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A homotopy-theoretic model of function extensionality in the effective topos

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We present a way of constructing a Quillen model structure on a full subcategory of an elementary topos, starting with an interval object with connections and a certain dominance. The advantage of this method is that it does not require the underlying topos to be cocomplete. The resulting model category structure gives rise to a model of homotopy type theory with identity types, Σ- and Π-types, and functional extensionality.
We apply the method to the effective topos with the interval object $\nabla 2$. In the resulting model structure we identify uniform inhabited objects as contractible objects, and show that discrete objects are fibrant. Moreover, we show that the unit of the discrete reflection is a homotopy equivalence and the homotopy category of fibrant assemblies is equivalent to the category of modest sets. We compare our work with the path object category construction on the effective topos by Jaap van Oosten.

1. Introduction

Any constructive proof implicitly contains an algorithm; realizability makes this algorithm explicit. For instance, realizability shows how from a constructive proof of a statement of the form

$$\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, \varphi(x, y),$$

one can extract an algorithm computing a suitable y given x as input. For this reason, realizability, as invented by Kleene in 1945 (Kleene 1945), has become an important tool in the study of formal systems for constructive mathematics. More recently, it has been used to provide semantics for various type theories, in particular polymorphic type theories, for which no set-theoretic models exist. For these more advanced applications a category-theoretic understanding of realizability is essential; indeed, around 1980 Martin Hyland discovered the effective topos, a topos whose internal logic is given by Kleene's realizability. In fact, various realizability interpretations exist besides Kleene's original variant, and many of these interpretations can be given a topos-theoretic formulation (for more on this, see van Oosten 2008). Here, it has to be understood that these realizability toposes are elementary toposes in the sense of Lawvere and Tierney: they are not Grothendieck toposes. In particular, realizability toposes are not cocomplete (for instance, in the effective topos the countably infinite coproduct of the terminal object does not exist), a point which will be important for us later.

The purpose of this paper is to make some first steps in applying ideas from realizability to homotopy type theory. Homotopy type theory refers to a recent influx of ideas from abstract homotopy theory and higher category to type theory. The starting point for these developments is the discovery by Hofmann and Streicher (1998) that Martin-Löf's identity type gives every type in type theory the structure of a groupoid; in fact, they give every type the structure of an ∞-groupoid, as shown in Lumsdaine (2010) and van den Berg and Garner (2011). Conversely, types in type theory can be interpreted as ∞-groupoids: this is what underlies Voevodsky's interpretation of type theory in simplicial sets, with the types interpreted as Kan complexes (Kapulkin and Lumsdaine 2016). Such a Kan complex is both understood as a combinatorial model for the homotopy type of a space (hence 'homotopy type theory') and a notion of ∞-groupoid. In his proof Voevodsky relies heavily on the fact that the category of simplicial sets carries a Quillen model structure in which the Kan complexes are precisely the fibrant objects. This model structure is also essential for the interpretation of the identity types, as in Awodey and Warren (2009).

Homotopy theory not only provides an unexpected interpretation of type theory, but it also gives one a new perspective on some old problems in type theory, such as the mysterious identity types and the problem of extensional constructs (Hofmann 1995). In addition, it suggests many new extensions of type theory, such as higher inductive types and the univalence axiom. In this paper, we will focus on a particular consequence of univalence: function extensionality. For more on these exciting new developments, we refer to The Univalent Foundations Program (2013).

So far realizability has only played a minor role in these developments (some exceptions are (Angiuli *et al.* 2016; van Oosten 2015)). However, given its prominent place in the study of constructive formal systems, it seems quite likely that realizability will be fruitful here as well. Also, the most important questions in homotopy type theory (such as Voevodsky's homotopy canonicity conjecture (Voevodsky 2010)) concern its computational behaviour. Since realizability aims to make the computational content of constructive formal systems explicit, a realizability interpretation of homotopy type theory would help us understand its computational content.

In this paper, we make a step in that direction by endowing a subcategory of the effective topos with a Quillen model structure. In fact, in the first half of this paper we show that in any elementary topos equipped with a suitable class of monomorphisms and an interval object one can define three classes of maps (cofibrations, fibrations and weak equivalences, respectively) such that on the full subcategory of fibrant objects these induce a model structure. To see that this can lead to non-trivial results, note that we can apply this theorem to the category of simplicial sets and we can choose our interval and cofibrations in such a manner that the fibrant objects will be the Kan complexes. In that case, we recover the model structure obtained by restricting the classical model structures on simplicial sets to the Kan complexes. Also, if we consider cubical sets as in the work by Coquand and others (Cohen *et al.* 2018), we recover their notion of a Kan cubical set and obtain a model structure on these Kan cubical sets.

This result is inspired by earlier works of Orton and Pitts (2016) and Gambino and Sattler (2017), which in turn is based on the work of Cohen *et al.* (2018) and earlier work by Cisinski (2002); indeed, with a few exceptions most steps in our proof of the model structure can be found in these earlier sources. The main innovation is that we do not assume cocompleteness of the underlying topos, so that our result can be applied to realizability toposes as well. This means that, unlike Cisinski, Gambino and Sattler, we do not rely on the small object argument to build our factorisations. (Other differences are that we work in a non-algebraic setting, as in part 1, but not part 2 of Gambino and Sattler (2017), and we work mostly externally with toposes, not inside a type theory as in Orton and Pitts (2016); related to this is that for us being a fibration, for instance, is a property of morphism, not additional structure.)

As we already mentioned, any model structure gives rise to a model of Martin-Löf's identity types. But they also provide a model for strong Σ-types and products. Again following Gambino and Sattler, we also show that in our setting one can interpret Π-types, which moreover satisfy function extensionality. Function extensionality says that two functions are equal if they give equal outputs on identical inputs, and this is one of those desirable principles which are valid on the homotopy-theoretic interpretation of type theory, but are unprovable in type theory proper. So, in our setting we are able to interpret basic type theory with function extensionality. We should point out that we will not consider univalent universes or higher inductive types in this paper, leaving this for future work. (In addition, we will ignore, like many authors, coherence issues related to substitution: for a possible solution, see Lumsdaine and Warren (2015).)

In some more detail, the precise contents of the first few sections are as follows. In Section 2 we recall some important categorical notions (like that of a model structure, a dominance and the Leibniz adjunction) that will be used throughout this paper. In Section 3 we present our axiomatic set-up for building model structures. We define cofibrations, fibrations and (strong) homotopy equivalences in this setting and we establish some basic properties of these classes of maps. This is then used in Section 4 to construct a model structure on the full subcategory of fibrant objects. We also show that the resulting model of type theory interprets extensional Π-types.

In the second part of this paper, we apply this general recipe for constructing model structures to Hyland's effective topos. For our class of monomorphisms we take the class of all monos and for our interval object we take $\nabla 2$. The latter choice is inspired by earlier work by van Oosten (2015). It also seems natural, because $\nabla 2$ contains two points, which are, however, computationally indistinguishable (because they have identical realizers). The analogy with the usual interval $[0, 1]$ is that its endpoints are distinct, but homotopy-theoretically indistinguishable.

So, in Section 5, we recall some basic facts about the effective topos and check that it fits into our axiomatic framework. In Section 6, we make some progress in characterizing contractible objects and maps in the effective topos and we show that these are closely related to the uniform objects and maps. This also leads to a concrete criterion for characterizing the fibrant assemblies. In Section 7, we prove that discrete objects like the natural numbers object are fibrant and we show that the homotopy category of the full subcategory of fibrant assemblies is the category of modest sets.

In Section 6, we also compare our work with earlier work by van Oosten (2015). In his paper, Van Oosten constructs a path object category structure (in the sense of van den Berg and Garner (2012)) on the effective topos, resulting in a type-theoretic fibration category in the sense of Shulman (2015). This falls short of a full model structure, but it does provide an interesting interpretation of the identity types. The main difference is that Van Oosten's structure contains all objects of the effective topos and function extensionality does not hold (private communication). In this paper, we show that it is possible to use $\nabla 2$ as an interval, whilst obtaining a model of function extensionality by choosing our notion of fibrant object appropriately.

The present paper is written purely in the language of category theory and does not assume familiarity with homotopy type theory. We use **ZFC** as our metatheory and are aware that some of our results in the section on the effective topos make use of the axiom of choice. We leave it for future work to determine what can be said in a constructive metatheory (but see Frumin (2016)).

The contents of this paper are based on the Master thesis of the first author written under supervision of the second author (see Frumin (2016)). We thank Jaap van Oosten for useful comments on the thesis. In addition, we are grateful to the referees for two exceptionally detailed and helpful referee reports.

2. Categorical definitions

In this section we recall, for the convenience of the reader, the definitions of a model structure, a dominance and the Leibniz adjunction.

Definition 2.1. Let f and g be two morphisms in some category $\mathcal{C}$. We will say that f has the *left lifting property (LLP)* with respect to g and g has the *right lifting property (RLP)* with respect to f, and write $f \pitchfork g$, if for any commuting square in $\mathcal{C}$

$$\begin{array}{ccc} D & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \longrightarrow & A \end{array}$$

there exists a map $h : C \to B$ (a *diagonal filler*) making the two resulting triangles commute. If $\mathcal{A}$ is some class of morphisms in $\mathcal{C}$, we will write $\mathcal{A}^{\pitchfork}$ for the class of morphisms in $\mathcal{C}$ having the RLP with respect to every morphism in $\mathcal{A}$, and ${}^{\pitchfork}\mathcal{A}$ for the class of morphisms in $\mathcal{C}$ having the LLP with respect to every morphism in $\mathcal{A}$.

Definition 2.2. A *weak factorisation system (WFS)* on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ consisting of two classes of maps in $\mathcal{C}$, such that

1. every map h in $\mathcal{C}$ can be factored as $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$,
2. $\mathcal{L}^{\pitchfork} = \mathcal{R}$ and ${}^{\pitchfork}\mathcal{R} = \mathcal{L}$.

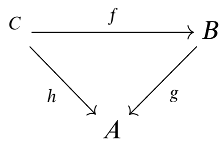
Lemma 2.3 (Retract argument). A pair $(\mathcal{L}, \mathcal{R})$ consisting of two classes of maps in a category $\mathcal{C}$ is a weak factorisation system if and only if the following conditions hold:

1. every map h in $\mathcal{C}$ can be factored as $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$,
2. for any $l \in \mathcal{L}$ and $r \in \mathcal{R}$ one has $l \pitchfork r$,
3. both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

Proof. See, for instance, Lemma 11.2.3 in Riehl (2014). □

Definition 2.4. A *(Quillen) model structure* on a category $\mathcal{C}$ consists of three classes of maps $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$, referred to as the cofibrations, the fibrations and the weak equivalences, respectively, such that the following hold:

1. both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems,
2. in any commuting triangle



if two of f, g, h are weak equivalences, then so is the third. (This is called 2-out-of-3 for weak equivalences.)

In this case, the maps in $\mathcal{C} \cap \mathcal{W}$ are referred to as the *trivial (or acyclic) cofibrations*, while the maps in $\mathcal{F} \cap \mathcal{W}$ are referred to as the *trivial (or acyclic) fibrations*.

Definition 2.5. Let $\mathcal{E}$ be an elementary topos and Σ be a class of monomorphisms in $\mathcal{E}$. Then Σ is called a *dominance* if

1. every isomorphism belongs to Σ and Σ is closed under composition,
2. every pullback of a map in Σ again belongs to Σ,
3. the category Σ_{cart} of morphisms in Σ and pullback squares between them has a terminal object.

One can show, using standard arguments, that for the terminal object $m : A \to B$ in Σ_{cart} we must have $A = 1$ and $B \subseteq \Omega$. We will also write Σ for the codomain of this classifying map, so that the classifying map is written $\top : 1 \to \Sigma$. This map $\top : 1 \to \Sigma$ is a pullback of the map $\top : 1 \to \Omega$ classifying all monomorphisms and determines the entire class. Indeed, a dominance can equivalently be defined as a subobject $\Sigma \subseteq \Omega$ satisfying the following principles in the internal logic of $\mathcal{E}$:

1. $\top \in \Sigma$,
2. $(\forall p, q \in \Omega)\,(\,(p \in \Sigma \wedge (p \Rightarrow (q \in \Sigma))) \Rightarrow p \wedge q \in \Sigma\,)$.

Proposition 2.6. If Σ is a dominance on an elementary topos $\mathcal{E}$, then $(\Sigma, \Sigma^{\pitchfork})$ is a weak factorisation system.

Proof. This seems to be well-known (e.g., Bourke and Garner (2016, Section 4.4)), but since we have not been able to locate this precise theorem in the literature, we include some details here. Both Σ and $\Sigma^{\pitchfork}$ are closed under retracts, so by the retract argument (Lemma 2.3) it suffices to prove that any map $h : B \to A$ can be factored as a map in Σ

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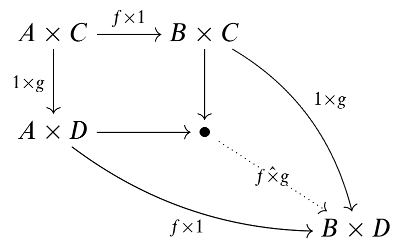
followed by a map in $\Sigma^{\pitchfork}$. This can be done as follows:

$$B \xrightarrow[f]{} \Sigma_{a\in A}\Sigma_{\sigma\in\Sigma}B_a^{\sigma} \xrightarrow[g]{} A,$$

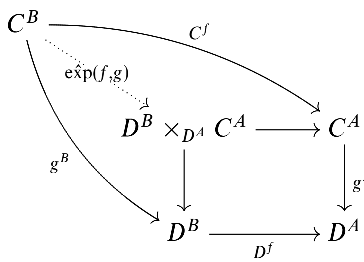
with $f(b) = (h(b), \top, \lambda x.b)$ and $g(a,\sigma,\tau) = a$. Here B^{σ} denotes an object of maps $\{* \mid \sigma\} \to B$. Note that in the case of $A = 1$, the object $\Sigma_{\sigma\in\Sigma}B^{\sigma}$ is isomorphic to the object $\widehat{B}$ of Rosolini (1986, Section 3.1) representing Σ-partial maps. By Rosolini (1986, Proposition 3.1.3), the inclusion $B \hookrightarrow \Sigma_{\sigma\in\Sigma}B^{\sigma}$ is a Σ-map; and the map $\Sigma_{\sigma\in\Sigma}B^{\sigma} \to 1$ has the right lifting property against Σ-maps by Rosolini (1986, Proposition 3.2.4). For the general case, a similar argument is performed in the slice over A. □

Definition 2.7. Suppose $f : A \to B$ and $g : C \to D$ are two maps in an elementary topos $\mathcal{E}$. Then the *Leibniz product* (or *pushout product*) of f and g is the unique map $f \hat{\times} g$ making



commute with the square being a pushout.

The *Leibniz exponential* of $f : A \to B$ and $g : C \to D$ is the unique map $\hat{\exp}(f,g)$ making



commute with the square being a pullback.

Proposition 2.8. The operations $\hat{\times}$ and $\hat{\exp}$ define bifunctors on $\mathcal{E}^{\to}$, and give rise to an adjunction

$$\mathcal{E}^{\to}(f \hat{\times} g, h) \cong \mathcal{E}^{\to}(f, \hat{\exp}(g,h)).$$

Also, for any triple of maps f, g, h we have $h \pitchfork \hat{\exp}(f,g)$ if and only if $f \hat{\times} h \pitchfork g$.

Proof. See Exercise 11.1.9 and Lemma 11.1.10 in Riehl (2014). □

3. An axiomatic set-up

In this section we will introduce our axiomatic set-up for building a model structure, which is inspired by earlier work by Orton and Pitts (2016) and Gambino and Sattler (2017). Following Gambino and Sattler, we define three classes of maps (cofibrations, fibrations

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and (strong) homotopy equivalences) and establish their basic properties. The results and proofs contain few surprises for the homotopy theorist, but we include them here, because we need to make sure that they can be established without using cocompleteness of the underlying category.

The setting in which we will be working will be the following:

1. We are given an elementary topos $\mathcal{E}$.
2. Within this topos $\mathcal{E}$ we are given an interval object $\mathbb{I}$, which here will mean that it comes equipped a monomorphism $[\partial_0, \partial_1] : 1 + 1 \to \mathbb{I}$ and *connections* $\wedge, \vee : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ satisfying:
$$i \wedge 0 = 0 \wedge i = 0, \quad i \wedge 1 = 1 \wedge i = i, \quad i \vee 0 = 0 \vee i = i, \quad i \vee 1 = 1 \vee i = 1.$$
3. A class $\mathcal{C}$ of monomorphisms in $\mathcal{E}$ satisfying the following axioms:
   a. $\mathcal{C}$ is a dominance.
   b. Elements in $\mathcal{C}$ are closed under finite unions (in other words, $\bot \in \mathcal{C}$ and $p, q \in \mathcal{C} \Rightarrow p \vee q \in \mathcal{C}$).
   c. The map $[\partial_0, \partial_1] : 1 + 1 \to \mathbb{I}$ belongs to $\mathcal{C}$.

   The elements of $\mathcal{C}$ will be referred to as the *cofibrations*.

It follows from these axioms that both ∂_i are cofibrations and that the cofibrations are closed under Leibniz products.

Example 3.1.

1. We could take for $\mathcal{E}$ the category of simplicial sets, in which we have an interval given by $\Delta[1]$ and the class of all monomorphisms is a class of cofibrations. In this case the constructed model structure is the restriction of the classical Quillen model structure to the Kan complexes (see Gambino and Sattler (2017)).
2. It is also possible to take the category of cubical sets as in Cohen *et al.* (2018) and Bezem *et al.* (2014). As discussed in Orton and Pitts (2016), this work fits into the present setting by taking for $\mathbb{I}$ the obvious representable and for $\mathcal{C}$ a special class of monos generated by the face lattice of Cohen *et al.* (2018).

Remark 3.2. In the example that we will work out below – the effective topos – the dominance $\mathcal{C}$ will be the class of all monomorphisms. However, in the axiomatic set-up we have decided to work with a general dominance, because the same proofs work in this more general setting and it will allow us to cover examples like the previous one. As another example we can mention the decidable monomorphism in a topos.

Within this setting we make the following definitions.

Definition 3.3. A morphism in $\mathcal{C}^{\pitchfork}$ will be referred to as a *trivial (or acyclic) fibration.*

By Proposition 2.6 we know that the cofibrations and trivial fibrations form a weak factorisation system on $\mathcal{E}$.

Definition 3.4. A morphism f in $\mathcal{E}$ is a *fibration* if it has the right lifting property with respect to maps of form $\partial_i \hat{\times} u$ with $u \in \mathcal{C}$ and $i \in \{0, 1\}$ (note that $\partial_i \hat{\times} u \in \mathcal{C}$, so that

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trivial fibrations are indeed fibrations). An object X will be called *fibrant* if the unique map $X \to 1$ is a fibration. We will write $\mathcal{E}_f$ for the full subcategory of $\mathcal{E}$ consisting of the fibrant objects.

Note that we are assuming that every map $0 \to X$ is a cofibration, so that every object in $\mathcal{E}$ is cofibrant in that sense.

Proposition 3.5.

1. If u is a cofibration and f is a (trivial) fibration, then $\hat{\exp}(u, f)$ is a (trivial) fibration as well.
2. A morphism f is a fibration if and only if both $\hat{\exp}(\partial_i, f)$ are trivial fibrations.

Proof. This is immediate from the Leibniz adjunction, the fact that the Leibniz product is associative and commutative, and the fact that cofibrations are closed under Leibniz products. (Note that the second point is Gambino and Sattler (2017, Proposition 3.4).) □

3.1. *The homotopy relation*

Definition 3.6. Two parallel arrows $f, g : B \to A$ will be called *homotopic* if there is a morphism $H : \mathbb{I} \times B \to A$, a *homotopy*, such that $f = H(\partial_0 \times B)$ and $g = H(\partial_1 \times B)$; in this case we will write $f \simeq g$, or $H : f \simeq g$, if we wish to stress the homotopy.

Proposition 3.7. The homotopy relation defines a congruence on $\mathcal{E}_f$.

Proof. This follows using a standard argument (see, for instance, Cisinski (2002, Lemma 2.2)). □

Definition 3.8. A morphism $f : B \to A$ is a *homotopy equivalence* if there is a morphism $g : A \to B$, a *homotopy inverse*, such that $gf \simeq 1_B$ and $fg \simeq 1_A$.

Corollary 3.9. On $\mathcal{E}_f$ the homotopy equivalences satisfy 2-out-of-3 (indeed, they satisfy 2-out-of-6) and they are closed under retracts.

3.2. *Strong homotopy equivalence*

For our purposes the following (somewhat non-standard) definition of a strong homotopy equivalence from Gambino and Sattler (2017, Definition 4.1) will be useful.

Definition 3.10. A homotopy equivalence $f : B \to A$ together with homotopy inverse g and homotopies $H : gf \simeq 1_B$ and $K : fg \simeq 1_A$ will be called *strong* if

$$\begin{array}{ccc} \mathbb{I} \times B & \xrightarrow{H} & B \\ {\scriptstyle \mathbb{I}\times f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{I} \times A & \xrightarrow[K]{} & A \end{array}$$

commutes. If H can be chosen to be π_B, we call f a *strong deformation retraction* and if K can be chosen to be π_A, we call f a *strong codeformation retraction*.

In what follows it will be convenient to use an alternative characterisation of the strong homotopy equivalences. For this we should observe that for any $f : B \to A$, there are maps $\theta_f : f \to \partial_0 \hat{\times} f$ and $\sigma_f : \hat{\exp}(\partial_0, f) \to f$ in $\mathcal{E}^{\to}$:

$$\begin{array}{ccc} B & \xrightarrow{i_1(\partial_1 \times B)} & A \cup_B \mathbb{I} \times B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \partial_0 \hat{\times} f} \\ A & \xrightarrow[\partial_1 \times A]{} & \mathbb{I} \times A \end{array} \qquad \begin{array}{ccc} B^I & \xrightarrow{B^{\partial_1}} & B \\ {\scriptstyle \hat{\exp}(\partial_0, f)}\downarrow & & \downarrow{\scriptstyle f} \\ B \times_A A^I & \xrightarrow[A^{\partial_1}\pi_2]{} & A \end{array}$$

Proposition 3.11. The following are equivalent for a morphism $f : B \to A$:

i. f is a strong homotopy equivalence.
ii. The map $\theta_f : f \to \partial_0 \hat{\times} f$ has a retraction in $\mathcal{E}^{\to}$.
iii. The map $\sigma_f : \hat{\exp}(\partial_0, f) \to f$ has a section in $\mathcal{E}^{\to}$.

Proof. If f is a strong homotopy equivalence, then $([g, H], K)$ is a retraction of θ_f as in

$$\begin{array}{ccccc} B & \xrightarrow{i_1(\partial_1 \times B)} & A \cup_B \mathbb{I} \times B & \xrightarrow{[g,H]} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \partial_0 \hat{\times} f} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\partial_1 \times A]{} & \mathbb{I} \times A & \xrightarrow[K]{} & A, \end{array}$$

and any retraction of θ_f must be of the form $([g, H], K)$ with g, H and K showing that f is a strong homotopy equivalence.

Similarly, if f is a strong homotopy equivalence, then $(\overline{H}, (g, \overline{K}))$ (where $\overline{H}$ and $\overline{K}$ refer to the exponential transposes of H and K) is a section of σ_f as in

$$\begin{array}{ccccc} B & \xrightarrow{\overline{H}} & B^I & \xrightarrow{B^{\partial_1}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{\exp}(\partial_0, f)} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[(g,\overline{K})]{} & B \times_A A^I & \xrightarrow[A^{\partial_1}\pi_2]{} & A, \end{array}$$

and any section of σ_f must be of the form $(\overline{H}, (g, \overline{K}))$ with g, H and K showing that f is a strong homotopy equivalence. □

Remark 3.12. The equivalence of (i) and (ii) in the previous proposition is Lemma 4.3 in Gambino and Sattler (2017). The equivalence of (ii) and (iii) also follows from the general fact that for any pointed adjunction $(u, v) : (\mathsf{Id}, \mathsf{Id}) \to (L, R)$, we have that $u_X : X \to LX$ is a split mono if and only if $v_X : RX \to X$ is a split epi.

Proposition 3.13. If $f : B \to A$ is a fibration and a homotopy equivalence between fibrant objects, then f is a strong codeformation retraction.

Proof. We follow a standard homotopy-theoretic argument (compare, for instance, Propositions 3.2.5 and 3.2.6 in Joyal and Tierney (2008)).

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Suppose $f : B \to A$ is a fibration and a homotopy equivalence between fibrant objects. This means that there is a homotopy inverse $g' : A \to B$ and there are homotopies $H : g'f \simeq 1_B$ and $K : fg' \simeq 1_A$. Therefore

$$\begin{array}{ccc} \{0\} \times A & \xrightarrow{g'} & B \\ {\scriptstyle \partial_0 \hat{\times} \perp_A} \downarrow & \overset{L}{\nearrow} & \downarrow {\scriptstyle f} \\ \mathbb{I} \times A & \xrightarrow[K]{} & A \end{array}$$

commutes and because f is a fibration, there will be a diagonal filler L. Writing $g = L(\partial_1 \times A)$, we see that g is a section of f with $g \simeq g'$. Hence, $\pi_A : \mathbb{I} \times A \to A$ is a homotopy $fg \simeq 1$ and because $gf \simeq g'f \simeq 1_B$, there is a homotopy $M : gf \simeq 1_B$ as well. Our aim is to modify this homotopy M to a homotopy N making

$$\begin{array}{ccc} \mathbb{I} \times B & \xrightarrow{N} & B \\ {\scriptstyle \mathbb{I} \times f} \downarrow & & \downarrow {\scriptstyle f} \\ \mathbb{I} \times A & \xrightarrow[\pi_A]{} & A \end{array}$$

commute.

For this we use the connections and the fact that

$$\begin{array}{ccc} \{0\} \times \mathbb{I} \cup \{1\} \times \mathbb{I} \cup \mathbb{I} \times \{0\} & \xrightarrow{[\overline{gfM},\overline{\pi_B},\overline{M}]} & B^B \\ {\scriptstyle [\partial_0,\partial_1] \hat{\times} \partial_0} \downarrow & \overset{F}{\nearrow} & \downarrow {\scriptstyle f^B} \\ \mathbb{I} \times \mathbb{I} & \xrightarrow[\vee]{} \mathbb{I} \xrightarrow[\overline{fM}]{} & A^B \end{array}$$

commutes. Since, $f^B = \hat{\exp}(\perp_B : 0 \to B, f : B \to A)$ is a fibration by Proposition 3.5.1, we obtain a diagonal filler $F : \mathbb{I} \times \mathbb{I} \to B^B$. Then, $N = \overline{F(\mathbb{I} \times \partial_1)}$ is the desired homotopy. □

Proposition 3.14. If $f : B \to A$ is a cofibration and a homotopy equivalence between fibrant objects, then f is a strong deformation retraction.

Proof. The proof of this proposition is very similar to the previous one. Suppose, $f : B \to A$ is a cofibration and a homotopy equivalence between fibrant objects. This means that there is a homotopy inverse $g' : A \to B$ and there are homotopies $H : g'f \simeq 1_B$ and $K : fg' \simeq 1_A$, resulting in a commuting square

$$\begin{array}{ccc} B & \xrightarrow{\overline{H}} & B^I \\ {\scriptstyle f} \downarrow & \overset{\overline{L}}{\nearrow} & \downarrow {\scriptstyle \hat{\exp}(\partial_0, !_B)} \\ A & \xrightarrow[g']{} & B. \end{array}$$

Because f is a cofibration and $\hat{\exp}(\partial_0, !_B)$ is a trivial fibration by Propostion 3.5.1, there will be a diagonal filler $\overline{L}$. Writing $g = L(\partial_1 \times A)$, we see that $gf = 1_B$ and $g \simeq g'$. Hence, $\pi_B : \mathbb{I} \times B \to B$ is a homotopy $gf \simeq 1$ and because $fg \simeq fg' \simeq 1_A$, there is a homotopy

$M : fg \simeq 1_A$ as well. Our aim is to modify this homotopy M to a homotopy N making

$$\begin{array}{ccc} \mathbb{I}\times B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \mathbb{I}\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{I}\times A & \xrightarrow[N]{} & A \end{array}$$

commute.

For this we use the connections and the fact that

$$\begin{array}{ccccc} \{0\}\times\mathbb{I}\cup\{1\}\times\mathbb{I}\cup\mathbb{I}\times\{0\} & & \xrightarrow{[\overline{Mfg},\overline{\pi_A},\overline{M}]} & & A^A \\ {\scriptstyle [\partial_0,\partial_1]\hat{\times}\partial_0}\big\downarrow & & \overset{F}{\nearrow} & & \big\downarrow{\scriptstyle A^f} \\ \mathbb{I}\times\mathbb{I} & \xrightarrow[\vee]{} & \mathbb{I} & \xrightarrow[\overline{Mf}]{} & A^B \end{array}$$

commutes. Since, $A^f = \hat{\exp}(f : B \to A, !_A : A \to 1)$ is a fibration by Proposition 3.5.1, we obtain a diagonal filler $F : \mathbb{I}\times\mathbb{I} \to A^A$. Then, $N = \overline{F(\mathbb{I}\times\partial_1)}$ is the desired homotopy. □

4. A model structure

We continue working in the setting of the previous section and we establish the existence of a model structure on the full subcategory of fibrant objects. In addition, we establish that the resulting model structure gives a model of type theory with Π-types satisfying function extensionality.

4.1. *A WFS with cofibrations and trivial fibrations*

Proposition 4.1. A morphism $f : B \to A$ is a trivial fibration if and only if it is a fibration and a strong homotopy equivalence.

Proof. This is again similar to Propositions 3.2.5 and 3.2.6 in Joyal and Tierney (2008). Suppose, $f : B \to A$ is a trivial fibration. Because $0 \to A$ is a cofibration, the square

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow[1]{} & A \end{array}$$

has a diagonal filler g. Therefore, f has a section g and $\pi_A : \mathbb{I}\times A \to A$ is a homotopy showing $fg \simeq 1$. Moreover, $[\partial_0, \partial_1]\times B = [\partial_0, \partial_1]\hat{\times}(0 \to B)$ is a cofibration as well, so also

$$\begin{array}{ccccc} \{0\}\times B+\{1\}\times B & & \xrightarrow{[gf,1_B]} & & B \\ {\scriptstyle [\partial_0,\partial_1]\times B}\big\downarrow & & \overset{H}{\nearrow} & & \big\downarrow{\scriptstyle f} \\ \mathbb{I}\times B & \xrightarrow[\mathbb{I}\times f]{} & \mathbb{I}\times A & \xrightarrow[\pi_A]{} & A \end{array}$$

has a diagonal filler, showing that f is a strong homotopy equivalence.

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Conversely, suppose f is both a fibration and a strong homotopy equivalence. Because f is a strong homotopy equivalence, it is a retract of $\hat{\exp}(\partial_0, f)$ by Proposition 3.11.iii, and because f is a fibration, it follows that $\hat{\exp}(\partial_0, f)$ is a trivial fibration by Proposition 3.5.1. Therefore, f is a trivial fibration as well. □

4.2. *A WFS with trivial cofibrations and fibrations*

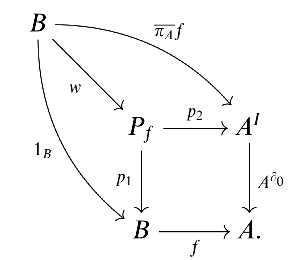
Proposition 4.2. If u is a cofibration and a strong homotopy equivalence and f is a fibration, then $u \pitchfork f$.

Proof. (This is Lemma 4.5.(ii) in Gambino and Sattler (2017).) If u is a strong homotopy equivalence, then u is a retract of $u \hat{\times} \partial_0$ by Proposition 3.11. So in order to show that $u \pitchfork f$ it suffices to show that $u \hat{\times} \partial_0 \pitchfork f$. But, that is immediate from the fact that u is a cofibration and f is a fibration. □

Proposition 4.3. Every morphism $f : B \to A$ between fibrant object factors as a map which is both a cofibration and a homotopy equivalence followed by a fibration.

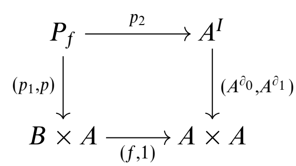
Proof. The idea of the proof is to build the factorisation in two steps. First, we use the co-cylinder factorisation to factor f as a homotopy equivalence w followed by a fibration. Second, we factor w as a cofibration followed by a trivial fibration using the factorisation system that we have already established (this idea is due to Andrew Swan (2015)). The precise details are as follows.

Construct the following diagram, in which the square is a pullback:



Since, $A^{\partial_0} = \hat{\exp}(\partial_0, A \to 1)$ it follows from Proposition 3.5.2 that this map is a trivial fibration. Since, trivial fibrations are stable under pullback, p_1 is a trivial fibration and hence a homotopy equivalence. Since, 1_B is a homotopy equivalence as well, it follows that w is a homotopy equivalence.

Next, consider the map $p := A^{\partial_1} p_2$. The square



is a pullback and because $(A^{\partial_0}, A^{\partial_1}) = \hat{\exp}([\partial_0, \partial_1], A \to 1)$ is a fibration by Proposition 3.5.1 and fibrations are stable under pullback, it follows that (p_1, p) is a fibration as well.

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In addition, B is fibrant, so $\pi_A : B \times A \to A$ is fibration and therefore $p = \pi_A(p_1, p)$ is as well.

So, $f = pw$ factors f as a homotopy equivalence w followed by a fibration p. Using the factorisation system that we have already established, we can write $w = w_1 w_0$, where w_1 is a trivial fibration and w_0 is a cofibration. So, pw_1 is a fibration, while w_0 is a homotopy equivalence, since both w and w_1 are. Therefore, $f = (pw_1)w_0$ factors f as a cofibration, which is also a homotopy equivalence followed by a fibration, as desired. □

Putting all the pieces together we can show:

Theorem 4.4. Let $\mathcal{E}$ be an elementary topos with an interval object $\mathbb{I}$ and a class of cofibrations $\mathcal{C}$ satisfying the conditions at the start of Section 2. Then the full subcategory of $\mathcal{E}$ on the fibrant objects carries a Quillen model structure in which the morphisms in $\mathcal{C}$ are the cofibrations, the fibrations as defined in Definition 3.4 are the fibrations and the homotopy equivalences are the weak equivalences.

Proof. Weak equivalences satisfy the 2-out-of-3 condition by Corollary 3.9.

We have defined trivial fibrations to be maps with the RLP with respect to cofibrations, but in a model structure they should be the maps which are both fibrations and weak equivalences. However, for maps between fibrant objects, these two notions coincide by Propositions 3.13 and 4.1. Therefore, by Proposition 2.6, cofibrations and trivial fibrations form a weak factorisation system on the full subcategory of fibrant objects.

To show that trivial cofibrations and fibrations form a weak factorisation system we use the retract argument (Lemma 2.3). The factorisation is given by Proposition 4.3. If u is a cofibration and a homotopy equivalence, and f is a fibration, then $u \pitchfork f$ by Propositions 3.14 and 4.2. The fibrations are closed under retracts because they are defined in terms of a lifting property, the cofibrations are closed under retracts because $\mathcal{C}$ is a dominance, and homotopy equivalences are closed under retracts by Corollary 3.9. □

4.3. Π-*types*

For the purpose of interpreting type theory in $\mathcal{E}_f$ we require Π- and Σ-types. The interpretation of Σ-types is trivial, as Σ_f is just composition with f, and fibrations are stable under composition.

To interpret Π-types, we have to be a bit careful. A standard construction (Seely 1984) allows us to leverage locally cartesian closed structure of a category to interpret Π-types. Despite the fact that $\mathcal{E}$ is a topos, and hence is locally cartesian closed, we do not necessarily know that $\mathcal{E}_f$ is locally cartesian closed. However, for the purposes of interpreting type theory, we do not need all adjunctions $\Sigma_f \dashv f^* \dashv \Pi_f$ to be present in $\mathcal{E}_f$; we only require $\Pi_f(g)$ to exist in $\mathcal{E}_f$ whenever f and g are fibrations, that is, we require an adjunction $f^* : (\mathcal{E}/A)_f \dashv (\mathcal{E}/B)_f : \Pi_f$ for a fibration $f : B \to A$ between fibrant objects. This follows from the following theorem:

Theorem 4.5. For any fibration $f : B \to A$ the right adjoint $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$ to pullback preserves fibrations.

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Proof. By a standard argument using the interaction between lifting properties and adjunctions, it suffices to prove that f^* preserves trivial cofibrations. But, this is Theorem 4.8 in Gambino and Sattler (2017). □

To show that we do not just have Π-types, but that they also satisfy function extensionality, we show the following proposition, which implies this principle by Lemma 5.9 in Shulman (2015).

Proposition 4.6. For any fibration $f : B \to A$ the right adjoint $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$ to pullback preserves trivial fibrations.

Proof. By the same standard argument as in the previous theorem, it suffices to prove that cofibrations are stable under pullback along fibrations. But, by assumption, cofibrations are stable along any map. □

5. The effective topos

For the remainder of this paper we work with the *effective topos* Eff. We briefly describe the effective topos and the category of assemblies, without giving any proofs. An interested reader is referred to a comprehensive book van Oosten (2008), the lecture notes Streicher (2008), and the original paper Hyland (1982) on the subject. We frequently conflate recursive functions and their Gödel codes, and we use standard notation $a \cdot b$ for Kleene application and standard notation $\lambda\langle x, y\rangle.t$ for pattern-matching in recursive functions.

The objects of Eff are pairs $(X, \sim)$, where X is a set and $\sim$ is a $\mathcal{P}(\omega)$-valued partial equivalence relation on X; that is $\sim$ is a mapping $X \times X \to \mathcal{P}(\omega)$. We denote $\sim (x, y)$ by $[x \sim y]$. We require the existence of computable functions s and tr, such that if $n \in [x \sim y]$, then $\mathsf{s}(n) \in [y \sim x]$ and if $m \in [y \sim z]$, then $\mathsf{tr}(n, m) \in [x \sim z]$.

A morphism $F : (X, \sim) \to (Y, \approx)$ is a $\mathcal{P}(\omega)$-valued functional relation between X and Y that respects $\sim$ and $\approx$. Specifically, F is a mapping $X \times Y \to \mathcal{P}(\omega)$ and we require the existence of computable functions st_X, st_Y, rel, sv and tot satisfying

— if $n \in F(x, y)$, then $\mathsf{st}_X(n) \in [x \sim x]$ and $\mathsf{st}_Y(n) \in [y \approx y]$;
— if $n \in F(x, y)$ and $k \in [x \sim x']$ and $l \in [y \approx y']$, then $\mathsf{rel}(n, k, l) \in F(x', y')$;
— if $n \in F(x, y)$ and $m \in F(x, y')$, then $\mathsf{sv}(n, m) \in [y \approx y']$;
— if $n \in [x \sim x]$, then $\mathsf{tot}(n) \in \bigcup_{y \in Y} F(x, y)$.

Two functional relations $F, G : X \times Y \to \mathcal{P}(\omega)$ are said to be equal if there is a computable function φ such that if $n \in F(x, y)$, then $\varphi(n) \in G(x, y)$. The identity arrow on $(X, \sim)$ is represented by the relation $\sim$ itself.

Given two sets $A, B \in \mathcal{P}(\omega)$, we write $A \wedge B$ for the set $\{\langle a, b\rangle \mid a \in A, b \in B\}$, where $\langle a, b\rangle$ is a surjective pairing of a and b. Then, the composition $G \circ F$ of two functional relations $F : (X, \sim) \to (Y, \approx)$ and $G : (Y, \approx) \to (Z, \approxeq)$ is defined as $(G \circ F)(x, z) = \bigcup_{y \in Y} F(x, y) \wedge G(y, z)$.

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5.1. *Constant objects functor*

The internal logic of Eff, as is the case with any topos, has a local operator $\neg\neg : \Omega \to \Omega$. Given an object $(A, \sim)$ and a subobject $(A', \sim_{A'})$, the latter is said to be $\neg\neg$-dense in $(A, \sim)$ if $\forall a : A(\neg\neg(A'(a)))$ holds; that is, if $A'(x)$ is non-empty whenever $[x \sim x]$ is non-empty. An object X is said to be a $\neg\neg$-sheaf if for any dense $A' \hookrightarrow A$ any map $A' \to X$ can be extended to a unique map $A \to X$. In the effective topos the $\neg\neg$-sheaves can be described, up to isomorphism, as objects in the image of a 'constant object functor' ∇.

Definition 5.1. The functor $\nabla : \text{Sets} \to \text{Eff}$ is defined on objects as $\nabla(X) = (X, \sim)$, where

$$[x \sim x'] = \begin{cases} \{0\} & \text{if } x = x' \\ \varnothing & \text{otherwise} \end{cases}$$

and on morphisms as

$$\nabla(f : X \to Y)(x, y) = [x \sim f(y)].$$

The functor ∇, together with the global sections functor $\Gamma(X) = \text{Hom}_{\text{Eff}}(1, X)$, forms a geometric morphism $\Gamma \dashv \nabla$, which embeds Sets into Eff. Note that in particular Γ preserves finite limits and arbitrary colimits (including preservation of monomorphisms and epimorphisms) and ∇ preserves arbitrary limits.

5.2. *Assemblies*

We say that an object A is $\neg\neg$-separated if $\forall x : A \forall y : A(\neg\neg(x \sim y) \to (x \sim y))$; that is, if we know that $[x \sim y]$ is non-empty and $n \in [x \sim x], m \in [y \sim y]$, then we can recursively find $\phi(n, m) \in [x \sim y]$. Just like $\neg\neg$-sheaves are objects in the essential image of the inclusion of Sets, the $\neg\neg$-separated objects can be described, up to isomorphism, as objects in the image of the inclusion of the category of *assemblies* into Eff.

Definition 5.2. An *assembly* is a pair (X, E_X) where X is a set, and $E_X : X \to \mathcal{P}(\omega)$ is a function, such that $E_X(x) \neq \varnothing$ for every $x \in X$. We will call such a function a *realizability relation* on X.

A morphism of assemblies $f : (X, E_X) \to (Y, E_Y)$ is a map $f : X \to Y$, such that there is a computable function φ and for every $x \in X$ and $n \in E_X(x)$, $\varphi(n)\downarrow$ and $\varphi(n) \in E_Y(f(x))$. In this case we say that φ *tracks* or *realizes* f.

We denote the category of assemblies and assembly morphisms as Asm. Sometimes we drop the realizability relation if it is obvious from the context. We also write $n \Vdash_X x$ for $n \in E_X(x)$.

Example 5.3. The natural numbers object $\mathbf{N}$ in Eff is an assembly $(\omega, E_{\mathbf{N}})$ with $E_{\mathbf{N}}(i) = \{i\}$ and the evident zero and successor maps.

Example 5.4. The terminal object 1 of Eff is an assembly $(\{*\}, E_1)$ with $E_1(*) = \{0\}$.

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The category of assemblies is a full subcategory of the effective topos via an inclusion which sends an assembly (X, E_X) to an object $(X, \sim_X)$ where

$$[x \sim_X x'] = \begin{cases} E_X(x) & \text{if } x = x' \\ \varnothing & \text{otherwise} \end{cases}$$

and which sends a map $f : (X, E_X) \to (Y, E_Y)$ to an induced relation

$$F(x, y) = [x \sim_X x] \wedge [y \sim_Y f(x)].$$

Example 5.5. Note that every $\nabla(X)$ is an assembly (X, E) with $E(x) = \{0\}$, and ∇ factors through Asm $\hookrightarrow$ Eff.

5.3. *Model structure on* Eff_f

In order to apply the result from Section 4 to the effective topos Eff, we must select an interval object $\mathbb{I}$ and a class of morphisms $\mathcal{C}$ satisfying certain conditions. Inspired by van Oosten (2015) and as discussed in the introduction, we take the interval object to be $\mathbb{I} = \nabla(2)$, which can be described as an assembly $(\{0,1\}, E)$ with $E(i) = \{0\}$. The connection structure $\wedge, \vee : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ is defined simply as

$$x \wedge y = \min(x, y) \qquad \text{tracked by } \lambda x.0$$
$$x \vee y = \max(x, y) \qquad \text{tracked by } \lambda x.0.$$

As our cofibrations, we have to choose a dominance on the effective topos. Several interesting dominances exist on the effective topos (see van Oosten (2008, Subsection 3.6.4)), but since we need the map $2 \to \nabla 2$ to belong to the dominance, the most natural choice seems to be to take the class of *all* monomorphisms as our dominance $\mathcal{C}$, as in simplicial sets. It is straightforward to verify that the class of monomorphisms satisfies the conditions outlined at the beginning of Section 3.

For the rest of this paper we use the following notation. We write s and t for the source and target maps $X^{\partial_0} : X^{\mathbb{I}} \to X$ and $X^{\partial_1} : X^{\mathbb{I}} \to X$, respectively. We write r for the "reflexivity" map $X^{!_{\mathbb{I}}} : X \to X^{\mathbb{I}}$.

Note that the interval object $\nabla(2)$ comes with 'twist' map $tw : \nabla(2) \to \nabla(2)$ which is a self-inverse and which satisfies $s \circ X^{tw} = t$, $t \circ X^{tw} = s$.

6. Contractible maps in Eff

In this section, we are going to characterize contractible objects in Eff as uniform fibrant objects with a global section (Propositions 6.3 and 6.4), and characterize trivial fibrations in Asm as uniform epimorphisms (Proposition 6.8). The latter characterization will allow us to give a concrete description of fibrant assemblies in terms of realizers (Theorem 6.10).

6.1. *Uniform objects and contractibility*

Definition 6.1. An object $(X, \sim)$ is said to be *uniform* if it is covered by a $\neg\neg$-sheaf, i.e., if there is an epimorphism $\nabla Y \to (X, \sim)$.

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Proposition 6.2. An object is uniform if it is isomorphic to an object $(X, \sim)$, such that there is a number $n \in \bigcap_{x \in X}[x \sim x]$.

Proof. By van Oosten (2008, Proposition 2.4.6). □

Recall that an object X is said to be *contractible* if the unique map $X \to 1$ is a trivial fibration.

Proposition 6.3. Every contractible object X is uniform and has a global element.

Proof. In our case, since the dominance $\mathcal{C}$ is exactly the class of monomorphisms, contractible objects are exactly the *injective* objects. So suppose X is an injective object in Eff. Because X embeds into $\mathcal{P}(X)$ via the singleton map, X is a retract of $\mathcal{P}(X)$. It has been shown in van Oosten (2008, Proposition 3.2.6) that every power object is uniform. Because X is covered by a uniform object, we can conclude that X is uniform as well.

A global element of X can be obtained by extending the unique map $0 \to X$ along the monomorphism $0 \to 1$, again using that X is injective. □

The converse of the previous proposition holds if we assume that X is fibrant.

Proposition 6.4. If $(X, \sim)$ is a fibrant object, which is uniform and has a global element $s : 1 \to (X, \sim)$, then $(X, \sim)$ is contractible.

Proof. We can assume that s is of the form $s(*, x) = [x \sim c]$ for some $c \in X$. We shall prove that s is a homotopy equivalence with homotopy inverse $!_X : (X, \sim) \to 1$.

The composition $!_X \circ s$ is the identity by the universal property of the terminal object. The homotopy $\theta : s \circ !_X \sim 1_X$ is constructed as follows.

$$\begin{cases} \theta(0, x, y) = s(*, y) = [y \sim c] \\ \theta(1, x, y) = [x \sim y] \end{cases}$$

Clearly, $\theta : \mathbb{I} \times (X, \sim) \to (X, \sim)$ is strict, single-valued and relational. To see that θ is total, it suffices to provide an element $\psi(n) \in \theta(0, x, y_0) \cap \theta(1, x, y_1) = [y_0 \sim c] \cap [x \sim y_1]$ for some y_0, y_1 given that $n \in [x \sim x]$. But, if we take $y_0 = c$ and $y_1 = x$, then the required element $\psi(n) \in [c \sim c] \cap [x \sim x]$ can be obtained from the uniformity of $(X, \sim)$ by Proposition 6.2. □

6.2. *Uniform maps and fibrant assemblies*

In the previous subsection we have discussed uniform objects. Now we move on to uniform maps.

Definition 6.5. A map $F : (Y, \approx) \to (X, \sim)$ is said to be *uniform* if it is covered by a $\neg\neg$-sheaf in the slice topos $\mathrm{Eff}/(X, \sim)$. That is, there is a map $\alpha : Z \to \Gamma(X, \sim)$, such that F is covered by the pullback $S : (X, \sim) \times_{\nabla\Gamma(X,\sim)} \nabla(Z) \to (X, \sim)$ of $\nabla(\alpha)$ along

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$\eta_X : (X, \sim) \to \nabla\Gamma(X, \sim)$, as depicted below.

$$\begin{array}{ccc} (X,\sim) \times_{\nabla\Gamma(X,\sim)} \nabla(Z) & \longrightarrow & \nabla(Z) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \nabla(\alpha)} \\ (X,\sim) & \xrightarrow[\eta_X]{} & \nabla\Gamma(X,\sim) \end{array} \qquad \begin{array}{ccc} (X,\sim) \times_{\nabla\Gamma(X,\sim)} \nabla(Z) & \overset{R}{\twoheadrightarrow} & (Y,\approx) \\ & {\scriptstyle S}\searrow & \downarrow{\scriptstyle F} \\ & & (X,\sim) \end{array}$$

Proposition 6.6. A map $F : (Y, \sim) \to (X, \approx)$ is uniform iff there are recursive functions α, β, such that for all $y \in Y$, $x \in X$, $n \in [x \approx x]$, $m \in F(y, x)$ there exists an $y' \in Y$ and

$$\begin{cases} \alpha(n) \in F(y', x) \\ \beta(n, m) \in [y \sim y']. \end{cases}$$

In particular a map $f : (Y, E_Y) \to (X, E_X)$ between assemblies is uniform iff there is a recursive α, such that

$$\forall x \in X\, \forall y \in Y\, \forall n \in E_X(x) \,(f(y) = x \to \alpha(n) \in E_Y(y)).$$

In other words, $\alpha(n) \in \bigcap_{y \in f^{-1}(x)}(E_Y(y))$ whenever $n \in E_X(x)$. In such a situation we say that every fiber of f is uniform and α witnesses the uniformity.

Proof. By van Oosten (2008, Proposition 3.4.6). □

The next proposition is aimed at generalizing Proposition 6.3 to uniform maps. We have not managed to extend the correspondence to arbitrary uniform maps. However, we can generalize the correspondence to the uniform maps in Asm (Proposition 6.8).

Theorem 6.7. Let $F : (Y, \approx) \to (X, \sim)$ be a map and let $(Y, \approx)$ be $\neg\neg$-separated in the slice $\mathrm{Eff}/(X, \sim)$. If F is a trivial fibration, then F is a uniform map.

Proof. Consider the following pullback

$$\begin{array}{ccc} (A, \asymp) & \longrightarrow & \nabla\Gamma(Y, \approx) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \nabla\Gamma F} \\ (X, \sim) & \xrightarrow[\eta]{} & \nabla\Gamma(X, \sim) \end{array}$$

The object $(A, \asymp)$ can be described as

$$A = \{\, ([y], x) \mid \nabla\Gamma(F)([y]) = [x] \,\},$$

where $[y]$ is the equivalence class of y', such that $[y \approx y']$ is non-empty, thus $\nabla\Gamma(F)([y]) = [x]$ means that $F(y, x)$ is non-empty; the realizability relation on A is

$$([y], x) \asymp ([y'], x') = \begin{cases} [x \sim x'] & \text{if } [y] = [y'] \text{ i.e. } [y \approx y'] \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

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and the maps from $(A, \asymp)$ to $\nabla\Gamma(Y, \approx)$ and $(X, \sim)$ are the evident projections. Then, consider a map $S : (Y, \approx) \to (A, \asymp)$ defined as $S = \langle F, \eta_Y \rangle$. Explicitly:

$$S(y, [y'], x) = F(y, x) \wedge \{0 \mid y \in [y'],\ \text{i.e.}, [y \approx y'] \neq \varnothing\}$$

If $(Y, \approx)$ is $\neg\neg$-separated in the slice over $(X, \sim)$, then S is a mono. To see this, suppose $\langle m_1, m_2 \rangle \in S(y_1, [y], x) \wedge S(y_2, [y], x)$. We are to provide an element of $[y_1 \approx y_2]$. Because $m_1 \in S(y_1, [y], x)$, we know that $[y \approx y_1]$ is non-empty; the case for m_2 and y_2 is similar. Then, from m_1 and m_2 we can get realizers for $[y_1 \approx y_1]$ and $[y_2 \approx y_2]$. Then, $[y_1 \approx y_2]$ follows from $\neg\neg$-separation, as both y_1 and y_2 lie over the same $x \in (X, \sim)$. Then, because S is a mono and F is a trivial fibration, the square below has a filler $H : (A, \asymp) \to (Y, \sim)$.

$$\begin{array}{ccc} (Y, \approx) & = & (Y, \approx) \\ {\scriptstyle S}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle F} \\ (A, \asymp) & \xrightarrow[\pi]{} & (X, \sim) \end{array}$$

Then, H is an epimorphism, as it is a retraction, and hence, F is a uniform map. □

Now we can show:

Proposition 6.8. A map f is a trivial fibration between assemblies if it is a uniform epimorphism between assemblies.

Proof. ($\Leftarrow$) For the 'if' direction, suppose f is a uniform epimorphism, with uniformity witnessed by α (in the sense of Definition 6.6), and we have the following commutative diagram in which i is a monomorphism:

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[h]{} & X \end{array}$$

Because Γ is the inverse part of a geometric morphism it preserves monomorphisms and epimorphisms. We can find a filler $\Gamma B \to \Gamma Y$ for the diagram above in Sets, under the image of Γ (such a filler exists by axiom of choice). Then, by the adjunction we get a map $k : B \to \nabla\Gamma Y$. Because Y is an assembly, in order to extend k to a map $B \to Y$ and to fill in the diagram above in Eff it suffices to find a realizer for k. One can check that the realizer is provided by $\lambda n.\alpha(\underline{h} \cdot n)$, where $\underline{h}$ is a realizer for h.

($\Rightarrow$) The "only if" direction follows from Theorem 6.7. □

Using Proposition 6.8 we can characterize the fibrant assemblies in recursion-theoretic terms. For this, we need to introduce a notion of path-connectedness.

Definition 6.9. Let X be an assembly, and let $x \in X$. A *path-connected component* of x, denoted as $[x]$, is a set of $y \in X$, such that there is a map $p : \mathbb{I} \to X$ such that $p(0) = x$ and $p(1) = y$. We also say that y is path-connected to x.

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Theorem 6.10. An assembly X is fibrant if for every $n \in E_X(x)$ one can uniformly find $\alpha(n)$ that realizes the path-connected component of x, i.e., $\alpha(n) \in \bigcap_{y\in[x]} E_X(y)$.

Proof. By Proposition 3.5, an assembly X is fibrant iff both $s = \hat{\exp}(\partial_0, X \to 1), t = \hat{\exp}(\partial_1, X \to 1) : X^{\mathbb{I}} \to X$ are trivial fibrations. So, the statement follows by applying Proposition 6.8 to the maps $s, t : X^{\mathbb{I}} \to X$. □

6.3. *Assemblies and the path object construction*

As an application of Theorem 6.10, we would like to present a comparison with the path object construction of van Oosten (2015). Van Oosten presented a path object category (van den Berg and Garner 2012) structure on the effective topos. In his setting, the object of paths in $(X, \sim)$ is represented not by an exponent $(X, \sim)^I$, but by a different object $\mathsf{P}(X, \sim)$, which is built out of paths of 'various length' I_n defined below. Whilst such an object is generally different from $(X, \sim)^{\mathbb{I}}$, we can show that both constructions are equivalent if X is a fibrant assembly. We refer the reader to the original paper for detailed definitions.

Definition 6.11. The assembly I_n is defined to be an underlying set $\{0, \ldots, n\}$ with the realizability relation $E(i) = \{i, i+1\}$. Note that I_1 is isomorphic to $\mathbb{I}$.

Definition 6.12 (van Oosten (2015, Definition 2.3)). A map $\sigma : I_n \to I_m$ is order and endpoint preserving iff

1. $\sigma(i) \leqslant \sigma(j)$ whenever $i \leqslant j$
2. $\sigma(0) = 0$ and $\sigma(n) = m$.

Definition 6.13. Given an object $(X, \sim)$ the path object $\mathsf{P}(X, \sim)$ (denoted as $\mathsf{P}(X)$ when unambiguous) is an object with the underlying set of pairs (n, f), where $n \in \mathbb{N}$ and $f \in (X, \sim)^{I_n}$. The realizability relation $[(n, f) \approx (m, g)]$ is a set of triples $\langle a, s, b \rangle$, such that

1. $a \in E_{X^{I_n}}(f)$ and $b \in E_{X^{I_m}}(g)$;
2. there is $k \in \mathbb{N}$ and a commutative square

$$\begin{array}{ccc} I_k & \xrightarrow{\tau} & I_m \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle g} \\ I_n & \xrightarrow[f]{} & (X, \sim) \end{array}$$

   where σ and τ are order and endpoint preserving maps, and $s \in [f\sigma \sim g\tau]$ in the sense of $(X, \sim)^{I_k}$.

If $(X, \sim)$ is an assembly, then $\mathsf{P}(X, \sim)$ is an assembly as well. Its underlying set is a quotient of $\{(n, f) \mid f : I_n \to X\}$ by an equivalence relation stating that (n, f) and (m, g) are related if there is a span of order and endpoint preserving maps, as in item (2) in Definition 6.13. The realizability relation given by

$$E_{\mathsf{P}(X)}([(n, f)]) = \bigcup_{(m,g)\in[(n,f)]} E_{X^{I_m}}(g).$$

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Remark 6.14. Note that Definition 6.13 is different from the one in van Oosten (2015). We require a span of order and endpoint preserving maps between I_n and I_m, whereas in *loc. cit.* it is required that there is a single order and endpoint preserving map $I_n \to I_m$. Unfortunately, with such a requirement, the relation $\approx$ of the path object is not transitive in general. The corrected definition that we present is sufficient to develop the theory in van Oosten (2015).

Proposition 6.15. Suppose X is a fibrant assembly. Then $\mathsf{P}(X)$ is homotopy equivalent to $X^{\mathbb{I}}$.

Proof. Given an n-path $[(n, q)] \in \mathsf{P}(X)$ one can, by repeated application of the composition, obtain a path $\mathfrak{p}(q) : \mathbb{I} \to X$, such that $\mathfrak{p}(q)(0) = q(0)$ and $\mathfrak{p}(q)(1) = q(n)$. Furthermore, by Theorem 6.10, there is a recursive α, such that for any $n \in E_X(x)$ the element $\alpha(n)$ is a common realizer of all elements in the path-connected component of x. That means that given a realizer $m \in E_X(q(0))$, the term $\lambda x.\alpha(m)$ tracks $\mathfrak{p}(q)$. One can obtain such m using the realizer for the original q.

This defines a map $\mathfrak{p} : \mathsf{P}(X) \to X^{\mathbb{I}}$, for a fibrant assembly X. One can check that a map $i : X^{\mathbb{I}} \to \mathsf{P}(X)$ that embeds $X^{\mathbb{I}}$ into the path object $\mathsf{P}(X)$ by sending $p : \mathbb{I} \to X$ to $[(1, p)] \in \mathsf{P}(X)$ is a right inverse of $\mathfrak{p}$. We can show that it is also a left homotopy inverse of $\mathfrak{p}$.

We do so by defining a homotopy $\theta : \mathbb{I} \times \mathsf{P}(X) \to \mathsf{P}(X)$ as $\theta(0, [(n, q)]) = [(n, q)]$ and $\theta(1, [(n, q)]) = [(1, \mathfrak{p}(q))]$. What remains is to provide a common realizer for $[(n, q)]$ and $[(1, \mathfrak{p}(q))]$ uniformly, given a realizer for $[(n, q)]$. From a realizer of $[(n, q)]$ one can find a realizer $k \in E_X(q(0)) = E_X(\mathfrak{p}(q)(0))$. Using the fibrancy of X one can find a realizer $\alpha(k) \in \bigcap_{x' \in [q(0)]} E_X(x')$. Then, $\lambda x.\alpha(k)$ realizes both $q : I_n \to X$ and $\mathfrak{p}(q) : \mathbb{I} \to X$. □

Remark 6.16. We do not know if Proposition 6.15 holds for all fibrant objects.

7. Discrete objects and discrete reflection

In this section, we describe the reflexive subcategory of discrete objects in Eff and show that every discrete object is fibrant. We also prove that the unit of the discrete reflection of a fibrant assembly is a homotopy equivalence, which allows us to concretely characterize the homotopy category of fibrant assemblies as the category of modest sets (Proposition 7.11).

7.1. *Discrete objects and discrete maps*

Definition 7.1. An object of Eff is said to be *discrete* if it is a quotient of a subobject of the natural numbers object.

Discrete objects can be characterized as objects which have no non-constant paths.

Lemma 7.2. An object X is discrete if and only if it is internally true that all paths in X are constant,*i.e.*, the diagonal map $p : X \to X^{\mathbb{I}}$ is an isomorphism.

Proof. By van Oosten (2008, Propositions 3.2.21 and 3.2.22). □

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Proposition 7.3. If $H : \mathbb{I} \times A \to X$ is a morphism and X is discrete, then there is a map $h : A \to X$ with $H = h \circ \pi_A$.

Proof. The map h is given as a composite $p^{-1} \circ \bar{H}$, where p^{-1} is the inverse of the canonical map from Lemma 7.2. □

Proposition 7.4. Every map $F : Y \to X$ between discrete objects is a fibration.

Proof. Let $F : Y \to X$ be a morphism between discrete objects and consider the following lifting problem

$$\begin{array}{ccc} (\{0\} \times B) \cup (\mathbb{I} \times A) & \xrightarrow{[\alpha_0, \alpha_1]} & Y \\ {\scriptstyle \delta_0 \hat{\times} u} \big\downarrow & & \big\downarrow {\scriptstyle F} \\ \mathbb{I} \times B & \xrightarrow[\beta]{} & X. \end{array}$$

From the previous proposition, it follows that there are maps $a : A \to Y$ with $\alpha_1 = a \circ \pi_A$ and $b : B \to X$ with $\beta = b \circ \pi_B$. Defining $\gamma : \mathbb{I} \times B \to Y$ to be $\gamma = \alpha_0(!_\mathbb{I} \times B)$, we have

$$F\gamma = F\alpha_0(!_\mathbb{I} \times B) = \beta(\delta_0 \times B)(!_\mathbb{I} \times B) = b\pi_B(\delta_0 !_\mathbb{I} \times B) = b\pi_B = \beta$$

and

$$\gamma(\mathbb{I} \times u) = \alpha_0(!_\mathbb{I} \times B)(\mathbb{I} \times u) = \alpha_0(\{0\} \times u)(!_\mathbb{I} \times A) = \alpha_1(\delta_0 \times A)(!_\mathbb{I} \times A) = a\pi_A(\delta_0 !_\mathbb{I} \times A) = a\pi_A = \alpha_1$$

as well as

$$\gamma(\delta_0 \times B) = \alpha_0(!_\mathbb{I} \times B)(\delta_0 \times B) = \alpha_0.$$

Hence, γ is a diagonal filler for the square above. □

The following corollary follows from the fact that the terminal object is discrete.

Corollary 7.5. Every discrete object X is fibrant.

Example 7.6. Recall that a *modest set* is a discrete assembly. Examples of modest sets are $\mathbf{N}$ and 1; in fact, all finite types in Eff are modest sets.

We denote the full subcategory of modest sets as Mod $\hookrightarrow$ Eff. By Corollary 7.5 every modest set is fibrant.

Definition 7.7. A map $F : (Y, \approx) \to (X, \sim)$ is *discrete* if it is a quotient of a subobject of the natural numbers object in Eff$/(X, \sim)$, which is represented by a map $(X, \sim) \times \mathbf{N} \to (X, \sim)$.

Note that every map between discrete objects is discrete (if $f : Y \to X$ is a map and Y is covered by $A \rightarrowtail \mathbf{N}$ in Eff, then A can be viewed as subobject of $X \times \mathbf{N}$ in the slice Eff$/X$, covering f). Proposition 7.4 cannot be strengthened to the one saying that all discrete maps are fibrations, even if we restrict the attention to the category of assemblies. For this consider the following counterexample.

Example 7.8. Note that if we restrict our attention to the category of assemblies, then a map $f : (Y, E_Y) \to (X, E_X)$ is discrete iff every fiber $f^{-1}(x)$ is discrete (van Oosten 2008, Proposition 3.4.4). Consider the inclusion map $\{0\} \to \mathbb{I}$. It is discrete and an

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acyclic cofibration. The intersection of the trivial cofibrations and fibrations is the class of isomorphisms (as in any weak factorisation system), so if it would also be a fibration, it would have to be an isomorphism, which is clearly not the case.

7.2. *Path contraction and discrete reflection*

The inclusion of discrete objects in the effective topos has a left adjoint called the *discrete reflection*, see van Oosten (2008, Proposition 3.2.19). It was noted in van Oosten (2015) that discrete reflection can be seen internally as a set of path-connected components.

Proposition 7.9. The discrete reflection X_d of an object X is a coequalizer of the diagram

$$X^{\mathbb{I}} \underset{t}{\overset{s}{\rightrightarrows}} X \overset{q}{\twoheadrightarrow} X_d$$

Proof. First, we check that X_d is discrete. For this, we reason in the internal logic. Let $\pi : \mathbb{I} \to X_d$ be a path. We will show that it is trivial, i.e., $\pi = \pi \circ \partial_0 \circ !_{\mathbb{I}}$. Because $\mathbb{I} = \nabla(2)$ is internally projective (van Oosten 2008, Proposition 3.2.7 and 3.2.8), there is a map $p : \mathbb{I} \to X$, such that $q \circ p = \pi$. Define $P = \overline{p \circ \wedge} : \mathbb{I} \to X^{\mathbb{I}}$. Then, $qsP = qtP$, as q coequalizes s and t. But $tP = p$ and $sP = p \circ \partial_0 \circ !_{\mathbb{I}}$. Hence, $\pi = qp = qp\partial_0 !_{\mathbb{I}} = \pi \circ \partial_0 \circ !_{\mathbb{I}}$.

Therefore, X_d is discrete. To see that it satisfies the universal property, let $f : X \to D$ be a map into a discrete object D. Then, $f \circ s = s \circ f^{\mathbb{I}}$ and $f \circ t = t \circ f^{\mathbb{I}}$, by naturality. By Lemma 7.2, $s = t : D^{\mathbb{I}} \to D$, hence $f \circ s = f \circ t$. As q is the coequalizer of s and t, there is a unique map $\bar{f}$ such that $f = \bar{f} \circ q$. □

It is known that in a model category where every object is cofibrant, every fibrant object can be equipped with a weak groupoid structure. We will need the path composition operation of the groupoid for the characterization of discrete reflection. Specifically, there is a composition operation $c : X^{\mathbb{I}} \times_X X^{\mathbb{I}} \to X^{\mathbb{I}}$ satisfying

— $s \circ c = s \circ \pi_1$ and $t \circ c = t \circ \pi_2$;
— $c\langle rs, \mathsf{id}\rangle \sim \mathsf{id}$;
— $c\langle \mathsf{id}, rt\rangle \sim \mathsf{id}$;
— $c\langle c \times_X X^I\rangle \sim c\langle X^I \times_X c\rangle$.

See, e.g., van den Berg (2016, Appendix A.1) for explicit constructions.

It follows, using the composition operation, that for a fibrant object X, the image of $\langle s, t\rangle : X^{\mathbb{I}} \to X \times X$ is an equivalence relation. Thus, for a fibrant assembly (X, E_X) the discrete reflection X_d can be described as an assembly $(X/\sim_p, E)$ where

$$x \sim_p y \iff \exists p : \mathbb{I} \to X(p(0) = x \wedge p(1) = y)$$

and $E([x]) = \bigcup_{y \in [x]} E_X(y)$. One can check directly that X_d is indeed the discrete reflection of X with the unit $\eta_X : x \mapsto [x]$ tracked by $\lambda x.x$.

Using this explicit description we can prove the following statement.

Proposition 7.10. For a fibrant assembly X, the unit of the discrete reflection unit $\eta : X \to X_d$ is a homotopy equivalence.

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Proof. Using the axiom of choice, one can pick for each $x \in X$ a canonical representative $g([x]) \in [x]$ of each equivalence class $[x] \in X_d$. By Theorem 6.10, there is a partial recursive function α, such that for each $n \in E_X(x)$ the element $\alpha(n)$ realizes the path-connected component of x, i.e., $\alpha(n) \in \bigcap_{y\in[x]} E_X(y)$: such a function α tracks $g : X_d \to X$.

Clearly, $\eta \circ g = \mathsf{id}_{X_d}$. We are to show that there is a homotopy $g \circ \eta \sim \mathsf{id}_X$. Intuitively, this is the case because $g([x]) \in [x]$, and thus $g([x])$ must be connected to x by some path. The homotopy Θ is thus given by

$$\begin{cases} \Theta(0, x) = x \\ \Theta(1, x) = g([x]) \end{cases}$$

and is tracked by $\lambda\langle i, n\rangle.\alpha(n)$. □

Proposition 7.10 actually gives us a concrete description of the homotopy category of fibrant assemblies. Since every assembly is homotopy equivalent to a modest set (the discrete reflection), fibrant assemblies and fibrant modest sets are identified in $\mathrm{Ho}(\mathcal{A}\mathrm{sm}_f)$. This immediately gives us:

Proposition 7.11. The homotopy category of the category of fibrant assemblies $\mathrm{Ho}(\mathcal{A}\mathrm{sm}_f)$ is equivalent to the category of modest sets.

Proof. By Proposition 7.10, every fibrant assembly X is homotopy equivalent to X_d. Furthermore, every modest set is fibrant by Corollary 7.5, and $X_d \in \mathcal{A}sm_f$. It is thus the case that $\mathrm{Ho}(\mathcal{A}\mathrm{sm}_f) \simeq Ho(\mathrm{Mod})$. By Proposition 7.3, the category Mod has no non-trivial homotopies, therefore, $Ho(\mathrm{Mod}) \simeq \mathrm{Mod}$. As a result, the homotopy category of fibrant assemblies is the category of modest sets. □

8. Conclusions and future research directions

8.1. *Summary*

We have presented a way of obtaining a model structure on a full subcategory of a general topos, starting from an interval object $\mathbb{I}$ and a dominance Σ, which contains the endpoint inclusion map $2 \to \mathbb{I}$. The resulting model structure is sufficient for interpreting Martin-Löf type theory with intensional identity types – which are interpreted with the help of the interval object. The resulting model of type theory supports Π- and Σ-types, and functional extensionality holds for Π-types.

We have worked out the construction in the case of the effective topos Eff. For this model structure we have obtained some results characterizing contractible objects and maps, as well as fibrant assemblies.

8.2. *Future research questions*

There remain several directions which can be further explored. One of the most interesting questions would be extending the model category structure on $\mathcal{E}_f$ to the whole topos $\mathcal{E}$. The mapping co-cylinder construction in Section 4.2 would not carry over directly, so one

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would have to find another way of constructing an (acyclic cofibrations, fibrations) weak factorisation system.

In this work, we have decided to politely side-step the issues of coherence (as discussed in, e.g., Curien (1993)). The authors expect that it is possible to resolve the coherence issues by considering algebraic counterparts of the homotopy-theoretic notions considered in this paper, such as algebraic weak factorisation systems (as done in the work of Gambino and Sattler (2017)) and algebraic model structures Riehl (2011), but this issue should be investigated further.

In addition, there are several open questions regarding the concrete model Eff_f presented in this paper. The most embarrassing open problem is whether there is an object in Eff_f that has non-trivial higher homotopies. All the examples of fibrant objects that we could find are 0-truncated (or 'h-sets'). In fact, because the inclusion $2 \to \nabla(2)$ is an epimorphism in Asm, any two paths $P, Q : \mathbb{I} \to X$ are equal whenever they have the same endpoints and X is an assembly. Therefore, examples of fibrant objects that are not h-sets will have to live outside the category of assemblies. This excludes some natural candidates like the circles C_n (with $n > 2$), as in van Oosten (2015). Also, Van Oosten's object C_2 will not work: it is uniform, so if it would be fibrant, it would have to be contractible (see Proposition 6.4). Extending the model to the whole of Eff might solve this problem, as taking fibrant replacements of these C_n might result in circles with non-trivial homotopies. More generally, there is the question whether it is possible to construct higher inductive types in the model. And if so, could the discrete reflection play the role of 0-truncation?

Another interesting aspect of the effective topos is the existence of an internal small complete category of modest sets Hyland (1988), which is represented by a *universal family of modest sets*. Such an internal category can be used as a type universe for interpreting second-order λ-calculus (Streicher 1991). The natural question to ask is then the following: does there exist a map u which is a fibration, discrete, and has a fibrant codomain, such that every discrete map that is a fibration is a pullback of u? And if so, is this universal fibration univalent?

Finally, it remains to be seen how much of the theory carries over to other realizability toposes.

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