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ALGEBRAIC EHP SEQUENCES REVISITED

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Abstract The algebraic EHP sequences, algebraic analogues of the EHP sequences in homotopy theory, are important tools in algebraic topology. This note will outline two new proofs of the existence of the algebraic EHP sequences. The first proof is derived from the minimal injective resolution of the reduced singular cohomology of spheres, and the second one follows Bousfield's idea using the loop functor of unstable modules.

Keywords: EHP sequence; homotopy groups of spheres; James model; Steenrod algebra; unstable modules

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1. Introduction

The James model, $\Omega\Sigma X$, of the (based) loop space of the suspension of a connected space X allows us to define the Hilton–Hopf invariants, $\Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$, which induce the famous theorem of Milnor and Hilton:

$$\Sigma\Omega\Sigma X \simeq \bigvee_{n\geq 1} \Sigma X^{\wedge n}.$$

When X is the sphere S^n, the second Hilton–Hopf invariant induces a fibration sequence after localization at the prime 2 [**5**]:

$$S^n \to \Omega\Sigma S^n \to \Omega\Sigma(S^n \wedge S^n). \tag{1.1}$$

At an odd prime p, matters depend on the parity of n. For the even case, we have

$$\Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1},$$

so the case of an even-dimensional sphere is reduced to the case of odd spheres. Now, for the odd case, localized at p, there is a fibre sequence [**14**]:

$$X \to \Omega S^{2m+1} \to \Omega S^{2pm+1}, \tag{1.2}$$

where

$$X = S^{2m} \cup \left(\bigcup_{i=2}^{p-1} e^{2im} \right)$$

is the $(2pm-1)$-skeleton of ΩS^{2m+1}. Localized at p, there is also a fibration [**14**]:

$$S^{2m-1} \to \Omega X \to \Omega S^{2pm-1}. \tag{1.3}$$

The long exact sequences of homotopy groups associated with the fibration sequences (1.1), (1.2) and (1.3) are known as the EHP sequences and provide an inductive method for computing $\pi_{n+k}(S^n)$, beginning with our knowledge of $\pi_*(S^1)$. The homotopy groups of spheres can also be computed *via* another algebraic invariant (which is simpler and well understood): the reduced singular cohomology. These computations are carried out with the help of the *unstable Adams spectral sequence* (UnASS), introduced by Massey and Peterson in [**8**], generalized by Bousfield and Curtis in [**2**], and generalized further by Bousfield and Kan in [**3**]. Denoting by $\Sigma^n\mathbb{F}_p$ the reduced cohomology $\tilde{\mathrm{H}}^*(S^n;\mathbb{F}_p)$, the UnASS is formulated as follows:

$$E_2^{s,t}(S^n) = \mathrm{Ext}^s_{\mathscr{U}}(\Sigma^n\mathbb{F}_p, \Sigma^t\mathbb{F}_p) \Longrightarrow \pi_{t-s}(S^n)^\wedge_p.$$

Here, $\mathscr{U}$ is the category of unstable modules over the Steenrod algebra $\mathcal{A}_p$. In [**2**, **4**], it is shown that the E_2 page of the UnASS for S^n is isomorphic to the homology of a certain differential bigraded module $\Lambda(n)$, which is a submodule of the Lambda algebra Λ. At the prime 2, for each non-negative integer n, there is a short exact sequence

$$0 \to \Lambda(n) \to \Lambda(n+1) \to \Lambda(2n+1) \to 0$$

whose associated long exact sequence is

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2n+1}) \xrightarrow{P} E_2^{s,t}(S^n) \xrightarrow{E} E_2^{s,t+1}(S^{n+1}) \xrightarrow{H} E_2^{s-1,t}(S^{2n+1}) \xrightarrow{P} \cdots. \tag{1.4}$$

At odd primes, there are also long exact sequences:

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2pn+1}) \xrightarrow{P} E_2^{s,t}(S^{2n}) \xrightarrow{E} E_2^{s,t+1}(S^{2n+1}) \xrightarrow{H} E_2^{s-1,t}(S^{2pn+1}) \xrightarrow{P} \cdots, \tag{1.5}$$

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2pn-1}) \xrightarrow{P} E_2^{s,t}(S^{2n-1}) \xrightarrow{E} E_2^{s,t+1}(S^{2n}) \xrightarrow{H} E_2^{s-1,t}(S^{2pn-1}) \xrightarrow{P} \cdots. \tag{1.6}$$

The sequences (1.4), (1.5) and (1.6) are called the algebraic EHP sequences.

In [**9**], the author gave an algorithm, called the BG algorithm, to compute the minimal injective resolution of $\Sigma^t\mathbb{F}_p$, in the category $\mathscr{U}$, based on the Mahowald short exact sequences. In this paper, we will give a slightly different presentation of this algorithm to construct injective resolutions of ΣN, where N is an unstable module, and use this to construct the algebraic EHP sequences.

Bousfield's method gives an abstract construction of the algebraic EHP sequences. Bousfield observes that the key to the existence of these sequences lies in the simple form of the reduced singular cohomology of spheres: they are the suspension of unstable modules (an unstable module is a suspension if it is of the form $\Sigma M := \Sigma\mathbb{F}_p \otimes M$ for some unstable module M). The suspension functor $\Sigma : \mathscr{U} \to \mathscr{U}$ is exact and admits a left adjoint, denoted by Ω (also known as the loop functor of unstable modules). Therefore, Ω is right exact and preserves projective unstable modules. In [**13**], the functor Ω, its k-fold iterate Ω^k and their left-derived functors are studied. In particular, the left-derived functors of Ω^k, denoted by Ω^k_*, are zero in homological degrees greater than k. Moreover, if M is an unstable module, then ΩM and $\Omega_1 M$ fit in an exact sequence:

$$0 \to \Sigma\Omega_1 M \to \Phi M \xrightarrow{\lambda_M} M \xrightarrow{\sigma_M} \Sigma\Omega M \to 0.$$

Here, Φ is an avatar for the Frobenius twist of the category $\mathscr{U}$ and σ_M is the unit of the adjunction $(\Omega \dashv \Sigma)$. (See § 2 for the construction of this sequence.) This property serves as the main ingredient in Bousfield's proof of the existence of the general algebraic EHP sequences.

Theorem 3.1. *For all unstable modules M and N, there exists a long exact sequence*

$$\cdots \to \mathrm{Ext}^{s-2}_{\mathscr{U}}(\Omega_1 M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N) \to \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Omega_1 M, N) \to \cdots$$

of Ext*-groups.*

1.1. Organization of the paper

We begin with some basic definitions and notation. In § 2, we recall the Steenrod algebra $\mathcal{A}_p$ and unstable $\mathcal{A}_p$-modules. We also recall the loop functor of unstable modules and study its left-derived functors.

Bousfield's construction is described in § 3, and we study a special case where the algebraic EHP sequence splits into short exact sequences.

We recall the BG algorithm in § 4 and use this to show the existence of the algebraic EHP sequence in § 5.

2. Unstable modules and the loop functor

Following Adem [**1**], the Steenrod algebra $\mathcal{A}_p$ at the prime p is generated by the stable cohomology operations P^i of degree $2i(p-1)$, $i \geq 0$, and the Bockstein β of degree 1, subject to the Adem relations. At the prime 2, the generators of the Steenrod algebra $\mathcal{A}_2$ are the Steenrod squares Sq^i of degree $i \geq 0$.

Definition 2.1 (unstable modules). An unstable module M is an $\mathbb{N}$-graded $\mathbb{F}_p$-vector space over the Steenrod algebra satisfying the instability condition:

- for $p = 2$: $\forall x \in M^n$, $Sq^i x = 0$ if $i > n$;
- for $p > 2$: $\forall x \in M^n$, $\beta^e P^i x = 0$ if $e + 2i > n$, where $e \in \{0, 1\}$.

Let $\mathscr{U}$ denote the category of unstable modules. Denote by $\Sigma^n\mathbb{F}_p$ the reduced singular cohomology of the sphere S^n, we write $\Sigma^n M$ for the tensor product $\Sigma^n\mathbb{F}_p \otimes M$. Then the

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correspondence $M \mapsto \Sigma M$, for all $M \in \mathscr{U}$, defines an endofunctor of $\mathscr{U}$, denoted by Σ and called the suspension functor.

Proposition 2.2 (see [10]). *The functor Σ is exact and admits a left adjoint, denoted by Ω, as well as a right adjoint, denoted by $\tilde{\Sigma}$.*

The category $\mathscr{U}$ is an abelian category with enough injectives and projectives. For a non-negative integer n, the injective envelope $J(n)$ of $\Sigma^n \mathbb{F}_p$, called the nth Brown–Gitler module, satisfies natural isomorphisms

$$\mathrm{Hom}_{\mathscr{U}}(M, J(n)) \cong \mathrm{Hom}_{\mathbb{F}_p}(M^n, \mathbb{F}_p).$$

Therefore, the Brown–Gitler modules form a system of injective co-generators for $\mathscr{U}$.

We now define a system of projective generators of $\mathscr{U}$. Instead of taking the injective envelope of $\Sigma^n \mathbb{F}_p$, we consider its projective cover $F(n)$. These $F(n)$ satisfy natural isomorphisms

$$\mathrm{Hom}_{\mathscr{U}}(F(n), M) \cong M^n.$$

Hence, the $F(n)$ form a system of projective generators for $\mathscr{U}$.

In what follows, we study the morphism $\sigma(F(n)) : F(n) \to \Sigma\Omega F(n)$, where $\sigma : Id \to \Sigma\Omega$ is the unit of the adjunction $(\Omega \dashv \Sigma)$. For this purpose, we recall the functor Φ. Let M be an unstable module and $x \in M^n$, we define:

$$\mathrm{Sq}_0 x = Sq^n x \quad \text{and} \quad \mathrm{P}_0 x = \begin{cases} P^k x & \text{if } n = 2k, \\ \beta P^k x & \text{if } n = 2k+1. \end{cases}$$

We write ΦM for the unstable module, concentrated in even degrees, such that

$$(\Phi M)^{2n} \cong M^n, \text{for } p = 2, \qquad \text{and} \qquad (\Phi M)^{2n} \cong \begin{cases} M^{2k} & \text{if } n = pk, \\ M^{2k+1} & \text{if } n = pk+1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } p > 2,$$

and the action of the Steenrod algebra is given by the following:

- for $p = 2$:

$$Sq^n \Phi x = \begin{cases} \Phi Sq^k x & \text{if } n = 2k, \\ 0 & \text{otherwise;} \end{cases}$$

- for $p > 2$:

$$\beta \Phi x = 0,$$

$$P^n \Phi x = \begin{cases} \Phi P^k x & \text{if } n = pk, \\ \Phi P^k x & \text{if } n = pk+1 \text{ and } |x| \equiv 1(2), \\ 0 & \text{otherwise.} \end{cases}$$

This defines an exact functor $\Phi : \mathscr{U} \to \mathscr{U}$. The correspondences $\Phi x \mapsto \mathrm{P}_0 x$ at odd primes and $\Phi x \mapsto \mathrm{Sq}_0 x$ at the prime 2 yield a natural transformation λ from the functor Φ to the identity functor. It follows from [10, Proposition 1.7.3] that there is an isomorphism from $\Phi F(n)$ to the kernel of $\sigma(F(n))$.

Proposition 2.3 (see [10]). *For each positive integer n, the sequence*

$$0 \to \Phi F(n) \xrightarrow{\lambda(F(n))} F(n) \xrightarrow{\sigma(F(n))} \Sigma\Omega F(n) \to 0 \tag{2.1}$$

is exact.

Throughout this note, we abbreviate $\lambda(M)$ as λ_M and $\sigma(M)$ as σ_M. On the one hand, the functor Φ is exact, and, on the other hand, $F(n)$, $n \geq 0$, form a system of projective generators of $\mathscr{U}$; thus, we can use the exact sequences (2.1) to describe the transformation λ. It is well known that the left-derived functors of Ω, denoted by Ω_s, $s \geq 0$, are zero on homological degrees greater than one.

Proposition 2.4 (see [10]). *Let M be an unstable module. Then $\Omega_s M$ are trivial for all $s > 1$. Moreover, $\Omega_1 M$ and ΩM fit in the following exact sequence:*

$$0 \to \Sigma\Omega_1 M \to \Phi M \xrightarrow{\lambda_M} M \xrightarrow{\sigma_M} \Sigma\Omega M \to 0.$$

Corollary 2.5. *Let M be an unstable module such that $\Omega_1 M$ is trivial. Then the functor Ω sends a projective resolution of M to a projective resolution of ΩM.*

Proof. This follows directly from Proposition 2.4. □

Remark 2.6.

- For all unstable modules M, the morphism $\lambda_{\Sigma M}$ is trivial. Hence,

$$\Omega\Sigma M \cong M \quad \text{and} \quad \Sigma\Omega_1\Sigma M \cong \Phi\Sigma M, \quad \forall\, M \in \mathscr{U}.$$

- The loop of σ_M is the identity of ΩM. As the loop functor Ω is right exact, then $\Omega\lambda_M$ is trivial.

Lemma 2.7. *There are natural isomorphisms of unstable modules*

$$\Phi\Sigma\Omega M \cong \Sigma\Omega\Phi M,$$
$$\Phi\Sigma\Omega_1 M \cong \Sigma\Omega_1\Phi M.$$

Proof. It follows from the definition of P_0 and Sq_0 that for all unstable modules M and all $x \in M$, we have

$$\begin{aligned} \mathrm{Sq}_0\Phi x &= \Phi\mathrm{Sq}_0 x && \text{if } p = 2,\\ \mathrm{P}_0\Phi x &= \Phi\mathrm{P}_0 x && \text{if } p > 2. \end{aligned}$$

Hence, $\Phi\lambda_M = \lambda_{\Phi M}$ for all unstable modules M. Applying the exact functor Φ to the sequence

$$0 \to \Sigma\Omega_1 M \to \Phi M \xrightarrow{\lambda_M} M \xrightarrow{\sigma_M} \Sigma\Omega M \to 0,$$

we obtain the desired isomorphisms. □

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Lemma 2.8. *Let M be an unstable module and let $\{P_k, \partial_k\}_{k\geq 0}$ be a projective resolution of M. Denote by C the co-kernel $\operatorname{Coker}(\Omega\partial_1 : \Omega P_2 \to \Omega P_1)$. Then, $\{\Omega P_k, \Omega\partial_k\}_{k\geq 1}$ is a projective resolution of C. Moreover, C fits in the short exact sequence:*

$$0 \to \Omega_1 M \to C \to \frac{\Omega P_1}{\operatorname{Ker}(\Omega\partial_0)} \to 0.$$

Proof. It follows from Proposition 2.3 that $\{\Omega P_k, \Omega\partial_k\}_{k\geq 1}$ is a resolution of C. As Ω is left adjoint to Σ, which is an exact functor, Ω sends a projective module to a projective one. Therefore, $\{\Omega P_k, \Omega\partial_k\}_{k\geq 1}$ is a projective resolution of C. The other conclusion follows from the fact that

$$C = \frac{\Omega P_1}{\operatorname{Im}(\Omega\partial_1)} \quad \text{and} \quad \Omega_1 M = \frac{\operatorname{Ker}(\Omega\partial_0)}{\operatorname{Im}(\Omega\partial_1)}.$$

□

3. Projective resolutions and the algebraic EHP sequences

An interesting fact about the algebraic EHP sequence: it can be derived in a completely abstract way. That is, it can be derived without the construction of special projective or injective resolutions and without any computation whatsoever. Bousfield explained to me how to do this, about 45 years ago. Here is the key idea. One has a 'loop functor' on the category of unstable Steenrod modules. It is left adjoint to the suspension. This functor is right exact, and has non-trivial left-derived functors. The key is to notice that these left-derived functors are zero, in homological degrees greater than one. The existence of the long exact EHP sequence follows immediately.

William M. Singer (private communication, January 2016)

Theorem 3.1 (Bousfield's construction of the algebraic EHP sequences). *For all unstable modules M and N, there exists a long exact sequence*

$$\cdots \to \operatorname{Ext}^{s-2}_{\mathscr{U}}(\Omega_1 M, N) \to \operatorname{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \operatorname{Ext}^{s}_{\mathscr{U}}(M, \Sigma N) \to \operatorname{Ext}^{s-1}_{\mathscr{U}}(\Omega_1 M, N) \to \cdots,$$

where the morphism

$$\operatorname{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \operatorname{Ext}^{s}_{\mathscr{U}}(M, \Sigma N)$$

is the composition

$$\operatorname{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \operatorname{Ext}^{s}_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) \to \operatorname{Ext}^{s}_{\mathscr{U}}(M, \Sigma N)$$

of the morphism induced by the unit $M \to \Sigma\Omega M$ of the adjunction $(\Omega \dashv \Sigma)$ and the one induced by the exact functor Σ.

Proof. Let

$$\{P_i, \partial_i : P_{i+1} \to P_i, i \geq 0\},$$

abbreviated as $P_\bullet$, be a projective resolution of M. Since ΩP_0 is projective, the long exact sequence of Ext-groups associated with the short exact sequence

$$0 \to \frac{\Omega P_1}{\mathrm{Ker}\,(\Omega\partial_0)} \to \Omega P_0 \to \Omega M \to 0$$

splits into an exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{\mathscr{U}}(\Omega M, N) \longrightarrow \mathrm{Hom}_{\mathscr{U}}(\Omega P_0, N) \\ \qquad\qquad\downarrow \\ \mathrm{Hom}_{\mathscr{U}}\left(\frac{\Omega P_1}{\mathrm{Ker}\,(\Omega\partial_0)} N\right) \longrightarrow \mathrm{Ext}^1_{\mathscr{U}}(\Omega M, N) \longrightarrow 0 \end{array}$$

and isomorphisms

$$\mathrm{Ext}^s_{\mathscr{U}}\left(\frac{\Omega P_1}{\mathrm{Ker}\,(\Omega\partial_0)} N\right) \xrightarrow{\sim} \mathrm{Ext}^{s+1}_{\mathscr{U}}(\Omega M, N),$$

for all $s \geq 1$. Now, because $\{\Omega P_i,\ \Omega\partial_i,\ i \geq 1\}$ is a projective resolution of C (see Lemma 2.8), for every $s \geq 1$ we have:

$$\begin{aligned} \mathrm{Ext}^s_{\mathscr{U}}(C, N) &\cong \mathrm{H}^{s+1}(\mathrm{Hom}_{\mathscr{U}}(\Omega P_\bullet, N), (\Omega\partial_\bullet)^*) \\ &\cong \mathrm{Ext}^{s+1}_{\mathscr{U}}(M, \Sigma N). \end{aligned}$$

Therefore, the long exact sequence of Ext-groups associated with the short exact sequence

$$0 \to \Omega_1 M \to C \to \frac{\Omega P_1}{\mathrm{Ker}\,(\Omega\partial_0)} \to 0$$

is the general algebraic long exact EHP sequence. Moreover, note that if $Q_\bullet$ is a projective resolution of $(\Omega P_1)/(\mathrm{Ker}\,(\Omega\partial_0))$ then the epimorphism $C \to (\Omega P_1)/(\mathrm{Ker}\,(\Omega\partial_0))$ lifts to a morphism of complexes $\Omega P_{\bullet+1} \to Q_\bullet$. The commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{U}}(Q_k, N) & \longrightarrow & \mathrm{Hom}_{\mathscr{U}}(\Omega P_{k+1}, N) \\ & & \downarrow \wr \\ \wr \downarrow & & \mathrm{Hom}_{\mathscr{U}}(\Sigma\Omega P_{k+1}, \Sigma N) \\ & & \downarrow \wr \\ \mathrm{Hom}_{\mathscr{U}}(\Sigma Q_k, \Sigma N) & \longrightarrow & \mathrm{Hom}_{\mathscr{U}}(P_{k+1}, \Sigma N) \end{array}$$

shows that

$$\mathrm{Ext}^s_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^s_{\mathscr{U}}(M, \Sigma N)$$

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is the composition

$$\mathrm{Ext}^s_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^s_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) \to \mathrm{Ext}^s_{\mathscr{U}}(M, \Sigma N),$$

where the first arrow is induced by the exact functor Σ and the second one is induced by the unit $M \to \Sigma\Omega M$ of the adjunction $(\Omega \dashv \Sigma)$. □

Remark 3.2. In his original proof, Bousfield used the Grothendieck spectral sequence associated with the composite functor $\mathrm{Hom}_{\mathscr{U}}(\Omega, -)$ to obtain the abstract construction of the algebraic EHP sequence. As the left-derived functors of Ω are zero on homological degrees greater than one, the spectral sequence collapses at E_2, giving rise to the above exact sequence.

Let M be $\Sigma^n \mathbb{F}_p$ and let N be $\Sigma^t \mathbb{F}_p$. If $n \geq 1$, then the morphism $\lambda_M : \Phi M \to M$ is trivial. Therefore,

$$\Omega M \cong \Sigma^{n-1} \mathbb{F}_p$$

and

- for $p = 2$:

$$\Omega_1 M \cong \Sigma^{2n-1} \mathbb{F}_2;$$

- for $p > 2$:

$$\Omega_1 M \cong \begin{cases} \Sigma^{2pk-1} \mathbb{F}_p & \text{if } n = 2k, \\ \Sigma^{2pk+1} \mathbb{F}_p & \text{if } n = 2k+1. \end{cases}$$

A reformulation of Bousfield's long exact sequence, in this case, yields the algebraic EHP sequence for S^n.

Theorem 3.3. *For every positive integer n, there exist long exact sequences:*

- *at the prime 2,*

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2n+1}) \xrightarrow{P} E_2^{s,t}(S^n) \xrightarrow{E} E_2^{s,t+1}(S^{n+1}) \xrightarrow{H} E_2^{s-1,t}(S^{2n+1}) \xrightarrow{P} \cdots;$$

- *at odd primes,*

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2pn+1}) \xrightarrow{P} E_2^{s,t}(S^{2n}) \xrightarrow{E} E_2^{s,t+1}(S^{2n+1}) \xrightarrow{H} E_2^{s-1,t}(S^{2pn+1}) \xrightarrow{P} \cdots,$$

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2pn-1}) \xrightarrow{P} E_2^{s,t}(S^{2n-1}) \xrightarrow{E} E_2^{s,t+1}(S^{2n}) \xrightarrow{H} E_2^{s-1,t}(S^{2pn-1}) \xrightarrow{P} \cdots.$$

Here, $E_2^{s,t}(S^n) := \mathrm{Ext}^s_{\mathscr{U}}(\Sigma^n \mathbb{F}_p, \Sigma^t \mathbb{F}_p)$.

3.1. Application

In this subsection, we use the loop functor Ω to study a special case of the algebraic EHP sequences.

If $\{C_i,\, \partial_i : C_{i+1} \to C_i,\, i \geq 0\}$ is a complex, denote by $C_\bullet[1]$ the complex:

$$C_\bullet[1]_i = \begin{cases} C_{i-1} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0, \end{cases}$$

$$\partial[1]_i = \begin{cases} \partial_{i-1} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

Let M be an unstable module such that $\Omega_1 M$ is trivial. Fix $\{P_\bullet,\, \partial_i : P_{i+1} \to P_i,\, i \geq 0\}$, abbreviated as $P_\bullet$, a projective resolution of M, and fix $\{Q_\bullet,\, \delta_i : Q_{i+1} \to Q_i,\, i \geq 0\}$, abbreviated as $Q_\bullet$, a projective resolution of ΦM. The natural transformation $\lambda : \Phi \to Id$ gives rise to a morphism of complexes: $\lambda_{P_\bullet} : \Phi P_\bullet \to P_\bullet$. On the other hand, the identity of ΦM yields a morphism of complexes: $\omega : Q_\bullet \to \Phi P_\bullet$. Therefore, the composition map $\omega \circ \lambda_{P_\bullet}$ makes the following diagram commute:

$$\begin{array}{ccc} Q_\bullet & \xrightarrow{\omega\circ\lambda_{P_\bullet}} & P_\bullet \\ \downarrow & & \downarrow \\ \Phi M & \xrightarrow[\lambda_M]{} & M \end{array}$$

Now, we can consider $\omega \circ \lambda_{P_\bullet} : Q_\bullet \to P_\bullet$ as a double complex with two non-trivial columns $Q_\bullet$ and $P_\bullet$. Denote by $T_\bullet$ the total complex of this double complex. As $Q_\bullet$ is a resolution of ΦM and $P_\bullet$ is a resolution of M, the homology groups of $T_\bullet$ are computed as follows:

$$\mathrm{H}_i(T_\bullet) \cong \begin{cases} \mathrm{Coker}\,(\lambda_M) = \Sigma\Omega M & \text{if } i = 0, \\ \mathrm{Ker}\,(\lambda_M) = \Sigma\Omega_1 M & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\Omega_1 M$ is trivial by hypothesis, and $T_\bullet$ is a projective resolution of $\Sigma\Omega M$. We now compute $\Omega T_\bullet$. It follows from Remark 2.6 that $\Omega\,(\omega \circ \lambda_{P_\bullet})$ is trivial. We then have:

$$\Omega T_\bullet \cong \Omega P_\bullet \bigoplus \Omega Q_\bullet[1].$$

We also deduce from Remark 2.6 that $\Omega P_\bullet$ is a projective resolution of ΩM, and that $\Omega Q_\bullet$ is a projective resolution of $\Omega\Phi M$.

Lemma 3.4. *Let M be an unstable module such that $\Omega_1 M$ is trivial. For all unstable modules N, we have an isomorphism of* Ext*-groups*

$$\mathrm{Ext}^s_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) \cong \mathrm{Ext}^s_{\mathscr{U}}(\Omega M, N) \bigoplus \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Omega\Phi M, N),$$

for all $s \geq 0$. (Here, by convention, the Ext*-groups of degree -1 are trivial.)*

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Proof. The Ext-groups $\mathrm{Ext}^*_{\mathscr{U}}(\Sigma\Omega M, \Sigma N)$ can be computed as follows.

$$\begin{aligned}\mathrm{Ext}^s_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) &\cong \mathrm{H}^s(\mathrm{Hom}_{\mathscr{U}}(T_\bullet, \Sigma N))\\ &\cong \mathrm{H}^s(\mathrm{Hom}_{\mathscr{U}}(\Omega T_\bullet, N))\\ &\cong \mathrm{H}^s\Big(\mathrm{Hom}_{\mathscr{U}}\Big(\Omega P_\bullet \bigoplus \Omega Q_\bullet[1], N\Big)\Big)\\ &\cong \mathrm{Ext}^s_{\mathscr{U}}(\Omega M, N) \bigoplus \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Omega\Phi M, N).\end{aligned}$$

We can then conclude the lemma. □

Applying Lemma 3.4 to $M = \Phi^n F(1)$, we have the following result.

Theorem 3.5 (James's splitting). *For all non-negative integers n and all unstable modules N, there are isomorphisms of $\mathbb{F}_p$-vector spaces:*

$$\mathrm{Ext}^s_{\mathscr{U}}(\Sigma^{2p^n}\mathbb{F}_p, \Sigma N) \cong \mathrm{Ext}^s_{\mathscr{U}}(\Sigma^{2p^n-1}\mathbb{F}_p, N) \bigoplus \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Sigma^{2p^{n+1}-1}\mathbb{F}_p, N).$$

Proof. Note that, after Lemma 2.7, we have natural isomorphisms

$$\begin{aligned}\Phi\Sigma\Omega M &\cong \Sigma\Omega\Phi M,\\ \Phi\Sigma\Omega_1 M &\cong \Sigma\Omega_1\Phi M.\end{aligned}$$

Now, Propositions 2.3 and 2.4 show that $\Omega_1 F(1) = 0$, whence $\Omega_1\Phi^n F(1) = 0$ for all natural numbers $n \geq 1$. Moreover, as $\Omega F(1) = F(0) = \mathbb{F}_p$, we have

$$\Omega\Phi^n F(1) \cong \Sigma^{2p^{n-1}-1}\mathbb{F}_p.$$

Then, the conclusion follows from Lemma 3.4. □

4. Injective resolutions of the suspension of an unstable module

Constructing injective resolutions is a basic problem in homological algebra. This section aims at the construction of injective resolutions of the suspension of an unstable module.

First, we recall how Brown–Gitler modules fit in the Mahowald short exact sequences. This will be carried out with the help of Φ and Σ. In fact, following [10], these functors admit a right adjoint. We denote the right adjoint of Φ by $\tilde{\Phi}$ and that of Σ by $\tilde{\Sigma}$. The morphisms $M \to \tilde{\Phi}M$, adjoint to λ_M, induce a natural transformation $\tilde{\lambda} : Id \to \tilde{\Phi}$. The natural transformations $\tilde{\sigma} : \Sigma\tilde{\Sigma} \to Id$ and $\tilde{\lambda} : Id \to \tilde{\Phi}$ give rise to the following natural exact sequence.

Theorem 4.1 (see [10]). *There is a natural exact sequence of unstable modules*

$$0 \to \Sigma\tilde{\Sigma}M \xrightarrow{\tilde{\sigma}_M} M \xrightarrow{\tilde{\lambda}_M} \tilde{\Phi}M \to \Sigma R^1\tilde{\Sigma}M \to 0. \tag{4.1}$$

Here, $R^1\tilde{\Sigma}M$ is the right-derived functor of $\tilde{\Sigma}$ in cohomological degree 1.

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Proof. The sequence (4.1) is obtained by applying the functor $\mathrm{Hom}_{\mathscr{U}}(-, M)$ to the sequence

$$0 \to \Phi F(n) \xrightarrow{\lambda_{F(n)}} F(n) \xrightarrow{\sigma_{F(n)}} \Sigma\Omega F(n) \to 0$$

and identifying $\mathrm{Ext}^1_{\mathscr{U}}(\Sigma\Omega F(n), M)$ with $\Sigma R^1\tilde{\Sigma}M$. □

If M is an injective unstable module, then $R^1\tilde{\Sigma}M = 0$ and the sequence (4.1) becomes a short exact sequence.

Theorem 4.2. *If I is an injective unstable module, then the following sequence is exact*

$$0 \to \Sigma\tilde{\Sigma}I \xrightarrow{\tilde{\sigma}_I} I \xrightarrow{\tilde{\lambda}_I} \tilde{\Phi}I \to 0. \tag{4.2}$$

Because Σ and Φ are exact, $\tilde{\Sigma}$ and $\tilde{\Phi}$ preserve injective unstable modules. More precisely,

$$\tilde{\Sigma}J(n) \cong J(n-1), \quad \forall n \geq 1,$$

and the module $\tilde{\Phi}J(n)$ depends on p and on the parity of n:

- for $p = 2$:
$$\tilde{\Phi}J(n) \cong J\left(\frac{n}{2}\right) := \begin{cases} J(k) & \text{if } n = 2k, \\ 0 & \text{otherwise,} \end{cases}$$

- for $p > 2$:
$$\tilde{\Phi}J(n) \cong \begin{cases} J(2k) & \text{if } n = 2pk, \\ J(2k+1) & \text{if } n = 2pk+2, \\ 0 & \text{otherwise.} \end{cases}$$

We get the classical Mahowald short exact sequences.

Theorem 4.3 (see [7, 10]). *For every non-negative integer n, there is a short exact sequence of unstable modules*

$$0 \to \Sigma J(n-1) \to J(n) \to \tilde{\Phi}J(n) \to 0. \tag{4.3}$$

Theorem 4.3 implies that the suspension of $J(n-1)$ is of injective dimension at most 1 and the Mahowald short exact sequence is in fact an injective resolution of $\Sigma J(n)$. In fact, this property remains true for the suspension of all injective unstable modules. Indeed, note that if I is an injective unstable module, then so are $\tilde{\Sigma}I$, $\tilde{\Phi}I$. It follows from Theorem 4.2 that $I \to \tilde{\Phi}I \to 0$ is an injective resolution of $\Sigma\tilde{\Sigma}I$. It turns out that every injective unstable module is isomorphic to $\tilde{\Sigma}I$ for some injective unstable module I.

Theorem 4.4 (see [10]). *Every injective unstable module is isomorphic to a direct sum of unstable modules of the form $J(n) \otimes L$, where $n \geq 0$ is a natural number and L is an indecomposable direct summand of H^*V for some elementary abelian p-group V.*

Corollary 4.5. *Every injective unstable module is isomorphic to $\tilde{\Sigma}I$ for some injective unstable module I.*

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Proof. As the functor $\tilde{\Sigma}$ commutes with direct sums, we can suppose that the injective unstable module is of the form $J(n) \otimes L$, where $n \geq 0$ is a natural number and L is an indecomposable direct summand of H^*V for some elementary abelian p-group V. Note that

$$J(n) \otimes L \cong \tilde{\Sigma} J(n+1) \otimes L.$$

On the other hand, it follows from [10] that the morphism

$$\tilde{\Sigma} J(n+1) \otimes L \to \tilde{\Sigma}(J(n+1) \otimes L),$$

adjoint to

$$\Sigma(\tilde{\Sigma} J(n+1) \otimes L) \cong \Sigma\tilde{\Sigma} J(n+1) \otimes L \xrightarrow{\tilde{\sigma}_{J(n+1)} \otimes id} J(n+1) \otimes L,$$

is an isomorphism. Therefore, we have

$$J(n) \otimes L \cong \tilde{\Sigma}(J(n+1) \otimes L),$$

whence the conclusion. □

We fix the following notation.

Definition 4.6. Let I be an injective unstable module. Denote by $\tilde{I}$ an injective unstable module such that $I \cong \tilde{\Sigma}\tilde{I}$ and by i_I the composition

$$\Sigma I \xrightarrow{\sim} \Sigma\tilde{\Sigma}\tilde{I} \xrightarrow{\tilde{\sigma}_{\tilde{I}}} \tilde{I}.$$

Corollary 4.7. *If I is an injective unstable module, then the sequence*

$$\tilde{I} \xrightarrow{\tilde{\lambda}_{\tilde{I}}} \tilde{\Phi}\tilde{I} \to 0$$

is an injective resolution of ΣI.

Now, we come back to the construction of injective resolutions of the suspension of an unstable module. Observe that if $I^\bullet$ is an injective resolution of an unstable module N, then $\Sigma I^\bullet$ is a resolution of ΣN. Although this resolution is no longer injective, we can resolve each ΣI^k by an injective resolution of length at most 1. The method we describe below allows for combining these resolutions into one of ΣN.

Proposition 4.8 (see [9]). *Let $(I^\bullet, \partial^\bullet)$ be an injective resolution of an unstable module N and let $\alpha^k : \tilde{I}^k \to \tilde{I}^{k+1}$ be an extension of ∂^k. Then, there exist morphisms*

$$\delta^k : \tilde{\Phi}\tilde{I}^k \to \tilde{I}^{k+2}$$

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such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \tilde{I}^{n-1}\oplus\tilde{I}^{n} & \xrightarrow{f^{n-1}} & \tilde{I}^{n}\oplus\tilde{I}^{n+1} & \xrightarrow{f^{n}} & \tilde{I}^{n+1}\oplus\tilde{I}^{n+2} & \xrightarrow{f^{n+1}} & \cdots \\
 & & \big\downarrow{\scriptstyle h^{n-1}} & & \big\downarrow{\scriptstyle h^{n}} & & \big\downarrow{\scriptstyle h^{n+1}} & & \\
\cdots & \longrightarrow & \tilde{\Phi}\tilde{I}^{n-1}\oplus\tilde{I}^{n} & \xrightarrow{g^{n-1}} & \tilde{\Phi}\tilde{I}^{n}\oplus\tilde{I}^{n+1} & \xrightarrow{g^{n}} & \tilde{\Phi}\tilde{I}^{n+1}\oplus\tilde{I}^{n+2} & \xrightarrow{g^{n+1}} & \cdots
\end{array} \tag{4.4}$$

where

$$f^{n}=\begin{pmatrix}\alpha^{n} & (-1)^{n}id\\ (-1)^{n}\alpha^{n+1}\circ\alpha^{n} & \alpha^{n+1}\end{pmatrix},\quad g^{n}=\begin{pmatrix}\tilde{\Phi}\alpha^{n} & (-1)^{n}\tilde{\lambda}_{\tilde{I}^{n+1}}\\ (-1)^{n}\delta^{n} & \alpha^{n+1}\end{pmatrix},\quad h^{n}=\begin{pmatrix}\tilde{\lambda}_{\tilde{I}^{n}} & 0\\ 0 & id\end{pmatrix}$$

is a double complex whose associated total complex is an injective resolution of ΣN.

Proof. As the kth column of Diagram (4.4) is an injective resolution of ΣI^{k}, it suffices to prove that Diagram (4.4) is a double complex. For this, we must construct $\delta^{\bullet}$ such that

$$f^{n+1}\circ f^{n}=0,\quad g^{n+1}\circ g^{n}=0,\quad g^{n}\circ h^{n}=h^{n+1}\circ f^{n}.$$

That is, we need to verify the following identities

$$\begin{aligned}
\alpha^{n+1}\circ\alpha^{n}&=\delta^{n}\circ\tilde{\lambda}_{\tilde{I}^{n}},\\
\alpha^{n+1}\circ\delta^{n-1}&=\delta^{n}\circ\tilde{\Phi}\alpha^{n-1},\\
\tilde{\lambda}_{\tilde{I}^{n+1}}\circ\delta^{n-1}&=\tilde{\Phi}\alpha^{n}\circ\tilde{\Phi}\alpha^{n-1},\\
\tilde{\Phi}\alpha^{n}\circ\tilde{\lambda}_{\tilde{I}^{n}}&=\tilde{\lambda}_{\tilde{I}^{n+1}}\circ\alpha^{n}.
\end{aligned}$$

First, as $\tilde{I}^{n}$ is an injective unstable module for all $n\geq 0$, the existence of an extension α^{n} of ∂^{n} is clear. Since $\tilde{\lambda}$ is a natural transformation from the identity functor to $\tilde{\Phi}$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\Sigma I^{n} & \longrightarrow & \tilde{I}^{n} & \xrightarrow{\tilde{\lambda}_{\tilde{I}^{n}}} & \tilde{\Phi}\tilde{I}^{n}\\
\big\downarrow{\scriptstyle \Sigma\partial^{n}} & & \big\downarrow{\scriptstyle \alpha^{n}} & & \big\downarrow{\scriptstyle \tilde{\Phi}\alpha^{n}}\\
\Sigma I^{n+1} & \longrightarrow & \tilde{I}^{n+1} & \xrightarrow[\tilde{\lambda}_{\tilde{I}^{n+1}}]{} & \tilde{\Phi}\tilde{I}^{n}
\end{array}$$

It is evident that we get the identity

$$\tilde{\Phi}\alpha^{n}\circ\tilde{\lambda}_{\tilde{I}^{n}}=\tilde{\lambda}_{\tilde{I}^{n+1}}\circ\alpha^{n}.$$

The construction of $\delta^{\bullet}$ goes as follows. Denote by i^{k} the inclusion $\Sigma I^{k}\to\tilde{I}^{k}$. Because of the identity

$$\alpha^{k+1}\circ\alpha^{k}\circ i^{k}=i^{k+2}\circ\partial^{k+1}\circ\partial^{k},$$

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the composition $\alpha^{k+1} \circ \alpha^k \circ i^k$ is trivial. It follows that there exists $\delta^k : \tilde{\Phi}\tilde{I}^k \to \tilde{I}^{k+2}$ such that

$$\alpha^{k+1} \circ \alpha^k = \delta^k \circ \tilde{\lambda}_{\tilde{I}^k}.$$

Therefore, for all natural numbers $n \geq 1$, we have

$$\begin{aligned}\tilde{\lambda}_{\tilde{I}^{n+1}} \circ \delta^n \circ \tilde{\lambda}_{\tilde{I}^{n-1}} &= \tilde{\lambda}_{\tilde{I}^{n+1}} \circ \alpha^n \circ \alpha^{n-1} \\ &= \tilde{\Phi}\alpha^n \circ \tilde{\Phi}\alpha^{n-1} \circ \tilde{\lambda}_{\tilde{I}^{n-1}}.\end{aligned}$$

As $\tilde{\lambda}_{\tilde{I}^{n-1}}$ is surjective, we obtain the identity

$$\tilde{\lambda}_{\tilde{I}^{n+1}} \circ \delta^{n-1} = \tilde{\Phi}\alpha^n \circ \tilde{\Phi}\alpha^{n-1}.$$

Similarly, since

$$\alpha^{n+1} \circ \delta^{n-1} \circ \tilde{\lambda}_{\tilde{I}^{n-1}} = \delta^n \circ \tilde{\Phi}\alpha^{n-1} \circ \tilde{\lambda}_{\tilde{I}^{n-1}},$$

we get the identity

$$\alpha^{n+1} \circ \delta^{n-1} = \delta^n \circ \tilde{\Phi}\alpha^{n-1}.$$

The conclusion follows. □

Remark 4.9. The resolution constructed in Theorem 4.8 is bigger than that given by the pseudo-hyperresolution [**9**]. However, the advantage of this construction is that it allows us to apply the spectral sequence of double complexes to compute Ext-groups, as we will see in the next section.

5. Injective resolutions and the algebraic EHP sequences

In this section, we use the results on injective resolutions of the suspension of an unstable module to construct the algebraic EHP sequences.

Theorem 5.1. *For all unstable modules M and N, there exists a long exact sequence*

$$\cdots \to \mathrm{Ext}^{s-2}_{\mathscr{U}}(\Omega_1 M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N) \to \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Omega_1 M, N) \to \cdots,$$

where the morphism

$$\mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N)$$

is the composition

$$\mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N)$$

of the morphism induced by the unit $M \to \Sigma\Omega M$ of the adjunction $(\Omega \dashv \Sigma)$ and the one induced by the exact functor Σ.

Proof. Let $(I^\bullet, \partial^\bullet)$ be an injective resolution of N and let $\alpha^k : \tilde{I}^k \to \tilde{I}^{k+1}$ be an extension of ∂^k. We are now in a position to apply Proposition 4.8. Take Diagram (4.4) as the double complex whose associated total complex is an injective resolution of ΣN. Applying the functor $\mathrm{Hom}_{\mathscr{U}}(M,-)$ to Diagram (4.4) yields a double complex:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Hom}_{\mathscr{U}}(M, \tilde{I}^{n-1}\oplus \tilde{I}^{n}) & \xrightarrow{f_*^{n-1}} & \mathrm{Hom}_{\mathscr{U}}(M, \tilde{I}^{n}\oplus \tilde{I}^{n+1}) & \xrightarrow{f^n} \cdots \\ & \downarrow{\scriptstyle h_*^{n-1}} & & \downarrow{\scriptstyle h_*^{n}} & \\ \cdots \longrightarrow & \mathrm{Hom}_{\mathscr{U}}(M, \tilde{\Phi}\tilde{I}^{n-1}\oplus \tilde{I}^{n}) & \xrightarrow{g_*^{n-1}} & \mathrm{Hom}_{\mathscr{U}}(M, \tilde{\Phi}\tilde{I}^{n}\oplus \tilde{I}^{n+1}) & \xrightarrow{g_*^n} \cdots \end{array} \tag{5.1}$$

The cohomology of the total complex of the double complex (5.1) is $\mathrm{Ext}^*_{\mathscr{U}}(M,\Sigma N)$. Note that the cohomology of the complex

$$\mathrm{Hom}_{\mathscr{U}}(M, \tilde{I}^{n}\oplus \tilde{I}^{n+1}) \xrightarrow{h_*^n} \mathrm{Hom}_{\mathscr{U}}(M, \tilde{\Phi}\tilde{I}^{n}\oplus \tilde{I}^{n+1}) \to 0$$

is isomorphic to the cohomology of the complex

$$\mathrm{Hom}_{\mathscr{U}}(M, \tilde{I}^{n}) \xrightarrow{\lambda_M^*} \mathrm{Hom}_{\mathscr{U}}(\Phi M, \tilde{I}^{n}) \to 0,$$

and therefore it is isomorphic to 0 in cohomological degrees greater than 2 and isomorphic to $\mathrm{Hom}_{\mathscr{U}}(\Omega M, \tilde{\Sigma}\tilde{I}^n)$ and $\mathrm{Hom}_{\mathscr{U}}(\Omega_1 M, \tilde{\Sigma}\tilde{I}^n)$ in cohomological degrees 0 and 1, respectively. Recall that $\tilde{\Sigma}i^n : I^n \to \tilde{\Sigma}\tilde{I}^n$ is an isomorphism, where i^n is the inclusion $\Sigma I^n \to \tilde{I}^n$. Therefore, we can identify $\left(\tilde{\Sigma}\tilde{I}^\bullet, \tilde{\Sigma}\alpha^\bullet\right)$ with $(I^\bullet, \partial^\bullet)$. Now, filter $\mathrm{Tot}(\mathcal{C})$ by row degrees. Then the associated spectral sequence of the double complex (5.1) collapses at E_2, giving rise to the following long exact sequence

$$\cdots \to \mathrm{Ext}^{s-2}_{\mathscr{U}}(\Omega_1 M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N) \to \mathrm{Ext}^{s-1}_{\mathscr{U}}(\Omega_1 M, N) \to \cdots,$$

where the morphism

$$\mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N)$$

is the corner homomorphism of the spectral sequence and, hence, is the composition

$$\mathrm{Ext}^{s}_{\mathscr{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(\Sigma\Omega M, \Sigma N) \to \mathrm{Ext}^{s}_{\mathscr{U}}(M, \Sigma N),$$

where the first arrow is induced by the exact functor Σ and the second one is induced by the unit $M \to \Sigma\Omega M$ of the adjunction $(\Omega \dashv \Sigma)$. □

Taking $M = \Sigma^n\mathbb{Z}/p$ and $N = \Sigma^t\mathbb{Z}/p$, we recover Theorem 3.3.

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References

1. J. Adem, *The relations on Steenrod powers of cohomology classes, in Algebraic geometry and topology, a symposium in honor of S. Lefschetz* (Princeton University Press, Princeton, NJ, 1957) pp. 191–238.
2. A. K. Bousfield and E. B. Curtis, A spectral sequence for the homotopy of nice spaces, *Trans. Amer. Math. Soc.* **151** (1970), 457–479.
3. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Volume 304 (Springer-Verlag, Berlin–New York, 1972).
4. E. B. Curtis, Simplicial homotopy theory, *Adv. Math.* **6** (1971), 107–209.
5. I. M. James, On the suspension sequence, *Ann. of Math.* **65**(2) (1957), 74–107.
6. M. Mahowald, On the double suspension homomorphism, *Trans. Amer. Math. Soc.* **214** (1975), 169–178.
7. M. Mahowald, A new infinite family in ${}_2\pi_*^s$, *Topology* **16** (1977), 249–256.
8. W. S. Massey and F. P. Peterson, *The mod 2 cohomology structure of certain fibre spaces,* Memoirs of the American Mathematical Society (AMS, 1967) p. 97.
9. T. C. Nguyen, *Some homological algebra computations in the category of unstable modules,* Thesis, Université Paris 13, July 2014.
10. L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1994).
11. W. M. Singer, The algebraic EHP sequence, *Trans. Amer. Math. Soc.* **201** (1975), 367–382.
12. W. M. Singer, Squaring operations and the algebraic EHP sequence, *J. Pure Appl. Algebra* **6** (1975), 13–29.
13. W. M. Singer, Iterated loop functors and the homology of the Steenrod algebra. II. A chain complex for $\Omega_s^k M$, *J. Pure Appl. Algebra* **16** (1980), 85–97.
14. H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, Volume 49 (Princeton University Press, Princeton, NJ, 1962).

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