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The Blow-up Lemma

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To the memory of Paul Erdős

Extremal graph theory has a great number of conjectures concerning the embedding of large sparse graphs into dense graphs. Szemerédi's Regularity Lemma is a valuable tool in finding embeddings of small graphs. The Blow-up Lemma, proved recently by Komlós, Sárközy and Szemerédi, can be applied to obtain approximate versions of many of the embedding conjectures. In this paper we review recent developments in the area.

This paper is based on my lectures at the DIMANET Mátraháza Workshop, October 22–28, 1995. On my transparencies, I wrote, 'For more details see the survey of Komlós–Simonovits in *Paul Erdős is 80* [20]. Solutions to the conjectures mentioned today will be presented in the Bolyai volume *Paul Erdős is 90*.' As you can tell, at that time I expected EP (who was sitting in the front row) to live to be 90 and more. The loss is obvious to all of us, and it will certainly deepen further in time.

1. Introduction

Our concern in this paper is how Szemerédi's Regularity Lemma can be applied to packing (or embedding) problems. In particular, we discuss a lemma that is a powerful weapon in proving the existence of embeddings of large sparse graphs into dense graphs.

After a brief passage in which we fix the notation, we start in Section 2 by recalling some of the fundamental results and conjectures. Section 3 is about the Regularity Lemma itself; we also demonstrate its power by reconstructing the elegant proof of Ruzsa and Szemerédi for Roth's theorem on arithmetic progressions of length 3.

In Section 4 we show how the Regularity Lemma can be applied to embedding a large sparse graph into a dense graph. We describe a method of constructing such an embedding in five separate phases; in this way we isolate the application of the Regularity Lemma from the other issues that typically need to be dealt with.

Section 5 is about the Blow-up Lemma, a result designed to overcome some of the unpleasant technical difficulties often arising in applications of Szemerédi's Regularity Lemma. The main benefit of the Blow-up Lemma is that in suitable contexts we may regard sufficiently regular pairs of subsets of vertices as spanning a complete bipartite subgraph. We will illustrate this principle by a concrete example and mention recent progress concerning major problems in the area.

Definitions and notation

We will write $v(G)$ for the order of G, G_n for an n-graph (order n), and $e(G)$ for the size of G, that is, for the number of edges in G; we denote the minimum, the maximum, and the average degree in G by $\delta(G), \Delta(G)$, and $t(G)$, respectively. Also, $\deg(v)$ is the degree of v, the number of edges incident with v, and $\deg(v, X)$ is the number of edges from v to X. We write $\chi(G)$ for the chromatic number of G, and $\overline{G}$ for the complement of G. Given a set U of vertices, $G|_U$ is the restriction of G to U. As usual, $H \subset G$ means that H is a subgraph of G, or at least that G has a subgraph isomorphic to H; we also say that H embeds into G, that is, there is a one-to-one map (injection) $\varphi : V(H) \to V(G)$ such that $\{x, y\} \in E(H)$ implies $\{\varphi(x), \varphi(y)\} \in E(G)$. We denote by $\|H \to G\|$ the number of (labelled, but not necessarily induced) isomorphic copies of H in G. The graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ *pack* (or *can be packed*) if they can be placed without overlapping edges. More precisely, they pack if there is a bijection $\varphi : V \to V$ such that $\{x, y\} \in E_1$ implies $\{\varphi(x), \varphi(y)\} \notin E_2$. This is clearly the same as $G_1 \subset \overline{G_2}$.

Finally, given a graph R and a positive integer t, let us write $R(t)$ for the graph obtained by replacing the vertices $v_i \in V(R)$ by (pairwise disjoint) t-sets V_i and replacing the edges $\{v_i, v_j\} \in E(R)$ by complete bipartite graphs between V_i and V_j.

2. Classical embedding results and conjectures

The overall form of all such results is the same: if G_1 and G_2 are 'small' then they pack. But I will mostly use the embedding language: if H is 'small' and G is 'large' then H can be embedded into G. Much of classical extremal graph theory can be stated in these loose terms and it is the various specific meanings of 'small' and 'large' in different theorems that give them different flavours.

Below I give a short list of some classical embedding results and conjectures. This list is not at all complete and I usually mention only the simplest forms of these results. (For relevant definitions and more information see Bollobás's book [7] and Simonovits's survey [25].) The list is divided into five categories according to the order of H (the graph to be embedded) relative to n, the order of the *host-graph* G:

(1) **fixed *H***
(2) ***H* of order $o(n)$**
(3) **medium-size *H*** (*i.e.*, $v(H) \approx cn$, $0 < c < 1$)
(4) **large *H*** (order $(1 - o(1))n$ – almost perfect packing)
(5) **full-size *H*** (order n – spanning subgraphs).

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2.1. Fixed H

The first theorem here, by Turán in 1941 [28], was the starting point of extremal graph theory, and the second one, by Erdős and Stone in 1946 [14] and Erdős and Simonovits in 1966 [12], summarizes its most important feature.

Theorem 1 (Turán [28]). *If* $e(G_n) > \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2}$ *then* $K_r \subset G_n$.

Theorem 2 (Erdős, Stone and Simonovits [14, 12]). *For every* $\varepsilon > 0$ *and* H *there is an* n_0 *such that, if* $n \geqslant n_0$ *and*

$$e(G_n) > \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right) \frac{n^2}{2},$$

then $H \subset G_n$. *In fact,* $\|H \to G_n\| > \alpha n^{v(H)}$, *where* $\alpha = \alpha(H, \varepsilon) > 0$.

Hence, for the Turán number $\mathrm{ex}(n, H)$, *one has*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}.$$

The main message from Theorem 2 is that in extremal graph theory $\chi(H)$ is the relevant quantity. Contrast this to random graph theory, where the critical quantity is the average degree $t(H)$ or, more precisely, the maximum average degree $\mathrm{MAD}(H) = \max_{H' \subset H} t(H')$. For a fixed H, one needs $e(G_n) \sim c(H) n^{2-2/\mathrm{MAD}(H)}$ to ensure that most n-graphs G_n contain a subgraph isomorphic to H.

2.2. H of order $o(n)$

The following two theorems show the tremendous power of the Regularity Lemma in extremal graph theory. They estimate the Turán and Ramsey numbers for very general classes of graphs. They are both from the same landmark paper by Chvátal, Rödl, Szemerédi and Trotter in 1983 [8] (the first one is only implicit in the paper and is quoted by Alon, Duke, Leffman, Rödl and Yuster [3] as such).

Theorem 3 (Chvátal, Rödl, Szemerédi and Trotter [8]). *For every* Δ *and* $c > 0$ *there is an* $\alpha > 0$ *such that, if* $e(G_n) > cn^2$ *and* H *is any bipartite graph of order less than* αn *and maximum degree* $\Delta(H) \leqslant \Delta$, *then* $H \subset G_n$.

Theorem 4 (Ramsey numbers of bounded degree graphs are linear [8]). *For every* Δ *there is an* $\alpha > 0$ *such that, if* G *is an* n*-graph and* H *is any graph of order less than* αn *and maximum degree* $\Delta(H) \leqslant \Delta$, *then* $H \subset G$ *or* $H \subset \overline{G}$. *In other words,* $r(H) \leqslant c(\Delta(H)) v(H)$ *for all graphs* H.

2.3. Medium-size H

The classical conjectures in this category are about embedding trees. In this subsection, T denotes trees while G are arbitrary graphs. The greedy algorithm proves the following trivial claim: $\delta(G) \geqslant e(T)$ implies $T \subset G$. A classical conjecture of Erdős and T. Sós from 1963 [13] says that minimum degree here can be replaced by average degree.

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Conjecture 5 (Erdős and Sós [13]). *If $t(G) > e(T) - 1$ then $T \subset G$.*

That is, if k-stars are forced, then so are all trees of size k.

(Since the above trivial claim implies that $T \subset G$ whenever $t(G) \geqslant 2e(T)$, the conjecture amounts to gaining a factor 2 in the condition for $t(G)$.)

The following conjecture, on the other hand, is fairly recent.

Conjecture 6 (Loebl [11]). *If G_n has at least $n/2$ vertices of degree at least $n/2$, then it contains all trees of size at most $n/2$.*

And here is a generalization by Komlós and T. Sós (see [2]). It says that the average degree in Conjecture 5 can probably be replaced by the median of the degrees (median degree, for short). (The number x is a median of the numbers $d_1, \ldots, d_n$ if at least half of them exceed x and at least half of them do not exceed x.)

Conjecture 7 (Komlós and Sós [2]). *A graph G contains all trees of size not exceeding the median degree of G.*

I am not going to discuss these tree conjectures in the rest of the paper.

2.4. Large *H*

The interpretation of 'large' G is different here. Large $e(G_n)$ is no longer sufficient: we need lower bounds on all degrees.

Theorem 8 (Alon and Yuster [5]). *For every $\varepsilon > 0$ and H there is an n_0 such that, if $n \geqslant n_0$ and*

$$\delta(G_n) \geqslant \left(1 - \frac{1}{\chi(H)}\right) n,$$

then G_n contains at least $(1-\varepsilon)n/|V(H)|$ vertex-disjoint copies of H.

(Note the change of $\chi - 1$ to χ. We want a very large H-factor; for a number between the two we would still get an H-factor covering a certain positive proportion of the vertices of G_n.)

The following two easy theorems are about embedding almost spanning trees, cycles and paths. For stronger statements see the next subsection.

Theorem 9. *For every $\varepsilon > 0$ there are $\alpha > 0$ and n_0 such that, if $n \geqslant n_0$, $\delta(G_n) \geqslant \frac{1}{2}n$, and T is a tree with $e(T) < (1-\varepsilon)n$ and $\Delta(T) < \alpha n$, then $T \subset G_n$.*

Theorem 10. *For every $\varepsilon > 0$ there is an n_0 such that, if $n \geqslant n_0$, $\delta(G_n) \geqslant \frac{2}{3}n$, $|V(H)| < (1-\varepsilon)n$, and $\Delta(H) \leqslant 2$, then $H \subset G_n$.*

2.5. Full-size *H*

This is certainly the most interesting and hardest class.

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Table 1

Powers of a cycle	Bounded degree trees	Bounded degree graphs
If $\delta(G_n) \geqslant n/2$ then G_n contains a Hamiltonian cycle	If $\delta(G_n) \geqslant (n-1)/2$ then G_n contains a Hamiltonian path	If $\delta(G_n) \geqslant (n-1)/2$ then G_n contains a 'perfect' matching
Pósa & Seymour (Conjecture 11)	Bollobás (Conjecture 12)	Hajnal & Szemerédi (Theorem 13) Bollobás & Eldridge (Conjecture 14) Alon & Yuster (Conjecture 15)

The mother of all full-size packing problems is Dirac's theorem of 1952 [10]. Here are various forms and problems that can be considered their generalizations. In Table 1 we arrange them in tabular form: the column headers indicate the main feature of the graph H to be embedded. Below, we state them one by one.

Conjecture 11 (Pósa and Seymour [24]). *If $\delta(G_n) \geqslant \frac{r}{r+1}n$, then G_n contains the rth power of a Hamiltonian cycle.* (*This would actually imply Theorem 13 below.*)

Conjecture 12 (Bollobás [7]). *For every $\varepsilon > 0$ and Δ there is an n_0 such that, if $n \geqslant n_0$, $\delta(G_n) > (\frac{1}{2} + \varepsilon)n$, and T is a tree of order n with maximum degree $\Delta(T) \leqslant \Delta$, then $T \subset G_n$.*

The following beautiful (and hard!) classical theorem is one of the main tools in embedding algorithms. (The original formulation was about colouring. The case $r = 3$ was solved by Corrádi and Hajnal [9].)

Theorem 13 (Hajnal and Szemerédi [15]). *If $\delta(G_n) \geqslant (1 - \frac{1}{r})n$, then G_n contains $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r.*

Theorem 13 would follow from the following very hard packing conjecture.

Conjecture 14 (Bollobás and Eldridge [7]). *If $v(G_1) = v(G_2) = n$ and*

$$(\Delta(G_1) + 1)(\Delta(G_2) + 1) \leqslant n + 1,$$

then G_1 and G_2 pack.

(For $\Delta(G_1) = 2$, this is about the union of cycles and paths, and was conjectured by Sauer and Spencer [23], and proved by Aigner and Brandt [1] as well as by Alon and Fischer [4].)

Theorems 2 and 8 suggest that the chromatic number should be the right quantity, not the maximum degree. Here is another beautiful conjecture expressing this.

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Conjecture 15 (Alon and Yuster). *For every H there are n_0 and K such that, if $n \geqslant n_0$ and*

$$\delta(G_n) \geqslant \left(1 - \frac{1}{\chi(H)}\right) n, \tag{2.1}$$

then G_n contains at least $(n-K)/|V(H)|$ vertex-disjoint copies of H.

Alon and Yuster [6] give an approximate solution by showing that, for every $\varepsilon > 0$ and H, there is an n_0 such that, if $n \geqslant n_0$,

$$\delta(G_n) \geqslant \left(1 - \frac{1}{\chi(H)} + \varepsilon\right) n, \quad \text{and} \quad |V(H)| \text{ divides } n,$$

then G_n can be perfectly tiled by copies of H.

Remark. The conjecture is false for $\varepsilon = 0$, that is, without the extra term εn. However, Alon and Yuster conjectured that an extra $C = C(H)$ suffices. In 1990, Erdős and Faudree conjectured that this latter conjecture is true with $C(H) = 0$ for the special case $H = C_4$.

Since chromatic number is the relevant quantity for embedding problems, the following would be a natural extension of Theorem 8.

Replace $\Delta(G_1)+1$ in Conjecture 14 by $\chi(G_1)$. That is, for every $\varepsilon > 0$, r and Δ there is an n_0 such that, if $v(G_1) = v(G_2) = n \geqslant n_0$, $\chi(G_1) \leqslant r$, $\Delta(G_1) \leqslant \Delta$, and $\chi(G_1)\Delta(G_2) \leqslant (1-\varepsilon)n$, then G_1 and G_2 pack. Alternatively, in embedding form: if $v(H) = v(G) = n \geqslant n_0$, $\chi(H) \leqslant r$, $\Delta(H) \leqslant \Delta$, and $\delta(G) \geqslant \left(1 - \frac{1}{r} + \varepsilon\right) n$, then $H \subset G$.

Unfortunately, this is false even for $r = 2$. Let G be the union of two cliques with only a little overlap, and H a random bipartite graph (an expander).

Why did it work for copies of H? It may have something to do with band-width. Let us write $w(G)$ for the band-width of the graph G.

Conjecture 16 (Bollobás and Komlós). *For every $\varepsilon > 0$, r and Δ, there are $\alpha > 0$ and n_0 such that, if $v(H) = v(G) = n \geqslant n_0$, $\chi(H) \leqslant r$, $\Delta(H) \leqslant \Delta$, $w(H) < \alpha n$, and $\delta(G) \geqslant \left(1 - \frac{1}{r} + \varepsilon\right) n$, then $H \subset G$.*

Another way to sharpen Theorem 8 and Conjecture 15 is to observe that the minimum degree $(1-1/r)n$ (where $r = \chi(H)$) is necessary only for graphs similar to K_r. For example, a special case of a famous conjecture of El-Zahar [29] says that when $H = C_k$ (cycle of length k) the minimum degree needed to get an H-factor in an n-graph is $n/2$ when k is even (as it should be) but only $(1/2 + 1/(2k))n$ when k is odd, rather than $(2/3)n$ as in Conjecture 15. The question naturally arises as to what quantity χ' can replace χ in (2.1).

An obvious obstruction is that one cannot embed a graph H into a graph G_n of lower chromatic number. That is, we certainly cannot replace $r = \chi$ in (2.1) with anything less than $\chi - 1$, as the example of $G_n = K_{r-1}(n/(r-1))$ shows, but there may be room for improvement between χ and $\chi - 1$.

Here is a more subtle obstruction. For an r-chromatic graph H, let us define the i-independence number α_i as the maximum possible sum of i colour-class sizes in any r-colouring of H, and the i-chromatic number as

$$\chi_i(H) = i\,\frac{v(H)}{\alpha_i(H)}.$$

(Thus, $\chi_i > i$ for $i < \chi$ and $\chi_i = i$ for $i \geqslant \chi$.)

Now, let G_n be a complete r-partite graph with the following colour-class sizes (n is large): i classes of size greater than n/χ_i each, and one left-over class (of size less than $n(1-i/\chi_i)$). Then, clearly, $H \not\subset G_n$. Thus, one could attempt to replace χ in (2.1) by the largest of the numbers χ_i, $i < \chi$. But it is easy to see that

$$\frac{v(H)}{\alpha(H)} \leqslant \chi_1 \leqslant \chi_2 \leqslant \ldots \leqslant \chi_r = \chi(H),$$

and thus χ_{r-1} is the largest one. Hence I suggest the following quantity:

$$\chi'(H) = (\chi(H)-1)v(H)/\alpha'(H), \tag{2.2}$$

where α' is the maximum possible sum of $\chi(H)-1$ colour-class sizes in any colouring of H with $\chi(H)$ colours. (It is easy to see that $\chi - 1 < \chi' \leqslant \chi$, and $\chi' = \chi = r$ can hold only for graphs in which every r-colouring has equal colour-sizes.) In other words, I propose the following conjecture.

Conjecture 17. *For every H there is a K such that, if*

$$\delta(G_n) \geqslant \left(1 - \frac{1}{\chi'(H)}\right) n, \tag{2.3}$$

then G_n contains an H-factor covering all but at most K vertices.

The above example shows that the conjecture – if true – is best possible for any H.

All the above were about two types of H: *bounded degree graphs* (fixed H, or the union of its copies, powers of a Hamilton cycle, *etc.*) and *trees* (forests) (with degrees up to αn). In the rest of the paper I'll concentrate on bounded degree subgraphs.

3. The Regularity Lemma

In a bipartite graph $G = (A, B, E)$ (A and B are the colour classes), the *density* is defined as

$$d(A, B) = \frac{e(A, B)}{|A| \cdot |B|}.$$

We say that $G = (A, B, E)$ is an ε-*regular* pair (or more often we say that (A, B) is an ε-regular pair) if

$$X \subset A, |X| > \varepsilon|A|,\ Y \subset B, |Y| > \varepsilon|B| \quad \text{imply} \quad |d(X, Y) - d(A, B)| < \varepsilon.$$

We say that $G = (A, B, E)$ is an (ε, δ)-*super-regular* pair if

$$X \subset A, |X| > \varepsilon|A|,\ Y \subset B, |Y| > \varepsilon|B| \quad \text{imply} \quad e(X, Y) > \delta|X||Y|;$$

furthermore, $\deg(a) > \delta|B|$ for all $a \in A$, and $\deg(b) > \delta|A|$ for all $b \in B$.

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The following fact is the most important property of regular pairs.

Fact. (Most degrees into a large set are large) Let (A, B) be an ε-regular pair with density d. Then (with $\delta = d - \varepsilon$), for any $Y \subset B$, $|Y| > \varepsilon|B|$,

$$\#\{x \in A : \deg(x, Y) \leqslant \delta|Y|\} \leqslant \varepsilon|A|.$$

Naturally, a similar bound holds for the number of vertices x with $\deg(x, Y) \geqslant (d+\varepsilon)|Y|$. In particular, most pairs of vertices in A have about the right number of common neighbours (about $d^2|B|$) and the same holds for B. This latter property turns out to be equivalent to regularity ([26] on pseudo-random graphs). That is, *the property that a bipartite graph is a regular pair is equivalent to its bipartite adjacency matrix being close to the sum of an orthogonal matrix and a constant matrix.*

An ε-regular pair of density d nicely imitates random bipartite graphs, provided ε is small enough in terms of d (often $\varepsilon \leqslant d/2$ is enough). Let me list a few such random-like properties in an informal way: all small trees are subgraphs; all small bipartite graphs of bounded degree are subgraphs; large chunks of a regular pair are regular; an (ε, δ)-super-regular pair (with ε small enough) has diameter at most 4.

Now, Szemerédi's Regularity Lemma (Theorem 18 below) says that *every (dense) graph is the union of a small number of regular pairs plus a little noise.*

Theorem 18 (Regularity Lemma [27]). *For every $\varepsilon > 0$ and m there are M and n_0 with the following property: for every graph $G = (V, E)$ with $n \geqslant n_0$ vertices there is a partition of the vertex set into $k+1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + \ldots + V_k,$$

such that

- $m \leqslant k \leqslant M$
- $|V_1| = |V_2| = \ldots = |V_k|$
- $|V_0| < \varepsilon n$
- *all but at most εk^2 of the pairs $\{V_i, V_j\}$ are ε-regular.*

This is not a very transparent theorem, but it grows on you with time. The following is a simple consequence.

Theorem 19 (Regularity Lemma: degree form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that, if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $k+1$ clusters $V_0, V_1, \ldots, V_k$, and there is a subgraph $G' \subset G$ with the following properties:*

- $k \leqslant M$
- $|V_0| \leqslant \varepsilon|V|$
- *all clusters V_i, $i \geqslant 1$, are of the same size $N \leqslant \lceil \varepsilon|V| \rceil$*
- $\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)|V|$ *for all $v \in V$*
- *for each $i \geqslant 1$, $G'|_{V_i}$ is empty*
- all *pairs $G'|_{V_i \times V_j}$ $(1 \leqslant i < j \leqslant k)$ are ε-regular, each with a density either 0 or exceeding d.*

A graph, such as $G'' = G'|_{V-V_0}$, satisfying the last two properties will be called *pure*.

The reduced graph

Given an ε-regular partition and a number $d \geqslant 0$, we define the *reduced graph* R on $\{V_1, V_2, \dots, V_k\}$ by connecting V_i and V_j if (V_i, V_j) is an ε-regular pair with density $> d$.

The graph R reflects many properties of G. For instance, R contains a triangle if and only if G contains many of them (cn^3). For much stronger properties see the Key Lemma and the Blow-up Lemma later.

For the graph G'' in the *degree form*, V_i and V_j are adjacent in the reduced graph if and only if there are *any* edges of G'' between them. But then, automatically, there are *many* ($dN^2 = cn^2$) edges of G'' between them, since the density $d(V_i, V_j) > d$. Here is a beautiful illustration of this.

Theorem 20 (Roth [21]). *Every subset of $\mathbb{Z}$ of positive density contains an arithmetic progression of three terms.*

Roth's original proof uses Fourier analysis. Here is an elegant proof by Ruzsa and Szemerédi from 1976 (see [22]), based on the Regularity Lemma.

Proof. Let $R = r_3(n)$, and let $(1 \leqslant)a_1 < a_2 < \cdots < a_R(\leqslant n)$ be a maximum length sequence without a three-term arithmetic progression. Define a bipartite graph $G = G_{5n} = (A, B, E)$ as follows. $A = [2n]$, $B = [3n]$ and

$$E \subset A \times B, \quad E = \{(x + a_i, x + 2a_i) : x \in [n],\ i \in [R]\}.$$

G_{5n} is the union of the n matchings

$$M_x = \{(x + a_i, x + 2a_i) : i \in [R]\}.$$

We say that a subgraph $H \subset G$ is induced in G if the restriction of G to $V(H)$ equals H.

Claim. *The matchings M_x are induced in G.*

Here we sketch two different ways to see this.

Geometrically: if we had a 'cross-edge', the slopes would form an arithmetic progression.

Algebraically: $x + 2a_i = y + 2a_k$ and $x + a_j = y + a_k$ would imply $2a_i - a_j = a_k$, an arithmetic progression.

Now the main tool for proving Theorem 20 is the following result.

Theorem 21 (Induced matchings). *If G_n is the union of n induced matchings, then $e(G_n) = o(n^2)$.*

Now this indeed proves Theorem 20, since we have $Rn = e(G_{5n}) < 2\varepsilon(5n)^2 + n\varepsilon(5n)$ provided $5n \geqslant 2M(\varepsilon)/\varepsilon^2$. Hence, $R = r_3(n) < 55\varepsilon n$. □

Theorem 21, in turn, is a corollary of the next lemma (used with $d = 2\varepsilon$).

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Lemma 22. *Let G_n be* pure *with parameters ε, d, write $\beta = d - \varepsilon$, and assume $N \geqslant 1/(\beta\varepsilon)$ (which is certainly satisfied if $n \geqslant 2M(\varepsilon)/(\beta\varepsilon)$). If* IM *is an induced matching in G_n, then* $|\text{IM}| \leqslant \varepsilon n$.

Proof. Write $U = V(\text{IM})$ for the vertex set of IM, and $U_i = U \cap V_i$. Define $I = \{i : |U_i| > \varepsilon|V_i|\}$, and set $L = \cup_{i\in I} U_i$ and $S = U \setminus L$. Clearly $|S| \leqslant \varepsilon n$. Hence, if we had $|U| > 2\varepsilon n$, then we would have $|L| > |U|/2$, and thus there would exist two vertices $u, v \in L$ adjacent in IM. Let $u \in V_i$ and $v \in V_j$. We would thus have an edge between V_i and V_j, and hence, by *purity*, a density more than $\beta + \varepsilon$ between them. The sets U_i and U_j, being of size larger than εN each, would have a density more than β between them. This means more than $\beta|U_i||U_j| \geqslant \min\{|U_i|, |U_j|\}$ edges, a contradiction with IM being induced. □

4. General framework for embedding H into G

A typical embedding procedure using the Regularity Lemma has several distinct phases: the last one is our main focus here.

(A) Prepare H
(B) Prepare G
(C) Assignment
(D) Making connections
(E) Piecewise embedding

Here are these five steps in more detail. We will use Alon and Yuster's proof of Theorem 8 to illustrate this structure. Since our notation H may be confused with the H there, we will use H for the whole graph to be embedded, so that in the Alon and Yuster example H is a vertex-disjoint union of copies of a fixed graph F.

(A) Prepare *H*. That is, chop it into (a constant number of) small pieces. Since H in Theorem 8 is a union of cn copies of F, the subdivision here is simply grouping them into clusters of εn copies.

(B) Prepare *G*. This has a number of steps, as follows.

(B1) Apply the Regularity Lemma to G.
(B2) Clean up the 'noise' to make it *pure* (that is, apply the degree form of the Regularity Lemma – degrees don't drop much), and define the reduced graph R.

We often need the following step.

(B3) Select a large (or perfect) matching in R (using standard matching theory) or, in general, a large K_r-factor (using Theorem 13).

In the example, we use $r = \chi(F)$, since every graph of chromatic number r is a subgraph of $K_r(\ell)$ for some ℓ. In fact, the union of r vertex-disjoint copies of F is a spanning subgraph of $K_r(\ell)$ with $\ell = v(F)$. Since there is a slack ε in Theorem 8, the degrees in the reduced graph R are still over $(1 - 1/r)v(R)$, so Theorem 13 can be applied.

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(B4) Also, we sometimes move some vertices around to change some of the cluster sizes a little to match in size those of the small pieces of H, and/or to make some of the regular pairs super-regular.

For steps (B3) and (B4), the following trivial lemma is usually sufficient. If $\delta(G_n) \geqslant \left(1 - \frac{1}{r} + \varepsilon\right) n$ (where n is large), then G contains a tiling with super-regular r-cliques with only a constant number of left-over vertices. In the example, we have to make all cluster sizes divisible by $v(F)$, which can be achieved by discarding a constant number of vertices.

In the much harder cases of full-size H and with no slack in the degree conditions, we need a structural lemma that guarantees a Hajnal–Szemerédi structure in the reduced graph even under the weaker condition $\delta(G_n) \geqslant \left(1 - \frac{1}{r}\right) n$ *unless* G_n is very special. This is not the topic of the present paper so we only give an illustration here for the case $r = 2$ (for larger r it is too technical to spell out). A 'special' graph is an almost complete bipartite graph plus some left-over vertices; more precisely, a graph $G = (V, E)$, $|V| = n$, such that V contains two disjoint subsets A and B with

(1) $|A|, |B| > (1 - \varepsilon)n/2$
(2) $\deg(a, B) > (1 - \varepsilon)|B|$ for all $a \in A$ and the same for B.

For such special graphs, the theorems have to be proved directly.

(C) Assignment. An assignment is a map

$$\psi : V(H) \to \{V_1, \dots, V_k\},$$

such that

$$|\psi^{-1}(V_i)| \leqslant |V_i| \quad \text{for all } i, \quad \text{and}$$
$$\{x, y\} \in E(H) \Rightarrow \{\psi(x), \psi(y)\} \in E(R)$$

In other words, we need to find an embedding $\varphi : V(H) \to V(G)$ of H into the closure G^c of G, where G^c is defined by replacing, for all $i < j$, the regular pair between V_i and V_j by the complete bipartite graph between V_i and V_j.

(This is the step in which we formally assign the small pieces of H to those of G.) This step is often just a simple problem about integers. In our example, we just assign vertices of the colour classes of $K_r(\ell)$ to corresponding clusters in some regular r-cliques of G.

(D) Making connections. That is, make the necessary connections *between* the (constant number of) pieces of G. This step is often easy, but sometimes tricky or even very hard.

Blissfully, H in the Alon and Yuster example is disconnected, and this step is missing. The example of embedding small powers of a Hamilton cycle is more typical. Here the small pieces are obtained by chopping the base cycle into a constant number of intervals, and there are a few edges going across these pieces. (See the remarks about band-width above, especially in Conjecture 16.)

(E) Embedding individual pieces. In principle, the whole embedding procedure is about finding an injection

$$\varphi : V(H) \to V(G)$$

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such that

$$\varphi(x) \in \psi(x) \quad (\varphi \text{ is compatible with } \psi)$$
$$\{x, y\} \in E(H) \Rightarrow \{\varphi(x), \varphi(y)\} \in E(G) \quad \text{(embedding)}.$$

But we usually do it *piece by piece*, and this phase is about embedding the small parts $\tilde{H}_i$ of H into the corresponding parts $\tilde{G}_i$ of G. This is typically the hardest part, and the rest of the paper is only about embedding the small pieces $\tilde{H}$ into the small parts $\tilde{G}$.

5. The Blow-up Lemma

The classical papers using the Regularity Lemma have some unpleasant technical details that seem to recur again and again. A close inspection of those proofs makes it possible to distil the essence of those computations. Indeed, all the quoted packing theorems about bounded degree H that did not involve 'full-size packing' (meaning $|V(H)| = |V(G)|$) can be described as follows: first the Regularity Lemma is applied, then some kind of matching theorem is used for the reduced graph (this is often Theorem 13), and then the following simple lemma completes the proof.

Lemma 23 (Key Lemma). *Given $\delta, \varepsilon > 0$, a graph R, and a positive integer N, let us construct a graph G by replacing every vertex of R by N vertices, and replacing the edges of R with ε-regular pairs of density $\geqslant \delta + \varepsilon$ (possibly different pairs for different edges). Let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$, and let $\varepsilon_0 = \delta^\Delta/(2+\Delta)$.*

If $\varepsilon \leqslant \varepsilon_0$ and $t - 1 \leqslant \varepsilon_0 N$ then $H \subset G$. In fact,

$$\|H \to G\| \geqslant (\varepsilon_0 N)^h.$$

Remark. Note that $|V(R)|$ hasn't played any role here.

Using the fact that a large chunk of a regular pair is still regular (and changing ε_0), it is easy to replace the condition $H \subset R(\varepsilon_0 N)$ with the assumptions that

(*) $H \subset R((1 - \varepsilon_0)N)$

(**) every component of H is smaller than $\varepsilon_0 N$.

Typically, R is a complete graph of fixed order (*e.g.* a triangle).

While this is not very deep, it makes proofs short and transparent (and hence it's a great educational tool). Let us illuminate this by describing the main structure of a few classical proofs phrased in terms of the Key Lemma.

Example 1: Theorem 3.

(a) Apply the Regularity Lemma.
(b) Find a regular edge with a density greater than c.
(c) Apply the Key Lemma.

Remark. One can find much larger regular edges (about $ne^{-1/c}$) than those provided by the Regularity Lemma.

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Example 2: Theorem 4.

(a) Apply the Regularity Lemma.
(b) Two-colour the edges of the reduced graph R according to whether they represent regular pairs of density less than or more than $1/2$ (disregard the few irregular pairs). Use classical Ramsey theory to find a monochromatic r-clique in R.
(c) Apply the Key Lemma.

Example 3: Theorem 8.

(a) Apply the Regularity Lemma.
(b) Apply Theorem 13 to find a covering with regular r-cliques.
(c) Apply the Key Lemma (the union of many disjoint copies of a fixed graph is a bounded degree graph).

The only use of the Key Lemma is to make classical proofs more compact. However, if we could get rid of the strong restrictions ($*$) and ($**$), we would have a strong new tool at hand. We describe this tool – the Blow-up Lemma – later, but first here is the proof of the Key Lemma.

Proof. We prove the following more general estimate.

$$\text{If} \quad t-1 \leqslant (\delta^\Delta - \Delta\varepsilon)N \quad \text{then} \quad \|H \to G\| > \left[(\delta^\Delta - \Delta\varepsilon)N - (t-1)\right]^h. \tag{5.1}$$

We embed the vertices $v_1, \dots, v_h$ of H into G by picking them one by one. For each v_j not yet picked, we keep track of an ever-shrinking set C_{ij} that v_j is confined to, and we only make a final choice for the location of v_j at time j. At time 0, C_{0j} is the full N-set that v_j is *a priori* restricted to in the natural way. Hence $|C_{0j}| = N$ for all j. The algorithm at time $i \geqslant 1$ consists of two steps.

Step 1: picking v_i. We pick a vertex $v_i \in C_{i-1,i}$ such that

$$\deg_G(v_i, C_{i-1,j}) > \delta|C_{i-1,j}| \quad \text{for all} \quad j > i,\ \{v_i, v_j\} \in E(H). \tag{5.2}$$

Step 2: updating the C_js. We set, for each $j > i$,

$$C_{ij} = \begin{cases} C_{i-1,j} \cap N(v_i) & \text{if } \{v_i, v_j\} \in E(H), \\ C_{i-1,j} & \text{otherwise.} \end{cases}$$

For $i < j$, let $d_{ij} = \#\{\ell \in [i] : \{v_\ell, v_j\} \in E(H)\}$.

Fact. If $d_{ij} > 0$ then $|C_{ij}| > \delta^{d_{ij}} N$. (If $d_{ij} = 0$ then $|C_{ij}| = N$.)

Thus, for all $i < j$, $|C_{ij}| > \delta^\Delta N \geqslant \varepsilon N$, and hence, when choosing the exact location of v_i, all but at most $\Delta\varepsilon N$ vertices of $C_{i-1,i}$ satisfy (5.2). At most $t-1$ of them have been used up before (and this is the pessimistic step here!) and consequently, we have at least

$$|C_{i-1,i}| - \Delta\varepsilon N - (t-1) > (\delta^\Delta - \Delta\varepsilon)N - (t-1)$$

free choices for v_i, proving the claim. □

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The interesting (and hard) problems don't have the above-mentioned two properties:

(*) small components of H

(**) slack: $|V(H)| < (1-\varepsilon)|V(G)|$.

The following strengthening is a powerful general tool for embedding bounded degree graphs (and often trees, too).

Lemma 24 (Blow-up Lemma: short form). *Given $\delta, \varepsilon > 0$, a graph R, and a positive integer N, let us construct a graph G by replacing every vertex of R by N vertices, and replacing the edges of R with (ε, δ)-super-regular pairs. Let H be a subgraph of $R(N)$ with maximum degree $\Delta > 0$. If $\varepsilon \leqslant \varepsilon_0(\delta, \Delta)$ then $H \subset G$.*

Lemma 25 (Blow-up Lemma: full form [18, 19]). *Given a graph R of order r and positive parameters δ, Δ, there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices $v_1, v_2, \ldots, v_r$ of R with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex set $V = \cup V_i$. The first graph $\mathbf{R}$ is obtained by replacing each edge $\{v_i, v_j\}$ of R with the complete bipartite graph between the corresponding vertex sets V_i and V_j. A sparser graph G is constructed by replacing each edge $\{v_i, v_j\}$ with an (ε, δ)-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \leqslant \Delta$ is embeddable into $\mathbf{R}$ then it is already embeddable into G.*

In short, *regular pairs behave like complete bipartite graphs from the point of view of bounded degree subgraphs.*

Example: the Hajnal–Szemerédi set-up. Let $\mathbf{R} = K_r$ and G a regular r-clique: that is, let the vertex set of G consist of r disjoint sets of size (arbitrary) N each, with (ε, δ)-super-regular connections between any two (where $\varepsilon \leqslant \varepsilon_0(r, \delta)$). Then G contains N vertex-disjoint K_rs. (Note that arbitrarily small (but fixed) densities δ are sufficient: we don't need large degrees as in Theorem 13.)

The Blow-up Lemma is not easy to prove. It can be proved by using a probabilistic version of the greedy algorithm used above for proving the Key Lemma, but the execution is technically complicated. For a full proof see [18]; for an algorithmic version see [19].

Remark. When using the Blow-up Lemma, we usually need the following strengthened version. Given $c > 0$, there are positive numbers $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ and $\alpha = \alpha(\delta, \Delta, r, c)$, such that the Blow-up Lemma in the equal-size case (all $|V_i|$ are the same) remains true if, for every i, there are certain vertices x to be embedded into V_i whose images are *a priori* restricted to certain sets $C_x \subset V_i$ provided that

(i) each C_x within a V_i is of size at least $c|V_i|$

(ii) the number of such restrictions within a V_i is not more than $\alpha|V_i|$.

Remark. The condition that H is of bounded degree can be relaxed to p-arrangeability.

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Some classical full-size embedding results easily follow from the Blow-up Lemma. Here are some recent results using (certain preliminary versions of) the Blow-up Lemma:

- a proof of Bollobás's conjecture (Conjecture 12) appears in [16] (actually, the bounded degree condition can be changed to degrees at most $cn/\log n$)
- a proof of an *approximate version* of Seymour's conjecture (Conjecture 11) appears in [17] (that of Pósa was proved earlier, in 1994, by Fan and Kierstead).

Here are some even more recent results using the Blow-up Lemma (all by Komlós, Sárközy and Szemerédi):

- a proof of Pósa's conjecture (for large n)
- a proof of Seymour's conjecture (for fixed r and large n)
- a proof of Alon and Yuster's conjecture (Conjecture 15).

We hope to get an approximate version of Conjecture 14 (Bollobás and Eldridge) and Conjecture 16 (Bollobás and Komlós).

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