Desargues, Pascal and Kirkman

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1. *Introduction*

In this article, we are going to give a geometrical construction of an outer automorphism of S_6. The terms will be explained below. Very little group theory will be needed: just some elementary facts about permutation groups, which can be found in [1, pp.31-40], for instance. It is a remarkable fact, due to Hölder (see [2]), that among the symmetric groups S_n (permutations of n symbols) the only one to possess any outer automorphisms is S_6. See also [1, pp.132-133]. It is this fact that led J. A. Todd to write his paper [3], whose title I would have loved to use for this article, but Todd got there first.

Here is an outline of the construction. First, Desargues' Theorem is about two triangles in perspective, from a point and from a line. See Figure 1. The diagram consists of ten points and ten lines, with three points on each line and three lines through each point. As an aside, we shall look at the combinatorial symmetries of this configuration.

Next, Pascal's Theorem is about a hexagon inscribed in a conic, and the existence of a certain line called the Pascal line of the hexagon. See Figure 3. Now the same six points can be ordered in various ways to produce sixty different hexagons, and hence sixty Pascal lines. These intersect by threes in sixty Kirkman points, of which there are three on each Pascal line. These sixty lines and points fall into six disjoint Desargues configurations. A permutation of the six hexagon vertices induces a permutation of the six Desargues configurations, and this gives rise to an automorphism of the symmetric group S_6.

A group automorphism $\varphi : G \to G$ is called *inner* if it is given by conjugation, that is, if there is an element $h \in G$ such that $\varphi(g) = h^{-1}gh$, for all $g \in G$. An automorphism which is not inner is called *outer*. Now conjugation in a permutation group preserves cycle structure, so we recognise an automorphism of a permutation group as being an outer automorphism if it does not preserve cycle structure. In our final section we shall show that the automorphism described above is in fact an outer automorphism. There are constructions of outer automorphisms of S_6 scattered around the literature, but they tend to be heavily algebraic, and a purely geometrical construction like the one offered here is a comparatively rare beast. See [4, 5, 6, 7].

In addition to the Pascal lines and Kirkman points mentioned above, Pascal's diagram also gives rise to Steiner points, Cayley lines, Salmon points and Plücker lines, though these will not concern us here. The complete set of points and lines is often called the *Hexagrammum Mysticum*, and a thoroughly readable account of it, together with its history, can be found in [8]. There is also a full account in [9, pp. 379-383], though

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the notation is cumbersome and there are no diagrams. Here we mostly cover just the parts of the theory needed for the construction of the promised automorphism, so there is nothing new in the next four sections, but the final section may possibly be new.

2. *Desargues' Theorem*

Two triangles ABC and $A'B'C$ (see Figure 1) are said to be in *perspective from a point* O if the lines joining corresponding vertices, AA', BB' and CC', are concurrent at O, the *vertex* of perspective. The triangles are said to be *in perspective from a line* ℓ if the meets of corresponding sides, $P = BC \cdot B'C'$, $Q = CA \cdot C'A'$ and $R = AB \cdot A'B'$, lie on ℓ, the *axis* of perspective. Desargues' Theorem says that two triangles are in perspective from a point if, and only if, they are in perspective from a line. The theorem is self-dual: specifically, the 'if' part is the dual of the 'only if' part. There are numerous proofs of it in the literature, by various methods: by multiple applications of Menelaus' Theorem, by homogeneous coordinates, or by three-dimensional methods, for example, and we shall not give a proof here. See [10, p. 70], [11, p. 80] and [12, pp. 121-122] respectively.

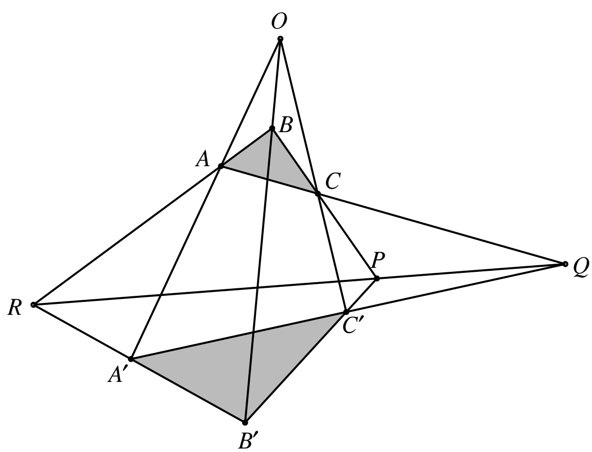

FIGURE 1

Desargues' Theorem is a theorem of projective geometry, and if one wants to use it in a Euclidean context, one needs to take care about the possibility of lines being parallel, with their meets at infinity. So, for example, if ΔABC and $\Delta A'B'C'$ are such that $BC \parallel B'C'$, $CA \parallel C'A'$ and $AB \parallel A'B'$, then they are in perspective from the line at infinity, so must be in perspective from a point. Conversely, if they are in perspective from a point, and $BC \parallel B'C'$, and $CA \parallel C'A'$, then also $AB \parallel A'B'$. Such triangles are called *homothetic*, and the vertex of perspective is their *homothetic centre*.

We now look at the symmetries of the Desargues configuration. These symmetries are not isometries, but rather permutations of the ten vertices that preserve the configuration. So any one of the ten vertices can be taken as the vertex O, and this determines three lines through O, and two triangles each with one vertex on each of these three lines. Choose one of these two triangles, and then attach the labels A, B, C to its vertices in one of the $3! = 6$ possible ways. This now determines the labelling of the rest of the figure. So altogether we can arrange the labels in $10 \times 2 \times 6 = 120 = 5!$ ways, and one immediately suspects that the group G of symmetries might be isomorphic to S_5. To see that this is indeed the case, we look for five objects within the Desargues configuration that are permuted by the symmetries.

In Figure 2, the ten vertices have been relabelled $X_0, X_1, \ldots, X_9$. Looking at the vertices $\{X_0, X_1, X_2, X_3\}$, we see that each is joined to the other three by a line of the configuration, so that they form a complete quadrangle, Q_1. There are four other complete quadrangles within the figure, namely $Q_2 = \{X_0, X_4, X_5, X_6\}$, $Q_3 = \{X_1, X_4, X_7, X_9\}$, $Q_4 = \{X_2, X_5, X_7, X_8\}$ and $Q_5 = \{X_3, X_6, X_8, X_9\}$. See Figure 2 again, where the Q_i are shown bold. So we obtain a homomorphism $\varphi : G \rightarrow S_5$, elements of the first group being thought of as permutations of the subscripts i of the ten points X_i, and the second as permutations of the subscripts j of the five quadrangles Q_j. It is clear that the permutation (14)(25)(36) of X-subscripts belongs to G, swapping Q_1 with Q_2, and mapping each of Q_3, Q_4 and Q_5 to itself, that is, $\varphi\big((14)(25)(36)\big) = (12)$. In like manner, $\varphi\big((04)(27)(39)\big) = (13)$, $\varphi\big((05)(17)(38)\big) = (14)$, and $\varphi\big((06)(19)(28)\big) = (15)$. But the 2-cycles (12), (13), (14), (15) generate S_5, so that φ is surjective. Since $|G| = |S_5|$, it is also injective, and thus we have an isomorphism, $G \cong S_5$.

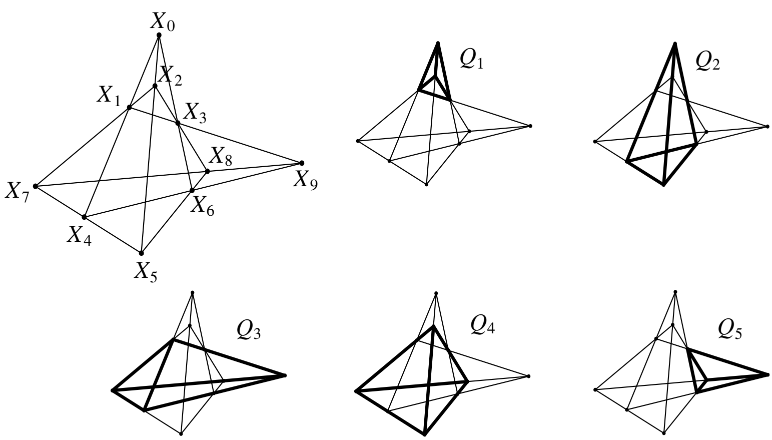

FIGURE 2

There are many obvious ways in which we could identify S_5 with a subgroup of S_{10}, as the set of permutations fixing any chosen five of the ten symbols and permuting the other five. Our subgroup G is not at all like this, however: it is a *transitive* subgroup of S_{10}, that is, for any i and j there is a

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permutation in G sending i to j. The search for transitive subgroups of symmetric groups is ongoing; see, for example, [13, p. 268].

3. *Pascal's Theorem*

Pascal's Theorem is about a hexagon $ABCDEF$ inscribed in a conic (see Figure 3) and says that, if $L = AB \cdot DE$, $M = BC \cdot EF$ and $N = CD \cdot FA$ (the meets of opposite sides of the hexagon), then L, M, N are collinear. The line LMN is the *Pascal line* of $ABCDEF$. As with Desargues, there are numerous proofs in the literature, by various methods: by multiple applications of Menelaus' Theorem, by projective methods, or by three-dimensional methods, for example. See [10, pp. 74-75], [11, p. 143], and [14, pp. 380-383] respectively. There is a proof by Jan van Yzeren [15] which is both elementary and elegant, and which appeared as recently as 1993. It deserves to be better known, so is worth outlining here.

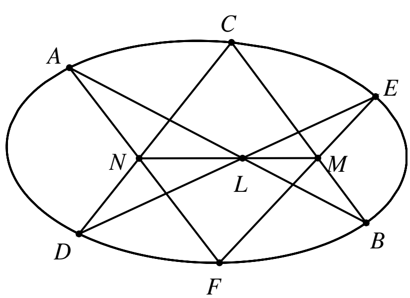

FIGURE 3

First project the conic into a circle, and then put in a second circle, through A, D and L. See Figure 4. Let the lines AF, CD meeet this second circle again in P and Q, respectively, and join PQ and AD. An easy angle-chase, indicated by the marked angles in Figure 4, shows that ΔPQL and ΔFCM are homothetic. Since PF and QC meet at N, this is their homothetic centre, and hence LM passes through N also, as required. This proof is diagram-dependent, and the reader is invited to draw one or two cases where $A, \ldots, F$ lie on the circle in a different order, and adapt the argument appropriately. It is also instructive to look at the case where N is at infinity, that is, when $AF \parallel CD$; here you need to show that $LM \parallel CD$. (If two of L, M, N are at infinity, so is the third; this says that if two pairs of opposite sides of $ABCDEF$ are parallel, so is the third, and the proof of this is easy.)

The dual of Pascal's Theorem is Brianchon's Theorem. This says that if six lines a, b, c, d, e, f touch a conic, and if lines ℓ, m, n are the joins of $a \cdot b$ to $d \cdot e$, $b \cdot c$ to $e \cdot f$, $c \cdot d$ to $f \cdot a$, respectively, then ℓ, m, n are concurrent. Said more concisely, if the hexagon $ABCDEF$ circumscribes a conic, then the diagonals AD, BE, CF are concurrent. The point of concurrency is the *Brianchon point* of the hexagon.

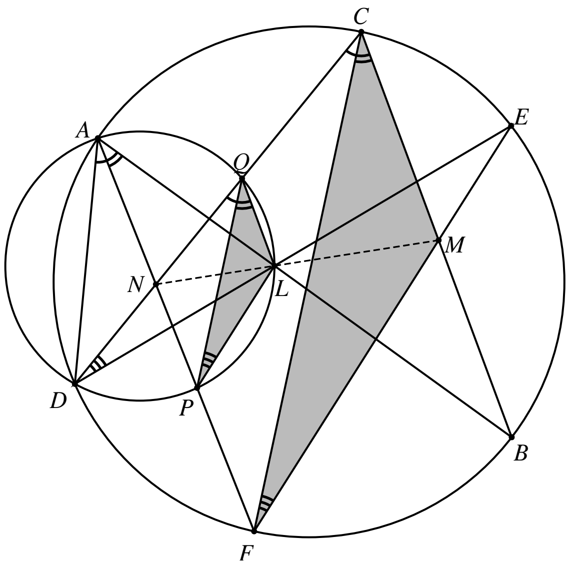

FIGURE 4

The converse of Pascal's theorem is also true, and one can use this (most easily with a dynamic geometry package) to draw a conic through five given points. This is an exercise for the reader. Likewise, the converse of Brianchon's theorem allows one to construct the envelope of a conic touching five given lines. To construct this as a locus instead, first apply Brianchon's theorem to the degenerate hexagon consisting of the six lines a, a, b, c, d, e, which will determine where the conic meets the line a. Do likewise for the other four lines, and then put in the conic through the five points of contact.

4. *The sixty Pascal lines and Kirkman points*

Given six points A, B, C, D, E, F on a conic, they can be ordered in $6! = 720$ ways to form a hexagon; but the labels of each hexagon can be read off starting at any one of its six points, and going around in either direction, which is $6 \times 2 = 12$ ways. So the number of different hexagons is $720 \div 12 = 60$, and hence the six points give rise to 60 Pascal lines. Alternatively, the six points can be joined pairwise to form ${}^6C_2 = 15$ lines, and these lines meet pairwise (other than at the six points) in ${}^6C_2 \times {}^4C_2 \times \frac{1}{2} = 45$ points. Each of the 45 points lies on four Pascal lines: for example, $AB \cdot DE$ lies on the Pascal lines of $ABCDEF$, $ABCEDF$, $BACDEF$ and $BACEDF$. Also, each Pascal line contains three of the 45 points, and so the number of Pascal lines is $45 \times \frac{4}{3} = 60$, as before.

We shall now prove Kirkman's theorem, that the Pascal lines meet by threes in 60 Kirkman points, of which there are three on each Pascal line. First, we need a notation for Pascal lines, and we shall simply write

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$[ABCDEF]$ for the Pascal line of the hexagon *ABCDEF*. This has the merit of being compact, but the disadvantage that there are 12 ways to write the same Pascal line: for example, $[ABCDEF] = [BCDEFA] = [FEDCBA]$.

Let A, B, C, D, E, F lie on a conic. With $L_i, M_i, N_i, i = 1, 2, 3$, as in Figure 5, we have $[ACEBFD] = L_1M_1N_1$, $[CEADBF] = L_2M_2N_2$, and $[EACFDB] = L_3M_3N_3$. Since ΔACE and ΔBDF are inscribed in a conic, their sides touch a second conic. (This is [11, p. 146, Theorem 20], essentially Poncelet's Porism.) The hexagon $L_1M_2L_3M_1L_2M_3$ is circumscribed to this second conic, and its Brianchon point X, the intersection of its diagonals L_1M_1, L_2M_2 and L_3M_3, is the required Kirkman point, the intersection of the three Pascal lines $[ACEBFD]$, $[CEADBF]$ and $[EACFDB]$.

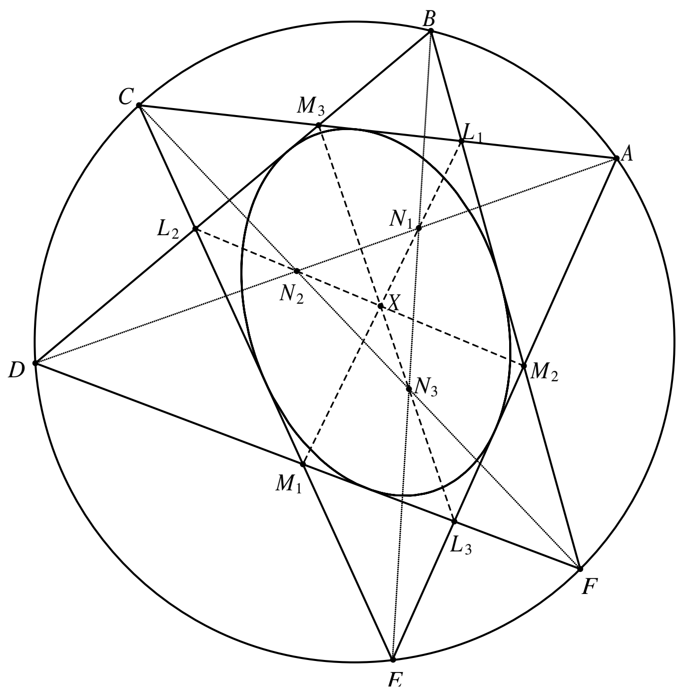

FIGURE 5

The three hexagons *ACEBFD*, *CEADBF* and *EACFDB* are shown separately in Figure 6. Notice that each of them is made up entirely from *diagonals*, not sides, of the hexagon *ABCDEF*. Following [8], we say that two hexagons on the same six vertices are *disjoint* if they do not have a side in common, that is, if one is made up entirely of diagonals of the other, in either order. It is a simple matter, which we leave the reader to verify, that *ACEBFD*, *CEADBF* and *EACFDB* are the *only* hexagons on A, B, C, D, E, F which are disjoint from *ABCDEF*, and in consequence of this it makes sense to use the notation $\langle ABCDEF\rangle$ for the Kirkman point which is their

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intersection. Just as with Pascal lines, there are 12 ways of writing the same Kirkman point: for example, $\langle ABCDEF\rangle = \langle BCDEFA\rangle = \langle FEDCBA\rangle$.

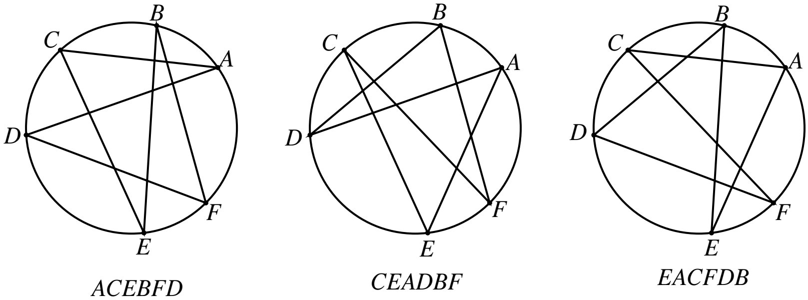

FIGURE 6

So, there are 60 Kirkman points, one corresponding to each hexagon on A, B, C, D, E, F. Further, a particular Kirkman point lies on a particular Pascal line if, and only if, their corresponding hexagons are disjoint. And, just as there are three Pascal lines, $[ACEBFD]$, $[CEADBF]$ and $[EACFDB]$, through the Kirkman point $\langle ABCDEF\rangle$, so there are three Kirkman points, $\langle ACEBFD\rangle$, $\langle CEADBF\rangle$ and $\langle EACFDB\rangle$, on the Pascal line $[ABCDEF]$.

5. *The six Desargues configurations*

We shall now show that the 60 Pascal lines and 60 Kirkman points fall into six disjoint sets, each one a Desargues configuration of 10 lines and 10 points. To make comparisons easier, we shall write each hexagon in the first, lexicographically, of its 12 possible representations. So we start at A and then move around the hexagon in the direction of whichever of its immediate neighbours comes earlier in the alphabet.

So if we start with the Kirkman point $\langle ABCDEF\rangle$, this lies on the three Pascal lines $[ACEBFD]$, $[ADBFCE]$ and $[ACFDBE]$. On the first of these lie $\langle ABDCFE\rangle$ and $\langle AEDBCF\rangle$, on the second lie $\langle ABEFDC\rangle$ and $\langle ACBEDF\rangle$, and on the third lie $\langle ABFECD\rangle$ and $\langle ADECBF\rangle$, and so on. The relationships are shown schematically in Figure 7, where we have relabelled the hexagons $a = ABCDEF$, etc, as in the Key, and verification of the remaining details is left to the reader. We clearly have a Desargues configuration here; note in particular that if we choose a Kirkman point, say $\langle a\rangle = \langle ABCDEF\rangle$ as vertex of perspective, then the axis of perspective is the Pascal line corresponding to the same hexagon, in this case $[a] = [ABCDEF]$.

Since, in Figure 7, we have used up all the Pascal lines through each of the Kirkman points in the diagram, and all of the Kirkman points on each of the Pascal lines in the diagram, then if we start again with a Kirkman point not in Figure 7, we shall obtain a second Desargues configuration disjoint from the first. And so on, until we have used up all 60 points and lines, and thus we obtain our six disjoint Desargues configurations, as promised.

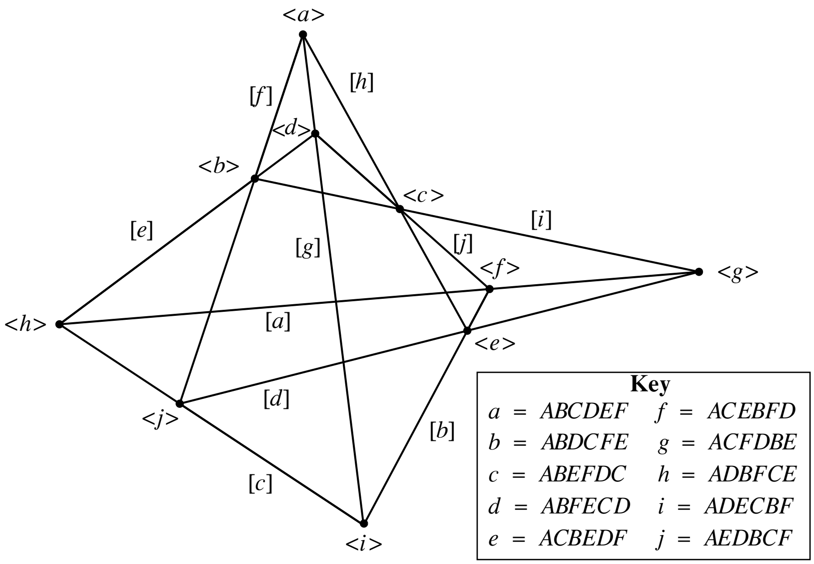

FIGURE 7

There are 10 hexagons corresponding to the Desargues configuration in Figure 7, each occurring once as a point and once as a line. They are, in lexicographic order,

$$\begin{array}{ccccc} ABCDEF, & ABDCFE, & ABEFDC, & ABFECD, & ACBEDF, \\ ACEBFD, & ACFDBE, & ADBFCE, & ADECBF, & AEDBCF. \end{array} \qquad (1)$$

Notice that, in the above set of 10 hexagons, the vertex B has as its neighbours A and C only in the first hexagon. Now, from the list of ten hexagons, one can choose a hexagon and then a vertex of that hexagon (with its neighbours) in $10 \times 6 = 60$ ways. On the other hand, given the six vertices, one can choose one of them and then two others to be its neighbours, in $6 \times {}^5C_2 = 60$ ways. We leave the reader to check that each of these 60 ways, just like B with neighbours A and C, occurs *precisely once* in the list (1) of ten hexagons. As a consequence, the six Pascal lines

$$[ABCDEF], \quad [ABCDFE], \quad [ABCEDF],$$
$$[ABCEFD], \quad [ABCFDE], \quad [ABCFED]$$

(and the six corresponding Kirkman points) will all occur in *different* Desargues configurations, and so one in each, since the number of these configurations is also six. We can use hexagons to label the different Desargues configurations, and we shall write $\widehat{ABCDEF}$, for example, to denote the configuration depicted in Figure 7, or, more specifically, to denote the set of ten hexagons listed in (1), above. However, because there are 10 Pascal lines in each Deasargues configuration, and 12 ways of writing

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each hexagon, there are now 120 notations for the each Desargues configuration. It will help to have an abbreviation, and so we shall write $\hat{1} = \widehat{ABCDEF}$, and so on; specifically

$$\hat{1} = \widehat{ABCDEF}, \quad \hat{2} = \widehat{ABCDFE}, \quad \hat{3} = \widehat{ABCEDF}, \qquad (2)$$
$$\hat{4} = \widehat{ABCEFD}, \quad \hat{5} = \widehat{ABCFDE}, \quad \hat{6} = \widehat{ABCFED}.$$

6. *An outer automorphism of* S_6

Let G be the group of permutations of $\{A, B, C, D, E, F\}$, and let H be the group of permutations of $\{\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}\}$. Obviously G and H are isomorphic, by identifying $A, B, \ldots$ with $\hat{1}, \hat{2}, \ldots$ in some order; this gives one of $6! = 720$ different isomorphisms. Note that under any of these isomorphisms, any permutation in G and its image in H have the same cycle structure. We are about to use the foregoing geometry to construct an isomorphism $\varphi : G \to H$ which does *not* preserve cycle structure, so is not one of the above 720 isomorphisms.

Write the action of $\sigma \in G$ on A, for example, as $A \to A^\sigma$. Then σ induces a permutation of our sixty hexagons by acting on all six vertices, by $ABCDEF \to |ABCDEF|^\sigma = A^\sigma B^\sigma C^\sigma D^\sigma E^\sigma F^\sigma$, for example. (We write $|ABCDEF|$ in place of $(ABCDEF)$ to avoid confusion with the 6-cycle $(ABCDEF)$.) If P, K are hexagons on $A, \ldots, F$ then we write $[P] \to [P]^\sigma$ and $\langle K\rangle \to \langle K\rangle^\sigma$ for the action on the corresponding Pascal lines and Kirkman points. It is clear that two hexagons are disjoint if, and only if, their images under this action are disjoint, and hence $\langle K\rangle$ lies on $[P]$ if, and only if, $\langle K\rangle^\sigma$ lies on $[P]^\sigma$. Consequently, if $P \in \hat{i}$ and $P^\sigma \in \hat{j}$, for some $\hat{i}, \hat{j}$, then we can write $\hat{i}^{\bar{\sigma}} = \hat{j}$, and we have a well-defined permutation $\bar{\sigma} \in H$. Define $\varphi : G \to H$ by $\varphi : \sigma \to \bar{\sigma}$.

It is obvious that φ is a homomorphism, and we shall show shortly that it is an isomorphism. But first consider the 2-cycle $\sigma = (EF)$. We have $|ABCDEF|^\sigma = ABCDFE$, so that, looking at (2), we obtain $\hat{1}^{\bar{\sigma}} = \hat{2}$, and likewise $\hat{3}^{\bar{\sigma}} = \hat{5}$ and $\hat{4}^{\bar{\sigma}} = \hat{6}$. Since $\sigma^2 = 1$, we have $\overline{(EF)} = (\hat{1}\hat{2})(\hat{3}\hat{5})(\hat{4}\hat{6})$, so that φ does *not* preserve cycle-structure.

Below are two more examples, which will be useful in what follows. But first, with one eye on (2), put $\tau_1 = 1$, $\tau_2 = (EF)$, $\tau_3 = (DE)$, $\tau_4 = (DEF)$, $\tau_5 = (DFE)$ and $\tau_6 = (DF)$, so that $\hat{1}^{\bar{\tau}_i} = \hat{i}$, for $1 \leqslant i \leqslant 6$.

1. Let $\sigma = (AB)(CD)(EF)$. Now $ABCDEF \in \hat{1}$, and $|ABCDEF|^\sigma = BADCFE = ABEFCD$. Then $ABEFCD = |ABFECD|^{\tau_2}$. But, looking at (1), we see that $ABFECD \in \hat{1}$, and we know that $\hat{1}^{\bar{\tau}_2} = \hat{2}$, so we deduce that $\hat{1}^{\bar{\sigma}} = \hat{2}$.

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Again, $ABCDFE \in \hat{2}$, and $|ABCDFE|^{\sigma} = BADCEF = ABFECD$. But, from (1), $ABFECD \in \hat{1}$, so $\hat{2}^{\bar{\sigma}} = \hat{1}$.

Continuing, we apply σ to the other polygons listed in (2); we have

$$|ABCEDF|^{\sigma} = ABECFD = |ABDCFE|^{\tau_3}$$

$$|ABCEFD|^{\sigma} = ABCEFD$$

$$|ABCFDE|^{\sigma} = ABFCED = |ABDCFE|^{\tau_5}$$

$$|ABCFED|^{\sigma} = ABCFED.$$

Since $ABDCFE \in \hat{1}$, it follows that the hexagons on the right are in $\hat{3}, \hat{4}, \hat{5}, \hat{6}$ respectively, so that $\bar{\sigma}$ fixes $\hat{3}, \hat{4}, \hat{5}$ and $\hat{6}$. We conclude that $\bar{\sigma} = (\hat{1}\hat{2})$.

2. Let $\sigma = (AE)(CFD)$. We have

$$|ABCDEF|^{\sigma} = ACFBED = |ACEBFD|^{\tau_2}$$

$$|ABCDFE|^{\sigma} = ADCFBE = |ADBFCE|^{\tau_3}$$

$$|ABCEDF|^{\sigma} = ACDEBF = |ACFDBE|^{\tau_4}$$

$$|ABCEFD|^{\sigma} = ADCEBF = |ADBFCE|^{\tau_5}$$

$$|ABCFDE|^{\sigma} = ACDFBE = |ACFDBE|^{\tau_6}$$

$$|ABCFED|^{\sigma} = ACEBFD = |ACEBFD|^{\tau_1}.$$

Since here the hexagons on which τ_i acts are all in $\hat{1}$, it follows that, in this case, $\bar{\sigma} = (\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6})$.

We now identify G and H with the symmetric group S_6 of all permutations of $\{1, 2, 3, 4, 5, 6\}$, by putting $A = 1$, $B = 2$, etc, and $\hat{1} = 1$, $\hat{2} = 2$, etc. This makes φ into an endomorphism of S_6 (a homomorphism $S_6 \to S_6$), and, by the above calculations,

$$\varphi((12)(34)(56)) = (12) \qquad \text{and} \qquad \varphi((15)(364)) = (123456).$$

But these two permutations generate S_6, and it follows that φ is surjective, and hence is an automorphism; and, because it does not preserve cycle structure, it is an outer automorphism of S_6, as promised.

As a parting shot, we note that the theory above gives another way of identifying the symmetries of the Desargues configuration with S_5. For these symmetries can be identified with the subgroup of G which permutes the ten hexagons listed in (1), that is, the hexagons belonging to $\hat{1}$, and this maps by φ to the stabiliser of $\hat{1}$ in H, the permutations that fix $\hat{1}$. But this is the same

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as the subgroup consisting of all permutations of $\hat{2}$, $\hat{3}$, $\hat{4}$, $\hat{5}$, $\hat{6}$, which is obviously isomorphic to S_5, done.

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10.1017/mag.2020.14

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