

ARITHMETIC DENSITY

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Abstract We give measure estimates for sets appearing in the study of dynamical systems, such as preimages of Diophantine classes.

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1. Introduction

The notion of *pull-back* of a family is well established in algebraic geometry: starting with a flat morphism of schemes

$$\pi: X \rightarrow B$$

and given a mapping $f: B' \rightarrow B$, we construct a new family over B' :

$$\begin{array}{ccc} X \times_B B' & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array}$$

where \times_B denotes the fibred product over B .

The absence of such a general notion in Kolmogorov–Arnold–Moser (KAM) theory is a major problem. For, as a general rule, B is not a manifold but rather some closed subset in Euclidean space. Therefore, its preimage under a mapping defined on the ambient space might even be reduced to a single point. The aim of this paper is to provide local measure estimates that guarantee that such a phenomenon does not occur.

In order to be more explicit, let us go back to the origin of KAM theory: Kolmogorov’s note of 1954 [10]. Kolmogorov constructed Hamiltonian systems with invariant tori that are robust under perturbation. These invariant tori form a smooth family in phase space

$$\pi: X := T^*M \rightarrow B := \Omega_{C,\tau}$$

parametrized by some *Diophantine class* in the Euclidean space \mathbb{R}^n :

$$\Omega_{C,\tau} = \left\{ v \in \mathbb{R}^n : \forall i \in \mathbb{Z}^n \setminus \{0\}, |(v, i)| \geq \frac{C}{\|i\|^\tau} \right\}$$

for C small enough and τ sufficiently large. Here (\cdot, \cdot) denotes the Euclidean scalar product in \mathbb{R}^n . Arnold gave accurate estimates that relate the constant C to the size of the neighbourhood in which the theorem applies. In this way, using the fact that the set

$$\Omega_\tau = \bigcup_{C>0} \Omega_{C,\tau}$$

is of full measure for $\tau > n - 1$, he proved that the invariant tori form a set of positive measure, as predicted by Kolmogorov [1].

In the late 1960s, Arnold and Pyartli gave general measure estimates for the preimages of the sets $\Omega_{C,\tau}$ and Ω_τ under a map

$$f: U \rightarrow \mathbb{R}^n,$$

where $U \subset \mathbb{R}^d$ is an open subset* [12]. Back in 1932, Mahler studied the analogous affine problem, where Ω_τ is replaced by

$$\tilde{\Omega}_{C,\tau} := \left\{ v \in \mathbb{R}^n : \forall i \in \mathbb{Z}^n \setminus \{0\} \forall j \in \mathbb{Z}, |(v, i) + j| \geq \frac{C}{\|i\|^\tau} \right\},$$

$$\tilde{\Omega}_\tau = \bigcup_{C>0} \tilde{\Omega}_{C,\tau}.$$

He proved that the preimage of $\tilde{\Omega}_\tau$ under the rational map

$$\mathbb{R} \mapsto \mathbb{R}^n, \quad t \mapsto (t, t^2, \dots, t^n)$$

has full measure for τ big enough, and conjectured that the minimal bound was $\tau > n$ [11]. The conjecture was proved in the 1960s by Sprindzhuk [14].

More recently, Kleinbock and Margulis obtained the following generalization of the Arnold–Pyartli theorem.

Theorem 1.1 (Kleinbock and Margulis [9]). *Consider a mapping curved in \mathbb{R}^n*

$$f = (f_1, \dots, f_n): U \rightarrow \mathbb{R}^n,$$

where U is an open subset of \mathbb{R}^n . For any $\tau > n$, the set $f^{-1}(\tilde{\Omega}_\tau)$ is of full measure in U .

The notion of curvedness, which is used in the theorem, contains the previous notions of non-degeneracy, introduced by various authors, as particular cases. Kleinbock extended the notion of curvedness to arbitrary subspaces and generalized the above theorem to arbitrary curved mappings [8]. In what follows, we will always use this refined version of curvedness. We recall it in § 2.

* According to Pyartli, Arnold lectured on similar results but did not publish them. Margulis is credited in Pyartli's paper for the first lemma.

Note that in Diophantine analysis, like in Sprindzhuk’s theorem, one is interested in the *critical exponent*, that is, the infimum of τ for which one can guarantee that some subset is of full measure. In KAM theory, the situation is different: for any τ big enough, the sets $\Omega_{C,\tau}$ parametrize invariant tori. Moreover, for $\tau' > \tau \geq n - 1$, we have a strict inclusion

$$\Omega_{C,\tau} \subset \Omega_{C,\tau'}$$

If we let τ increase, then, away from the critical exponent, the set becomes bigger. However, this phenomenon is compensated by the fact that the KAM theorem applies in a smaller neighbourhood when τ becomes larger. Consequently, *if we disregard the size of the neighbourhood in which the KAM theorem applies, then τ can be chosen arbitrarily large.*

So, we wish to find a statement similar to the Kleinbock–Margulis theorem for the sets $\Omega_{C,\tau}$ but with an important difference: we do not require the value of τ to be optimal. But even if we consider f to be the identity, this seems rather difficult since the sets $\Omega_{C,\tau}$ can have locally zero measure. Thus, to carry out measure estimates for $\Omega_{C,\tau}$ seems hopeless. To bypass this difficulty, we compare the sets $\Omega_{C,\tau}$ between themselves for different values of C, τ^* .

Before doing that, we recall the definition of density, in the measure theoretical sense. For $\alpha \in \mathbb{R}^n$ let $B(\alpha, r)$ be the ball centred at α with radius r . The *density* of a measurable subset $K \subset \mathbb{R}^n$ at a point α is the limit (if it exists)

$$\lim_{r \rightarrow 0} \frac{\text{Vol}(K \cap B(\alpha, r))}{\text{Vol}(B(\alpha, r))},$$

where $\text{Vol}(\cdot)$ denotes the Lebesgue measure. The aim of this paper is to prove the following result.

Theorem 1.2. *Consider an l -curved mapping at the origin*

$$f = (f_1, \dots, f_n): \mathbb{R}^d \supset U \rightarrow \mathbb{R}^n$$

such that $f(0)$ belongs to a Diophantine class $\Omega_{C,\tau}$. For any

$$\tau' > \tau + ndl$$

the density of the set $f^{-1}(\Omega_{C',\tau'})$ at the origin is equal to 1, where $C' = C/2^{\tau'}$.

In fact, we will even prove a stronger statement (Theorem 3.2).

Theorem 1.2 does not give the critical value for τ' : by taking f equal to the identity and reproducing the proof of the theorem, we obtain that the condition

$$\tau' > \tau + \frac{n}{n-1}$$

is sufficient, while the theorem only gives $\tau' > \tau + n^2$. Like for Mahler’s problem, it would be interesting to find out the critical exponent.

* In 2010, during a short informal discussion, J.-C. Yoccoz suggested to me to compare these sets for different values of C and fixed τ .

Now, in the context of the KAM theorem, we begin with a quasi-periodic motion with Diophantine frequency. From the number theoretical point of view, this means simply that there exist C, τ such that $f(0) \in \Omega_{C,\tau}$. After choosing $\tau' > \tau + ndl$, the KAM theorem provides a set of invariant tori in a small neighbourhood of the origin parametrized by $f^{-1}(\Omega_{C',\tau'})$, $C' = C/2^{\tau'}$. The above theorem shows that this set has positive measure and this proves directly the existence of a positive measure set of invariant tori.

The same idea goes far beyond the standard KAM theorem: Theorem 1.2 shows that many of the sets constructed by abstract KAM theory have positive measure. This result is used to prove the Herman conjecture [3–7].

2. Curved mappings

From the whole apparatus developed in Diophantine analysis, we shall retain only basic facts on ‘good functions’. In this section we denote by $U \subset \mathbb{R}^d$ an open neighbourhood of the origin. The origin does of course not play a particular role and can be replaced by any point.

For a subset $K \subset \mathbb{R}^d$ and a function

$$F: K \rightarrow \mathbb{R}$$

we define

$$\|F\|_K := \sup_{x \in K} |F(x)|$$

(which might be infinite) and use the conventions $1/0 = +\infty$, $1/+\infty = 0$.

Definition 2.1 (Kleinbock and Margulis [9]). A continuous function $F: U \rightarrow \mathbb{R}$ is (C, τ) -good if, for any open ball $B \subset U$ and any $\varepsilon > 0$, the following estimate holds:

$$\text{Vol}(\{x \in B: |F(x)| \leq \varepsilon\}) \leq C \left(\frac{\varepsilon}{\|F\|_B} \right)^\tau \text{Vol}(B).$$

The following definition is due to Kleinbock and Margulis for $V = \mathbb{R}^n$ and was extended by Kleinbock to arbitrary subspaces.

Definition 2.2 (Kleinbock [8]; Kleinbock and Margulis [9]). A C^l -map

$$f: U \rightarrow \mathbb{R}^n, x = (x_1, \dots, x_d) \mapsto (f_1(x), \dots, f_n(x))$$

is called l -curved in a vector subspace $V \subset \mathbb{R}^n$ at a point $y \in \mathbb{R}^d$ if:

- (i) $f(U) \subset f(y) + V$;
- (ii) the vector space V is spanned by the partial derivatives $\{\partial^j f(y): |j| \leq l\}$, where $j = (j_1, j_2, \dots, j_d)$ is a multi-index and $|j| = j_1 + j_2 + \dots + j_d$.

Such mappings are usually called *non-degenerate* in the literature. This terminology is problematic in the context of KAM theory, because these non-degenerate mappings of Diophantine analysis can be degenerated in the sense of Kolmogorov, Arnold or

Rüssmann. Therefore, we will call them *curved* since maps with vanishing derivatives at all orders such as

$$x \mapsto e^{-1/x^2}$$

give a prototype of a non-curved map at a point (here, the origin).

Note that

- (1) the condition of being curved at a point is open,
- (2) the vector space V in which a map is curved is unique,
- (3) any analytic mapping is curved.

Because of (2), we sometimes neglect to specify V . Sometimes we also omit l and say that a mapping

$$f: U \rightarrow \mathbb{R}^n$$

is curved at a point if there exists some $l \geq 0$ and some vector space $V \subset \mathbb{R}^n$ such that f is l -curved in V .

A mapping is *curved* if it is curved at all of his points. However due to the fact that this condition is open and that we are concerned only with local problems, we only need to consider curved mappings at a given point.

Proposition 2.3 (Kleinbock [8, Corollary 3.2]; Kleinbock and Margulis [9, Proposition 3.4]). *For any map*

$$f = (f_1, \dots, f_n): U \rightarrow \mathbb{R}^n$$

that is l -curved at the origin, there exist a neighbourhood U' of the origin and a constant C such that for any $c = (c_0, c_1, \dots, c_n) \in \mathbb{R}^{n+1}$ the restriction of the function $c_0 + c_1 f_1 + c_2 f_2 + \dots + c_n f_n$ to U' is $(C, 1/dl)$ -good.

We will use this statement as follows.

Corollary 2.4. *Let*

$$f = (f_1, \dots, f_n): U \rightarrow \mathbb{R}^n$$

be l -curved at the origin. There exist constants $C, R > 0$ such that for any $a > 0$ and $i \in \mathbb{Z}^n \setminus \{0\}$ that satisfies $|(f(0), i)| \geq a$, and any $r \leq R$, we have the estimate

$$\text{Vol}(\{x \in B(0, r) : |(f(x), i)| \leq a'\}) \leq C \left(\frac{a'}{a}\right)^{1/dl} \text{Vol}(B(0, r)),$$

where $B(0, r) \subset \mathbb{R}^d$ denotes the ball centred at the origin with radius r .

3. Arithmetic density for Diophantine classes

In his original paper, Kolmogorov used the Diophantine condition to ensure the convergence of formal power series involving small denominators. It was realised by Bruno, in a non-Hamiltonian context, that the Diophantine condition can be relaxed [2] (see also [13]). Arithmetic classes, that we shall now define, give a practical generalization of Diophantine and Bruno classes.

For any vector $\alpha \in \mathbb{R}^n$ we define the sequence $\sigma(\alpha)$ by

$$\sigma(\alpha)_k := \min\{ |(\alpha, i)| : i \in \mathbb{Z}^n \setminus \{0\}, \|i\| \leq 2^k \}.$$

Definition 3.1. The arithmetic class in \mathbb{R}^n associated with a real positive decreasing sequence $a = (a_k)$ is the set

$$\mathcal{C}(a) := \{ \alpha \in \mathbb{R}^n : \sigma(\alpha)_k \geq a_k \ \forall k \in \mathbb{N} \}.$$

An arithmetic class is a closed subset in \mathbb{R}^n with the property

$$\alpha \in \mathcal{C}(a) \implies \lambda \alpha \in \mathcal{C}(a) \quad \forall \lambda > 1.$$

It cannot be of full measure unless it is equal to \mathbb{R}^n itself. We may now state the main theorem of this paper.

Theorem 3.2. Consider a real positive decreasing sequence $a = (a_k)$ and let $\rho = (\rho_k)$ be a real positive sequence such that the sequence

$$(2^{kn} \rho_k^{1/dl})$$

is summable and $\rho_k < 1$ for all k . For any l -curved mapping at the origin

$$f = (f_1, \dots, f_n) : \mathbb{R}^d \supset U \rightarrow \mathbb{R}^n$$

such that $f(0) \in \mathcal{C}(a)$, the density of the set $f^{-1}(\mathcal{C}(\rho a))$ at the origin is equal to 1.

We start by proving the following assertion.

Proof that Theorem 3.2 \implies Theorem 1.2. For $a = (C2^{-\tau k})$, the arithmetic class $\mathcal{C}(a)$ is related to the Diophantine class $\Omega_{C,\tau}$ by the chain of inclusions

$$\Omega_{C,\tau} \subset \mathcal{C}(a) \subset \Omega_{C/2^\tau,\tau}.$$

In particular, $f(0) \in \Omega_{C,\tau}$ implies that $f(0) \in \mathcal{C}(a)$. Choose $\varepsilon > 0$ and define the sequence ρ by

$$\rho_k := 2^{-kndl - k\varepsilon}.$$

The sequence with terms

$$2^{kn} \rho_k^{1/dl} = 2^{-k\varepsilon/dl}$$

is summable. Thus, according to the theorem, $f^{-1}(\mathcal{C}(\rho a))$ has density 1 at the origin. Moreover, the previous chain of inclusion shows that

$$\mathcal{C}(\rho a) \subset \Omega_{C',\tau'}$$

with $\tau' = \tau + ndl + \varepsilon$ and $C' = C/2^{\tau'}$. This proves the assertion. □

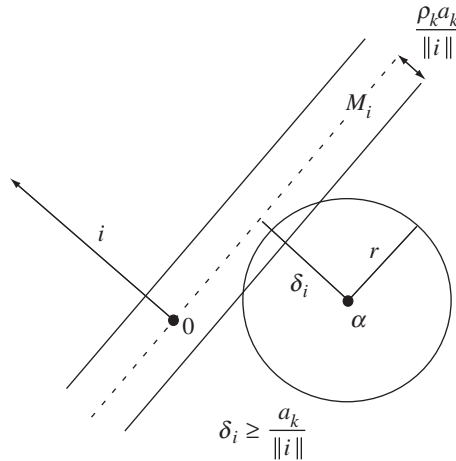


Figure 1. Proof of Lemma 4.1.

4. Proof of Theorem 3.2

Denote by $[\cdot]$ the integer part and consider the map

$$\varphi: \mathbb{Z}^n \rightarrow \mathbb{N}, \quad i \mapsto [\log_2 \|i\|] + 1.$$

For $i \in \mathbb{Z}^n$, $\varphi(i)$ is the smallest natural number such that i is contained in the ball of radius $2^{\varphi(i)}$ centred at the origin.

Fix $i \in \mathbb{Z}^n$ and put $k := \varphi(i)$. The set

$$M_i := \{\beta \in \mathbb{R}^n : |(\beta, i)| < \rho_k a_k\}$$

is a band of width $2\rho_k a_k / \|i\|$ and the union over $i \in \mathbb{Z}^n$ of the subsets M_i is the complement of the arithmetic class $\mathcal{C}(\rho a)$:

$$\mathbb{R}^n \setminus \mathcal{C}(\rho a) = \bigcup_{i \in \mathbb{Z}^n} M_i.$$

Lemma 4.1. *Let α be a vector in $\mathcal{C}(a)$. There exists R such that, for any $r \leq R$ and any $i \in \mathbb{Z}^n$, if the set M_i intersects the ball $B(\alpha, r)$, then the vector $i \in \mathbb{Z}^n$ satisfies*

$$\frac{a_k}{2^{k+1}} < r$$

with $k := \varphi(i)$.

Proof. As (ρ_j) is summable, there exists N such that

$$j \geq N \implies \rho_j < \frac{1}{2}.$$

We define

$$R := \min_{j \leq N} \frac{(1 - \rho_j)a_j}{2^j}.$$

Denote by δ_i the distance from α to the hyperplane orthogonal to the vector $i \in \mathbb{Z}^n$ (see Figure 1).

Take $r \leq R$ and assume that the intersection of the set M_i with the ball $B(\alpha, r)$ is non-empty. In such a case,

$$r > \delta_i - \frac{\rho_k a_k}{\|i\|}$$

with $k = \varphi(i)$. As $\alpha \in \mathcal{C}(a)$, we have

$$|(\alpha, i)| \geq a_k,$$

and thus

$$\delta_i \geq \frac{a_k}{\|i\|} \geq \frac{a_k}{2^k},$$

and therefore

$$\frac{(1 - \rho_k)a_k}{2^k} < r.$$

As $r \leq R$, this condition implies that $k > N$, and in this case $\rho_k < \frac{1}{2}$. Thus, if the ball $B(\alpha, r)$ intersects M_i , then

$$\frac{a_k}{2^{k+1}} < r$$

with $k = \varphi(i)$. This proves the lemma. \square

Consider the set

$$I_r := \left\{ i \in \mathbb{Z}^n : \frac{a_k}{2^{k+1}} < r, k := \varphi(i) \right\}.$$

According to the lemma, when r tends to zero the elements $i \in I_r$ define values of $\varphi(i)$ that go to infinity.

We may now conclude the proof of the theorem. As the mapping f is l -curved, we can use Corollary 2.4. We obtain a constant C such that

$$\text{Vol}(B(0, r) \cap f^{-1}(M_i)) \leq C \rho_k^{1/dl} \text{Vol}(B(0, r))$$

for any r small enough.

As the map

$$f: \mathbb{R}^d \rightarrow \mathbb{R}^n$$

is differentiable, there exists a constant κ such that for any sufficiently small r ,

$$f(B(0, r)) \subset B(\alpha, \kappa r), \quad f(0) = \alpha.$$

In particular,

$$f^{-1}(M_i) \cap B(0, r) \neq \emptyset \implies i \in I_{\kappa r}$$

for any sufficiently small r .

This shows that the measure of the complement of $f^{-1}(\mathcal{C}(a'))$ in $B(0, r)$ is bounded from above by

$$C \text{Vol}(B(0, r)) \sum_{i \in I_{\kappa r}} \rho_{\varphi(i)}^{1/dl}.$$

We have

$$\#\{\varphi(i) = k\} = \#\{\varphi(i) \leq k\} - \#\{\varphi(i) \leq k-1\} \leq 2^{(k+1)n},$$

where the symbol $\#$ stands for the cardinality; thus,

$$\sum_{i \in \mathbb{Z}^n} \rho_{\varphi(i)}^{1/dl} \leq \sum_{k \geq 0} 2^{(k+1)n} \rho_k^{1/dl} < +\infty.$$

Therefore, the partial sums

$$\sum_{i \in I_{kr}} \rho_{\varphi(i)}^{1/dl}$$

converge to zero as r tends to zero. This concludes the proof of the theorem.

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