdots\vec{A}_k} = 0$ is obtained.

Case 3: Remaining Case.
The proof is the same as that of Case 3 of Proposition 6. □

By virtue of Theorem 3 the following theorem can now be proved.

Theorem 4 (Completeness theorem for a binary sequent). If a binary sequent $\vdash \Gamma$ is not provable in MLL, then

$$Dinat^{\mathscr{T}}_{\mathscr{RTA}}(\ \vdash \Gamma) = \mathbf{0}.$$

Proof. Take an arbitrary θ from the left-hand side of the above. By left composing weak distributivities (*cf.* the paragraph below Theorem 1 of Section 2) to θ, we obtain $\tilde{\theta} \in Dinat^{\mathscr{T}}_{\mathscr{RTA}}(\ \vdash \tilde{\Gamma})$ such that $\vdash \tilde{\Gamma}$ is semisimple (and of course is not provable). Theorem 3 implies that $\tilde{\theta} = 0$. Since the weak distributivities in $\mathscr{RTA}$ are monic, we have $\theta = 0$. □

Now, combining the unique interpretation theorem (Theorem 2 of Section 2) and the completeness theorem (Theorem 4 of Section 3), we immediately achieve our full completeness theorem for MLL (without the Mix rule). Our semantic framework for an interpretation of an arbitrary sequent is the Blute–Scott notion of *associated binary space* (Blute and Scott 1996) defined as follows.

Definition 5 ($\mathscr{ABS}$ (*cf.* Blute and Scott (1996))). For a balanced sequent $\Gamma \vdash \Delta$, there

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exists the finite list $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$ of binary sequents of which $\Gamma \vdash \Delta$ is a substitution instance. Then we define a new R-module $\mathscr{ABS}(\Gamma \vdash \Delta)$ by

$$\mathscr{ABS}^{\mathscr{D}}_{\mathscr{C}}(\Gamma \vdash \Delta) := \coprod_{i=1}^{n} Dinat^{\mathscr{D}}_{\mathscr{C}}(\Gamma_i \vdash \Delta_i).$$

Then the following is our full completeness theorem for MLL without the Mix rule.

Theorem 5 (Full completeness theorem for MLL). The R-module $\mathscr{ABS}^{\mathscr{T}}_{\mathscr{RTA}}(\Gamma \vdash \Delta)$ has a basis consisting of the interpretations of the cut free proofs of $\Gamma \vdash \Delta$ in MLL.

Proof. If $\Gamma_i \vdash \Delta_i$ is provable, then $\mathtt{dim}_R(Dinat_{\mathscr{RTA}}(\Gamma_i \vdash \Delta_i)) = 1$ by Theorem 2. If $\Gamma_i \vdash \Delta_i$ is not provable, then $\mathtt{dim}_R(Dinat^{\mathscr{T}}_{\mathscr{RTA}}(\Gamma_i \vdash \Delta_i)) = 0$ by Theorem 4. □

4. Future work

Recently, Barr (Barr 1999) presented an elegant new Chu construction that works uniformly to obtain almost all the $*$-autonomous categories (including $\mathscr{RTA}$) of his monograph Barr (1979). To capture our results in terms of this new Chu construction may help to improve our results, especially our restriction that the sequents considered are binary, which is also the case for Blute and Scott (Blute and Scott 1996). Improvements of the Blute–Scott results in $\mathscr{RTVEC}$ (as a model of $MLL + Mix$) were achieved by Tan (Tan 1997).

An extension of our method in this paper to non-commutative linear logic would be interesting since the only known non-commutative full completeness is Blute and Scott's for Cyclic Linear Logic + Mix (Blute and Scott 1999), which is also analyzed in the author's Hamano (1998).

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