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Calabi–Yau Quotients of Hyperkähler Four-folds

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Abstract. The aim of this paper is to construct Calabi–Yau 4-folds as crepant resolutions of the quotients of a hyperkähler 4-fold X by a non-symplectic involution α. We first compute the Hodge numbers of a Calabi–Yau constructed in this way in a general setting, and then we apply the results to several specific examples of non-symplectic involutions, producing Calabi–Yau 4-folds with different Hodge diamonds. Then we restrict ourselves to the case where X is the Hilbert scheme of two points on a K3 surface S, and the involution α is induced by a non-symplectic involution on the K3 surface. In this case we compare the Calabi–Yau 4-fold Y_S, which is the crepant resolution of X/α, with the Calabi–Yau 4-fold Z_S, constructed from S through the Borcea–Voisin construction. We give several explicit geometrical examples of both these Calabi–Yau 4-folds, describing maps related to interesting linear systems as well as a rational 2:1 map from Z_S to Y_S.

1 Introduction

By the famous decomposition theorem of Beauville [4] and Bogomolov [6], compact Kähler Ricci flat varieties decompose, after an étale cover, into the product of three fundamental building blocks: complex tori, hyperkähler manifolds, and Calabi–Yau manifolds. The aim of this paper is to construct a relation between two of these blocks in dimension 4; indeed, our starting point is the observation that the presence of a non-symplectic involution α on a hyperkähler 4-fold X allows one to construct a Calabi–Yau 4-fold as crepant resolution $\widetilde{X/\alpha}$ of the quotient X/α.

We observe that several quotients of hyperkähler varieties have been deeply investigated both in the case of symplectic and non-symplectic actions and in particular in low dimension. In dimension 2, Calabi–Yau varieties and hyperkähler varieties collapse to the same class of surfaces, the K3 surfaces; in fact, the symplectic form and the volume form coincide. In the case of automorphisms acting on K3 surfaces, it is well-known that the quotient by a symplectic automorphism (which by definition preserves both the symplectic and the volume form) gives, after a minimal resolution, a K3 surface again; see [43]. This is known not to be the case in general for higher dimensional hyperkähler manifolds; indeed, given a hyperkähler variety with a symplectic automorphism α, there is in general no resolution of X/σ where the natural quotient symplectic form on the smooth locus is preserved, and partial resolutions

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give singular irreducible holomorphic symplectic varieties (see, for example, [22]). Up to now, the only known case where a symplectic resolution of the quotient exists is described in [33].

In dimension 2, the quotient of a K3 surface by a non-symplectic involution can not have trivial canonical bundle, and indeed it is a smooth surface, either rational or Enriques. In higher dimensions there are examples of automorphisms that do not preserve the symplectic structure (so the quotient is not a symplectic variety) but preserve the volume form, and the quotient admits a resolution that has trivial canonical bundle. So generically, the quotient of a hyperkähler variety by a finite automorphism does not produce a hyperkähler variety, but, as we noticed above, it can produce, under some conditions, a Calabi–Yau variety, and this is the main topic of this paper.

In Section 3, we consider the quotients of a hyperkähler variety X, of dimension $2n$, by a finite automorphism α, of prime order p. When α acts freely on X, the quotient manifold X/α has numerically but not rationally trivial canonical bundle, and it has been studied in [9, 49]. Here we ask when it is possible to obtain a quotient X/α that has rationally trivial canonical bundle and when it is possible to construct a crepant resolution of X/α by blowing up its singular locus. The main result of this section is Theorem 3.6, where we state that the good candidates are hyperkähler varieties of dimension $2p$ that admit a non-symplectic automorphism of order p. We also observe that there is a condition on the dimension of the components of the fixed locus, which has to be p. This condition automatically excludes the natural non-symplectic automorphisms of order p on $S^{[p]}$ unless $p = 2$. This is one of the motivations for our attention to the case $p = 2$. So we restrict our attention to hyperkähler varieties of dimension 4 admitting non-symplectic involutions. The study of non-symplectic involutions on hyperkähler varieties is the topic of several papers: in [5] a topological classification is given, in [8, 40] a lattice theoretical classification of automorphisms on two different type of hyperkähler 4-folds is presented with many explicit examples. Other explicit examples are given in [21, 29, 41, 45, 50]. All of these works provide a large set of explicit examples of non-symplectic involutions α defined on hyperkähler 4-folds X, and thus one is able to effectively construct Calabi–Yau 4-folds as quotients.

In Section 4 we consider a hyperkähler 4-fold X with a non-symplectic involution α, and we observe that a crepant resolution of X/α is simply given by blowing up the singular locus. If one knows the action of the non-symplectic involution α both on the cohomology of X and on X (more precisely if one knows the topology of the fixed locus of α on X), one is able to compute the Hodge numbers of the Calabi–Yau $\widetilde{X/\alpha}$. This is done in the general context in Theorem 4.1 and in some specific examples in Section 4.2.1, Proposition 4.3, and Section 4.2.3. Then in Section 5 we restrict ourselves to a particular type of hyperkähler 4-folds: denoted by S a K3 surface, the Hilbert scheme of two points of S, denoted $S^{[2]}$, is a hyperkähler 4-fold. The 4-fold $S^{[2]}$ is the blow-up of $(S \times S)/\sigma$ in its singular locus, where $\sigma \in \operatorname{Aut}(S \times S)$ is the automorphism switching the two copies of S. If S admits a non-symplectic involution ι_S, then the involution $\iota_S \times \iota_S \in \operatorname{Aut}(S \times S)$ induces a non-symplectic involution on $S^{[2]}$, denoted by ${\iota_S}^{[2]}$ and called the natural non-symplectic involution of (S, ι_S). By the construction described above, this allows one to produce Calabi–Yau 4-folds, denoted by Y_S, as crepant resolutions of $S^{[2]}/\iota_S{}^{[2]}$ and to compute their Hodge numbers, which depend only on the topological properties of the fixed locus of ι_S on S, as shown in Theorem 5.1. Since it is quite a natural question, we have to remark here that neither the mirror symmetry at the level of the K3 surface S nor the lattice theoretic mirror symmetry at the level of the hyperkähler 4-fold $S^{[2]}$ produces a Calabi–Yau 4-fold that is mirror symmetric to Y_S (see Section 5.6).

Essentially, we produce a Calabi–Yau 4-fold, Y_S, by the data (S, ι_S), where S is a K3 surface and ι_S is a non-symplectic involution acting on it. On the other hand, there is a very well known and natural way to produce a (different) Calabi–Yau 4-fold starting from these data: the Borcea–Voisin construction; *cf.* [14, 18]. This construction is recalled in Section 5.2 and reduces in our case to the blow-up of $(S \times S)/(\iota_S \times \iota_S)$ in its singular locus, producing another smooth Calabi–Yau 4-fold, denoted in the sequel by Z_S. By our construction, one immediately finds that there is a 2:1 rational map $Z_S \dashrightarrow Y_S$, indeed Y_S is birational to $(S \times S)/\langle \sigma, \iota_S \times \iota_S\rangle$ and Z_S is birational to $(S \times S)/\langle \iota_S \times \iota_S\rangle$, so the covering involution of the 2:1 map $Z_S \dashrightarrow Y_S$ is induced on Z_S by the action of σ on $S \times S$, as shown in Section 5.3. So we prove that Y_S is a Calabi–Yau 4-fold that is 2:1 covered both by a hyperkähler 4-fold and by a Calabi–Yau 4-fold and in fact it has a bidouble cover that is $S \times S$. Since the construction of Y_S is quite explicit, we are also able to describe a $\mathbb{Q}$-basis of its Picard lattice and to identify two 2-divisible divisors: one which is associated with the double cover $\widetilde{S^{[2]}} \to Y_S$ where $\widetilde{S^{[2]}}$ is the blow-up of $S^{[2]}$ in the fixed locus of $\iota_S{}^{[2]}$; the other is associated to the rational double cover $Z_S \dashrightarrow Y_Z$. This is done in Section 5.5.

In order to better describe the varieties constructed, we observe that by our assumptions the group generated by σ, $\iota_S \times \mathrm{id}, \mathrm{id} \times \iota_S \in \mathrm{Aut}(S \times S)$ acts on $S \times S$ and is isomorphic to the dihedral group of order 8. In Section 6 we describe the quotients of $S \times S$ by subgroups of this group. If W is the smooth surface S/ι_S, then $(S \times S)/\langle \sigma, \iota_S \times \mathrm{id}, \mathrm{id} \times \iota_S\rangle$ is $W^{(2)}$ (where $W^{(2)}$ is the quotient of $W \times W$ by the automorphism that switches the two copies of W and thus birational to the Hilbert scheme $W^{[2]}$). So the 4-folds Y_S and Z_S are birational to (possibly singular) 4-folds that are respectively 2:1 and 2^2:1 covers of $W^{(2)}$. The 4-fold $(S \times S)/\langle \sigma, \iota_S \times \iota_S\rangle$ is by construction birational to Y_S and it is also birational to the blow-up, $\widetilde{Z_S/\sigma_Z}$, of Z_S/σ_Z in its singular locus, where σ_Z is induced by σ on Z_S. We will prove that the 4-folds Y_S and $\widetilde{Z_S/\sigma_Z}$ are isomorphic (and not only birational) if the involution ι_S is free on S. When ι_S fixes exactly one irreducible curve on S, $(S \times S)/\langle \sigma, \iota_S \times \iota_S\rangle$ is singular along three surfaces meeting transversally in a curve and the 4-folds Y_S and $\widetilde{Z_S/\sigma_Z}$ are obtained by blowing up these singular surfaces in a different order.

In Section 7 we give a detailed geometric descriptions of Y_S, Z_S, and $S^{[2]}$ and of linear systems on them, under some conditions on (S, ι_S). Indeed, we first explain how to compute the dimension of certain linear systems induced on these varieties by nef and big divisors on S in Theorem 7.5, and then we explicitly describe projective models associated to certain linear systems. In particular, we show that if S is a 2:1 cover of $\mathbb{P}^2$, Y_S is a 2:1 cover of $(\mathbb{P}^2)^{(2)}$ embedded in $\mathbb{P}^5$ and Z_S is a 2:1 cover of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in $\mathbb{P}^8$ by the Segre embedding; see Proposition 7.8.

If S admits a genus 1 fibration, then $S^{[2]}$ admits a Lagrangian fibration whose smooth fibers are abelian surfaces (generically isomorphic to a product of two elliptic curves), Z_S and Y_S admit fibrations whose smooth fibers are the Kummer surfaces of the fibers of the Lagrangian fibration. Moreover, Z_S has an elliptic fibration and a fibration in Calabi–Yau 3-folds of Borcea–Voisin type, see Propositions 7.9, 7.10, and 7.11.

2 Preliminaries

In this section we collect some known results that are useful in the sequel.

Definition 2.1 Let Y be a smooth compact Kähler manifold of dimension n. Then Y is called a *Calabi–Yau variety* if

- the canonical bundle of Y is trivial and
- $h^{i,0}(Y) = 0$ for every $i = 1, \dots, n-1$.

We emphasize that we do not require a Calabi–Yau variety to be simply connected.

Definition 2.2 Let X be a smooth compact Kähler manifold. Then X is called a hyperkähler variety or, equivalently, *IHS variety* if

- X is simply connected
- $H^{2,0}(X) = \mathbb{C}\omega_X$, where ω_X is a symplectic form.

We observe that the existence of a symplectic form on a hyperkähler variety X implies that the canonical bundle of X is trivial and the complex dimension of X is even.

Definition 2.3 Let X be a smooth manifold. If X admits a volume form Ω_X, we say that an automorphism $\alpha \in \mathrm{Aut}(X)$ is volume preserving if $\alpha^*(\Omega_X) = \Omega_X$. If X admits a symplectic form ω_X, we say that an automorphism $\alpha \in \mathrm{Aut}(X)$ is symplectic if $\alpha^*(\omega_X) = \omega_X$ and non-symplectic otherwise.

Clearly, if X is a hyperkähler a symplectic automorphism is always volume preserving, but the converse is false as soon as the dimension of X is at least 4.

Fundamental examples of hyperkähler manifolds were discovered by Beauville [4]; he produced two families of hyperkähler manifolds in every even dimension that is constructed as follows. Let S be a $K3$ surface and let $S^{[n]}$ denote the Hilbert scheme of length n zero dimensional subschemes of S. Then $S^{[n]}$ is a resolution of the n-th symmetric product $S^{(n)}$ and it has a unique symplectic form up to a scalar multiple. Kähler deformations of $S^{[n]}$ are called manifolds of $K3^{[n]}$ type. Similarly, if A is an abelian surface, $A^{[n+1]}$ has a symplectic form and a fibre $K_n(A)$ of the Albanese map is hyperkähler and is called a *generalized Kummer manifold.*

The existence of a symplectic form provides a canonically defined integral quadratic form on the second cohomology of hyperkähler manifolds, which is given in terms of the top self intersection of divisors. This form is usually called the Beauville–Bogomolov–Fujiki form and gives a lattice structure to the second integral cohomology. In the above two examples, the lattices are

$$H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta \cong U^3 \oplus E_8(-1)^2 \oplus (-2n+2)$$

and

$$H^2(K_n(A),\mathbb{Z}) \cong H^2(A,\mathbb{Z}) \oplus \mathbb{Z}\delta \cong U^3 \oplus (-2n-2),$$

where 2δ is the class of the exceptional divisor of the map $S^{[n]} \to S^{(n)}$ or of the Albanese fibre of $A^{[n+1]} \to A^{(n+1)}$, respectively.

In the following we will concentrate on 4-dimensional hyperkähler varieties, so we fix here some useful notation.

Notation We denote by $\sigma \in \mathrm{Aut}(S \times S)$ the map $\sigma\colon (P,Q) \mapsto (Q,P)$, for each $(P,Q) \in S \times S$, *i.e.,* σ is the map that switches the two factors in $S \times S$. If g is an automorphism of a K3 surface S (resp. an abelian surface A), then $g \times g \in \mathrm{Aut}(S \times S)$ (resp. $g \times g \times g \in \mathrm{Aut}(A \times A \times A)$) induces a unique automorphism on $S^{[2]}$ (resp. $K_2(A)$), called the *natural automorphism induced by g*.

The natural automorphism of $S^{[2]}$ induced by g is denoted by $g^{[2]}$.

We denote by ι_S a non-symplectic involution on S and thus by ${\iota_S}^{[2]}$ the natural non-symplectic involution induced by ι_S on $S^{[2]}$.

3 Quotients of Hyperkähler Varieties by Automorphisms

In the following we will consider quotients of hyperkähler varieties by certain finite order automorphisms and crepant resolutions of these quotients. The main theorem of this section is that there exists a crepant resolution of the quotient of a hyperkähler variety of dimension $2p$ by a non-symplectic automorphism of prime order p whenever the fixed locus is pure of dimension p; this hypothesis is surely satisfied for non-symplectic involutions on hyperkähler 4-folds. First, we recall some basic definitions and known results.

Definition 3.1 Let V be a smooth variety of dimension m and let $\alpha \in \mathrm{Aut}(V)$ be an automorphism of order p with a non-empty fixed locus, which is necessarily smooth. We denote by A be the matrix that linearizes α near a component C of its fixed locus and let $(\zeta_p^{a_1},\dots,\zeta_p^{a_m})$, with $0 \le a_i < p$, be its eigenvalues. The age of α, denoted $\mathrm{age}(\alpha)$, near C is defined as $(\sum_{i=1}^m a_i)/p$.

Proposition 3.2 (see, for example, [31, Theorem 6.4.3 and Proposition 6.4.4]) *With the same notation as before, let us assume that $A \in SL(\mathbb{C},m)$. The quotient V/α has canonical non-terminal singularities on the image of C if and only if the age of α near C is* 1.

If V/α has terminal singularities, then it does not admit a crepant resolution.

When $G = \langle\alpha\rangle$ acts non-symplectically and is cyclic of prime order p, it is always possible to find a generator, that we keep on denoting α, such that $\alpha^*\omega_V = \zeta_p\omega_V$ for ζ_p a primitive p-th root of unity.

Recall the following standard fact about the components of the fixed locus.

Lemma 3.3 *Let V be a hyperkähler manifold of dimension* $2n$ *and let α be a non-trivial automorphism of finite order acting on X. Then the fixed locus has codimension at least* 2 *if α is symplectic, and at least n if α is non-symplectic.*

Proof When α is symplectic this follows from the fact that every connected component of the fixed locus is symplectic, as shown in [22, Proposition 2.6].

When α is non-symplectic, suppose on the contrary that there exists a connected component of codimension $< n$. Then the eigenspace relative to the eigenvalue 1 of A would be an isotropic subspace of dimension $> n$, and this is impossible. ∎

Lemma 3.4 *Let V be a hyperkähler variety of dimension $2n$ and let $\alpha \in \mathrm{Aut}(V)$ be an automorphism of order p with a non-empty fixed locus. Let A be the matrix that linearizes α near a component C of its fixed locus and let $(\zeta_p^{a_1}, \dots \zeta_p^{a_{2n}})$, with $0 \le a_i < p$, be its eigenvalues.*

Then there exist local coordinates $(x_1, \dots, x_{2n})$ in an open neighbourhood containing C such that:

(i) *if α is symplectic, the spectrum of A is the union of n pairs of the form $(\zeta_p^{a_j}, \zeta_p^{p-a_j})$ with $a_j \ge 0$, for $j = 1, \dots, n$;*
(ii) *if α is non-symplectic, then:*
   - *if $p = 2$, then $\alpha^*(\omega_V) = -\omega_V$ and the spectrum of A is the union of n pairs of the form $(1, -1)$;*
   - *if $p \ne 2$, then, without loss of generality, $\alpha^*(\omega_V) = \zeta_p\omega_V$ and the spectrum of A is the union of $s \le n$ pairs of the form $(1, \zeta_p)$ or $(\zeta_p^{a_j}, \zeta_p^{p+1-a_j})$ with $a_j > 0$, for $j = 1, \dots, s$, plus the eigenvalue $\zeta_p^{p+1/2}$ with multiplicity $2n - 2s$.*

Proof Fix a component of the fixed locus; we choose local coordinates $(x_1, \dots, x_{2n})$, such that the symplectic form ω_V is represented by the standard symplectic matrix J.

Since α is an automorphism of V, it preserves the Hodge decomposition of $H^2(V, \mathbb{C})$ and so $\alpha^*(\omega_V) = \lambda_\alpha\omega_V$ where $\lambda_\alpha \in \mathbb{C}^*$. Moreover, since α has order p, $\lambda_\alpha^p = 1$. If α is a symplectic automorphism, then $\lambda_\alpha = 1$; otherwise, without loss of generality, we can assume that $\lambda_\alpha = \zeta_p$, where $\zeta_p = e^{2i\pi/p}$.

Let $\mu = \zeta_p^{a_j}$ be one of the eigenvalues of A; then $\lambda_\alpha\mu^{-1}$ is an eigenvalue of A with the same multiplicity, since A and $\lambda_\alpha A^{-1}$ are conjugated by J and so have the same characteristic polynomial (see also [8, Remark 7.2]).

Thus, we obtain the following possibilities:

(i) $p = 2$: in this case, all the eigenvalues satisfy $\mu^2 = 1$. If α is symplectic, all the connected components of the fixed locus are symplectic [22, Proposition 2.6], hence even-dimensional, and this implies that ± 1 occur both with even multiplicity, so that $\det A = 1$. If α is non-symplectic, all the connected components are Lagrangian submanifolds [5, Lemma 1], so that the multiplicity of $\mu = 1$ is exactly n and $\det A = (-1)^n$.
(ii) $p > 2$, $\lambda_\alpha = 1$: the eigenvalues of α are n pairs of the form $(\zeta_p^{a_j}, \zeta_p^{p-a_j})$ with $a_j \ge 0$, for $j = 1, \dots, n$, and the determinant is $\det A = \zeta_p^{\sum_{j=1}^n (a_j + p - a_j)} = 1$.
(iii) $p > 2$, $\lambda_\alpha = \zeta_p$: the eigenvalues of α are $s \le n$ pairs of the form $(1, \zeta_p)$ or $(\zeta_p^{a_j}, \zeta_p^{p+1-a_j})$ with $a_j > 0$, for $j = 1, \dots, s$, plus the eigenvalue $\zeta_p^{p+1/2}$ with multiplicity $2n - 2s$. Here $\det A = \zeta_p^{\sum_{j=1}^s (a_j + p - a_j + 1) + (n-s)(p+1)} = \zeta_p^n$.

∎

Singular quotients of hyperkähler manifolds have already been studied in the literature, although the accent has always been on using quotients by symplectic automorphisms to construct singular symplectic manifolds and look for possible desingularizations. The following results also include results previously obtained by Fujiki [22] and Menet [38, Proposition 2.39].

Proposition 3.5 *Let V be a hyperkähler variety of dimension $2n$ and let $\alpha \in \mathrm{Aut}(V)$ be a symplectic automorphism of order p with a non-empty fixed locus. Let A be the matrix that linearizes α near a component of its fixed locus and let $(\zeta_p^{a_1}, \dots \zeta_p^{a_{2n}})$, with $0 \le a_i < p$, be its eigenvalues.*

In this case α preserves the volume form of V, and the singularities of V/α are canonical and not terminal if and only if all the components of the fixed locus have dimension $2n-2$.

Proof Let C be a connected component of the fixed locus of α and let $(x_1, \dots, x_{2n})$ be local coordinates as in Lemma 3.4.

We recall that the volume form Ω of V is a complex multiple of ω_V^n; we can assume that $\Omega := k\omega_V^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}$ for a certain constant $k \in \mathbb{C}^*$. This implies that the action of α^* on Ω is the multiplication by the determinant of A, so we have $\alpha^*(\Omega) = \det(A)\Omega$.

Volume-preserving automorphisms are exactly those whose linearization A belongs to $SL(\mathbb{C}, 2n)$. Since α is symplectic, by Lemma 3.4 $\det(A) = 1$ and $\alpha^*(\Omega) = \Omega$.

Let us consider a component $C \subset V$ of the fixed locus of α; C has codimension greater than 1 by Lemma 3.3. The quotient V/α is singular in $q(C) \subset V/\alpha$, where $q\colon V \to V/\alpha$ is the quotient map. By Proposition 3.2, the singularity $q(C)$ is canonical but not terminal if and only if the age of α near C is 1, *i.e.*, $(\sum_{j=1}^{2n} a_j)/p = 1$. Denote by $2k$ the multiplicity of 1 as an eigenvalue; by Lemma 3.4(i), the age $(\sum_{j=1}^{2n} a_j)/p = p(n-k)/p = n-k$ equals 1 if and only if $k = n-1$. As a consequence, C has codimension two.

Vice versa, if every component C of the fixed locus of α has dimension $2n-2$, then there are exactly two eigenvalues A that are not equal to 1. Since α is symplectic, they are of the form $(\zeta_p^{a_j}, \zeta_p^{p-a_j})$ by Lemma 3.4. This implies that the singularities of V/α are canonical but not terminal. ∎

In [38, Proposition 2.39], the interested reader can find an explicit list of quotients of this kind and more details on the existence of a resolution of singularities of V/α.

Theorem 3.6 *Let V be a hyperkähler variety of dimension $2n$ and let $\alpha \in \mathrm{Aut}(V)$ be a non-symplectic automorphism of prime order p with a non-empty fixed locus.*

(i) *α preserves the volume form if and only if $p|n$.*
(ii) *If α is as in* (i)*, the singularities of V/α are canonical and not terminal if and only if $p = n$ and all the components of the fixed locus of α have dimension $p = n$.*

In particular, if V is a $2p$-dimensional hyperkähler variety and α is a non-symplectic automorphism of order p of V such that all the components of the fixed locus of α have dimension p, then the blow-up of V/α in its singular locus is a Calabi–Yau $2p$-fold.

Proof Let C be a connected component of the fixed locus of α and let $(x_1, \ldots, x_{2n})$ be local coordinates as in Lemma 3.4. If α is non-symplectic, Lemma 3.4 implies that $\alpha^*(\Omega) = \zeta_p^n \Omega$. Since $\zeta_p^n = 1$ if and only if $p|n$, we obtain that a non-symplectic automorphism of order p of V preserves the volume form if and only if $p|n$.

Let us consider a component $C \subset V$ of the fixed locus of α; C has codimension greater than $n-1$ by Lemma 3.3. The quotient V/α is singular in $q(C) \subset V/\alpha$, where $q\colon V \to V/\alpha$ is the quotient map. The singularity $q(C)$ is canonical but not terminal if and only if $(\sum_{j=1}^{2n} a_j)/p = 1$.

We know that $p \mid n$, so we can write n as $n = n'p$, $n' \in \mathbb{N}_{>0}$. If $p = 2$, there are r eigenvalues equal to -1 and $2n-r$ eigenvalues equal to 1, and we already observed that r is even (since $A \in SL(\mathbb{C}, 2n)$; see proof of Proposition 3.5) and $r \geq n$ (by Lemma 3.3). The condition $(\sum_{j=1}^{2n} a_j)/p = 1$ can be rewritten as $r/2 = 1$, which implies $r = 2$. Together with the condition $r \geq n$, and $n = 2n'$ this implies $n' = 1$, *i.e.*, $n = 2$.

If $p \neq 2$, we know that there are s pairs of distinct eigenvalues $(\zeta_p^{a_j}, \zeta_p^{a_{h_j}})$ of the form $(1, \zeta_p)$ or $(\zeta_p^{a_j}, \zeta_p^{p+1-a_j})$ with $a_j > 0$ and that $\zeta_p^{p+1/2}$ occurs with multiplicity $2n - 2s$. We can assume without loss of generality that $0 \leq a_j < p$ for every $j = 1, \ldots, 2s$ and that $a_j < a_{h_j}$. So $a_j + a_{h_j} = 1 + k_j p$ with $k_j = 0$ if $a_j = 0$ and $a_{h_j} = 1$, and $k_j = 1$ otherwise (*i.e.*, if $a_j > 0$ and so $a_{h_j} > 0$). Hence, $(\sum_{j=1}^{2n} a_j)/p = (\sum_{j=1}^{s}(a_j + a_{h_j}))/p + (n-s)(p+1)/p = \sum_{j=1}^{s}(1 + k_j p)/p + (n-s)(p+1)/p = n' + \sum_{j=1}^{s} k_j + n - s$. Now the condition $(\sum_{j=1}^{2n} a_j)/p = 1$ implies that $n' = 1$, $s = n$, and $k_i = 0$ for each $i = 1, \ldots, n$. This implies that $n = p$ and $a_j = 0$ for every $j = 1, \ldots, p$. So the eigenvalues of A are 1 and ζ_p, both with multiplicity p. Thus, locally, the fixed locus could be described by $x'_{2i} = 0$, $i = 1, \ldots, p$, its codimension is p and its dimension is p.

Vice versa, if α is a non-symplectic automorphism of a $2p$-dimensional hyperkähler variety V such that the fixed locus of α consists of subvarieties of dimension p, then each block A_i of the eigenvalues matrix is $A_i := \operatorname{diag}(1, \zeta_p)$ (see Lemma 3.4), and it is immediate to check that $\sum_{j=1}^{n} a_j/p = 1$.

Finally, let us show that there exists a crepant resolution Y of V/α. By our assumptions, the local action of α on V near any fixed component is given by a diagonal matrix with p eigenvalues 1 and p eigenvalues ζ_p. Let $\beta\colon \widetilde{V} \to V$ be the blow-up of V in the fixed locus $\operatorname{Fix}_\alpha(V)$ of α. By base change, the map $\alpha\colon V \to V$ induces a map $\widetilde{\alpha}\colon \widetilde{V} \to \operatorname{Bl}_{\alpha(\operatorname{Fix}_\alpha(V))}(V)$, where $\operatorname{Bl}_{\alpha(\operatorname{Fix}_\alpha(V))}(V)$ is the blow-up of V in the image by α of $\operatorname{Fix}_\alpha(V)$. By definition of $\operatorname{Fix}_\alpha(V)$, $\widetilde{V} = \operatorname{Bl}_{\alpha(\operatorname{Fix}_\alpha(V))}(V)$, so the action of α lifts to an automorphism of $\widetilde{V}$. Moreover β is equivariant, $\alpha \circ \beta = \beta \circ \widetilde{\alpha}$, and thus descends to the quotients.

We have the commutative diagram

$$
\begin{tikzcd}
V \arrow[d, "q"'] & \widetilde{V} \arrow[l, "\beta"'] \arrow[d, "\pi"] \\
V/\alpha & \widetilde{V}/\widetilde{\alpha}. \arrow[l, "\beta'"']
\end{tikzcd}
\tag{3.1}
$$

The local action of α near the fixed locus shows that on the normal bundle $N_{\operatorname{Fix}_\alpha(V)|V}$ α acts as a multiplication by ζ_p. So $\widetilde{\alpha}$ acts as the identity on the exceptional divisor

$\widetilde{E} := \mathbb{P}(N_{\mathrm{Fix}_\alpha(V)|V})$. Moreover, $\widetilde{\alpha}$ acts freely on $\widetilde{V} - \widetilde{E}$. So the quotient $\widetilde{V}/\widetilde{\alpha}$ is a smooth variety, which is isomorphic to the blow-up of V/α in its singular locus (see also [36, Section 3]). We now show that the canonical bundle of $\widetilde{V}/\widetilde{\alpha}$ is trivial. First, one computes

$$K_{\widetilde{V}} = \beta^*(K_V) + (p-1)\widetilde{E}$$

so that $K_{\widetilde{V}} = (p-1)\widetilde{E}$. Then one observes that the quotient map $\pi\colon \widetilde{V} \to \widetilde{V}/\widetilde{\alpha}$ exhibits $\widetilde{V}$ as $p{:}1$ cyclic cover of $\widetilde{V}/\widetilde{\alpha}$ branched on $E := \pi(\widetilde{E})$. Hence, there exists a divisor $L \in \mathrm{Pic}(\widetilde{V}/\widetilde{\alpha})$ such that $pL \simeq E$ and $K_{\widetilde{V}} = \pi^*(K_{\widetilde{V}/\widetilde{\alpha}} + (p-1)L)$. Multiplying both terms by p, one obtains

$$pK_{\widetilde{V}} = \pi^*(pK_{\widetilde{V}/\widetilde{\alpha}}) + \pi^*\big((p-1)pL\big).$$

Recalling that $pL \simeq E$ and $\pi^*(E) = p\widetilde{E}$, one has

$$p(p-1)\widetilde{E} = \pi^*(pK_{\widetilde{V}/\widetilde{\alpha}}) + p(p-1)\widetilde{E},$$

which implies that $\pi^*(pK_{\widetilde{V}/\widetilde{\alpha}})$ is trivial. On the other hand $\widetilde{V}/\widetilde{\alpha}$ is the blow-up of V/α in its singular locus and V/α has trivial canonical bundle (because it is the quotient of V, which has trivial canonical bundle, by a volume preserving automorphism). Let us denote by $\beta'\colon \widetilde{V}/\widetilde{\alpha} \to V/\alpha$ this blow-up. We obtain $K_{\widetilde{V}/\widetilde{\alpha}} = \beta'^*(K_{V/\alpha}) + hE = hE$ for a certain $h \in \mathbb{Q}$. Since $\pi^*(pK_{\widetilde{V}/\widetilde{\alpha}})$ is trivial, one has that $\pi^*(phE) = p^2h\widetilde{E}$ is trivial. The divisor $\widetilde{E}$ is effective, so $p^2h\widetilde{E} = 0$ implies $h = 0$. So β' is a crepant resolution of V/α.

Moreover, $h^{i,0}(\widetilde{V}) = h^{i,0}(V)$, because they are birational invariants, and on the other hand, $h^{i,0}(\widetilde{V}/\widetilde{\alpha}) = \dim H^{i,0}(\widetilde{V})^{\widetilde{\alpha}} = \dim H^{i,0}(V)^{\alpha} = 0$ for $0 < i < 2p$, since α does not preserve the symplectic form of V.

So $Y := \widetilde{V}/\widetilde{\alpha}$ (and any crepant resolution of V/α) is Calabi–Yau. ∎

Remark 3.7 By [22, Lemma 1.2], V/α is a simply connected singular variety. Hence the blow-up of V/α in its singular locus is a simply connected manifold.

If V is a hyperkähler $2n$-fold and α is a non-symplectic involution on V, then the components of the fixed locus of α are Lagrangian submanifolds of V, and thus in particular they have dimension n [5, Lemma 1]. So a non-symplectic involution on V preserves the canonical bundle and is such that V/α has canonical singularities if and only if $n = 2$. This will be the setting of the rest of the paper.

Remark 3.8 To the best of our knowledge there are no examples of pairs (V, α_V) that satisfy Theorem 3.6 such that $\dim V > 4$. We observe, for example, that the natural automorphism $\alpha^{[p]}$ of $S^{[p]}$ induced by the non-symplectic order p automorphism α on a K3 surface S does not satisfy this condition, even if the action of α on S does. The reason is that there is at least a 2-dimensional component of the fixed locus of $\alpha^{[p]}$ on $S^{[p]}$ that is isomorphic to S/α; see *e.g.*, [8].

4 Quotients of Hyperkähler 4-folds by Non-symplectic Involutions

From now on, V is a hyperkähler 4-fold and α is a non-symplectic involution (so $n = p = 2$). Hence, V/α admits a crepant resolution which is a Calabi–Yau 4-fold. The aim of this section is to describe explicitly the crepant resolution of V/α constructed in Theorem 3.6 that allows us to construct a Calabi–Yau 4-fold and to compute its Hodge diamond starting from some information on the action of α on V (Theorem 4.1). Then we apply these results to some specific hyperkähler 4-folds with a non-symplectic involution; in particular, we will consider quotients of hyperkähler 4-folds of generalized Kummer type and of $K3^{[2]}$ type in Sections 4.2.1 and 4.2.2, respectively.

4.1 The Computation of the Hodge Numbers

By Theorem 3.6 a crepant resolution of V/α is obtained by blowing up its singular locus. From now on we always consider this crepant resolution, and we denote it by Y. It is isomorphic to $\widetilde{V}/\widetilde{\alpha}$ by the proof of Theorem 3.6; see in particular diagram (3.1). In particular, we denote by B_j the irreducible components of the fixed locus of α: these are smooth disjoint surfaces. The blow-up $\beta'\colon Y \to V/\alpha$ introduces one divisor for each component B_j of the fixed locus, and this divisor is a $\mathbb{P}^1$-bundle over B_j.

The 4-fold Y is a Calabi–Yau variety and so its Hodge diamond is invariant under birational transformation (see [3, Theorem 1.1], [17], [30], and [56]). So we can deduce the Hodge diamond of any crepant resolution of V/α (and in particular of Y) from the computation of the $\widetilde{\alpha}$ invariant part of the cohomology of $\widetilde{V}$. After fixing some notation, we summarize the final outcome in Theorem 4.1.

Let $\coprod B_j$ be the fixed locus of α. Let $b := h^0(\coprod B_j)$; *i.e.*, b is the number of components of the fixed locus of α, and let

$$c := \sum_{j=1}^{b}\left(h^{1,0}(B_j)\right), \quad d := \sum_{j=1}^{b}\left(h^{2,0}(B_j)\right), \quad e := \sum_{j=1}^{b}\left(h^{1,1}(B_j)\right).$$

Moreover, let

$$t_{1,1} := \dim\left(H^2(V,\mathbb{C})\right)^{\alpha} = \dim\left(H^{1,1}(V)\right)^{\alpha},$$

where the last equality follows from the fact that α is non-symplectic. We also set

$$t_{2,1} := \dim(H^{2,1}(V)^{\alpha}), \quad t_{3,1} := \dim(H^{3,1}(V)^{\alpha}), \quad t_{2,2} := \dim(H^{2,2}(V)^{\alpha}).$$

We have $t_{2,1} = \frac{1}{2}\dim(H^3(V,\mathbb{C})^{\alpha})$, because $H^{3,0}(V) = 0$. Since $H^{4,0}(V)$ and $H^{0,4}(V)$ are invariant for α, we have the following relation: $\dim(H^4(V,\mathbb{C})^{\alpha}) = 2 + 2t_{3,1} + t_{2,2}$.

Theorem 4.1 *The Hodge diamond of any Calabi–Yau birational to Y is given by*

$$\begin{aligned} h^{0,0}(Y) &= h^{4,0} = 1, & h^{1,0}(Y) &= h^{2,0} = h^{3,0} = 0,\\ h^{1,1}(Y) &= t_{1,1} + b, & h^{2,1}(Y) &= t_{2,1} + c,\\ h^{2,2}(Y) &= t_{2,2} + e, & h^{3,1}(Y) &= t_{3,1} + d. \end{aligned}$$

Proof The crepant resolution Y of V/α is isomorphic to $\widetilde{V}/\widetilde{\alpha}$ by Section 4.1, and any other Calabi–Yau birational to Y has the same Hodge numbers as Y. The statement now follows from the fact that $H^{*,*}(Y) = H^{*,*}(\widetilde{V})^{\widetilde{\alpha}}$ (see [22, p. 80]).

By classical results on the cohomology of blow-ups (see [55, Theorem 7.31]), the map

$$\beta^* \oplus j_*\beta^*_{|\coprod B_j} \colon H^{p,q}(V) \oplus H^{p-1,q-1}\big(\coprod B_j\big) \to H^{p,q}(\widetilde{V})$$

is an isomorphism, where $j\colon \widetilde{E} \hookrightarrow \widetilde{V}$ is the inclusion of the union $\widetilde{E}$ of the exceptional divisors of the blow-up $\beta\colon \widetilde{V} \to V$.

Furthermore, this isomorphism is equivariant with respect to the action of α and $\widetilde{\alpha}$: we have

$$(\beta^* \oplus j_*\beta^*_{|\coprod B_j})(\alpha^* \oplus \mathrm{id}) = \widetilde{\alpha}^*(\beta^* \oplus j_*\beta^*_{|\coprod B_j}).$$

Indeed, the equivariance of β observed in the proof of Theorem 3.6 and the functoriality of pullback give $\beta^*\alpha^* = \widetilde{\alpha}^*\beta^*$ on $H^{p,q}(V)$, whereas the second equality $j_*\beta^*_{|\coprod B_j} = \widetilde{\alpha}^* j_*\beta^*_{|\coprod B_j}$ on $H^{p-1,q-1}(\coprod B_j)$ can be shown as follows. We observe that, by definition, the Gysin map j_* is the composition

$$H^{p-1,q-1}(\widetilde{E}) \xrightarrow{\Psi} (H^{n-p,n-q}(\widetilde{E}))^* \xrightarrow{(j^*)^t} (H^{n-p,n-q}(\widetilde{V}))^* \xrightarrow{\Psi} H^{p,q}(\widetilde{V}),$$

where Ψ denotes Poincaré duality and $(j^*)^t$ is the adjoint operator of the pullback j^*; hence, equivariance of Ψ and functoriality of j^* imply $\widetilde{\alpha}^* j_* = j_*\widetilde{\alpha}^*_{|H^{p-1,q-1}(\widetilde{E})}$. Since $\widetilde{\alpha}_{|\widetilde{E}} = \mathrm{id}$, we obtain that $\widetilde{\alpha}^* j_* = j_*$ on $H^{p-1,q-1}(\widetilde{E})$.

The fact that $\widetilde{E}$ is fixed by $\widetilde{\alpha}$ finally allows us to conclude that

$$\dim\big(H^{p,q}(\widetilde{V})^{\widetilde{\alpha}}\big) = \dim\big(H^{p,q}(V)^{\alpha}\big) + h^{p-1,q-1}\big(\coprod B_j\big). \qquad \blacksquare$$

4.2 Computations in Special Cases

The aim of this section is to apply Theorem 4.1 to special hyperkähler 4-folds V with a given non-symplectic involution α such that either there exist some relations among the numbers $t_{i,j}$, b, c, d, e, or some of these numbers are determined.

4.2.1 Generalized Kummer Four-folds

Non-symplectic involutions on generalised Kummer fourfolds have been recently classified in [40]. The cohomology of generalised Kummer fourfolds has been studied in detail by Hassett and Tschinkel [27] and Oguiso [46]; let us review the results relevant for our purposes.

The Hodge diamond of the generalized Kummer fourfold $K(A_2)$ is as follows:

$$\begin{array}{lllll}
h^{0,0}=1 & & & & \\
 & h^{0,1}=0 & & & \\
h^{1,1}=5 & & h^{0,2}=1 & & \\
 & h^{2,1}=4 & & h^{0,3}=0 & \\
h^{2,2}=96 & & h^{1,3}=5 & & h^{0,4}=1,
\end{array}$$

where we used that $H^2(K_2(A)) \simeq H^2(A) \oplus \mathbb{C}[E]$, with E the class of the exceptional divisor of the Albanese fibre of $A^{[3]} \to A^{(3)}$. Here, we only wrote the relevant Hodge numbers; the rest are obtained via the symmetries of the Hodge diamond.

Here, the third cohomology has trivial $H^{3,0}$ and four dimensional $H^{2,1}$, which, in the following special setting, can be constructed explicitly. Let A be an abelian surface and let $\mathbb{C}^2$ be its universal cover, with coordinates z and w. Let $A(2)$ be the subset of A^3 where all points sum to zero and let $A^{((2))}$ be the corresponding locus inside $\mathrm{Sym}^3(A)$. The universal cover of $A(2)$ is the closed submanifold of $(\mathbb{C}^2)^3$ cut out by the equations

$$z_1 + z_2 + z_3 = 0 \quad \text{and} \quad w_1 + w_2 + w_3 = 0.$$

The natural one forms dz_i, dw_i on $A(2)$ satisfy the same equations, and the cohomology classes of dz_1, dz_2, dw_1, dw_2 form a basis of $H^{1,0}(A(2))$. A basis of $H^{2,1}(K_2(A))$ is then given by the permutation orbits of the following forms:

$$dz_1 \wedge dw_1 \wedge d\bar{z}_1, \quad dz_1 \wedge dw_2 \wedge d\bar{z}_3, \quad dw_1 \wedge dz_1 \wedge d\overline{w}_1, \quad \text{and} \quad dw_1 \wedge dz_2 \wedge d\overline{w}_3.$$

Indeed, the permutation orbit of one of these four forms was studied in [46, Lemma 3.3], where Oguiso proves that it descends to a non-zero form on $A^{((2))}$, and the proof for the other three forms is analogous. As these forms are in different orbits for the action of $\mathfrak{S}_3$, they give independent forms on $A^{((2))}$, and, as the natural resolution map $\nu\colon K_2(A) \to A^{((2))}$ induces an injection of forms, these are a basis for $H^{2,1}(K_2(A))$.

This will allow us to compute $t_{2,1}$ of a natural automorphism only from the action of the automorphism on $H^1(A)$. The last tool we need is a determination of $H^4(K_2(A))/\operatorname{Sym}^2 H^2(K_2(A))$. This has the following geometric characterisation, where we denote by $A[3]$ the group of the 3-torsion points of A.

Lemma 4.2 ([27, Section 4]) *Let F be the subspace of $H^4(K_2(A))$ spanned by the classes Z_τ of preimages of $(\tau,\tau,\tau) \in A[3]^{(3)}$ under the Hilbert–Chow morphism $K_2(A) \to A^{(3)}$, for all $\tau \in A[3]$. Let $W := \sum_{\tau \in A[3]} Z_\tau$. Then $W^\perp \subset F$ is isomorphic to $H^4(K_2(A))/\operatorname{Sym}^2 H^2(K_2(A))$.*

As it is classically known ([23]), there are three families of non-symplectic involutions on abelian surfaces. The first family is given in terms of the product $E \times E'$ of two elliptic curves, and the involution acts as -1 on one curve and as 1 on the other. The elements of the second family are quotients of elements of the first family by an order two point t such that both projections of t on the elliptic curves E and E' are nontrivial. The third family is an iteration of this procedure with a further quotient by a point of order 2. Let $X_1 = K_2(E \times E')$, $X_2 = K_2(E \times E'/t)$ and $X_3 = K_2(E \times E'/\langle t, t'\rangle)$ and let φ_1, φ_2, and φ_3 be the three involutions induced on them. In all three cases, we have $t_{1,1} = 3$ and $t_{2,1} = 2$. The three actions give also the same action on order three points of A, with 36 non-trivial orbits and 9 fixed points. Therefore, we have $t_{2,2} = 54$, where 10 dimensions come from the symmetric power of H^2 and 45 arise from the action on order three points and the intersection of both contributions is generated by W. The fixed locus on X_1 is given by points of the form $\{(p,e),(q,f),(p+q,-e-f)\}$, where $p,q \in E[2]$ and $e,f \in E'$ or points of the form $\{(0,-2e),(a,e),(-a,e)\}$, where $a \in E$ and $e \in E'$. These cover six surfaces; one is a $\mathbb{P}^2$ (the fibre of the Albanese map for $(E')^{(3)}$, given by fixed subschemes of the form $\{(0,e),(0,f),(0,-e-f)\}_{e,f\in E'}$), three are $(E')^{(2)}$ (which are $\{(0,e),(p,f),(p,-e-f)\}_{e,f\in E',p\in E[2]-0}$), one is $E'\times E'$ (given as $\{(p,e),(q,f),(p+q,-e-f)\}_{e,f\in E',\{0,p,q,p+q\}=E[2]}$), and the last one is given by the invariant subschemes whose points are not fixed, and this is isomorphic to $E\times\mathbb{P}^1$ blown up along nine points, which are subschemes supported entirely on a triple point of A. Therefore, we get the following Hodge diamond for a crepant resolution Y_1 of X_1/φ_1:

$$\begin{array}{ccccc}
1 & & & & \\
& 0 & & & \\
9 & & 0 & & \\
& 8 & & 0 & \\
75 & & 5 & & 1.
\end{array}$$

In case $X_2 = K_2(E\times E'/t)$, the quotient of $E\times E'/t$ of $E\times E'$ identifies some of the components of the fixed locus, and we are left with only three surfaces. The first one is again $\mathbb{P}^2$. The second one is $(E')^{(2)}$, and the last one is the surface of invariant non-fixed subschemes, isomorphic to that of the previous case. Therefore, the Hodge diamond of a crepant resolution Y_2 of X_2/φ_2 is

$$\begin{array}{ccccc}
1 & & & & \\
& 0 & & & \\
6 & & 0 & & \\
& 4 & & 0 & \\
68 & & 4 & & 1.
\end{array}$$

Also on X_3 the fixed locus gets smaller, and only $\mathbb{P}^2$ and the surface of invariant non-fixed subschemes are left. Thus, the Hodge diamond of a crepant resolution Y_3 of X_3/φ_3 is

$$\begin{array}{ccccc}
1 & & & & \\
& 0 & & & \\
5 & & 0 & & \\
& 3 & & 0 & \\
66 & & 4 & & 1.
\end{array}$$

4.2.2 Four-folds of $K3^{[2]}$ Type

We want to apply the results of Theorem 4.1 if V is deformation equivalent to the Hilbert scheme of two points of a K3 surface. In this case, the Hodge diamond of V is known to be

$$\begin{array}{ccccc}
1 & & & & \\
& 0 & & & \\
21 & & 1 & & \\
& 0 & & 0 & \\
232 & & 21 & & 1.
\end{array}$$

The cohomology of a manifold V of $K3^{[2]}$ type has very nice properties. Indeed the cohomology group $H^3(V,\mathbb{C})$ is trivial and the cohomology group $H^4(V,\mathbb{C})$ is

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completely determined by $H^2(V,\mathbb{C})$. In particular, this implies that $t_{2,1}=0$ and that we can get $t_{3,1}$ and $t_{2,2}$ from $t_{1,1}$. Indeed, there is an isomorphism of Hodge structures $H^4(V,\mathbb{C})\cong \mathrm{Sym}^2(H^2(V,\mathbb{C}))$ (see [26]). Let us denote by $H^{1,1}(V,\mathbb{C})_{-1}$ the subgroup of $H^{1,1}(V,\mathbb{C})$ that is anti-invariant for α. Then

$$\begin{aligned}
t_{3,1} &= \dim(H^{3,1}(V,\mathbb{C})^{\alpha}) = \dim(H^{2,0}(V)\otimes H^{1,1}(V)_{-1}) = h^{1,1}-t_{1,1} = 21-t_{1,1}\\
t_{2,2} &= \dim(H^{2,2}(V)^{\alpha}) = \dim\left(H^{2,0}(V)\otimes H^{0,2}(V)\oplus \mathrm{Sym}^2\left(H^{1,1}(V)\right)\right)^{\alpha}\\
&= \dim\left(H^{2,0}(V)\otimes H^{0,2}(V)\oplus \mathrm{Sym}^2(H^{1,1}(V)^{\alpha})\oplus \mathrm{Sym}^2(H^{1,1}(V)_{-1})\right)\\
&= 1+\binom{t_{1,1}+1}{2}+\binom{22-t_{1,1}}{2} = 232+t_{1,1}^2-21t_{1,1}.
\end{aligned}$$

By considering the invariant and the anti-invariant parts of the cohomology of V, we obtain two sub-Hodge structures of weight k of $H^k(V,\mathbb{Q})$. In particular, we obtain the following two invariant and anti-invariant Hodge diamonds:

$$\begin{array}{lcccc|lccc}
1 & & & & & 0 & & & \\
 & 0 & & & & & 0 & & \\
t_{1,1} & & 0 & & & 21-t_{1,1} & & 1 & \\
 & 0 & & 0 & & & 0 & & 0\\
232+t_{1,1}^2-21t_{1,1} & & 21-t_{1,1} & & 1 & -t_{1,1}^2+21t_{1,1} & & t_{1,1} & 0.
\end{array}$$

This allows an easy computation of the trace of α on $H^*(V,\mathbb{C})$ and, by means of the topological trace formula, Beauville, in [5], deduces information on the Hodge diamond of the fixed locus of α (in [5] the trace of α on $H^{1,1}(V)$ is denoted by t). The relation between t and $t_{1,1}$ is clearly $t=t_{1,1}-(21-t_{1,1})$, so $t=2t_{1,1}-21$. By [5, Theorem 2], once denoted by $F=\coprod B_j$ the fixed locus of α, we have

$$\chi(\mathcal{O}_F)=(t_{1,1}^2-21t_{1,1}+112)/2 \quad\text{and}\quad e(F)=2t_{1,1}^2-42t_{1,1}+232.$$

In particular, with the notations of Theorem 4.1, we have

$$\begin{aligned}
b-c+d &= (t_{1,1}^2-21t_{1,1}+112)/2,\\
2b-4c+2d+e &= 2t_{1,1}^2-42t_{1,1}+232,
\end{aligned}$$

from which it follows

$$\begin{aligned}
b &= (112-21t_{1,1}+2c-2d+t_{1,1}^2)/2,\\
e &= 120-21t_{1,1}+2c+t_{1,1}^2.
\end{aligned}$$

Proposition 4.3 *Let V be a hyperkähler 4-fold of $K3^{[2]}$ type admitting a non-symplectic involution α such that the fixed locus of α consists of the disjoint union of a finite number of surfaces B_j. As before, denote by:*

$$t_{1,1}:=\dim\left(H^{1,1}(V)^{\alpha}\right),\quad c:=\sum_j\left(H^{1,0}(B_j)\right),\quad d:=\sum_j\left(H^{2,0}(B_j)\right),$$

any crepant resolution of V/α is a Calabi–Yau variety with Hodge numbers:

$$\begin{array}{ll}
h^{0,0}=h^{4,0}=1, & h^{1,0}=h^{2,0}=h^{3,0}=0,\\
h^{1,1}=(112-19t_{1,1}+2c-2d+t_{1,1}^2)/2, & h^{2,1}=c,\\
h^{2,2}=352+2t_{1,1}^2-42t_{1,1}+2c, & h^{3,1}=21-t_{1,1}+d.
\end{array}$$

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4.2.3 Beauville's Non-natural Involution

Let H be a double EPW sextic and let ι_H be the covering involution; see [45] for the definition. It is well known that (H,ι_H) is deformation equivalent to $(S^{[2]},\iota_4)$, where S is a quartic in $\mathbb{P}^3$ not containing a line and ι_4 is the involution given by sending a subscheme $\{P,Q\}\in S^{[2]}$ to the subscheme $(l\cap S)\smallsetminus\{P,Q\}\in S^{[2]}$, where l is the line spanned by P and Q (possibly infinitesimally closed). If $\operatorname{Pic}(S)=\mathbb{Z}A$, where A is a polarization of degree 4, the fixed locus of ι_4 on $S^{[2]}$ consists of a smooth surface parametrizing the bitangents to S (see [21]) and ι_4 fixes only one class in $H^{1,1}(S^{[2]})$, so $t_{1,1}=1$ (see [5, Section 3]).

Let F be the surface of bitangents to S; it was proved in [37, (0.7)] that its Hodge numbers are

$$h^{0,0}(F)=1,\quad h^{1,0}(F)=0,\quad h^{2,0}(F)=45,\quad h^{1,1}(F)=100.$$

For a general double EPW sextic H, the fixed locus $\operatorname{Fix}_{\iota_H}(H)$ is then deformation equivalent to the surface F considered before, so that they have the same Hodge numbers. This allows us to compute the Hodge numbers of the crepant resolution Y of H/ι_H (as in the previous section):

$$\begin{aligned}
&h^{0,0}(Y)=h^{4,0}(Y)=1,\\
&h^{1,0}(Y)=h^{2,0}=h^{3,0}=0,\\
&h^{1,1}(Y)=\dim\big(H^{1,1}(H)^{\iota_H}\big)+\dim\big(H^{0,0}(F)\big)=1+1=2,\\
&h^{2,1}(Y)=\dim\big(H^{2,1}(H)^{\iota_H}\big)+\dim\big(H^{1,0}(F)\big)=0+0=0,\\
&h^{3,1}(Y)=\dim\big(H^{3,1}(H)^{\iota_H}\big)+\dim\big(H^{2,0}(F)\big)=20+45=65,\\
&h^{2,2}(Y)=\dim\big(H^{2,2}(H)^{\iota_H}\big)+\dim\big(H^{1,1}(F)\big)=212+100=312.
\end{aligned}$$

5 Non-symplectic Natural Involutions on $S^{[2]}$ and the Calabi–Yau 4-folds Y_S and Z_S

Here we consider the case of natural non-symplectic involutions on the Hilbert scheme of two points $S^{[2]}$ of a K3 surface S. As already discussed in Section 2, if the K3 surface S admits a non-symplectic involution ι_S, this induces a non-symplectic involution, denoted $\iota_S{}^{[2]}$, on $S^{[2]}$. Our first goal will be the construction of the Calabi–Yau variety Y_S, the crepant resolution of $S^{[2]}/\iota_S{}^{[2]}$, and the computation of its Hodge numbers, in Theorem 5.1. Then in Section 5.2 we recall the construction of other Calabi–Yau 4-folds obtained applying the Borcea–Voisin construction to $S\times S$: this Calabi–Yau will be denoted by Z_S and is the crepant resolution of $(S\times S)/\langle\iota_S\times\iota_S\rangle$. This allows us to show that there exists an involution on Z_S, denoted by σ_Z, such that Z_S/σ_Z is birational to Y_S; see Proposition 5.3. The existence of this involution depends on the fact that the group $(\mathbb{Z}/2\mathbb{Z})^2=\langle\iota_S\times\iota_S,\sigma\rangle$ acts on $S\times S$ and the 4-fold Y_S is birational to the quotient of $S\times S$ for the full group, while $S^{[2]}$ and Z_S are birational to the quotients of $S\times S$ by two specific subgroups of order 2 of $(\mathbb{Z}/2\mathbb{Z})^2$. In Proposition 5.4, we will prove that the quotient of $S\times S$ by the third cyclic subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$ admits a crepant resolution V that is isomorphic to $S^{[2]}$. In Section 5.5, we describe explicitly the Picard group of Y_S and the branch divisors associated to the three double covers $Z_S \dashrightarrow Y_S$, $S^{[2]} \dashrightarrow Y_S$ and $V \dashrightarrow Y_S$.

5.1 The Calabi–Yau 4-folds Y_S

We first recall the main results on non-symplectic involutions on $K3$ surfaces, due to Nikulin; see [44].

Let S be a K3 surface admitting a non-symplectic involution ι_S.

We denote by $r := \operatorname{rank}(H^2(S,\mathbb{Z})^{\iota_S})$ and by a the integer defined by

$$\left(H^2(S,\mathbb{Z})^{\iota_S}\right)^\vee / \left(H^2(S,\mathbb{Z})^{\iota_S}\right) \simeq (\mathbb{Z}/2\mathbb{Z})^a.$$

The fixed locus of ι_S consists of N disjoint curves. If $N = 0$, then $r = a = 10$. If the fixed locus of ι_S is the disjoint union of two genus 1 curves, then $r = 10$, $a = 8$, $N = 2$, and we set $N' := 2$.

Otherwise, if the fixed locus of ι_S is neither empty nor the disjoint union of two curves of genus one, let us consider the fixed curve with highest genus and denote it by C. Then $\operatorname{Fix}_{\iota_S}(S) = C \coprod_{i=1}^{N-1} R_i$ where R_i is a rational curve for all $i = 0,\dots,N-1$. Let us denote by $g := g(C)$, by $k := N-1$, by $N' = g(C)$.

The invariants just introduced satisfy the following relations:

$$\begin{aligned} &k = (r-a)/2, && g = (22-r-a)/2,\\ &N = (2+r-a)/2, && N' = (22-r-a)/2,\\ &r = 11+k-g = 10+N-N', && a = 11-k-g = 12-N-N'. \end{aligned} \tag{5.1}$$

We observe that if the fixed locus consists of two disjoint genus 1 curves, then $r = 10$, $a = 8$, $N = 2$, and $N' = 2$, so that the relation between (r,a) and (N,N') written above is still true.

From now on we suppose that the fixed locus of ι_S on S consists of a curve C of genus $g := g(C)$ and of k other rational curves. An application of [7, §4.2] shows that the fixed locus of $\iota_S{}^{[2]}$ on $S^{[2]}$ consists of the following surfaces:

- one surface isomorphic to $C^{[2]}$ whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = g$, $h^{2,0} = g(g-1)/2$, $h^{1,1} = 1+g^2$;
- k surfaces isomorphic to $C \times R_j \simeq C \times \mathbb{P}^1$, $j = 1,\dots k$, whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = g$, $h^{2,0} = 0$, $h^{1,1} = 2$;
- k surfaces isomorphic to $(\mathbb{P}^1)^{[2]}$, whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 0$, $h^{1,1} = 1$;
- $k(k-1)/2$ surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 0$, $h^{1,1} = 2$;
- one surface that is isomorphic to the smooth quotient surface S/ι_S, hence $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 0$ and $h^{1,1} = r$.

So the Hodge diamond of the fixed locus is:

$$h^{0,0} = 1 + k + k + k(k-1)/2 + 1, \qquad h^{1,0} = g + kg,$$
$$h^{1,1} = 1 + g^2 + 3k + k(k-1) + r, \qquad h^{2,0} = g(g-1)/2.$$

Note that the topological Euler characteristic of the fixed locus is $2(r^2 - 19r + 96)$ (where we used the relations (5.1)).

Now we consider the action of $\iota_S{}^{[2]}$ on the cohomology of $S^{[2]}$. We are exactly in the setting of Section 4.2.2 with $V := S^{[2]}$ and $\alpha := \iota_S{}^{[2]}$. Since $\iota_S{}^{[2]}$ is induced by ι_S and preserves the exceptional divisor in $S^{[2]}$, we have that $t_{1,1} = r + 1$.

Hence, applying Proposition 4.3, we obtain the Hodge diamond of any crepant resolution Y_S of $S^{[2]}/\iota_S{}^{[2]}$:

Theorem 5.1 *Let S be a K3 surface and let ι_S be a non-symplectic involution on S, whose fixed locus is associated with the values (N, N'). Let Y_S be the blow-up of $S^{[2]}/\iota_S{}^{[2]}$ in its fixed locus. Then the Hodge numbers of any Calabi–Yau 4-fold birational to Y_S are: $h^{i,0} = 0$, $i = 1, 2, 3$, $h^{0,0} = h^{4,0} = 1$, and*

$$\begin{aligned}
h^{1,1} &= (24 + 3N - 2N' + N^2)/2,\\
h^{2,1} &= NN',\\
h^{3,1} &= (20 - 2N + N' + N'^2)/2,\\
h^{2,2} &= 132 + 2N - 2N' + 2N^2 - 2NN' + 2N'^2.
\end{aligned}$$

Remark 5.2 Since there are 64 possible choices for the pairs (N, N') associated with ι_S we obtain 64 different Hodge diamonds for the Calabi–Yau 4-folds Y_S. In all the admissible pairs, $1 \le N \le 10$ and $0 \le N' \le 10$. The complete list of the Hodge diamonds of the Calabi–Yaus Y_S is given in Appendix A.

5.2 The Calabi–Yau of Borcea–Voisin Type and Z_S

Let B_1 and B_2 be two Calabi–Yau varieties of dimension n_1 and n_2, respectively. Let us assume that B_i admits an involution ι_i which does not preserve the volume form of B_i. Let us consider the involution $\iota_1 \times \iota_2$ on $B_1 \times B_2$. If all the components of the fixed locus of $\iota_1 \times \iota_2$ on $B_1 \times B_2$ have codimension 2, then there exists a crepant resolution of $(B_1 \times B_2)/(\iota_1 \times \iota_2)$ that is a Calabi–Yau manifold of dimension $(n_1 + n_2)$. The Calabi–Yau manifolds constructed in this way are said to be of Borcea–Voisin type, after the original independent papers by Borcea [11] and Voisin [54]; the generalization that we have just reviewed is a result by Cynk and Hulek (see [14, Proposition 2.1]), where the authors refer to the construction as the "Kummer construction".

In our setting it is quite natural to consider a Calabi–Yau 4-fold of Borcea–Voisin type, by choosing $B_1 = B_2 = S$ and $\iota_1 = \iota_2 = \iota_S$, with the same notation of the previous section. The quotient $(S \times S)/(\iota_S \times \iota_S)$ is singular along some surfaces, and the blow-up of these surfaces gives a smooth Calabi–Yau 4-fold, that we denote by Z_S. In this section we compute the Hodge numbers of Z_S, which do not depend on the Calabi–Yau birational model of Z_S that we choose; these Hodge numbers were computed in [18] by using orbifold cohomology.

Let S be a K3 surface and ι_S be a non-symplectic involution of S whose fixed locus consists of a curve of genus g and k rational curves.

The Hodge diamond of $S \times S$ is

$$\begin{array}{ccccc}
1 & & & & \\
 & 0 & & & \\
40 & & 2 & & \\
 & 0 & & 0 & \\
404 & & 40 & & 1.
\end{array}$$

First, we consider the action of $(\iota_S \times \iota_S)^*$ on $H^*(S \times S, \mathbb{Q})$. The invariant and the anti-invariant part of $H^*(S \times S, \mathbb{Q})$ give two sub-Hodge structures of $H^*(S \times S, \mathbb{Q})$ whose Hodge diamonds are respectively

$$\begin{array}{ccccc|ccccc}
1 & & & & & 0 & & & & \\
 & 0 & & & & & 0 & & & \\
2r & & 0 & & & 40-2r & & 2 & & \\
 & 0 & & 0 & & & 0 & & 0 & \\
2r^2-40r+404 & & 40-2r & & 1, & 40r-2r^2 & & 2r & & 0.
\end{array}$$

The fixed locus of $\iota_S \times \iota_S$ on $S \times S$ consists of

- one surface isomorphic to $C \times C$ whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = 2g$, $h^{2,0} = g^2$, $h^{1,1} = 2 + 2g^2$;
- $2k$ surfaces isomorphic to $C \times \mathbb{P}^1$ whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = g$, $h^{2,0} = 0$, $h^{1,1} = 2$;
- k^2 surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, whose Hodge numbers are $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 0$, $h^{1,1} = 2$.

It follows that the Hodge diamond of the fixed locus is

$$h^{0,0} = 1 + 2k + k^2, \qquad h^{1,0} = 2g(1+k),$$
$$h^{1,1} = 2 + 2g^2 + 4k + 2k^2, \qquad h^{2,0} = g^2.$$

Hence, using relations (5.1), the Hodge numbers of Z_S, and so of any crepant resolution of $(S \times S)/(\iota_S \times \iota_S)$, are

$$\begin{aligned}
h^{1,1} &= 20 + 2N - 2N' + N^2,\\
h^{2,1} &= 2NN',\\
h^{3,1} &= 20 - 2N + 2N' + N'^2,\\
h^{2,2} &= 204 + 4N^2 - 4NN' + 4N'^2.
\end{aligned}$$

5.3 The Quotient of Z_S, Birational to Y_S

The group $\langle \sigma, \iota_S \times \iota_S \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is contained in $\mathrm{Aut}(S \times S)$. In particular, the automorphism σ commutes with $\iota_S \times \iota_S$, and thus it induces an automorphism of $(S \times S)/(\iota_S \times \iota_S)$, denoted by σ'. Since the singular locus of $(S \times S)/(\iota_S \times \iota_S)$ is the image, under the quotient map, of $\mathrm{Fix}_{\iota_s}(S) \times \mathrm{Fix}_{\iota_s}(S)$, and since $\mathrm{Fix}_{\iota_s}(S) \times \mathrm{Fix}_{\iota_s}(S)$

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is preserved by σ, the automorphism σ extends to an automorphism σ_Z of Z_S. Moreover, σ_Z preserves the volume form on Z_S, because σ preserves the $(4,0)$ form on $S \times S$.

So we have now a Calabi–Yau 4-fold Z_S with an involution, σ_Z, which preserves the volume form of Z_S.

Proposition 5.3 *The quotient 4-fold Z_S/σ_Z is birational to Y_S.*

Proof The statement follows by the commutativity of the following diagram:

$$\begin{tikzcd}
 & & & S \times S \arrow[dll, "q_3"'] \arrow[drr, "q_1"] \arrow[dd] & & & \\
Z_S \arrow[r, dashed, "\beta'"] \arrow[dr] & (S \times S)/(\iota_S \times \iota_S) \arrow[drr, "q_4"'] & & & & (S \times S)/\sigma \arrow[dll, "q_2"] & S^{[2]} \arrow[l, dashed] \arrow[dl] \\
 & Z_S/\sigma_Z \arrow[rr, dashed] & & (S \times S)/\langle \sigma, \iota_S \times \iota_S \rangle & & S^{[2]}/\iota_S^{[2]} \arrow[ll, dashed] & Y_S \arrow[l, dashed]
\end{tikzcd}$$

where all dash arrows are the birational maps induced by the chosen crepant resolutions and the other arrows are quotient maps. ∎

We show now that the 4-fold Z_S/σ_Z is singular along surfaces, and so the local action of σ_Z is given by a diagonal matrix with two eigenvalues equal to $+1$ and two eigenvalues equal to -1. Hence, it admits a crepant resolution $\widetilde{Z_S/\sigma_Z}$, which is a Calabi–Yau variety and is birational to Y_S. In particular, the Hodge numbers of $\widetilde{Z_S/\sigma_Z}$ coincide with those of Y_S.

We explicitly determine the fixed locus of the automorphism σ' induced by σ on $(S \times S)/(\iota_S \times \iota_S)$. Let $q_3 \colon S \times S \to (S \times S)/(\iota_S \times \iota_S)$ be the quotient map. The automorphism $\sigma \colon S \times S \to S \times S$ acts by sending (P,Q) to (Q,P). The points on $(S \times S)/(\iota_S \times \iota_S)$ are denoted by $(\overline{P},\overline{Q})$, where $(\overline{P},\overline{Q})$ is the common image of $(P,Q) \in S \times S$ and $(\iota_S(P),\iota_S(Q)) \in S \times S$ under the quotient map (*i.e.*, $(\overline{P},\overline{Q}) = q_3(P,Q) = q_3(\iota_S(P),\iota_S(Q))$). Thus, the condition $\sigma'(\overline{P},\overline{Q}) = (\overline{P},\overline{Q})$ implies that either $P = Q$ or $P = \iota_S(Q)$. Hence, the surfaces

$$\Sigma_1 := q_3(\{(P,P) | P \in S\}) \quad \text{and} \quad \Sigma_2 := q_3(\{(P,\iota_S(P)) | P \in S\})$$

are fixed by σ'.

Let $\beta' \colon Z_S \to (S \times S)/(\iota_S \times \iota_S)$ be the blow-up of $(S \times S)/(\iota_S \times \iota_S)$ in its singular locus. The two surfaces $\Sigma_1 = q_3(\{(P,P) | P \in S\})$ and $\Sigma_2 = q_3(\{(P,\iota_S(P)) | P \in S\})$ intersect transversally exactly in the curve $q_3(\Delta_{\mathrm{Fix}_{\iota_S}(S) \times \mathrm{Fix}_{\iota_S}(S)})$ inside the singular locus of $(S \times S)/(\iota_S \times \iota_S)$. The fixed locus of σ_Z maps to $\Sigma_1 \cup \Sigma_2 \subset (S \times S)/(\iota_S \times \iota_S)$.

In order to show that the fixed locus of σ_Z on Z_S has dimension two, we consider the surfaces $\Delta_S := \{(P,P) | P \in S\}$ and $\Gamma_S := \{(P,\iota_S(P)) | P \in S\}$ in $S \times S$. Let us blow-up $S \times S$ in $\mathrm{Fix}_{\iota_S \times \iota_S}(S \times S)$, call it $\widetilde{S \times S}$. The automorphisms $\iota_S \times \iota_S$ and σ induce automorphisms on $\widetilde{S \times S}$, called $\widetilde{\iota_S \times \iota_S}$ and $\widetilde{\sigma}$, respectively. The quotient $\widetilde{S \times S}/\widetilde{\iota_S \times \iota_S}$ is Z_S, so the automorphism σ_Z is induced by $\widetilde{\sigma}$ on the quotient. The surface $\Delta_S \subset S \times S$ is the fixed locus of σ, and its strict transform on $\widetilde{S \times S}$ is contained in the fixed locus

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of $\widetilde{\sigma}$. Considering the explicit equation in local charts of the blow-up of $\widetilde{S\times S}\to S\times S$, one observes that $\widetilde{\sigma}$ fixes the strict transform of Δ_S and a curve inside the exceptional divisor, mapped to $\Delta_{\mathrm{Fix}_{\iota_S}(S)\times \mathrm{Fix}_{\iota_S}(S)}$ by the blow-up $\widetilde{S\times S}\to S\times S$. This curve is the intersection between the strict transforms of Γ_S and $(\iota_S\times\iota_S)(\Gamma_S)$. Due to the quotient by $\iota_S\times\iota_S$, σ_Z fixes also the image of the strict transform of Γ_S.

5.4 The Other 2:1 Quotient of $S\times S$

The group $\langle\sigma,\iota_S\times\iota_S\rangle\simeq(\mathbb{Z}/2\mathbb{Z})^2\subset\mathrm{Aut}(S\times S)$ contains three distinct copies of $\mathbb{Z}/2\mathbb{Z}$: the one generated by σ, which gives rise to $S^{[2]}$, the one generated by $\iota_S\times\iota_S$, which gives rise to Z_S, and the one generated by $\sigma\circ(\iota_S\times\iota_S)$.

Proposition 5.4 *The blow-up of $(S\times S)/(\sigma\circ(\iota_S\times\iota_S))$ in its singular locus is isomorphic to $S^{[2]}$.*

Proof We consider the isomorphism $\phi=\iota_S\times\mathrm{id}_S\colon S\times S\to S\times S$; given the graph Γ_{ι_S} of ι_S, we have $\phi^{-1}(\Gamma_{\iota_S})=\Delta_S$, the diagonal in $S\times S$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl}_{\Delta_S}(S\times S) & \xrightarrow{\tilde{\phi}} & \mathrm{Bl}_{\Gamma_{\iota_S}}(S\times S)\\ \downarrow & & \downarrow\\ S\times S & \xrightarrow{\phi} & S\times S. \end{array}$$

We have thus deduced the existence of an isomorphism between the two blow-ups. On the other hand, we remark that $\mathrm{Fix}_\sigma(S\times S)=\Delta_S$ and $\mathrm{Fix}_{\sigma\circ(\iota_S\times\iota_S)}(S\times S)=\Gamma_{\iota_S}$, hence we get induced involutions on the blow-ups that we still denote as on $S\times S$. Moreover, $\phi\circ\sigma=(\sigma\circ(\iota_S\times\iota_S))\circ\phi$, so everything is equivariant and induces an isomorphism also between the smooth quotients; *i.e.,* we obtain an isomorphism

$$S^{[2]}\longrightarrow \mathrm{Bl}_{\Gamma_{\iota_S}}S^2/\big(\sigma\circ(\iota_S\times\iota_S)\big). \qquad\blacksquare$$

5.5 The Picard Group of Y_S

Since Y_S is a Calabi–Yau variety, $H^2(Y_S,\mathbb{Z})=\mathrm{Pic}(Y_S)$, so in order to determine a $\mathbb{Q}$-basis of $\mathrm{Pic}(Y_S)$, it suffices to find a basis of $H^2(Y_S,\mathbb{Q})$. This follows directly from the previous description of $H^{1,1}(Y_S)$. Let us assume that S is general among the K3 surfaces admitting a non-symplectic involution ι_S with a given fixed locus: this is equivalent to require that $\mathrm{NS}(S)=H^2(S,\mathbb{Z})^{\iota_S}$, or equivalently that $\rho(S)=\mathrm{rank}\,\mathrm{NS}(S)=r$. Let us denote by $D_S^{(1)},\dots,D_S^{(r)}$ a basis of $\mathrm{NS}(S)=\mathrm{Pic}(S)$. Let us now consider $S\times S$. A basis of $H^{1,1}(S)^{\iota_S}\otimes H^{0,0}(S)$ is given by ${D_S}^{(i)}\times[S]$ for $i=1,\dots,r$. Each divisor ${D_S}^{(i)}\times[S]$ is sent by σ^* to $[S]\times{D_S}^{(i)}\in H^{0,0}(S)\otimes H^{1,1}(S)^{\iota_S}$. In particular, the class ${D_S}^{(i)}\times[S]+[S]\times{D_S}^{(i)}$ is preserved by $\langle\iota_S\times\iota_S,\sigma\rangle$, and thus it corresponds to a class in $\mathrm{Pic}(Y_S)$, denoted by ${D_Y}^{(i)}$. The other generators of $\mathrm{Pic}(Y_S)$ come from the desingularization of the quotients of order two.

We recall once again the construction of our desingularizations by a diagram:

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(5.2)

$$\begin{tikzcd}
& \widetilde{S\times S} \arrow[r, "\beta_{\mathrm{Fix}_{\iota_S\times\iota_S}(S\times S)}"] \arrow[d, "/\iota_S\times\iota_S=:\pi_3"'] & S\times S \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd] & \widetilde{\widetilde{S\times S}} \arrow[l, "\beta_\Delta"'] \arrow[d, "/\sigma=:\pi_1"'] & \\
\widetilde{Z_S} \arrow[r, "\beta_{\mathrm{Fix}_{\sigma_Z}(Z_S)}"'] \arrow[d, "/\sigma_Z=:\pi_4"'] & Z_S & & S^{[2]} & \widetilde{S^{[2]}} \arrow[l, "\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}"'] \arrow[d, "/\iota_S^{[2]}=:\pi_2"'] \\
\widetilde{Z_S/\sigma_Z} \arrow[rr] & & (S\times S)/\langle\iota_S\times\iota_S,\sigma\rangle & & Y_S. \arrow[ll]
\end{tikzcd}$$

We will denote the quotient maps as follows: $\pi_1\colon \widetilde{\widetilde{S\times S}} \to S^{[2]}$ and $\pi_2\colon \widetilde{S^{[2]}} \to Y_S$. We study the divisors introduced by the blow-ups β_Δ and $\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}$ and identify the ones which are preserved by the quotient maps.

The blow-up β_Δ introduces one divisor, which is the exceptional divisor over the diagonal and is also the branch divisor of the quotient $\pi_1\colon \widetilde{\widetilde{S\times S}} \to S^{[2]}$. The natural involution $\iota_S{}^{[2]}$ preserves this divisor, called the exceptional divisor of $S^{[2]}$, (*cf.* [10, Theorem 1]), so its image under the quotient map π_2 is a divisor in $\mathrm{Pic}(Y_S)$, denoted by E_Δ and isomorphic to the exceptional divisor of $S^{[2]} \to S^{(2)}$. The other divisors of Y_S come from the blow-up $\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}$. They are as follows:

- $E_{C\times C}$, the exceptional divisor over $C^{[2]}$ (which is a surface fixed by $\iota_S{}^{[2]}$).
- $E_{R_i\times R_i}$ for $i = 1,\dots,k$, the exceptional divisor over $R_i^{[2]}$ (which is a surface fixed by $\iota_S{}^{[2]}$).
- $E_{C\times R_i}$, for $i = 1,\dots,k$: $C\times R_i \subset S\times S$ is a surface that is sent to $R_i \times C$ by σ, the common image of these surfaces is fixed by $\iota_S{}^{[2]}$, and so it is blown up by $\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}$, and $E_{C\times R_i}$ is the exceptional divisor of this blow-up.
- $E_{R_i\times R_j}$, for $i, j = 1,\dots k$, $i<j$: $R_i \times R_j \subset S\times S$ is a surface that is sent to $R_j \times R_i$ by σ, and the common image of these surfaces is fixed by $\iota_S{}^{[2]}$ and so it is blown up by $\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}$; $E_{R_i\times R_j}$, with $i<j$, is the exceptional divisor of this blow-up;
- E_{S/ι_S}: $\iota_S{}^{[2]}$ fixes a surface inside the exceptional divisor of $S^{[2]}$, which is given by the image of points $(P, \iota_S(P)) \in S\times S$; this surface is isomorphic to S/ι_S, and so $\beta_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}$ introduces an exceptional divisor on it, denoted by E_{S/ι_S}.

Proposition 5.5 *With the same notation as above, let*

$$\mathcal{S} = \{D_Y^{(h)}, E_\Delta, E_{S/\iota_S}, E_{C\times C}, E_{C\times R_i}, E_{R_i\times R_i}, E_{R_i\times R_j}\},$$

where $h = 1,\dots,r$, $i, j = 1,\dots k$ and $i<j$. Then $\mathcal{S}$ is a $\mathbb{Q}$-basis of $\mathrm{NS}(Y_S)$, $\mathrm{NS}(Y_S)\otimes\mathbb{Q}$ is isomorphic to $\mathrm{NS}(\widetilde{Z_S/\sigma_Z})\otimes\mathbb{Q}$, and so $\mathcal{S}$ is a $\mathbb{Q}$-basis of $\mathrm{NS}((\widetilde{Z_S/\sigma_Z})$. The divisors

$$B_{\iota_S^{[2]}} := \sum_{i=1}^{k}\Big(E_{C\times R_i} + E_{R_i\times R_i} + \sum_{j=i+1}^{k} E_{R_i\times R_j}\Big) + E_{C\times C} + E_{S/\iota_S}$$
$$B_{\sigma_Z} := E_\Delta + E_{S/\iota_S},$$

and $B_{\iota_S^{[2]}} + B_{\sigma_Z}$ *are 2-divisible in* $\mathrm{NS}(Y_S)$*, and indeed they are associated with three different double covers of* Y_S*.*

Proof By construction, $\mathcal{S} \subset \mathrm{NS}(Y_S)$ and the divisors in $\mathcal{S}$ are linearly independent. The cardinality of $\mathcal{S}$ coincides with $h^{1,1}(Y_S)$, so $\mathcal{S}$ is a $\mathbb{Q}$-basis of $\mathrm{NS}(Y_S)$.

By [32, p. 420], we know that $\widetilde{Z_S/\sigma_Z}$ and Y_S are related by a sequence of flops; therefore, they are isomorphic in codimension 1, and this allows one to identify the same basis $\mathcal{S}$ as a $\mathbb{Q}$-basis of $\mathrm{NS}(\widetilde{Z_S/\sigma_Z})$ by taking pullbacks along the sequence of flops. In particular, the birational map is well defined outside of all intersections of the exceptional divisors listed before, which are the divisors introduced by blow-ups in (5.2) and their intersection is the support of the flops.

We explicitly describe the isomorphism between $\mathrm{NS}(\widetilde{Z_S/\sigma_Z}) \otimes \mathbb{Q} \simeq \mathrm{NS}(Y_S) \otimes \mathbb{Q}$: the divisors $D_Y^{(h)}$, $h = 1, \dots, r$ are induced by the cohomology of $S \times S$ and do not depend on the desingularization that we are considering. The other divisors come from the blow-ups $\beta_{\mathrm{Fix}_{\iota_S \times \iota_S}(S \times S)}$ and $\beta_{\mathrm{Fix}_{\sigma_Z}(Z_S)}$. The first blow-up introduces exceptional divisors over the curves $C \times C$, $R_i \times R_i$, $R_i \times R_j$, $C \times R_i$, $R_i \times C$. The exceptional divisors over $C \times C$ and $R_i \times R_i$ are preserved by σ. The exceptional divisors over $R_i \times R_j$ (resp. $C \times R_i$) are identified with the ones over $R_j \times R_i$ (resp. $R_i \times C$) by the quotient by σ_Z. This gives the divisors $E_{C \times C}$, $E_{C \times R_i}$ for $i = 1, \dots, k$, $E_{R_i \times R_i}$ for $i = 1, \dots, k$, $E_{R_i \times R_j}$ for $i, j = 1, \dots k$, $i < j$ on Y_S. The blow-up $\beta_{\mathrm{Fix}_{\sigma_Z}(Z_S)}$ introduces two other divisors over the fixed locus of σ_Z and we already proved that $\mathrm{Fix}_{\sigma_Z}(Z_S)$ consists of two surfaces: the strict transforms of the images Σ_1 and Σ_2 of $\{(P, P) \in S \times S\}$ and of $\{(P, \iota_S(P)) \in S \times S\}$. We conclude that $\beta_{\mathrm{Fix}_{\sigma_Z}(Z_S)}$ introduces the divisors E_Δ and E_{S/ι_S} on Y_S. Moreover, this shows that the support of the divisor B_{σ_Z} is the branch locus of the 2:1 cover $\widetilde{Z_S} \to \widetilde{Z_S/\sigma_Z} \sim Y_S$. Hence it is a 2-divisible divisor on Y_S.

The support of the divisor $B_{\iota_S^{[2]}}$ is exactly the branch locus of the 2:1 cover $\pi_2\colon \widetilde{S^{[2]}} \to Y_S$.

Since the divisors $B_{\iota_S^{[2]}}$ and B_{σ_Z} are 2-divisible, $B_{\iota_S^{[2]}} + B_{\sigma_Z}$ is also 2-divisible in $\mathrm{NS}(Y_S)$. This means that Y_S admits a 2:1 cover branched along

$$\coprod_{i=1}^{k}(E_{C \times R_i}) \coprod_{i=1}^{k}(E_{R_i \times R_i}) \coprod_{i,j=1, i<j}^{k}(E_{R_i \times R_j}) \amalg E_{C \times C} \amalg E_\Delta.$$

This cover is naturally birational to $(S \times S)/((\iota_S \times \iota_S) \circ \sigma)$. ■

Since there are divisors that are 2-divisible in $\mathrm{NS}(Y_S)$ but not 2-divisible in $\mathcal{S}$, we conclude that $\mathcal{S}$ cannot be a $\mathbb{Z}$-basis of $\mathrm{NS}(Y_S)$.

5.6 Remarks on Complex Deformations and Mirror Symmetry

5.6.1 Complex Deformations

We make some remarks on the dimensions of the families of 4-folds that we are constructing.

Remark 5.6 Table 1 lists the dimensions of local complex deformations of the families of 4-folds constructed in the previous sections.

Object	Dimension of complex deformations space	Dimension in terms of (N, N')
(S, ι_S)	$\dim((H^{1,1}(S)^{\iota_S})^\perp \cap H^{1,1}(S))$	$10 - N + N'$
$(S \times S, \iota_S \times \iota_S)$	$\dim((H^{1,1}(S)^{\iota_S})^\perp \cap H^{1,1}(S))^{\oplus 2}$	$20 - 2N + 2N'$
Z_S	$h^{3,1}(Z_S)$	$20 - 2N + 2N' + N'^2$
$(S^{[2]}, \iota^{[2]})$	$\dim((H^{1,1}(S^{[2]})^{\iota_S^{[2]}})^\perp \cap H^{1,1}(S^{[2]}))$	$10 - N + N'$
$(S \times S, \iota_S \times \iota_S, \sigma)$	$\dim((H^{1,1}(S)^{\iota_S})^\perp \cap H^{1,1}(S))$	$10 - N + N'$
Y_S	$h^{3,1}(Y_S)$	$(20 - 2N + N' + N'^2)/2$
$\widetilde{Z_S/\sigma_Z}$	$h^{3,1}(\widetilde{Z_S/\sigma_Z})$	$(20 - 2N + N' + N'^2)/2$

Table 1: Dimension of complex deformation spaces

Let us explain the computations on the local deformation space of a pair (X, f), denoted $\mathrm{Def}(X, f)$. The fact that the dimension of $\mathrm{Def}(X, f)$ for X hyperkähler and $f \in \mathrm{Aut}(X)$ a non-symplectic involution, equals the dimension of $H^{1,1}(X) \cap (H^{1,1}(X)^f)^\perp$ is proved in [8, §4] and implies the statement both for (S, ι_S) and for $(S^{[2]}, {\iota_S}^{[2]})$.

In order to describe $\mathrm{Def}(S \times S, \iota_S \times \iota_S)$ and $\mathrm{Def}(S \times S, \iota_S \times \iota_S, \sigma)$, one observes that the same proof of *loc. cit.* yields, for any smooth complex manifold X such that $H^0(X, T_X) = 0$ and $c_1(X) = 0$ and any automorphism $f \in \mathrm{Aut}(X)$, that the dimension of the family $\mathrm{Def}(X, f)$ coincides with $\dim H^1(X, T_X)^{df}$, where df is the differential of f. Since $T_{S\times S} \cong T_S \boxplus T_S$, we have indeed that $H^0(S \times S, T_{S\times S}) = 0$, so that by [25, Theorem 14.10] $\mathrm{Def}(S \times S) \cong H^1(S \times S, T_{S\times S})$ as germs over 0; moreover, $H^1(S\times S, T_{S\times S})^{d(\iota_S\times\iota_S)} \cong (H^1(S, T_S)^{d\iota_S})^{\oplus 2}$, hence the statement for $\mathrm{Def}(S\times S, \iota_S\times\iota_S)$. Iteration of this reasoning finally gives the deformations of $(S \times S, \iota_S \times \iota_S, \sigma)$, since $d\sigma$ permutes the two summands of $(H^1(S, T_S)^{d\iota_S})^{\oplus 2}$.

The fact that local complex deformations of a smooth Calabi–Yau manifold X are given by an open set inside $H^{n-1,1}(X)$ is the well-known Tian–Todorov's theorem for smooth Kähler manifolds with trivial canonical bundle [25, Theorem 6.8.1]. Since Y_S and $\widetilde{Z_S/\sigma_Z}$ are birational their Hodge numbers coincide.

The third column is now an easy consequence of the previous computations.

We will say that a Calabi–Yau 4-fold is of Borcea–Voisin type if it is the desingularization of the quotient $(S_1 \times S_2)/(\iota_1 \times \iota_2)$ where S_i are K3 surfaces and ι_i is a non-symplectic involution on S_i and we generalize the definition [13, Definition 3.6] saying that a Borcea–Voisin maximal family is a family of Calabi–Yau 4-folds such that the general member of this family is of Borcea–Voisin type.

Corollary 5.7 *Given a pair (S, ι_S) the following hold:*

- $\dim(\mathrm{Def}(S \times S, \iota_S \times \iota_S, \sigma)) = \dim(\mathrm{Def}(S, \iota_S))$;

- $\dim(\mathrm{Def}(Z_S)) \geq \dim(\mathrm{Def}(S \times S, \iota_S \times \iota_S))$, *and the equality holds if and only if* $N' = 0$ *(*i.e., *if* ι_S *fixes on* S *only rational curves);*
- $\dim(\mathrm{Def}(S^{[2]}, \iota_S{}^{[2]})) = \dim(\mathrm{Def}(S, \iota_S))$, *but* $\dim(\mathrm{Def}(S^{[2]})) = \dim(\mathrm{Def}(S)) + 1$*;*
- $\dim(\mathrm{Def}(Y_S)) \geq \dim(\mathrm{Def}(S^{[2]}, \iota_S{}^{[2]})) = \dim(\mathrm{Def}(S, \iota_S))$*; the equality holds if and only if either* $N' = 0$ *or* $N' = 1$*.*
- $\dim(\mathrm{Def}(Y_S)) \leq \dim(\mathrm{Def}(Z_S))$*; the equality holds if and only if* ι_S *fixes exactly* 10 *rational curves,* i.e., *if* (S, ι_S) *is rigid.*

In particular, it follows from $\dim(\mathrm{Def}(Z_S)) \geq \dim(\mathrm{Def}(S \times S, \iota_S \times \iota_S))$ that not all the Calabi–Yau 4-folds that deform Z_S are of Borcea–Voisin type. Indeed, the deformations of Z_S are all of Borcea–Voisin type if and only if the fixed locus of ι_S on S is rigid. In this case, the family of Z_S is a Borcea–Voisin maximal family, in analogy with [13, Proposition 3.7]. This implies that the deformations of the complex structure of Z_S depend only on the deformations of the complex structure of $(S \times S, \iota_S \times \iota_S)$.

By $\dim(\mathrm{Def}(S^{[2]}, \iota_S{}^{[2]})) = \dim(\mathrm{Def}(S \times S, \iota_S \times \iota_S, \sigma))$ it follows that all the deformations of $S^{[2]}$ that preserve the non-symplectic involution $\iota_S{}^{[2]}$ are dominated by $S \times S$, but there is one more deformation of $S^{[2]}$ if we do not require that $\iota_S{}^{[2]}$ deforms with $S^{[2]}$.

By $\dim(\mathrm{Def}(Y_S)) \geq \dim(\mathrm{Def}(S^{[2]}, \iota_S{}^{[2]}))$, it follows that not all the deformations of Y_S are dominated by $S^{[2]}$ and by $S \times S$. However, again, if the fixed locus of ι_S is rigid on S, then all the deformations of Y_S are obtained both as the crepant resolution of $S^{[2]}/\iota_S{}^{[2]}$ and as crepant resolution of $(S \times S)/\langle \iota_S \times \iota_S, \sigma\rangle$. In this case, the variation of the complex structure of Y_S depends only on the variation of the complex structure of S in $\mathrm{Def}(S, \iota_S)$. This is the analogue of the [13, Proposition 3.7]. A little bit more surprising is the fact that also if $N' = 1$, *i.e.*, if the fixed locus of ι_S contains a curve of genus 1 (which a priori can be deformed), then $\dim(\mathrm{Def}(Y_S)) = \dim(\mathrm{Def}(S^{[2]}, \iota_S{}^{[2]})) = \dim(\mathrm{Def}(S \times S, \iota_S \times \iota_S, \sigma))$.

Since $\dim(\mathrm{Def}(Y_S)) \leq \dim(\mathrm{Def}(Z_S))$, all the deformations of Y_S are dominated by deformations of Z_S. We observe that a general deformation of Z_S does not necessarily admit the automorphism σ_Z needed to construct Y_S as quotient.

5.6.2 Mirror Symmetry

Here we discuss the mirror symmetry of Y_S and Z_S, at least at the level of the Hodge diamond. For this reason we recall here the dimension of the space of small deformations of the Kähler structure, since we want to compare it with the dimension of the complex deformations, given in Table 1.

Under a mild condition on the Néron–Severi group, a lattice theoretic mirror symmetry between K3 surfaces is defined by Dolgachev in [19], extending work by Pinkham and Nikulin. Given a smooth K3 surface S and a primitive hyperbolic sublattice $M \subset \mathrm{NS}(S)$ of its Néron–Severi group, the K3 surface S is said to be M-polarized, and, if the orthogonal of M inside $H^2(S, \mathbb{Z})$ is of the form $U \oplus \check{M}$ with $\check{M}$ a hyperbolic sublattice, then a *mirror symmetric K3 surface for* S is any smooth K3 surface $\check{S}$ such that $\check{M} \subset \mathrm{NS}(\check{S})$. In particular, if S is a general K3 surface with a non-symplectic

Object	Dimension of Kähler deformations space	Dimension in terms of (N, N')
Z_S	$h^{1,1}(Z_S)$	$20 + 2N - 2N' + N^2$
$(S^{[2]}, \iota^{[2]})$	$\dim(H^{1,1}(S^{[2]})^{\iota_S^{[2]}})$	$11 + N - N'$
Y_S	$h^{1,1}(Y_S)$	$\left(24 + 3N - 2N' + N^2\right)/2$

Table 2: Dimension of Kähler deformations spaces

involution ι_S whose fixed locus contains exactly one curve of positive genus, then the invariant sublattice $H^2(S,\mathbb{Z})^{\iota_S} \subset \mathrm{NS}(S)$ is primitive hyperbolic and its orthogonal inside $H^2(S,\mathbb{Z})$ is of the form $U \oplus \check{M}$ with $\check{M}$ a hyperbolic sublattice; Voisin showed in [54] that S admits a mirror symmetric K3 surface $\check{S}$ with a non-symplectic involution $\check{\iota_S}$ such that $H^2(\check{S},\mathbb{Z})^{\check{\iota_S}} \simeq \check{M}$. Let N and N' be the invariants of the fixed locus of ι_S on S and let $\check{N}$ and $\check{N}'$ be the invariants of the fixed locus of $\check{\iota_S}$ on $\check{S}$; we have $N = \check{N}'$, $N' = \check{N}$. As already observed in [18, Proposition 8.1], this induces the mirror symmetry between the Hodge diamond of Z_S and that of $Z_{\check{S}}$ (which is a crepant resolution of $(\check{S} \times \check{S})/(\check{\iota_S} \times \check{\iota_S})$), *i.e.*, $h^{1,1}(Z_S) = h^{3,1}(Z_{\check{S}})$, $h^{3,1}(Z_S) = h^{1,1}(Z_{\check{S}})$ and $h^{2,2}(Z_S) = h^{2,2}(Z_{\check{S}})$. It is thus natural to ask whether the same holds for Y_S and $Y_{\check{S}}$; unfortunately, the answer is negative, as can be observed also from the last table in Appendix A.

Corollary 5.8 *Let $\check{S}$ be a K3 surface that is a lattice theoretic mirror of the K3 surface S. Let $\check{Y}_S$ be a Calabi–Yau 4-fold whose Hodge diamond is mirror to the one of Y_S. Then $\check{Y}_S$ is not birational to $Y_{\check{S}}$.*

Proof As observed above, we have $N = \check{N}'$, $N' = \check{N}$, so it follows from Theorem 5.1 that $h^{3,1}(Y_{\check{S}}) = (20 - 2N' + N + N^2)/2$, and this is different from $h^{1,1}(Y_S)$ in Table 2 for all possible values of (N, N'). ∎

On the other hand, it is known that the mirror symmetry between (S, ι_S) and $(\check{S}, \check{\iota_S})$ does not induce a lattice theoretic mirror symmetry between $S^{[2]}$ and $\check{S}^{[2]}$. This is because mirror symmetry for hyperkähler manifolds works differently from the case of Calabi–Yau manifolds (see Huybrechts' lecture notes for extensive explanations of this phenomenon [28]). The definition of lattice theoretic mirror symmetry is exactly the same as for K3 surfaces (see [12] for further details). Given a smooth hyperkähler 4-fold of $K3^{[2]}$ type V and a primitive hyperbolic sublattice $M \subset \mathrm{NS}(V)$ of its Néron–Severi group, if the orthogonal of M inside $H^2(V,\mathbb{Z})$ is of the form $U \oplus \check{M}$ with $\check{M}$ a hyperbolic sublattice, then a *mirror symmetric hyperkähler 4-fold of $K3^{[2]}$ type for V* is any smooth 4-fold of $K3^{[2]}$ type $\check{V}$ such that $\check{M} \subset \mathrm{NS}(\check{V})$. Again, this lattice theoretic mirror symmetry between two hyperkähler 4-folds of $K3^{[2]}$ type induces mirror symmetry between different families of 4-folds of $K3^{[2]}$ type endowed with a non-symplectic involution: in particular, in [12, Sect. 5.2] it is shown that, given ι_V a non-symplectic involution on V of $K3^{[2]}$ type satisfying some mild assumptions, a

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general mirror symmetric 4-fold of $K3^{[2]}$ type $\check{V}$ for V carries the action of another non-symplectic involution, denoted $\iota_V^{\vee}$. In the special case of a natural involution $(V, \iota_V) = (S^{[2]}, \iota_S{}^{[2]})$, a general mirror pair turns out to be always non-natural, *i.e.*, a pair $(\check{V}, \iota_V^{\vee})$ with $\iota_V^{\vee}$, which cannot be a natural involution on a Hilbert scheme of points of a K3 surface.

In view of this second mirror construction it is quite natural to ask whether the lattice theoretic mirror symmetry on hyperkähler varieties of $K3^{[2]}$ type can induce a mirror symmetry between the Calabi–Yau 4-folds resolutions of V/ι_V and $\check{V}/\iota_V^{\vee}$ respectively. The situation is more complicated in this case, since in order to compute the Hodge numbers of the 4-folds obtained as quotient of $\check{V}$ by a non-natural involution $\iota_V^{\vee}$, one needs a good description of the fixed locus (in order to compute the Hodge diamond of a resolution of $\check{V}/\iota_V^{\vee}$). In general this is not available, but testing the mirror symmetry in the unique explicit case present in the literature, we obtain a negative answer to the previous question. Indeed, the first possible test fails: let S be a general K3 surface with a non-symplectic involution ι_S such that the numbers associated with its fixed locus are $N = 10$ and $N' = 2$, in particular $H^2(S,\mathbb{Z})^{\iota_S} \simeq \mathrm{NS}(S) \simeq U \oplus E_8(-1) \oplus E_8(-1)$ and $T_S \simeq U \oplus U$. A lattice theoretic mirror of the pair $(S^{[2]}, \iota_S{}^{[2]})$ is the pair (X, ι_X), where X is a hyperkähler 4-fold of $K3^{[2]}$ type with $\mathrm{NS}(X) \simeq U$ and ι_X is the non-natural non-symplectic involution on it described by Ohashi and Wandel in [50]. We do not need any detail about the geometry of this example here, so we refer to *loc. cit.* for those, and we limit ourselves to recalling that for this involution $t_{1,1} = 2$ (indeed the subspace of $H^{1,1}(X)^{\iota_X}$ is the Néron–Severi group) and that the fixed locus of ι_X consists of two disjoint surfaces.

Corollary 5.9 *Let S be a general K3 surface with a non-symplectic involution ι_S such that the numbers associated with its fixed locus are $N = 10$ and $N' = 2$, as above, and let (X, ι_X) be the mirror pair of $(S^{[2]}, \iota_S{}^{[2]})$ given by Ohashi–Wandel's example. Let $\widetilde{X/\iota_X}$ be the Calabi–Yau constructed as in Section 4.1, and let $\check{Y}_S$ be a Calabi–Yau 4-fold whose Hodge diamond is mirror to the one of Y_S. Then $\check{Y}_S$ is not birational to $\widetilde{X/\iota_X}$.*

Proof Under these assumptions on (S, ι_S), Theorem 5.1 implies that $h^{1,1}(\check{Y}_S) = h^{3,1}(Y_S) = 3$. Moreover, Theorem 4.3 and the properties recalled above give $h^{1,1}(\widetilde{X/\iota_X}) = 2 + 2 = 4 \neq 3 = h^{1,1}(\check{Y}_S)$, showing that $\check{Y}_S$ is not birational to $\widetilde{X/\iota_X}$. ∎

The question whether a mirror symmetry is induced in other examples of non-natural non-symplectic involutions on 4-folds of $K3^{[2]}$-type remains open; in a certain sense though, it is not to be expected that the mirror symmetry of Y_S can be deduced from the mirror symmetry of $S^{[2]}$, exactly as it cannot be deduced from the mirror symmetry for S and Z_S as shown in Corollary 5.8. Indeed, the space of the complex (resp. Kähler) deformations of Y_S includes not only deformations that come from S but also some coming from deformations of Z_S and some coming from deformations of $S^{[2]}$. Since the two mirror constructions, of Z_S and of $S^{[2]}$ respectively, are not compatible, it seems reasonable that none of these two gives the right one for Y_S. As observed in Appendix A, there is no mirror relation between the explicit Hodge numbers of any two 4-folds Y_S either, with the exception of a few cases that could be self-mirror, so that the mirror of Y_S has to be looked for elsewhere.

6 Quotients of $S \times S$ and Covers of These Quotients

Since the 4-folds $S^{[2]}$, Y_S, Z_S and $\widetilde{Z_S/\sigma_S}$ are obtained as desingularizations of the quotients of $S \times S$ by an automorphism of $S \times S$, we now consider a subgroup of $\operatorname{Aut}(S\times S)$ and the quotient of $S\times S$ by this subgroup. Indeed, in our geometric setting, it is natural to look at the subgroup generated by $\mathrm{id} \times \iota_S$, $\iota_S \times \mathrm{id}$ and σ. By Proposition 6.1, due to Oguiso, this is also the "maximal" choice for a general K3 surface with an involution. Indeed, if S is general among the K3 surfaces with a non-symplectic involution ι_S, then $\operatorname{Aut}(S) = \langle \iota_S \rangle$, and thus the subgroup described coincides with $\operatorname{Aut}(S \times S)$.

For every K3 surface admitting a non-symplectic involution ι_S, we will consider the quotients by $\mathcal{D}_8 = \langle \iota_S \times \mathrm{id}, \mathrm{id} \times \iota_S, \sigma \rangle$ and by its subgroups. Let W be the smooth surface S/ι_S. We observe (see diagram (6.1)) that $(S \times S)/\mathcal{D}_8 \simeq W^{(2)}$. Then we describe the singular models of $S^{[2]}$, Y_S, Z_S as covers of $(S \times S)/\mathcal{D}_8$ under the assumption that the fixed locus of ι_S is connected (see Section 6.2). When ι_S is a fixed point free involution, so that W is an Enriques surface, this description allows us to prove that $\widetilde{Z_S/\sigma_Z}$ is in fact isomorphic (and not only birational) to Y_S (see Proposition 6.3) and to show that Y_S is indeed the universal cover of $W^{[2]}$ mentioned in [49, Theorem 3.1].

Proposition 6.1 ([48, Section 4]) *The automorphism group of $S \times S$ is $\operatorname{Aut}(S \times S) \simeq \langle \sigma \rangle \rtimes Aut(S)^2$.*

Remark 6.2 If $\operatorname{Aut}(S) \simeq \mathbb{Z}/2\mathbb{Z}$, then $\operatorname{Aut}(S \times S) \simeq \mathcal{D}_8$.

The group $\mathcal{D}_8 \simeq \langle \iota_S \times \mathrm{id}, \mathrm{id} \times \iota_S, \sigma \rangle$ contains the following elements:

$$\begin{aligned}
&g_1 := \mathrm{id} \times \mathrm{id} \colon (P,Q) \mapsto (P,Q), && g_2 := \sigma \colon (P,Q) \mapsto (Q,P),\\
&g_3 := \iota_S \times \mathrm{id} \colon (P,Q) \mapsto (\iota_S(P),Q), && g_4 := \sigma \circ (\iota_S \times \mathrm{id}) \colon (P,Q) \mapsto (Q,\iota_S(P)),\\
&g_5 := \mathrm{id} \times \iota_S \colon (P,Q) \mapsto (P,\iota_S(Q)), && g_6 := \sigma \circ (\mathrm{id} \times \iota_S) \colon (P,Q) \mapsto (\iota_S(Q),P),\\
&g_7 := \iota_S \times \iota_S \colon (P,Q) \mapsto (\iota_S(P),\iota_S(Q)) && g_8 := \sigma \circ (\iota_S \times \iota_S) \colon (P,Q) \mapsto (\iota_S(Q),\iota_S(P)).
\end{aligned}$$

We observe that g_2, g_3, g_5, g_7, g_8 have order two, while g_4 and g_6 have order 4 ($g_4^3 = g_6$) and their square is g_7. The subgroup $\langle g_7 \rangle$ is the center of the group; in particular, it is normal. The other normal subgroups are $\langle g_7, g_2 \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$, $\langle g_3, g_7 \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and $\langle g_4 \rangle \simeq \mathbb{Z}/4\mathbb{Z}$.

We denote by $\overline{g_i}$ the automorphisms induced by g_i on the quotients of $S \times S$ by a certain subgroup of $\mathcal{D}_8$, and by q_j the quotient maps, and we underline that q_1, q_2, q_3 and q_4 were already defined in Proposition 5.3. We obtain the following diagram, where all the arrows are quotients of order two:

(6.1)
$$
\begin{tikzcd}
 & & & & S\times S \arrow[dllll, "q_1:=/g_2"'] \arrow[dll, "/g_8"] \arrow[d, "q_3:=/g_7"'] \arrow[drr, "/g_5"] \arrow[drrrr, "/g_3"] & & & & \\
S^{(2)} \arrow[drr, "q_2:=/\overline{g_7}"'] & & (S\times S)/(\sigma\circ(\iota_S\times\iota_S)) \arrow[d, "/\overline{g_2}"'] & & (S\times S)/(\iota_S\times\iota_S) \arrow[dll, "q_4:=/\overline{g_2}"] \arrow[drr, "q_6:=/\overline{g_3}"'] & & S\times W \arrow[d, "/\overline{g_7}"'] & & W\times S \arrow[dll, "/\overline{g_7}"'] \\
 & & S^{(2)}/\iota_S^{(2)} \arrow[drr, "q_5:=/\overline{g_3}"'] & & & & W\times W \arrow[dll, "q_7:=/\overline{g_2}"'] & & \\
 & & & & W^{(2)} & & & &
\end{tikzcd}
$$

where $W^{(2)} = (W\times W)/\sigma$. This is a 4-fold, singular along a surface isomorphic to W, image of the diagonal under the quotient map $W\times W\to W^{(2)}$. The diagram (6.1) shows that $W^{(2)}$ is the quotient of $S\times S$ by the group $\langle\iota_S\times\mathrm{id},\mathrm{id}\times\iota_S,\sigma\rangle\simeq\mathcal{D}_8$.

The 4-folds $S^{(2)}$, $(S\times S)/(\sigma\circ(\iota_S\times\iota_S))$, $(S\times S)/(\iota_S\times\iota_S)$, and $W^{(2)}$ are singular along surfaces and admit the crepant resolutions $S^{[2]}$, $\widetilde{(S\times S)/(\sigma\circ(\iota_S\times\iota_S))}$, Z_S, and $W^{[2]}$, respectively. We already proved that $\widetilde{(S\times S)/(\sigma\circ(\iota_S\times\iota_S))}$ is isomorphic to $S^{[2]}$. The singular quotient $S^{(2)}/\iota_S^{(2)}$ is birational to the Calabi–Yau 4-fold Y_S. All the other 4-folds that appear in the diagram are smooth.

The 4:1 map $(S\times S)/(\iota_S\times\iota_S)\to W^{(2)}$ is the quotient map by the group $(\mathbb{Z}/2\mathbb{Z})^2\simeq\mathcal{D}_8/g_7$. There are also some 4:1 maps that are induced by this diagram, but that are not quotient maps (*i.e.*, the target space is not the quotient of the domain by the action of a group of order 4 defined over the domain): by the previous diagram, both $S^{(2)}\to W^{(2)}$ and $(S\times S)/(\sigma\circ(\iota_S\times\iota_S))\to W^{(2)}$ have order 4.

There is a 2:1 quotient map from $S^{(2)}/\iota_S^{(2)}$ to $W^{(2)}$.

There is a map $(S\times S)/(\iota_S\times\iota_S)\to W$, obtained by composing

$$(S\times S)/(\iota_S\times\iota_S)\to W\times W$$

with the projection on one factor $W\times W\to W$. The general fiber of $(S\times S)/(\iota_S\times\iota_S)\to W$ is isomorphic to S.

We will consider the quotient maps

$$
\begin{aligned}
&S\times S\xrightarrow{q_3}(S\times S)/(\iota_S\times\iota_S)\xrightarrow{q_6}W\times W\xrightarrow{q_7}W^{(2)},\\
&S\times S\xrightarrow{q_3}(S\times S)/(\iota_S\times\iota_S)\xrightarrow{q_4}S^{(2)}/\iota_S^{(2)}\xrightarrow{q_5}W^{(2)},\\
&S\times S\xrightarrow{q_1}S^{(2)}\xrightarrow{q_2}S^{(2)}/\iota_S^{(2)}\xrightarrow{q_5}W^{(2)},
\end{aligned}
\tag{6.2}
$$

and the following subspaces in $S\times S$:

$$\Delta_S:=\{(P,P)|P\in S\}\simeq S\quad\text{and}\quad\Gamma_S:=\{(P,\iota_S(P))|P\in S\}\simeq S.$$

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6.1 A Very Special Case: ι_S is an Enriques Involution

We now focus on the case where ι_S is an Enriques involution of S (which is by definition a non-symplectic involution such that the fixed locus of ι_S is empty). The quotient surface $W = S/\iota_S$ is an Enriques surface (a regular surface with the canonical bundle which is a 2-torsion bundle).

Proposition 6.3 *If ι_S is an Enriques involution on S, then Y_S is isomorphic to $\widetilde{Z_S/\sigma_Z}$ and they are the blow-up of the non-ramified double cover of $W^{(2)}$ in its singular locus.*

Proof Both Y_S and $\widetilde{Z_S/\sigma_Z}$ are desingularizations of $S^{(2)}/\iota_S^{(2)}$. The canonical bundle of $W^{(2)}$ is a 2-torsion bundle which induces the 2:1 cover $S^{(2)}/\iota_S^{(2)} \to W^{(2)}$; thus, $S^{(2)}/\iota_S^{(2)}$ is singular along two surfaces (mapped to $\mathrm{Sing}(W^{(2)})$). These surfaces are $q_4(q_3(\Delta_S))$ and $q_4(q_3(\Gamma_S))$. Since $S^{(2)}/\iota_S^{(2)} \to W^{(2)}$ is an unramified cover (or since $\mathrm{Fix}_{\iota_S}(S)$ is empty), the surfaces $q_4(q_3(\Delta_S))$ and $q_4(q_3(\Gamma_S))$ are disjoint.

In order to construct Y_S, one first constructs $S^{[2]}$ as blow-up of $S^{(2)}$ in its singular locus, which is $q_1(\Delta_S)$. Then one constructs the quotient $S^{[2]}/\iota_S{}^{[2]}$ and blows up its singular locus, which is mapped on $S^{(2)}/\iota_S^{(2)}$ to $q_2(q_1(\Gamma_S))$. So Y_S is isomorphic to $S^{(2)}/\iota_S^{(2)}$ blown up in the two disjoint surfaces $q_2(q_1(\Delta_S)) \simeq q_4(q_3(\Delta_S))$ and $q_2(q_1(\Gamma_S)) \simeq q_4(q_3(\Gamma_S))$.

In order to construct $\widetilde{Z_S/\sigma_Z}$, one constructs the smooth quotient $(S \times S)/(\iota_S \times \iota_S)$, then one considers the quotient $q_4\colon (S\times S)/(\iota_S\times\iota_S) \to S^{(2)}/\iota_S^{(2)}$, and finally one blows up the two singular surfaces $q_4(q_3(\Delta_S))$ and $q_4(q_3(\Gamma_S))$ of $S^{(2)}/\iota_S^{(2)}$.

So both Y_S and $\widetilde{Z_S/\sigma_Z}$ are the blow-ups of $S^{(2)}/\iota_S^{(2)}$ in its two disjoint singular surfaces. ∎

We remark that $Y_S \simeq \widetilde{Z_S/\sigma_S}$ (blow-up of $S^{(2)}/\iota_S^{(2)}$ in its singular locus) is exactly the universal cover of $W^{[2]}$ (blow-up of $W^{(2)}$ in its singular locus), whose existence was proved in [49, Theorem 3.1].

6.2 Covers

If the involution ι_S has a non-empty fixed locus, the intersection of $\mathrm{Fix}_{\iota_S\times\iota_S}(S \times S)$ and $\mathrm{Fix}_\sigma(S \times S)$ is non-trivial, both $(S \times S)/\sigma$ and $(S \times S)/(\iota_S \times \iota_S)$ are singular, and the fixed locus of the involution induced by $\iota_S \times \iota_S$ (resp. σ) on $(S \times S)/\sigma$ (resp. $(S \times S)/(\iota_S \times \iota_S)$) intersects the singular locus.

We now assume for simplicity that ι_S fixes only one smooth curve C on S. For simplicity we also denote by C the isomorphic image of C in the smooth quotient surface $W = S/\iota_S$. The purpose of this section is to reconstruct all the singular four-folds in diagram (6.1) by the data (W, C): in the following subsections we describe these constructions.

We will consider the following subspaces in $S \times S$:

$$\begin{aligned}
\Delta_C &:= \left\{ (P,P) | P \in \mathrm{Fix}_{\iota_S}(S) \right\} \simeq \mathrm{Fix}_{\iota_S}(S),\\
B_C &:= \left\{ (P,Q) | P \in \mathrm{Fix}_{\iota_S}(S),\ Q \in \mathrm{Fix}_{\iota_S}(S) \right\} \simeq \mathrm{Fix}_{\iota_S}(S) \times \mathrm{Fix}_{\iota_S}(S),\\
T_1 &:= \left\{ (P,Q) | P \in \mathrm{Fix}_{\iota_S}(S),\ Q \in S \right\} \simeq \mathrm{Fix}_{\iota_S}(S) \times S,\\
T_2 &:= \left\{ (P,Q) | P \in S,\ Q \in \mathrm{Fix}_{\iota_S}(S) \right\} \simeq S \times \mathrm{Fix}_{\iota_S}(S).
\end{aligned}$$

The map $\pi_{\mathrm{tot}} \colon S \times S \to W^{(2)}$ coincides with each of the compositions: $q_7 \circ q_6 \circ q_3$, $q_5 \circ q_4 \circ q_3$, and $q_5 \circ q_2 \circ q_1$ (see (6.2)).

We also remark that $\mathrm{Sing}(W^{(2)}) = \pi_{\mathrm{tot}}(\Delta_S) = \pi_{\mathrm{tot}}(\Gamma_S)$. In the following we will assume that $\mathrm{Fix}_{\iota_S}(S)$ consists of one smooth irreducible curve.

6.2.1 $S^{(2)}/\iota_S^{(2)}$ as Double Cover of $W^{(2)}$

The threefold $T_{\mathrm{tot}} := \pi_{\mathrm{tot}}(T_1) = \pi_{\mathrm{tot}}(T_2) \subset W^{(2)}$ meets the singular locus of $W^{(2)}$ in the curve $\pi_{\mathrm{tot}}(\Delta_C)$. Moreover we observe that it is a singular threefold and its singular locus is $\pi_{\mathrm{tot}}(B_C)$.

By diagram (6.1), the double cover of $W^{(2)}$ branched along T_{tot} is the 4-fold $S^{(2)}/\iota_S^{(2)}$. The inverse image of $\mathrm{Sing}(W^{(2)})$ consists of two surfaces, $q_4(q_3(\Delta_S))$ and $q_4(q_3(\Gamma_S))$, meeting along the inverse image of $T_{\mathrm{tot}} \cap \mathrm{Sing}(W^{(2)})$, which is the curve $q_5(q_4(q_3(\Delta_C)))$. These surfaces are singular for $S^{(2)}/\iota_S^{(2)}$ and intersects along a curve, which is $q_4(q_3(\Delta_C))$. They are both isomorphic to W.

But the singularities of $S^{(2)}/\iota_S^{(2)}$ do not consist only of the surfaces $q_4(q_3(\Delta_S))$ and $q_4(q_3(\Gamma_S))$. Indeed, we already remarked that the branch threefold T_{tot} is singular along a surface, which is $\pi_{\mathrm{tot}}(B_C) \notin \mathrm{Sing}(W^{(2)})$. Hence, the 2:1 cover $S^{(2)}/\iota_S^{(2)} \to W^{(2)}$ is singular along the inverse image of this surface. Thus, we have a third singular surface in $S^{(2)}/\iota_S^{(2)}$, $q_4(q_3(B_C))$, isomorphic to $C^{(2)}$. This third singular surface intersects the other two singular surfaces in their common intersection, *i.e.*, in the curve $q_4(q_3(\Delta_C))$. The intersection among these three surfaces is transversal, since they are images of surfaces on $S \times S$ that generically have no common tangent directions. Choosing two surfaces among these three as branch locus of double covers of $S^{(2)}/\iota_S^{(2)}$, we reconstruct the other 4-folds in diagram (6.1).

6.2.2 $S^{(2)}$ as Double Cover of $S^{(2)}/\iota_S^{(2)}$ (and Thus as 4:1 Cover of $W^{(2)}$)

The 4-fold $S^{(2)}/\iota_S^{(2)}$ is singular in three surfaces, which intersect in $q_4(q_3(\Delta_C))$ and are $q_4(q_3(\Delta_S))$, $q_4(q_3(\Gamma_S))$, and $q_4(q_3(B_C))$. The same surfaces can be described also as $q_2(q_1(\Delta_S))$, $q_2(q_1(\Gamma_S))$ and $q_2(q_1(B_C))$.

Let us now consider the double cover of $S^{(2)}/\iota_S^{(2)}$ branched along $q_2(q_1(\Gamma_S)) \cup q_2(q_1(B_C))$. By diagram (6.1), we obtain a 4-fold that is $S^{(2)}$. It is singular in $q_1(\Delta_S) \simeq S$, and the quotient map $q_2 \colon S^{(2)} \to S^{(2)}/\iota_S^{(2)}$ restricts to a 2:1 map between $q_1(\Delta_S) \simeq S$ and $q_2(q_1(\Delta_S)) \simeq W$, branched along $q_2(q_1(\Delta_C))$.

6.2.3 $(S\times S)/(\iota_S\times\iota_S)$ as Double Cover of $S^{(2)}/\iota_S^{(2)}$ (and Thus as $4{:}1$ Cover of $W^{(2)}$, via $S^{(2)}/\iota_S^{(2)}$)

Similarly to what we did in the previous paragraph, one can consider two surfaces among the three singular surfaces in $S^{(2)}/\iota_S^{(2)}$ and consider the double cover branched along these two surfaces. In particular, let us consider the double cover of $S^{(2)}/\iota_S^{(2)}$ branched along $q_4(q_3(\Delta_S))\cup q_4(q_3(\Gamma_S))$. We obtain a 4-fold that is $(S\times S)/(\iota_S\times\iota_S)$. It is singular in $q_3(B_C)\simeq C\times C$, and the quotient map $q_4\colon (S\times S)/(\iota_S\times\iota_S)\to S^{(2)}/\iota_S^{(2)}$ restricts to a 2:1 map between $q_3(B_C)\simeq C\times C$ and $q_4(q_3(B_C))\simeq C^{(2)}$, branched along $q_4(q_3(\Delta_C))$.

6.2.4 $(S\times S)/\langle(\iota_S\times\iota_S)\circ\sigma\rangle$ as Double Cover of $S^{(2)}/\iota_S^{(2)}$ (and Thus as $2^2{:}1$ Cover of $W^{(2)}$)

The third (and last) possible choice is to consider the double cover of $S^{(2)}/\iota_S^{(2)}$ branched along $q_4(q_3(B_C))\cup q_4(q_3(\Delta_S))$. We obtain a 4-fold $(S\times S)/\langle(\iota_S\times\iota_S)\circ\sigma\rangle$ that is isomorphic to $S^{(2)}$ by Proposition 5.4.

6.2.5 $(S\times S)/(\iota_S\times\iota_S)$ as $4{:}1$ Cover of $W^{(2)}$, via $W\times W$

Let us consider the double cover of $W^{(2)}$ branched along $\mathrm{Sing}(W^{(2)})$. This is the smooth fourfold $W\times W$. Let us consider the (singular) threefold $T_{\mathrm{tot}} := q_7(q_6(q_3(T_1))) = q_7(q_6(q_3(T_2)))$, which intersects the singular locus of $W^{(2)}$ in $q_7(q_6(q_3(\Delta_C)))\simeq C$. In the double cover $W\times W$, T_{tot} splits in the two threefolds $q_6(q_3(T_1))$ and $q_6(q_3(T_2))$, meeting along the surface $q_6(q_3(B_C))$ (which is in fact the surface mapped on the singularity of T_{tot}). The fourfold $(S\times S)/(\iota_S\times\iota_S)$ is the double cover of $W\times W$ branched along the union of the two threefolds over $q_6(q_3(T_1))$ and $q_6(q_3(T_2))$ (*i.e.*, along the two inverse images of T_{tot} in the double cover $W\times W\to W^{(2)}$). These two threefolds meet along the surface $(q_6(q_3(B_C))\simeq C\times C$; thus, the double cover $(S\times S)/(\iota_S\times\iota_S)$ is singular along the surface inverse image of $q_6(q_3(B_C))\simeq C\times C$.

7 Projective Models

The aim of this section is to describe some explicit models of the 4-folds constructed relating the geometric description given by the diagram (6.1) and in Section 6.2 with the description of the cohomology on $S^{[2]}$, Y_S, and Z_S, given in Sections 5.1 and 5.2.

The main results of this section are of two different types: first we consider divisors induced on the 4-folds $S^{[2]}$, Z_S and Y_S by nef and big divisors on S, and we compute their characteristic (Proposition 7.3) and in some cases the dimension of their linear systems (Theorem 7.5). Thus, we give an explicit formulation of Riemann–Roch theorem in our context. Then we apply these general results to specific examples of divisors and K3 surfaces S, in Sections 7.2, 7.3, and 7.4. In some cases we also give explicit equations for some of the 4-folds constructed; see Section 7.2.1 and proof of Propositions 7.10 and 7.11.

We consider an L-polarized K3 surface for $L \cong \langle 2 \rangle, U, U(2)$ (see Sections 7.2, 7.3, and 7.4 respectively). We notice that for each such choice of L, the L-polarized K3 surface admits exactly one of the following geometric models (see [51, Section 5]):

(a) S is a double cover of $\mathbb{P}^2$ and ι_S is the cover involution;
(b) S admits an elliptic fibration (with section) and ι_S is the elliptic involution (equivalently, S is a double cover of $\mathbb{F}_4$ and ι_S is the cover involution);
(c) S is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ and ι_S is the cover involution.

All the pairs (S, ι_S) admit at least one of these geometric models where the fixed locus can specialize.

7.1 Results on Divisors and Linear Systems

In the sequel we will apply the results of this subsection to study divisors on three different varieties, $S^{[2]}$, Z_S and Y_S, since all of them can be constructed as the crepant resolution X of the quotient of an appropriate 4-fold V with $c_1(V) = 0$ by a volume preserving involution α, we consider now the commutative diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r, hook] \arrow[d, "\beta_{|\widetilde{E}}"'] & \widetilde{V} \arrow[r, "\pi"'] \arrow[d, "\beta"'] & X \arrow[d, "\beta'"'] \\ \Sigma \arrow[r, hook, "j"'] & V \arrow[r, "\pi'"'] & V/\alpha \end{tikzcd}$$

with

- V a smooth 4-fold with $c_1(V) = 0$;
- $\alpha \in \mathrm{Aut}(V)$ a volume preserving involution;
- Σ the smooth surface fixed by α in V, embedded via j;
- $\beta\colon \widetilde{V} \to V$ the blow-up of V along Σ;
- $\widetilde{E}$ the exceptional divisor over Σ;
- $q\colon V \to V/\alpha$ the quotient map; V/α is singular in $q(\Sigma)$;
- $X = \widetilde{V}/\widetilde{\alpha}$, where $\widetilde{\alpha}$ is the involution induced by α on $\tilde{V}$ and $\pi\colon \widetilde{V} \to X$ is the quotient map;
- $\beta'\colon X \to V/\alpha$ the blow-up of V/α in its singular locus, $q(\Sigma)$.

The same proof as in Theorem 3.6 yields $c_1(X) = 0$, as we already know in all cases that interest us, *i.e.*, $X = S^{[2]}$, $X = Z_S$ or $X = Y_S$. The map $\pi\colon \widetilde{V} \to X$ is a double cover ramified along $\widetilde{E}$ and branched along the exceptional divisor E of the blow-up $\beta'\colon X \to V/\alpha$. In particular, $\pi(\widetilde{E}) \simeq E$.

Let D be a divisor on V invariant for α; we set $\widetilde{D} := \beta^* D$ and denote by D_X the divisor on X such that $\pi^* D_X = \widetilde{D} = \beta^* D$. For the sake of simplicity, we denote by D also the class of a divisor D in $H^2(X, \mathbb{Q})$.

Lemma 7.1 *If D is big and nef on V, then D_X is big and nef on X.*

Proof The nef (or ample) divisors on V that are invariant for an automorphism $\alpha \in \mathrm{Aut}(V)$, descend to nef (or ample) divisors on the quotient V/α. Moreover, bigness of nef divisors is preserved under finite quotient maps by [35, Proposition 2.61], as

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the sign of the top self intersection does not change. These divisors on the possibly singular quotient V/α induce nef divisors on X, and bigness is a birational invariant (see *e.g.*, [35, Definition 2.59 and Lemma 2.60]). ∎

For a divisor D on a fourfold X with $c_1(X) = 0$, the Riemann–Roch formula (see for example [24, Corollary 15.2.1]) is

$$\chi(D) = \frac{D^4}{24} + \frac{1}{24}D^2.c_2(X) + \chi(\mathcal{O}_X) \tag{7.1}$$

If X is Calabi–Yau, $\chi(\mathcal{O}_X) = 2$, and if $X = S \times S$, $\chi(\mathcal{O}_X) = 4$.

Moreover, for a divisor D on a fourfold of $K3^{[2]}$ type, the Riemann–Roch formula can be written in terms of the Beauville–Bogomolov–Fujiki quadratic form q on $H^2(S^{[2]}, \mathbb{Z})$ (see [25, Example 23.19]):

$$\chi(D) = \tfrac{1}{8}(q(D) + 4)(q(D) + 6).$$

We want now to compute $\chi(D_X)$ in terms of $\chi(D)$, and to do so we need to understand Chern classes of X in terms of Chern classes of V.

Proposition 7.2 *Under the above assumptions, up to torsion we have*

$$c_2(X) = \tfrac{1}{2}\pi_*\beta^* c_2(V) + \tfrac{1}{2}\pi_*\beta^* j_*[\Sigma] - \pi_*(\widetilde{E}^2).$$

Proof We apply the theory explained in [20, §3.5] (see also [15]): note that π is a double cover branched along the smooth divisor E. Then there is $L \in \operatorname{Pic}(X)$ such that $L^{\otimes 2} = \mathcal{O}(E)$ and $\pi_* T_{\widetilde{V}} = (T_X \otimes L^{-1}) \oplus T_X(\log E)$, and we have the short exact sequence

$$0 \longrightarrow T_X(\log E) \longrightarrow T_X \longrightarrow N_{E|X} \longrightarrow 0.$$

Since $c_1(X) = 0$, we have $c_1(T_X(\log E)) = -c_1(N_{E|X}) = -c_1(\mathcal{O}_E(E))$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_E(E) \longrightarrow 0,$$

we deduce $c_1(\mathcal{O}_E(E)) = E$, $c_2(\mathcal{O}_E(E)) = 0$.

Hence, $c_1(T_X(\log E)) = -E$ and

$$c_2\big(T_X(\log E)\big) = c_2(X) - c_2(N_{E|X}) - c_1(N_{E|X})c_1\big(T_X(\log E)\big) = c_2(X) + E^2.$$

Next, we use the following formula for the Chern character ch of the tensor product of a vector bundle W and a line bundle L: $\operatorname{ch}(W \otimes L) = \operatorname{ch}(W) \cdot \operatorname{ch}(L)$, from which:

$$\begin{aligned}
c_1(T_X \otimes L^{-1}) &= \operatorname{rank} T_X \cdot c_1(L^{-1}) + c_1(X) = -4c_1(L) = -2E,\\
c_2(T_X \otimes L^{-1}) &= c_2(X) + 3c_1(X)c_1(L^{-1}) + \tbinom{4}{2}c_1(L^{-1})^2 = c_2(X) + \tfrac{3}{2}E^2.
\end{aligned}$$

And we thus obtain

$$\begin{aligned}
c_1(\pi_* T_{\widetilde{V}}) &= c_1(T_X \otimes L^{-1}) + c_1\big(T_X(\log E)\big) = -3E,\\
c_2(\pi_* T_{\widetilde{V}}) &= c_2(T_X \otimes L^{-1}) + c_2\big(T_X(\log E)\big) + c_1(T_X \otimes L^{-1})c_1\big(T_X(\log E)\big)\\
&= 2c_2(X) + \tfrac{9}{2}E^2.
\end{aligned}$$

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Since π is finite, $R^i\pi_* T_{\widetilde{V}} = 0$ for $i > 0$, and we can apply Grothendieck–Riemann–Roch theorem [24, Theorem 15.2]:

$$\mathrm{ch}(\pi_* T_{\widetilde{V}})\,\mathrm{Td}(T_X) = \pi_*\big(\,\mathrm{ch}(\widetilde{V})\,\mathrm{Td}(T_{\widetilde{V}})\big),\ \ \textit{i.e.},$$

$$[8 - 3E + \tfrac{1}{2}(-3E)^2 - 2c_2(X) - \tfrac{9}{2}E^2 + \dots][1 + \tfrac{1}{12}c_2(X) + \dots] =$$
$$\pi_*\Big[\Big(4 + c_1(\widetilde{V}) + \tfrac{1}{2}c_1(\widetilde{V})^2 - c_2(\widetilde{V}) + \dots\Big)\Big(1 + \tfrac{1}{2}c_1(\widetilde{V}) + \tfrac{1}{12}c_1(\widetilde{V})^2 + \tfrac{1}{12}c_2(\widetilde{V}) + \dots\Big)\Big]$$

which yields in degree one $\pi_* c_1(\widetilde{V}) = -E$ and in degree two

$$c_2(X) = -\pi_*\big(c_1(\widetilde{V})^2\big) + \tfrac{1}{2}\pi_*\big(c_2(\widetilde{V})\big). \tag{7.2}$$

By $\pi_* c_1(\widetilde{V}) = -E$ and $\pi_*\widetilde{E} = E$, it follows $c_1(\widetilde{V}) = -\widetilde{E}$.

Finally, we remark that, by [24, Example 15.4.3], $c_2(\widetilde{V}) = \beta^* c_2(V) + \beta^* j_*[\Sigma]$. Substituting this in (7.2) one obtains the statement. ∎

Proposition 7.3 *With the notation above, we have*

$$\chi(D_X) = \tfrac{1}{2}\chi(D) + \tfrac{1}{16}(D_{|\Sigma})^2 - \tfrac{1}{2}\chi(\mathcal{O}_V) + \chi(\mathcal{O}_X).$$

Proof By Riemann–Roch (7.1), $\chi(D_X) = \frac{1}{24}D_X^4 + \frac{1}{24}D_X^2.c_2(X) + \chi(\mathcal{O}_X)$.

By [16, Proposition 1.10], $\widetilde{D}^4 = (\pi^* D_X)^4 = 2D_X^4$, and $\widetilde{D}^4 = D^4$, hence $D_X^4 = \frac{1}{2}D^4$.

We need now to compute $D_X^2.c_2(X)$. We use Proposition 7.2, the projection formula [24, Proposition 8.3(c)] and the fact that $\pi^* E = 2\widetilde{E}$ and $\pi_*\widetilde{E} = E$, getting:

$$D_X^2.\pi_*\beta^* c_2(V) = \pi_*\big(\widetilde{D}^2.\beta^* c_2(V)\big) = D^2.c_2(V),$$
$$D_X^2.\pi_*\beta^* j_*[\Sigma] = \pi_*\big(\widetilde{D}^2.\beta^* j_*[\Sigma]\big) = \pi_*\big(\beta^*(D^2.j_*[\Sigma])\big) = \pi_*\big(\beta^*((D_{|\Sigma})^2)\big),$$
$$D_X^2.\pi_*(\widetilde{E}^2) = \pi_*(\widetilde{D}^2.\widetilde{E}^2) = -\pi_*\big(\beta^*((D_{|\Sigma})^2)\big).$$

where the last equality follows from [1, Lemma 1.1]. We plug everything into (7.1) and obtain:

$$\chi(D_X) = \tfrac{1}{48}D^4 + \tfrac{1}{48}D^2.c_2(V) + \tfrac{1}{16}(D_{|\Sigma})^2 + \chi(\mathcal{O}_X).$$ ∎

Let H_S be a divisor on S. In the sequel we will use the following notation, building up on the notation of Diagram (5.2):

- $H_{i,S} \in \mathrm{NS}(S \times S)$ is the divisor $p_i^*(H_S)$ where $p_i\colon S \times S \to S$ is the projection on the i-th factor of $S \times S$;
- $H_{S^{[2]}}$ is the divisor on $S^{[2]}$, (resp. Z_S) such that $\pi_1^*(H_{S^{[2]}}) = \beta_\Delta^*(H_{1,S} + H_{2,S}) \subset \mathrm{NS}(\widetilde{S \times S})$;

If, moreover, H_S is invariant for ι_S we can also define the following divisors on Z_S and Y_S:

- H_Z is the divisor on Z_S such that $\pi_3^*(H_Z) = \beta^*_{\mathrm{Fix}_{\iota_S \times \iota_S}(S \times S)}(H_{1,S} + H_{2,S}) \subset \mathrm{NS}(\widetilde{S \times S})$;
- H_Y is the divisor on Y_S such that $\pi_2^*(H_Y) = \beta^*_{\mathrm{Fix}_{\iota_S^{[2]}}(S^{[2]})}(H_{S^{[2]}}) \subset \mathrm{NS}(\widetilde{S^{[2]}})$;
- $H_{1,Z}$, $H_{2,Z}$ are the divisors on Z_S such that $\pi_3^*(H_{i,Z}) = \beta^*_{\mathrm{Fix}_{\iota_S \times \iota_S}(S \times S)}(H_{i,S}) \subset \mathrm{NS}(\widetilde{S \times S})$.

A straightforward application of Lemma 7.1 yields the following corollary.

Corollary 7.4 *Let H_S be an ample (or big and nef) divisor on S. The divisor $H_{1,S}+H_{2,S}$ is ample (or big and nef) on $S \times S$, and $H_{S^{[2]}}$ is big and nef on $S^{[2]}$.*

If, moreover, we assume that $H_S \in \mathrm{NS}(S)$ is invariant for ι_S, then H_Z and H_Y are big and nef divisors.

To avoid confusion, in the following theorem we will denote the self intersection product between divisors by $(H \cdot H)$ (and not by H^2). Moreover, we will use the following notation: $h_{\Sigma_{S\times S}}$ (resp. $h_{\Sigma_{1,S\times S}}$, h_{Σ_Z}) to denote the number of points in the intersection of the surface $\Sigma_{S\times S} = \mathrm{Fix}_{\iota_S}(S)\times \mathrm{Fix}_{\iota_S}(S) \subset S\times S$ (resp. $\Sigma_{1,S\times S} = \mathrm{Fix}_{\iota_S}(S)\times \mathrm{Fix}_{\iota_S}(S) \subset S\times S$, $\Sigma_Z = \mathrm{Fix}_{\sigma_Z}(Z) \subset Z$) and a surface equivalent to $(H_{1,S}+H_{2,S})^2$ (resp. $H_{1,S}^2$, H_Z^2).

Theorem 7.5 *Let H be a nef and big ι_S-invariant divisor on S. Then*

$$h^0(H_{S^{[2]}}) = \tfrac{1}{8}\Big((H\cdot H)+4\Big)\Big((H\cdot H)+6\Big),$$
$$h^0(H_Z) = \tfrac{1}{2}\Big(h^0(H)\Big)^2 + \tfrac{1}{16}h_{\Sigma_{S\times S}},$$
$$h^0(H_Y) = \tfrac{1}{2}h^0(H_Z) + \tfrac{1}{16}h_{\Sigma_Z} + 1 = \tfrac{1}{4}\Big(h^0(H)\Big)^2 + \tfrac{1}{32}h_{\Sigma_{S\times S}} + \tfrac{1}{16}h_{\Sigma_Z} + 1.$$

Proof Since H is big and nef, by the Kawamata–Viehweg vanishing theorem $\chi(H) = h^0(H)$. Similarly, by Corollary 7.4, $\chi(H_X) = h^0(H_X)$ for $X = S^{[2]}, Z_S, Y_S$ since these divisors are big and nef. Now the theorem is a trivial application of Proposition 7.3. ∎

Remark 7.6 The divisor $H_{1,Z}$ is not necessarily big and nef, thus we cannot assume that $\chi(H_{1,Z}) = h^0(H_{1,Z})$. However, one can compute $\chi(H_{1,Z})$ by Riemann–Roch and Proposition 7.3. Thus, one obtains $\chi(H_{1,Z}) = \frac{1}{2}\chi(H) + \frac{1}{16}h_{\Sigma_{1,S\times S}}$.

Remark 7.7 The map induced by the linear systems of H_X, for $X = S^{[2]}, Z_S, Y_S$ and $H_{1,Z}$ contracts the exceptional divisors introduced by the blow-up β' on these varieties. So all of them factorizes through the singular models described in diagram (6.1). In particular, $\varphi_{|H_{S^{[2]}}|}$ defines a map on $S^{(2)}$; $\varphi_{|H_Z|}$ and $\varphi_{|H_{1,Z}|}$ define maps on $(S\times S)/(\iota_S \times \iota_S)$, and $\varphi_{|H_Y|}$ defines a map on $S^{(2)}/\iota_S^{(2)} \simeq (S\times S)/\langle \sigma, \iota_S\times\iota_S\rangle$. The target spaces of the map $\varphi_{|H_X|}$, for $X = S^{[2]}$ (resp. Z_S, Y_S) is a copy of $W^{(2)}$ (resp. $W \times W$, $W^{(2)}$) embedded in a projective space.

7.2 ι_S is the Covering Involution of the $2{:}1$ Map $S \to \mathbb{P}^2$

Let us now assume that ι_S fixes on S one curve C of genus 10. In this case, generically, $\mathrm{NS}(S) \simeq \langle 2\rangle \simeq \mathbb{Z}H$ and $\varphi_{|H|}\colon S \to \mathbb{P}^2$ is a $2{:}1$ cover branched along a smooth plane sextic denoted by B. So H is an ample divisor, $h^0(H) = 3$, $W = S/\iota_S \simeq \mathbb{P}^2$ and the class of C in $\mathrm{NS}(S)$ is $3H$. The automorphisms group $\mathrm{Aut}(S\times S)$ is $\mathcal{D}_8$ and the admissible quotients are described in Section 6.2.

The multiplication map $H^0(S,H)\otimes H^0(S,H)\to H^0(S\times S,p_1^*(H)+p_2^*(H))$ is an isomorphism by the Kunneth formula for cohomology of sheaves [34, Proposition 9.2.4] and hence the target space of $\varphi_{|H_{1,S}+H_{2,S}|}$ is the Segre embedding of the self product of the target space of $\varphi_{|H|}$.

Let us denote by $i_2\colon \mathbb{P}^2\times\mathbb{P}^2\to\mathbb{P}^5$ the map

$$\big((x_0:x_1:x_2),(y_0:y_1:y_2)\big)\mapsto(x_0y_0:x_0y_1+x_1y_0:x_0y_2+x_2y_0:x_1y_1:x_1y_2+x_2y_1:x_2y_2)$$

which exhibits $(\mathbb{P}^2)^{(2)}$ as subvariety of $\mathbb{P}^5$. We observe that i_2 is induced by the Segre embedding.

Proposition 7.8 *Let S be a $\langle 2\rangle$-polarized K3 surface as above, and let ι_S be a non-symplectic involution on S fixing one curve C of genus 10, so that $W=S/\iota_S\simeq\mathbb{P}^2$. The map $\varphi_{|H_{S^{[2]}}|}\colon S^{[2]}\to\mathbb{P}^5$ is a generically finite $2^2{:}1$ cover of $i_2((\mathbb{P}^2)^{(2)})\subset\mathbb{P}^5$ totally branched on the image of $i_2(B\times B)$ and whose branch locus of order 2 is the image of $i_2(\mathbb{P}^2\times B)$. It contracts the exceptional divisors, thus it factorizes through $S^{(2)}$ inducing a $4{:}1$ map $S^{(2)}\to W^{(2)}$ (*cf. *diagram (6.1)).*

*The map $\varphi_{|H_Z|}\colon Z_S\to\mathbb{P}^8$ is a generically finite $2{:}1$ map onto $\mathbb{P}^2\times\mathbb{P}^2$ embedded in $\mathbb{P}^8$ by the Segre embedding. The branch locus of $\varphi_{|H_Z|}\colon Z_S\to\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$ is the image of $B\times B$ by the Segre embedding. This map contracts the exceptional divisors, thus it factorizes through $(S\times S)/(\iota_S\times\iota_S)$ inducing the $2{:}1$ map $(S\times S)/(\iota_S\times\iota_S)\to(W\times W)$ (*cf. *diagram (6.1)).*

The map $\varphi_{|H_{1,Z}|}\colon Z_S\to\mathbb{P}^2$ is a fibration whose general fibers are isomorphic to S. This map contracts the exceptional divisors, thus it factorizes through $(S\times S)/(\iota_S\times\iota_S)$ inducing the map $(S\times S)/(\iota_S\times\iota_S)\to W\simeq S/\iota_S$.

*The map $\varphi_{|H_Y|}\colon Y_S\to\mathbb{P}^5$ is a generically finite $2{:}1$ map to $(\mathbb{P}^2)^{(2)}$, embedded in $\mathbb{P}^5$ by i_2. The branch locus of $\varphi_{|H_Y|}$ is $i_2(B\times B)$. This map contracts the exceptional divisors, thus it factorizes through $(S\times S)/\langle\iota_S\times\iota_S,\sigma\rangle$ inducing the $2{:}1$ map $(S\times S)/\langle\iota_S\times\iota_S,\sigma\rangle\to W^{(2)}$ (*cf. *diagram (6.1)).*

Proof By Remark 7.7 one obtains that the map described contracts the exceptional divisors and are defined on the singular models of the 4-folds considered.

By the commutativity of the diagram

$$\begin{tikzcd} S\times S \arrow[r, "\varphi_{|H|}\times\varphi_{|H|}", "4:1"'] \arrow[d, "2:1"'] & \mathbb{P}^2\times\mathbb{P}^2 \arrow[d, "2:1"] \\ S^{(2)} \arrow[r, "(\varphi_{|H|})^{(2)}", "4:1"'] & (\mathbb{P}^2)^{(2)} \end{tikzcd}.$$

we get a map $S^{(2)}\to(\mathbb{P}^2)^{(2)}$ induced by $H_{S^{[2]}}$. The number $h^0(H_{S^{[2]}})=6$ can be computed by Theorem 7.5, so the target space of $\varphi_{|H_{S^{[2]}}|}$ is $\mathbb{P}^5$, and since $H_{S^{[2]}}$ is induced by $p_1^*(H)+p_2^*(H)$ (under the quotient σ), the image of the map $\varphi_{|H_{S^{[2]}}|}$ is the image of $(\mathbb{P}^2)^{(2)}$ by i_2. So the map $\varphi_{|H_{S^{[2]}}|}\colon S^{[2]}\to\mathbb{P}^5$ is the composition

$$S^{[2]}\xrightarrow{\beta'}S^{(2)}\xrightarrow{(\varphi_{|H|})^{(2)}}(\mathbb{P}^2)^{(2)}\xrightarrow{i_2}\mathbb{P}^5.$$

In order to compute $h^0(H_Z)$ one has to recall that the class of $\mathrm{Fix}_{\iota_S}(S)\times\mathrm{Fix}_{\iota_S}(S)\subset S\times S$ is $(3H_{1,S})(3H_{2,S})$, so

$$h_{\Sigma_{S\times S}}=(H_{1,S}+H_{2,S})^2(3H_{1,S})(3H_{2,S})=9(H_{1,S}^3H_{2,S}+H_{2,S}^3H_{1,S}+2H_{1,S}^2H_{2,S}^2).$$

Since $H_{i,S}$ is the pull-back of a divisor on S, $H_{i,S}^3=0$ and $H_{1,S}^2H_{2,S}^2=2^2$ since $H^2=2$ on S. Thus $h_{\Sigma_{S\times S}}=9\cdot 2\cdot 4=72$. By Theorem 7.5, $h^0(H_Z)=\frac{9}{2}+\frac{72}{16}=9$. By the commutativity of the diagram

$$\begin{tikzcd} & S\times S \arrow[r, "\varphi_{|H|}\times\varphi_{|H|}", "4:1"'] \arrow[d, "2:1"'] & \mathbb{P}^2\times\mathbb{P}^2 \arrow[r, hook, "s"] & \mathbb{P}^8 \\ Z_S \arrow[r, "\beta'"] & (S\times S)/(\iota_S\times\iota_S), \arrow[ur, "2:1", "f"'] & & \end{tikzcd}$$

$\varphi_{|H_Z|}:=s\circ f\circ\beta'$. In order to compute $h^0(H_{1,Z})$, we observe that

$$\pi_3^*\colon H^0(Z_S,H_{1,Z})\to H^0(\widetilde{S\times S},\pi_3^*(H_{1,Z}))$$

is injective. Moreover, we recall that $\beta^*_{\mathrm{Fix}_{\iota_S\times\iota_S}(S\times S)}(H_{1,S})=\pi_3^*(H_{1,Z})$ so that

$$H^0\big(\widetilde{S\times S},\pi_3^*(H_{1,Z})\big)\simeq H^0\big(\widetilde{S\times S},\beta^*_{\mathrm{Fix}_{\iota_S\times\iota_S}(S\times S)}(H_{1,S})\big)\simeq H^0(S,H_S).$$

Since the sections in $H^0(S,H_S)$ are invariant for ι_S^*, their pullbacks by $\beta_{\mathrm{Fix}_{\iota_S\times\iota_S}(S\times S)}$ to $\widetilde{S\times S}$ descend to sections in $H^0(Z_S,H_{1,Z})$. Hence, π_3^* is an isomorphism, and $h^0(H_{1,Z})=h^0(H_S)=3$. So, the target space of the map $\varphi_{|H_{1,Z}|}$ is $\mathbb{P}^2$. The map $\varphi_{|H_{1,Z}|}$ contracts the exceptional divisors of $Z_S\to(S\times S)/(\iota_S\times\iota_S)$, so it factorizes through a map $g\colon(S\times S)/(\iota_S\times\iota_S)\to\mathbb{P}^2$. The properties of $\varphi_{|H_{1,Z}|}$ follow by the commutativity of the following diagram:

$$\begin{tikzcd} S\times S \arrow[r, "/\iota_S\times\mathrm{id}"] \arrow[d] \arrow[dr, "f"] & S/\iota_S\times S\ . \arrow[d] \\ (S\times S)/(\iota_S\times\iota_S) \arrow[r, "g"] & S/\iota_S \end{tikzcd}$$

Indeed, f is a fibration whose general fibers are isomorphic to S and induces the map g.

In order to compute $h^0(H_Y)$, one has to recall that σ_Z fixes on Z_S the image of $\Delta_S\subset S\times S$ and the image of $\Gamma_S\subset S\times S$. First, we observe that on $S\times S$ it holds $(H_{1,S}+H_{2,S})^2\Delta_S=(H_{1,S}+H_{2,S})^2\Gamma_S=4(H\cdot H)$, which in our case implies $(H_{1,S}+H_{2,S})^2\Delta_S=(H_{1,S}+H_{2,S})^2\Gamma_S=8$. Then we observe that, since H is a movable divisor, generically the points in $(H_{1,S}+H_{2,S})^2\Delta_S$ and $(H_{1,S}+H_{2,S})^2\Gamma_S$ are not contained in the fixed locus of $\iota_S\times\iota_S$, the involution $\iota_S\times\iota_S$ is non-trivial on these points and the quotient by $\iota_S\times\iota_S$ identifies pairs of these points. Hence the intersection between H_Z and the image of Δ_S (resp. Γ_S) in Z_S consists of 4 points and thus $h_{\Sigma_Z}=4+4=8$. By Theorem 7.5, $h^0(H_Y)=\frac{9}{2}+\frac{8}{16}+1=6$. By the commutativity of the diagram,

$$\begin{tikzcd}
S \times S \arrow[rr, "\varphi_{|H|}\times\varphi_{|H|}", "4:1"'] \arrow[d, "2:1"'] & & \mathbb{P}^2 \times \mathbb{P}^2 \arrow[r, "2:1"'] & (\mathbb{P}^2)^{(2)} \arrow[r, hook, "i_2"] & \mathbb{P}^5 \\
(S \times S)/\langle \iota_S \times \iota_S \rangle \arrow[r, "2:1"'] & (S \times S)/\langle \iota_S \times \iota_S, \sigma \rangle \arrow[urr, "g", "2:1"'] & & Y_S \arrow[ll, "\beta'"'] &
\end{tikzcd}$$

$\varphi_{|H_Y|}$ is the composition $i_2 \circ g \circ \beta'$. ∎

7.2.1 A Special Case: S is the Double Cover of $\mathbb{P}^2$ Branched on a Line and a (Possibly Reducible) Quintic

In this case the hypotheses of Section 6.2 are not satisfied, since the fixed locus of ι_S contains at least 2 curves. On the other hand, there is a nice birational description of the Calabi–Yau involved in our construction. So let us assume that S is the minimal resolution of the double cover S' of $\mathbb{P}^2$ branched along a line and a quintic. In this case, W is a blow-up of $\mathbb{P}^2$, and if the quintic is smooth and intersects the line transversally, it is a blow-up of $\mathbb{P}^2$ in 5 (collinear) points. We denote by $p_i\colon S \times S \to S$ the i-th projection. An equation of a birational (singular) model of $S \simeq p_1(S \times S)$ is given by

$$X^2 = x_0 f_5(x_0 : x_1 : x_2),$$

which exhibits S' as double cover of $\mathbb{P}^2_{(x_0:x_1:x_2)}$. Let us denote by $Y^2 = y_0 f_5(y_0 : y_1 : y_2)$ the analogous birational equation for $S \simeq p_2(S \times S)$. The action of $\iota_S \times \iota_S$ is given by $(X, (x_0 : x_1 : x_2); Y, (y_0 : y_1 : y_2)) \to (-X, (x_0 : x_1 : x_2); -Y, (y_0 : y_1 : y_2))$.

We now consider the affine equation of $p_1(S \times S)$ and $p_2(S \times S)$ obtained by putting $x_0 = 1$ and $y_0 = 1$. The invariant functions for $\iota_S \times \iota_S$ are $Z := XY$, $a_1 = x_1$, $a_2 = x_2$, $a_3 = y_1$, $a_4 = y_2$. Then a birational equation for $(S \times S)/(\iota_S \times \iota_S)$ (and thus a birational model of Z_S) is given by

$$Z^2 = f_5(1 : a_1 : a_2) f_5(1 : a_3 : a_4).$$

This equation exhibits Z_S as a double cover of the complement of $\{a_0 = 1\}$ in $\mathbb{P}^4_{(a_0:a_1:a_2:a_3:a_4)}$. It is clearly possible to introduce the variable a_0 in order to obtain a homogenous polynomial $F_{10}(a_0 : a_1 : a_2 : a_3 : a_4)$ of degree 10 that reduces to $f_5(1 : a_1 : a_2) f_5(1 : a_3 : a_4)$ if $a_0 = 1$. So Z_S is birational to a 2:1 cover of $\mathbb{P}^4$ branched along a (possibly singular) 3-fold of degree 10, denoted by B.

On $\mathbb{P}^4$, the map $\sigma_{\mathbb{P}^4}\colon (a_0 : a_1 : a_2 : a_3 : a_4) \mapsto (a_0 : a_3 : a_4 : a_1 : a_2)$ acts preserving the homogeneous polynomial $F_{10}(a_0 : a_1 : a_2 : a_3 : a_4)$. The map σ_Z is induced on Z_S by the projective map $\sigma_{\mathbb{P}^4}$. Denoted by π the quotient map $\pi\colon \mathbb{P}^4 \to \mathbb{P}^4/\sigma_{\mathbb{P}^4}$, we obtain that Z_S/σ_Z and Y_S are birational to a double cover of $\pi(\mathbb{P}^4)$ branched over $\pi(V(F_{10}(a_0 : a_1 : a_2 : a_3 : a_4)))$ (where $V(F_{10}(a_0 : a_1 : a_2 : a_3 : a_4))$ is the zero locus of the polynomial F_{10}).

In order to better describe $\mathbb{P}^4/\sigma_{\mathbb{P}^4}$ it is convenient to apply the changes of coordinates $b_0 := a_0$, $b_1 := (a_1 + a_2)/2$, $b_2 := (a_3 + a_4)/2$, $b_3 := (a_1 - a_2)/2$, $b_4 := (a_3 - a_4)/2$. With these new coordinates, $\sigma_{\mathbb{P}^4}$ is the map $\sigma_{\mathbb{P}^4}\colon (b_0 : b_1 : b_2 : b_3 : b_4) \mapsto (b_0 : b_1 : b_2 : -b_3 : -b_4)$, and $\mathbb{P}^4/\sigma_{\mathbb{P}^4}$ is mapped to the 4-dimensional singular subspace of $\mathbb{P}^8_{(z_0:\dots:z_8)}$ given by the set-theoretic complete intersection of 4 singular quadrics

$M := V(z_0z_1 = z_5^2,\ z_0z_2 = z_6^2,\ z_1z_2 = z_7^2,\ z_3z_4 = z_8^2)$, where $z_i := b_i^2$ for $i = 0,\dots,4$, $z_5 := b_0b_1$, $z_6 := b_0b_2$, $z_7 := b_1b_2$, $z_8 := b_3b_4$. The space M is singular, $\mathrm{Sing}(M) = (M \cap V(z_0,z_1,z_2,z_5,z_6,z_7)) \cup (M \cap V(z_3,z_4,z_8))$.

The image of $V(F_{10})$ under the quotient map is a 3-fold T of degree 5 in $\mathbb{P}^8$. The 4-fold Y_S is birational to a double cover of M branched along $M \cap T$.

7.3 ι_S is the Covering Involution of the $2{:}1$ Map $S \to \mathbb{P}^1 \times \mathbb{P}^1$

Let us now assume that ι_S fixes on S one curve of genus 9 isomorphic to the branch curve B, which has bidegree $(4,4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, generically, $\mathrm{NS}(S) \simeq U(2) \simeq \mathbb{Z}l \oplus \mathbb{Z}m$, and $\varphi_{|l+m|}\colon S \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is a $2{:}1$ map on the image, and $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ by the Segre embedding $s_{1,1}$. We denote $H := l + m \in \mathrm{NS}(S)$. The ramification divisor of the $2{:}1$ cover $S \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is thus represented by $2H$. We observe that $W \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

By [53, Lemma 4.6], if S is a K3 surface with $\mathrm{NS}(S) \simeq U(2)$, then $\mathrm{Aut}(S) \simeq \mathbb{Z}/2\mathbb{Z}$. So $\mathrm{Aut}(S \times S)$ is $\mathcal{D}_8$ and the admissible quotients are described in Section 6.2.

Let us denote by $i_3\colon \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^9$ the map

$$\begin{aligned}\big((x_0{:}x_1{:}x_2{:}x_3),(y_0{:}y_1{:}y_2{:}y_3)\big) &\longmapsto \\ &(x_0y_0{:}x_0y_1 + x_1y_0{:}x_0y_2 + x_2y_0{:}x_0y_3 + x_3y_0{:}x_1y_1{:}x_1y_2+\\ &\qquad x_2y_1{:}x_1y_3 + x_3y_1{:}x_2y_2{:}x_2y_3 + x_3y_2{:}x_3y_3),\end{aligned}$$

which exhibits $(\mathbb{P}^3)^{(2)}$ as a subvariety of $\mathbb{P}^9$.

Proposition 7.9 *Let S be a $U(2)$-polarized K3 surface as above, and let ι_S be a non-symplectic involution on S fixing one curve of genus 9, so that $W = S/\iota_S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The map $\varphi_{|H_{S^{[2]}}|}\colon S^{[2]} \to \mathbb{P}^9$ is a generically finite $2^2{:}1$ cover of $i_3((\mathbb{P}^3)^{(2)}) \subset \mathbb{P}^9$ totally branched on the image of $i_3(s_{1,1}(B) \times s_{1,1}(B))$ and whose branch locus of order 2 is the image of $i_3(s_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \times s_{1,1}(B))$. It contracts the exceptional divisors, thus it factorizes through $S^{(2)}$ inducing a $4{:}1$ map $S^{(2)} \to W^{(2)}$ (*cf. *diagram (6.1)).*

*The map $\varphi_{|H_Z|}\colon Z_S \to \mathbb{P}^{15}$ is a generically finite $2{:}1$ map onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \times \mathbb{P}^3$ embedded in $\mathbb{P}^{15}$ by the Segre embedding. The branch of $\varphi_{|H_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$ is the image of $B \times B$ by the Segre embedding. This map contracts the exceptional divisors, thus it factorizes through $(S \times S)/(\iota_S \times \iota_S)$ inducing the $2{:}1$ map $(S \times S)/(\iota_S \times \iota_S) \to (W \times W)$ (*cf. *diagram (6.1)).*

*The map $\varphi_{|H_Y|}\colon Y_S \to \mathbb{P}^9$ is a generically finite $2{:}1$ map to $(\mathbb{P}^3)^{(2)}$, embedded in $\mathbb{P}^9$ by i_3. The branch locus is $i_3(B \times B)$.This map contracts the exceptional divisors, thus it factorizes through $(S \times S)/\langle \iota_S \times \iota_S, \sigma\rangle$ inducing the $2{:}1$ map $(S \times S)/\langle \iota_S \times \iota_S, \sigma\rangle \to W^{(2)}$ (*cf. *diagram (6.1)).*

The proof is analogous to that of Proposition 7.8, so we omit it. On the other hand, in this case some explicit equations can be written and can be used to describe some maps, associated with divisors that are not necessarily big and nef.

The notation will be analogous to the one introduced above, where we substitute H by l or m. We observe that l and m are not big divisors on S, thus l_X, m_X for $X = S \times S, Z_S, Y_S, S^{[2]}$ are not necessarily big divisors.

An equation for the surface S, which exhibits S as a double cover of $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(x_2:x_3)}$, is

$$X^2 = f_{4,4}\big((x_0:x_1):(x_2:x_3)\big),$$

where $f_{4,4}$ is a homogeneous polynomials of bidegree $(4,4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Composing the map $S \to \mathbb{P}^1 \times \mathbb{P}^1$ with the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor, we obtain a map $S \to \mathbb{P}^1_{(x_0:x_1)}$, which is a genus 1 fibration. The fibers over a point $(\overline{x_0}:\overline{x_1})$ is the genus one curve $X^2 = f_{4,4}((\overline{x_0}:\overline{x_1}):(x_2:x_3))$. The map $S \to \mathbb{P}^1_{(x_0:x_1)}$ coincides with the map $\varphi_{|l_2|}$. Similarly one obtains another genus 1 fibration, projecting $\mathbb{P}^1 \times \mathbb{P}^1$ on the second factor. Here we describe some models and fibrations on $S^{[2]}$, Z_S and Y_S induced by the maps $S \to \mathbb{P}^1 \times \mathbb{P}^1$ and $S \to \mathbb{P}^1$. In particular we will prove the following proposition.

Proposition 7.10 *The hyperkähler 4-fold $S^{[2]}$ admits a Lagrangian fibration $f_{S^{[2]}} = \varphi_{|l_{S^{[2]}}|}: S^{[2]} \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ whose general fibers are the product of two non-isogenous elliptic curves.*

The Calabi–Yau 4-fold Z_S admits:

- *a fibration $\varphi_{|l_{1,Z}|}: Z_S \to \mathbb{P}^1$ whose general fibers are Calabi–Yau 3-folds that are the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of 5 curves of tridegree $(1,0,0)$, $(1,0,0)$, $(1,0,0)$, $(1,0,0)$, $(0,4,4)$;*
- *a fibration $\varphi_{|l_{1,Z}+m_{1,Z}|}: Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$ whose general fibers are isomorphic to S;*
- *a fibration $f_Z := \varphi_{|l_Z|}: Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$ whose general fibers are the Kummer surfaces of the product of two non-isogenous elliptic curves;*
- *a fibration $\varphi_{|l_Z+m_{1,Z}|}: Z_S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that the general fibers are smooth irreducible curves of genus 1.*

The Calabi–Yau 4-fold Y_S admits a fibrations $f_Y := \varphi_{|l_Y|}: Y_S \to (\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$ whose fibers are the Kummer surfaces of the product of two non-isogenous elliptic curves. The fibration f_Y is induced on Y_S by both $f_{S^{[2]}}$ and f_Z.

Proof The map $\varphi_{|l_{S^{[2]}}|}: S^{[2]} \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ gives Lagrangian fibrations on $S^{[2]}$ whose fibers are the product of the fibers of $\varphi_{|l_i|}: S \to \mathbb{P}^1$ of each factor in $S \times S$. We denote this fibration by $f_{S^{[2]}}$, following [52, Example 3.5]. The involution $\iota_S^{[2]}$ acts on the fibers of $\varphi_{|l_{S^{[2]}}|}: S^{[2]} \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ preserving each fibers, so it acts as an involution on each fiber. On the other hand, we know that the fixed locus of ${\iota_S}^{[2]}$ consists of two surfaces, one isomorphic to $(\mathrm{Fix}_{\iota_S}(S))^{[2]}$ and one isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The surface $(\mathrm{Fix}_{\iota_S}(S))^{[2]}$ intersects the fiber in 16 points, and indeed ${\iota_S}^{[2]}$ restricts to each fiber to the involution that sends each point of an abelian surface in its opposite. The surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in the fixed locus of $\iota_S^{[2]}$ maps to the singular locus of $(\mathbb{P}^1 \times \mathbb{P}^1)^{(2)}$, so it does not intersect the general fiber. Hence the fibration $\varphi_{|l_{i,S^{[2]}}|}: S^{[2]} \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ induces on $S^{[2]}/{\iota_S}^{[2]}$ a fibration $f: S^{[2]}/{\iota_S}^{[2]} \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ whose general fibers are Kummer surfaces. This fibration extends to a map $f_Y: Y_S \to \mathbb{P}^2 \simeq (\mathbb{P}^1)^{(2)}$ whose general fibers are Kummer surfaces. By construction, the map f_Y is induced on Y by the divisor l_Y, since $f_{S^{[2]}}$ is induced on $S^{[2]}$ by the divisor $l_{S^{[2]}}$.

Let us now consider Z_S. In order to describe the map in the statement, we first give an equation for the surface S, which exhibits S as double cover of $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(x_2:x_3)}$:

$$X^2 = f_{4,4}\big((x_0:x_1):(x_2:x_3)\big),$$

where $f_{4,4}$ is a homogeneous polynomials of bidegree $(4,4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The second copy of S in the product $S \times S$ is given by the equation $Y^2 = f_{4,4}((y_0:y_1):(y_2:y_3))$, which exhibits it as double cover of $\mathbb{P}^1 \times \mathbb{P}^1$.

The divisor $l_Z + m_Z$ is H_Z, and we already observed that $\varphi_{|H_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$. It exhibits Z_S as double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a threefold of multidegree $(4,4,4,4)$ (by [2, Lemma 17.1 Chapter I], this is indeed a 4-fold with a trivial canonical bundle). The involution $\iota_S \times \iota_S$ acts only on the coordinates X and Y, changing the sign, so we choose as invariant functions $Z := XY$, x_i and y_i, $i = 0, \dots, 4$. The equation of Z_S is then

$$Z^2 = f_{4,4}\big((x_0:x_1):(x_2:x_3)\big)\, f_{4,4}\big((y_0:y_1):(y_2:y_3)\big). \tag{7.3}$$

If we project (7.3) to the first three copies of $\mathbb{P}^1$, we obtain a fibration $Z_S \to \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(x_2:x_3)} \times \mathbb{P}^1_{(y_0:y_1)}$ whose general fibers are the genus 1 curves

$$Z^2 = k f_{4,4}\big((\overline{y_0}:\overline{y_1}):(y_2:y_3)\big),$$

where $\overline{y_i}$ are specific value for y_i and k is a constant that depends on the values of x_i. This fibration is induced on Z_S by the map $\varphi_{|l_Z+m_{1,Z}|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

If we project (7.3) to the first two copies of $\mathbb{P}^1$, we obtain a fibration $Z_S \to \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(x_2:x_3)}$ whose general fibers are isomorphic to S (indeed they are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the curve of bidegree $(4,4)$ given by $f_{4,4}((y_0:y_1):(y_2:y_3))$). This fibration is induced on Z_S by the map $\varphi_{|l_{1,Z}+m_{1,Z}|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$. By definition, $l_{1,Z} + m_{1,Z} = H_{1,Z}$.

If we project (7.3) to the first and to the third copy of $\mathbb{P}^1$, we obtain a fibration $f_Z\colon Z_S \to \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$ whose general fibers are K3 surfaces, not isomorphic to S. The fibers over a general point $((\overline{x_0}:\overline{x_1}):(\overline{y_0}:\overline{y_1}))$, are the double covers of $\mathbb{P}^1_{(x_2:x_3)} \times \mathbb{P}^1_{(y_2:y_3)}$ branched along the curve of bidegree $(4,4)$:

$$f_4\big((\overline{x_0}:\overline{x_1}):(x_2:x_3)\big)\, f_4\big((\overline{y_0}:\overline{y_1}):(x_2:x_3)\big).$$

But this curve splits in the union of 8 curves, 4 of bidegree $(1,0)$ and 4 of bidegree $(0,1)$. So the branch locus of this double cover is singular in 16 points, and thus the K3 surfaces obtained by blowing up these points contain 16 disjoint rational curves. This suffices to conclude that each fiber of the fibration f_Z is a Kummer surface (see [42]). To be more precise, the general fiber is a K3 surface that contains 24 rational curves (the pull back of the eight curves in the branch locus of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ and the 16 curves that resolve the singularities) that form a double Kummer configuration (see [47]). We conclude that the general fibers are Kummer surfaces of the product of two non-isogenous elliptic curves. The fibration $f_Z\colon Z_S \to \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$ is induced on Z_S by the map $\varphi_{|l_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$. We observe that the map σ_Z acts on the basis of this fibration, by switching the two copies of $\mathbb{P}^1$, and does not act on the fibers.

If we project (7.3) to the first copy of $\mathbb{P}^1$, we obtain a fibration $Z_S \to \mathbb{P}^1_{(x_0:x_1)}$ whose general fibers are Calabi–Yau 3-folds that are double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve of multidegree $(4,4,4)$ that indeed splits into the union of 5 curves of multidegree $(1,0,0)$,$(1,0,0)$,$(1,0,0)$,$(1,0,0)$,$(0,4,4)$, respectively. This fibration is induced on Z_S by the map $\varphi_{|l_{1,Z}|}\colon Z_S \to \mathbb{P}^1$.

The map

$$f_Y := \varphi_{|l_Y|}\colon Y_S \longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1)^{(2)}$$

is the fibration induced both by $\varphi_{|l_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$ on the quotient $S^{(2)}/\iota_S^{(2)}$ and by $\varphi_{|l_Z|}\colon Z \to \mathbb{P}^1 \times \mathbb{P}^1$ on the quotient Z_S/σ_Z, by the definition of the divisors l_Y, $l_{S^{[2]}}$ and l_Z.

Generically, the fiber of f_Y are the Kummer surfaces, obtained either as quotients of the fibers of the fibration $f_Z\colon Z_S \to \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$ by the involution acting on the fiber of the fibration as $((x_0:x_1):(y_0:y_1)) \mapsto ((y_0:y_1):(x_0:x_1))$ or as the quotients of the fibers of the fibration $f\colon S^{[2]} \to \mathbb{P}^2$ by the involution acting on the general fiber, which is an abelian surface, as the involution that sends each point in its opposite. ∎

7.4 ι_S is the Elliptic Involution on a General Elliptic Fibration on the K3 Surface S

In this case, S has an elliptic fibration with 24 fibers of type I_1 and $\mathrm{NS}(S) \simeq U$, ι_S restricts to the elliptic involution on each fiber of the elliptic fibration and $W \simeq \mathbb{F}_4$; see *e.g.*, [39, Section III.2]. Generically, $\mathrm{NS}(S)$ is generated by the class of a fiber, F, and by the class of the zero section, O, whose intersection properties are $F^2 = 0$, $O^2 = -2$, $FO = 1$. The divisor F is nef, and it is such that $f = \varphi_{|F|}\colon S \to \mathbb{P}^1$ is the elliptic fibration, so F is not big. We will denote by F_p the fiber of f over the point p, *i.e.*, $F_p \simeq f^{-1}(p) \subset S$. The involution ι_S fixes 2 curves; one is the rational curve O, section of the fibration; the other is a trisection, branched with multiplicity 2 on each singular fibers. It is denoted by C, it has genus 10 and its class in $\mathrm{NS}(S)$ is $6F + 3O$. We will denote it by $H := 4F + 2O$. The map $\varphi_{|H|}\colon S \to \mathbb{P}^5$ is a 2:1 map onto the image, which is a cone over a normal quartic rational curve; see *e.g.*, [39, Section III.2].

Proposition 7.11 *The map $\varphi_{|F_{S^{[2]}}|}\colon S^{[2]} \to (\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$ is a fibration whose general fibers are products of two (non-isogenous) elliptic curves.*

The map $\varphi_{|F_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is a fibration whose general fibers are Kummer surfaces of the product of two (non-isogeneous) elliptic curves.

The map $\varphi_{|F_Y|}\colon Y_S \to (\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$ is a fibration whose general fibers are Kummer surfaces of the product of two (non-isogeneous) elliptic curves.

The fibration $\varphi_{|F_{1,Z}|}\colon Z_S \to \mathbb{P}^1$ is a fibration whose general fiber is a Calabi–Yau 3-fold of Borcea–Voisin type.

Proof The 4-fold $S \times S$ admits a fibration $f \times f\colon S \times S \to \mathbb{P}^1 \times \mathbb{P}^1$ whose fiber over the point (p,q) is the product $F_p \times F_q$, and it coincides with the map $\varphi_{|F_{1,S\times S}+F_{2,S\times S}|}\colon S\times S \to \mathbb{P}^1 \times \mathbb{P}^1$. So the general fiber of $S \times S \to \mathbb{P}^1 \times \mathbb{P}^1$ is an abelian surface, which is the product of two elliptic curves (generically non-isogeneous). The section of the fibration f defines a section of $f \times f$, passing through the zero of the abelian surfaces. The involution $\iota_S \times \iota_S$ fixes this section and other three surfaces, which are two 3-sections and one 9-section. The involution $\iota_S \times \iota_S$ on $A \simeq F_p \times F_q$ sends each point to its inverse with respect to the group law. The automorphism σ does not preserve the fibers of the fibration $f \times f$: it acts on the basis, switching the two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ and sending the fiber $F_p \times F_q$ to the fiber $F_q \times F_p$. This allows us to describe the fibrations induced by $\varphi_{|F_{S\times S}|}$ on the quotients of $S \times S$ as follows: the hyperkähler fourfold $S^{[2]}$ naturally admits a Lagrangian fibration $f^{[2]}: S^{[2]} \to \mathbb{P}^2$, whose fibers are generically the product of the corresponding fibers on the K3 surface S. Indeed, $f \times f$ is equivariant with respect to the action of the exchange σ; hence, we get an induced fibration $f^{(2)}$ of $S^{(2)}$ over $(\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$. The so-called natural Lagrangian fibration $f^{[2]}$ is the composition $f^{(2)} \circ \beta_\Delta$, where β_Δ is the resolution $S^{[2]} \to (S \times S)/\sigma$, and it coincides with $\varphi_{|F_{S^{[2]}}|}$. Let $\Delta_S \subset S \times S$ be the diagonal and let E_{Δ_S} be the exceptional divisor on $S^{[2]}$. Moreover, $f^{[2]}(E_{\Delta_S})$ is one-dimensional, so that the general fiber of $f^{[2]}$ does not intersect E_{Δ_S} and is isomorphic to the general fiber of $f^{(2)}$, which is the common image of $F_p \times F_q$ and of $F_q \times F_p$, still isomorphic to $F_p \times F_q$.

The automorphism ι_S acts trivially on the basis of the fibration $f: S \to \mathbb{P}^1$, so the basis of the fibration induced by $f \times f$ on $(S \times S)/(\iota_S \times \iota_S)$ is $\mathbb{P}^1 \times \mathbb{P}^1$. The fiber over the general point $(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1$ of the fibration $(S \times S)/(\iota_S \times \iota_S) \to \mathbb{P}^1 \times \mathbb{P}^1$ are the quotients of the abelian surfaces $F_p \times F_q$, fibers of $f \times f: S \times S \to \mathbb{P}^1 \times \mathbb{P}^1$, by the involution that sends each point in its inverse with respect to the group law. The fibration $(S\times S)/(\iota_S\times\iota_S) \to \mathbb{P}^1\times\mathbb{P}^1$ induces a fibration $Z_S \to \mathbb{P}^1\times\mathbb{P}^1$ whose fiber over a general point (p, q) is the Kummer surface $\mathrm{Km}(F_p\times F_q)$, which is a desingularization of the singular fiber of $(S \times S)/(\iota_S \times \iota_S) \to \mathbb{P}^1 \times \mathbb{P}^1$. By construction, this fibration is given by the map $\varphi_{|F_Z|}: Z_S \to \mathbb{P}^1\times\mathbb{P}^1$. The strict transform of $O\times O$ is a section of the fibration and it meets the general fiber in a rational curve. We recall that a Kummer surface $\mathrm{Km}(A)$ contains 16 disjoint rational curves that are in 1:1 correspondence with the 2-torsion points in A. The "zero section" of the fibration $Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$ is the section that meets the smooth fibers (which are a Kummer surfaces $\mathrm{Km}(F_p \times F_q)$) in the rational point that correspond to the 0 of the abelian surface $F_p \times F_q$. Similarly the strict transform of $O \times C$ (resp. $C \times O$, $C \times C$) meets the smooth fibers in 3 (resp. 3, 9) rational curves, corresponding to other 3 (resp. 3,9) points of order 2 on $F_p \times F_q$.

The automorphism σ_Z acts on the basis of this fibration $\varphi_{|F_Z|}: Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$. Outside of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, σ_Z identifies two fibers (the fiber $\mathrm{Km}(F_p \times F_q)$ with the fiber $\mathrm{Km}(F_q \times F_p)$). This identification sends the 2-torsion point of $F_p \times F_q$ to the one of $F_q \times F_p$. So on Z_S/σ_Z we have a fibration whose general fibers are Kummer surfaces. In particular the map $\varphi_{|F_Y|}: S \to (\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$ defines a fibration whose general fibers are Kummer surfaces of the product of 2 non-isogenous elliptic curves. We observe that the general fibers of this fibration are not isomorphic, but all of them are polarized with the same lattice. This fibration has a section, induced by the section of the fibration $Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$.

The fibration $F_{1,Z}: Z_S \to \mathbb{P}^1_\tau$ exhibits Z_S as a fibration in Calabi–Yau 3-folds, and the fiber over a general point $\tau \in \mathbb{P}^1$ is the Borcea-Voisin of $S \times F_\tau$, *i.e.*, it is the desingularization of $(S \times F_\tau)/(\iota_S \times \iota_{F_\tau})$, where ι_{F_τ} is the elliptic involution on the elliptic curve F_τ, and it is the restriction of ι_S to the fiber F_τ of the fibration $S \to \mathbb{P}^1$. This easily follows by our construction but can also be written explicitly by using the equations

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of the fibrations. Let us write the equation of $S \to \mathbb{P}^1_\tau$ as $y^2 = x^3 + a(\tau)x + b(\tau)$ (where $a(\tau)$ and $b(\tau)$ are polynomials of degree 8 and 12 respectively). Then the second copy of S in $S \times S$ has an equation of the type $v^2 = u^3 + a(s)u + b(s)$, which can be written as $v^2 = (u^3 + a(s:t)uz^2 + b(s:t)z^3)z$. This exhibits this second copy of S as double cover of the Hirzebruch surface $\mathbb{F}_4$ with variables $(s:t:x:z)$ (*cf.* [13]). Since $\iota_S \times \iota_S$ changes the sign of y and v, the functions $Y := yv^3$, $X := xv^2$, τ, t, s are invariant, so with these coordinates the equation for $(S \times S)/(\iota_S \times \iota_S)$ is

$$Y^2 = X^3 + a(\tau)X\big(x^3 + a(s:t)xz^2 + b(s:t)z^3\big)^2 z^2 + b(\tau)\big(x^3 + a(s:t)xz^2 + b(s:t)z^3\big)^3 z^3.$$

For general choices of τ, this equation is the equation of the Borcea-Voisin Calabi–Yau 3-folds given in [13, Section 4.4]. ■

Remark 7.12 The fixed locus of ι_S on S is given by $O \cup C$, and the class of the fixed locus is $O + 6F + 3O = 6F + 4O$. This allows us to compute $\chi(F_Z) = \chi(F_{1,S\times S} + F_{2,S\times S})/2 + h_{\Sigma,Z}/16 = 2 + 2 = 4$. The base of the fibration $\varphi_{|F_Z|}\colon Z_S \to \mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^{\chi(F_Z)-1}$ by the Segre embedding. Similarly, one computes $\chi(F_Y) = 2 + 1 = 3$, $\chi(F_{S^{[2]}}) = 3$, and $\chi(F_{1,Z}) = 2$, and one observes that the bases of the fibrations $\varphi_{|F_Y|}$, $\varphi_{|F_{S^2}|}$ and $\varphi_{|F_{1,Z}|}$ are again $\mathbb{P}^{\chi-1}$. This suggests that χ is equal to h^0 for all the divisors involved in Proposition 7.11.

Proposition 7.13 *The map $\varphi_{|H_{S^{[2]}}|}\colon S^{[2]} \longrightarrow (\mathbb{P}^5)^{(2)} \subset \mathbb{P}^{20}$ is a generically finite $4{:}1$ map onto its image, where the inclusion $(\mathbb{P}^5)^{(2)} \subset \mathbb{P}^{20}$ is given by*

$$i_5\colon \{(x_0:\ldots:x_5),(y_0:\ldots y_5)\} \longmapsto (x_0y_0 : x_0y_1 + x_1y_0 : \ldots : x_iy_j + x_jy_i : \ldots x_5y_5),$$

with $i < j$.

The map $\varphi_{|F_Z|}\colon Z_S \to \mathbb{P}^5 \times \mathbb{P}^5 \subset \mathbb{P}^{35}$ is a generically finite $2{:}1$ map and $\mathbb{P}^5 \times \mathbb{P}^5 \subset \mathbb{P}^{35}$ is given by the Segre embedding.

The map $\varphi_{|F_Y|}\colon Y_S \to \mathbb{P}^{20}$ is a generically finite $2{:}1$ map and the inclusion $(\mathbb{P}^5)^{(2)} \subset \mathbb{P}^{20}$ is given by i_5.

The proof is analogous to the one of Proposition 7.8.

A Hodge Numbers of Calabi–Yau Four-folds Constructed

Here we collect the Hodge numbers of the Calabi–Yau four-folds constructed as quotients of hyperkähler four-folds by non-symplectic involutions, *i.e.*, of the Calabi–Yau four-folds constructed in Sections 4 and 5.1. In Sections 4.2.1 and 4.2.3 we constructed Calabi–Yau four-folds with Hodge numbers:

CY quotients of $K_2(A)$

$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$
9	8	5	75
6	4	4	68
5	3	4	66

CY quotients by Beauville's involution

$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$
2	0	65	312

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In Section 5.1 we constructed Calabi–Yau varieties Y_S starting from a K3 surface S with a non-symplectic involution ι_S whose fixed locus is either empty or contains N curves, and N' depends on the genera of these curves; see Section 5.1. In the following table we list the Hodge numbers of Y_S in terms of (N, N'):

N	N'	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$
0	0	12	0	10	132
1	0	14	0	9	136
1	1	13	1	10	134
1	2	12	2	12	136
1	3	11	3	15	142
1	4	10	4	19	152
1	5	9	5	24	166
1	6	8	6	30	184
1	7	7	7	37	206
1	8	6	8	45	232
1	9	5	9	54	262
1	10	4	10	64	296
2	0	17	0	8	144
2	1	16	2	9	140
2	2	15	4	11	140
2	3	14	6	14	144
2	4	13	8	18	152
2	5	12	10	23	164
2	6	11	12	29	180
2	7	10	14	36	200
2	8	9	16	44	224
2	9	8	18	53	252
2	10	7	20	63	284
3	0	21	0	7	156
3	1	20	3	8	150
3	2	19	6	10	148
3	3	18	9	13	150
3	4	17	12	17	156
3	5	16	15	22	166
3	6	15	18	28	180
3	7	14	21	35	198
4	0	26	0	6	172
4	1	25	4	7	164
4	2	24	8	9	160
4	3	23	12	12	160
4	4	22	16	16	164
4	5	21	20	21	172
4	6	20	24	27	184
5	0	32	0	5	192
5	1	31	5	6	182
5	2	30	10	8	176
5	3	29	15	11	174
5	4	28	20	15	176
5	5	27	25	20	182
5	6	26	30	26	192
6	0	39	0	4	216
6	1	38	6	5	204
6	2	37	12	7	196
6	3	36	18	10	192
6	4	35	24	14	192
6	5	34	30	19	196
6	6	33	36	25	204
7	0	47	0	3	244
7	1	46	7	4	230
7	2	45	14	6	220
7	3	44	21	9	214
8	0	56	0	2	276
8	1	55	8	3	260
8	2	54	16	5	248
9	0	66	0	1	312
9	1	65	9	2	294
9	2	64	18	4	280
10	0	77	0	0	352
10	1	76	10	1	332
10	2	75	20	3	316

One can directly check that there are no mirror pairs in this table, except for the self-mirror Calabi–Yau Y_S associated with the values $N' = N + 1$ for $N = 1, \ldots, 5$ (compare with Section 5.6).

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