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BOCHNER–RIESZ MEANS ON BLOCK-SOBOLEV SPACES IN COMPACT LIE GROUP

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Abstract

On a compact Lie group G of dimension n, we study the Bochner–Riesz mean $S_R^\alpha(f)$ of the Fourier series for a function f. At the critical index $\alpha = (n-1)/2$, we obtain the convergence rate for $S_R^{(n-1)/2}(f)$ when f is a function in the block-Sobolev space. The main theorems extend some known results on the m-torus $\mathbb{T}^m$.

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1. Introduction

Let $\mathbb{T}^m$ be the m-dimensional torus and let $[-1/2, 1/2)^m$ be the fundamental cube of $\mathbb{T}^m$. For $f \in L^1(\mathbb{T}^m)$,

$$f(x) \sim \sum_{k\in\mathbb{Z}^m} \widehat{f}(k) \exp(2\pi i\langle k, x\rangle)$$

denotes the (formal) Fourier series of f, where

$$\widehat{f}(k) = \int_{\mathbb{T}^m} f(x) \exp(-2\pi i\langle k, x\rangle)\, dx$$

is the kth Fourier coefficient of f, $\langle k, x\rangle$ is the inner product of vectors $k = (k_1, \ldots, k_m)$ and $x = (x_1, \ldots, x_m)$. For $\alpha \geq 0$ and $R > 0$, the Bochner–Riesz mean of order α of the Fourier series of f in [3] is defined as

$$S_R^\alpha(f)(x) = \sum_{|k|<R} \Big(1 - \frac{|k|^2}{R^2}\Big)^\alpha \widehat{f}(k) \exp(2\pi i\langle k, x\rangle), \quad R > 1.$$

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A fundamental subject of classical Fourier analysis is to study the convergence

$$\lim_{R\to\infty} S_R^{\alpha}(f)(x) = f(x) \tag{1-1}$$

in various function (or distribution) spaces. In this paper, we shall concentrate on (1-1) in the sense of almost everywhere (a.e.). We denote it as

$$\lim_{R\to\infty} S_R^{\alpha}(f)(x) = f(x), \quad \text{a.e.}$$

or $S_R^{\alpha}(f)(x) \to f(x)$, a.e. for simplicity of notation. The number $\alpha_0 = (m-1)/2$ is called the critical index of $S_R^{\alpha}(f)(x)$, since $S_R^{\alpha}(f)(x) \to f(x)$, a.e. for any $f \in L^1(\mathbb{T}^m)$ if $\alpha > \alpha_0$, while there is an $f \in L^1(\mathbb{T}^m)$ for which

$$\limsup_{R\to\infty} |S_R^{\alpha_0}(f)(x)| = \infty, \quad \text{a.e.}$$

For the detail, the reader can see [16, Ch. 7]. Therefore, it raises an interesting problem of looking for a suitable subspace of $f \in L^1(\mathbb{T}^m)$ related to the a.e. convergence of $S_R^{\alpha_0}(f)(x)$. It is well known that the Hardy space H^1 is a good substitute of L^1 in many situations. Surprisingly, Stein [14] found that there also exists an $f \in H^1(\mathbb{T}^m)$ for which $\limsup_{R\to\infty} |S_R^{\alpha_0}(f)(x)| = \infty$ holds almost everywhere. Following this project, Fefferman [9] introduced a class of entropy $\mathbf{J}(f)$ for an $f \in L^1(\mathbb{T}^m)$ (for a set $S \subseteq [0, 1/e]$, the entropy of S is defined by $E(S) = \inf_{S\subseteq\cup_{I_k}} \sum_k |I_k| \log 1/|I_k|$, where the infimum is taken over all sequences of intervals $I_k \subseteq [0, 1/e]$ which cover S. The entropy, $\mathbf{J}(f)$ of a function f, roughly speaking, is the integral of f with respect to the set function E) and conjectured that

$$\lim_{R\to\infty} S_R^{\alpha_0}(f)(x) = f(x), \quad \text{a.e.}$$

provided $\mathbf{J}(f) < \infty$. Motivated by Fefferman's conjecture, Taibleson and Weiss [20] further introduced the block space $B_q(\mathbb{T}^m)$ with $1 < q \le \infty$, and Lu, Taibleson and Weiss [12] then showed that $\lim_{R\to\infty} S_R^{\alpha_0}(f)(x) = f(x)$, a.e. for any $f \in B_q(\mathbb{T}^m)$. The theorem of Lu–Taibleson–Weiss improves Fefferman's conjecture, since $\mathbf{J}(f) < \infty \Rightarrow f \in B_q(\mathbb{T}^m)$ (and more significantly $(B_q)^* = L^\infty$), but it is not necessarily true vice versa. Below, we shall introduce the definition of block space $B_q(X)$ in a more general space of homogeneous type.

Let X be a space of homogeneous type in the sense of Coifman and Weiss [6] with metric d and measure $d\nu$. Let $1 < q \le \infty$. A measurable function b on X is called a q-block if there exists a positive ρ for which b satisfies

(i) the support condition: $\operatorname{supp}(b) \subset B(x,\rho) = \{y \in X, d(x,y) < \rho\}$,
(ii) the size condition: $\|b\|_{L^q} = (\int_X |b|^q \, d\nu)^{1/q} \le |B(x,\rho)|^{-1+1/q}$.

The block space $B_q(X)$ is the function space that consists of all functions f of the form

$$f = \sum_k c_k b_k \quad \text{with } N(\{c_k\}) < \infty,$$

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where each b_k is a q-block and

$$N(\{c_k\}) = \sum_k |c_k|\Big(1 + \log\Big(\frac{\sum_j |c_j|}{|c_k|}\Big)\Big).$$

Obviously, $B_q(\mathbb{T}^m)$ can be regarded as a special case of $B_q(X)$.

In parallel to the Bochner–Riesz mean on the torus $\mathbb{T}^m$, the Bochner–Riesz mean $\widetilde{S_R^\alpha}(f)$ on the Euclidean space $\mathbb{R}^m$ is defined by

$$\widetilde{S_R^\alpha}(f)(x) = \int_{|\xi|<R} \Big(1 - \frac{|\xi|^2}{R^2}\Big)^\alpha \widehat{f}(\xi) \exp(2\pi i\langle \xi, x\rangle)\, d\xi,$$

where $\widehat{f}$ denotes the Fourier transform of a function f.

Lu and Wang [13] introduced the smooth block space $\widetilde{B_q^\gamma}(\mathbb{R}^m)$ in order to study the convergence rate of $\widetilde{S_R^\alpha}(f)(x)$. For $\gamma \geq 0$ and $1 < q \leq \infty$, a (q, γ)-block is a function b supported on a cube Q satisfying $\|b\|_{L_\gamma^q(\mathbb{R}^m)} \leq |Q|^{1/q-1}$, where $L_\gamma^q(\mathbb{R}^m)$ denotes the Bessel potential space

$$L_\gamma^q(\mathbb{R}^m) = \{f : \mathcal{J}_{-\gamma}(f) \in L^q(\mathbb{R}^m)\}$$

and $\mathcal{J}_{-\gamma}(f)$ is the Bessel potential defined via the Fourier transform

$$\widehat{\mathcal{J}_{-\gamma}(f)}(\xi) = (1 + |\xi|^{2\gamma})^{1/2} \widehat{f}(\xi).$$

The smooth block space $\widetilde{B_q^\gamma}(\mathbb{R}^m)$ is the function space that consists of all functions f of the form

$$f = \sum_k c_k b_k$$

satisfying $N(\{c_k\}) < \infty$, where each b_k is a (q, γ)-block.

In order to study the rate of almost everywhere convergence for the Bochner–Riesz mean on smooth functions, Lu and Wang [13] introduced two maximal functions

$$\mathcal{M}_\gamma^\alpha(f)(x) = \sup_{R>0} R^\gamma |S_R^\alpha(f)(x) - f(x)|$$

and

$$\widetilde{\mathcal{M}_\gamma^\alpha}(f)(x) = \sup_{R>0} R^\gamma |\widetilde{S_R^\alpha}(f)(x) - f(x)|.$$

In [13], the authors obtained the following two theorems.

Theorem A. *Let* $0 \leq \gamma \leq 2$, $1 < q < \infty$, *and*

$$\alpha > (m - 1)|1/q - 1/2|.$$

Then

$$\|\widetilde{\mathcal{M}_\gamma^\alpha}(f)\|_{L^q(\mathbb{R}^m)} \lesssim \|f\|_{L_\gamma^q(\mathbb{R}^m)} \quad \textit{for } f \in L_\gamma^q(\mathbb{R}^m)$$

and

$$\|\mathcal{M}_\gamma^\alpha(f)\|_{L^q(\mathbb{T}^m)} \lesssim \|f\|_{L_\gamma^q(\mathbb{T}^m)} \quad \textit{for } f \in L_\gamma^q(\mathbb{T}^m).$$

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THEOREM B. *Let* $1 < q \leq \infty$ *and* $\alpha_0 = (m-1)/2$. *If* $f \in \widetilde{B_q^1}(\mathbb{R}^m)$, *then*

$$\widetilde{S_R^{\alpha_0}}(f)(x) - f(x) = o(R^{-1}) \quad a.e.\ as\ R \to \infty.$$

The authors in [13] gave the proof of Theorem A only for $\widetilde{\mathcal{M}_\gamma^\alpha}(f)$ and made a remark on the boundedness of $\mathcal{M}_\gamma^\alpha(f)$, see Remark 1 in [13] or [11]. Note that Theorem B only states the result on $\mathbb{R}^m$ for $\gamma = 1$. The general case $0 < \gamma \leq 2$ in $\mathbb{R}^m$ was studied in our recent paper [7]. Also, following the proof in [7], one can easily obtain analogous results of $S_R^{\alpha_0}(f)$ on the torus $\mathbb{T}^m$. We notice that $\mathbb{T}^m$ is a special compact Lie group and a compact Lie group G is essentially a torus if G is abelian. Thus, it is interesting to study the Bochner–Riesz mean on a noncommutative compact Lie group (see [2, 5, 21, 22]). As pointed out in the final remark of [5], to study the convergence problem of the Bochner–Riesz mean on a compact Lie group, it suffices to make the investigation on a compact semisimple Lie group. Therefore, the main aim of this paper is, on a compact semisimple Lie group G, to study the Bochner–Riesz means of the Fourier series. More specifically, on G we will use a quite different method from [13] to establish an analogy of Theorem A, and extend Theorem B to any $0 < \gamma \leq 2$. A noncommutative G has a quite different structure from the torus $\mathbb{T}^m$ so that we must execute some nontrivial modifications in the estimates used on the torus. To this end, we shall adopt some basic estimates on the Bochner–Riesz mean on G obtained in [5].

This paper is organized as follows. In Section 2 we shall introduce some necessary notations and definitions and state our main theorems. The proofs of theorems will be presented in Section 3. In this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \leq CB$. We also use the notation $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

2. Notations, definitions and main theorems

Let G denote an n-dimensional connected, simply connected, compact semisimple Lie group with Lie algebra $\mathfrak{g}$ and let $\mathbb{T}^m$ denote an m-dimensional maximal torus of G with Lie algebra $\mathfrak{t}$. Let $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be a system of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, so that $\mathrm{card}(\triangle) = (n-m)/2$, and let $\delta = (\sum_{a \in \triangle} a)/2$.

Let $|\cdot|$ be the norm of $\mathfrak{g}$ induced by the negative of the Killing form B on $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Notice that $|\cdot|$ induces a bi-invariant metric d on G. Since $B|_{\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}}$ is nondegenerate, for any given $\lambda \in (\mathfrak{t}_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C})$, there is a unique ξ_λ in $\mathfrak{t}_{\mathbb{C}}$ such that $\lambda(\xi) = B(\xi, \xi_\lambda)$ for each $\xi \in \mathfrak{t}_{\mathbb{C}}$. We shall identify elements in $(\mathfrak{t}_{\mathbb{C}})^*$ with elements in $\mathfrak{t}_{\mathbb{C}}$ by means of this canonical isomorphism. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be the inner product and norm transferred from $\mathfrak{t}$ to $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, i\mathbb{R})$ by means of this canonical isomorphism.

Let $\Gamma = \{\xi \in \mathfrak{t}, \exp \xi = e\}$, where e is the identity in G. The weight lattice P is defined by

$$P = \{\lambda \in \mathfrak{t} : \langle \lambda, k \rangle \in 2\pi\mathbb{Z} \text{ for any } k \in \Gamma\}$$

with dominant weights defined by

$$\Lambda = \{\lambda \in P, \langle \lambda, a \rangle \geq 0 \text{ for any } a \in \triangle\}.$$

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With this, Λ provides a full set of parameters for the equivalence classes of unitary irreducible representations of G: for $\lambda \in \Lambda$, the representation U_λ has dimension

$$d_\lambda = \prod_{a\in\triangle} \frac{\langle \lambda + \delta, a\rangle}{\langle \delta, a\rangle},$$

and its associated character is

$$\chi_\lambda(\exp \xi) = \frac{\sum_{w\in W} \epsilon(w) \exp(i\langle w(\lambda+\delta), \xi\rangle)}{D(\exp(\xi))}, \quad \xi \in \mathfrak{t}$$

where W is the Weyl group, which acts on $\mathbb{T}^m$ and $\mathfrak{t}$, $\epsilon(w)$ is the signature of $w \in W$, and

$$D(\exp(\xi)) = \sum_{w\in W} \epsilon(w) \exp(i\langle w(\delta), \xi\rangle) = (2i)^{(n-m)/2} \prod_{a\in\triangle} \sin\Big(\frac{\langle a, \xi\rangle}{2}\Big) \tag{2-1}$$

is the Weyl denominator. Since $|D(\exp \xi)|$ is Γ-periodic and W-invariant, we have $|D(x)| = |D(\exp \xi)|$ if $\exp \xi \in \mathbb{T}^m$ is conjugate to $x \in G$ (we shall denote $x \sim \exp \xi$). By the Peter–Weyl theorem, any function $f \in L^1(G)$ has the formal Fourier series expansion

$$f(x) \sim \sum_{\lambda\in\Lambda} d_\lambda \chi_\lambda * f(x).$$

Let Q be a fixed fundamental domain for the exponential map up to conjugacy: any element $y \in G$ is conjugate to exactly one element in $\exp(\overline{Q})$ and $0 \in \overline{Q}$.

On G, the Bochner–Riesz mean of the Fourier series is defined by (see [5, Theorem 1])

$$S_R^\alpha(f)(x) = \sum_{\lambda\in\Lambda} \Big(1 - \frac{\|\lambda+\delta\|^2}{R^2}\Big)_+^\alpha d_\lambda \chi_\lambda * f(x), \quad R > 1,$$

where

$$\|\lambda+\delta\|^2 = \langle \lambda+\delta, \lambda+\delta\rangle, \quad \alpha > -1.$$

We can write $S_R^\alpha(f)$ as a convolution operator

$$S_R^\alpha(f)(x) = B_R^\alpha * f(x),$$

and define the associated maximal operator S_*^α by

$$S_*^\alpha(f)(x) = \sup_{R>0} |S_R^\alpha(f)(x)|,$$

where the kernel

$$B_R^\alpha(x) = \sum_{\lambda\in\Lambda} \Big(1 - \frac{\|\lambda+\delta\|^2}{R^2}\Big)_+^\alpha d_\lambda \chi_\lambda(x)$$

is a central kernel, that is $B_R^\alpha(x) = B_R^\alpha(\exp \xi)$ for any x, which is conjugate to the element $\exp \xi$ in a fixed maximal torus of G. The Bochner–Riesz mean on G has many similar behaviors to its counterpart on the torus $\mathbb{T}^m$. The number $\alpha_0 = (n-1)/2$ is the critical index of S_R^α. When $\alpha > (n-1)/2$, Clerc [5] proved that $S_R^\alpha(f)(x)$ converges to $f(x)$ almost everywhere for any $f \in L^1(G)$ as $R \to \infty$, while Chen and Fan [4] recently showed that there exists an $f \in L^1(G)$ for which

$$\lim_{R\to\infty} \sup |S_R^{(n-1)/2}(f)(x)| = \infty, \quad \text{a.e.}$$

On the other hand, being inspired by [12], Zaloznik [22] proved that $S_R^{(n-1)/2}(f)(x)$ converges to $f(x)$ almost everywhere as $R \to \infty$ whenever $f \in B_q(G)$. Based on these observations, the main aim of this paper is to study the convergence of $S_R^\alpha(f)(x)$ from the view of approximation theory. Precisely, we shall investigate the relation between the smoothness imposed on blocks and the rate of almost everywhere convergence of the Bochner–Riesz means on compact Lie groups.

Let $\gamma \in \mathbb{R}$. The fractional derivative $I_{-\gamma}$ of order γ is defined by the formal Fourier series expansion

$$I_{-\gamma}(f)(x) \sim \sum_{\lambda\in\Lambda} \|\lambda + \delta\|^\gamma d_\lambda \chi_\lambda * f(x), \quad x \in G.$$

For a function space X, the space $I_\gamma(X)$ is the space of all functions f satisfying $I_{-\gamma}(f) \in X$. That is, $f \in I_\gamma(X)$ if and only if $I_{-\gamma}(f) \in X$. The space $I_\gamma(X)$ is called the Sobolev space of order γ based on X (see [18, 19]). In this paper, we are mainly concerned with spaces $X = L^q(G)$ and $X = B_q(G)$. We denote $I_\gamma(L^q(G))$ by $\dot{L}_\gamma^q(G)$ and $I_\gamma(B_q(G))$ by $B_q^\gamma(G)$. The aim of the paper is to study the rate of speed for $B_R^\delta * f(x) - f(x) \to 0$, a.e. as $R \to \infty$ when f belongs to the spaces $\dot{L}_\gamma^q(G)$ or $B_q^\gamma(G)$.

Define the maximal function

$$M_\gamma^\alpha(f)(x) = \sup_{R>0} R^\gamma |B_R^\delta * f(x) - f(x)|.$$

The following two theorems are the main results in the paper.

Theorem 2.1. *Suppose that* $0 \le \gamma \le 2$, $1 < q < \infty$ *and*

$$\alpha > (n-1)|1/q - 1/2|.$$

If $f \in \dot{L}_\gamma^q(G)$, *then*

$$\|M_\gamma^\alpha(f)\|_{L^q(G)} \lesssim \|f\|_{\dot{L}_\gamma^q(G)}.$$

Theorem 2.2. *Let* $1 < q \le \infty$ *and* $\alpha_0 = (n-1)/2$. *If* $f \in B_q^\gamma(G)$ *for* $0 \le \gamma < 2$, *then*

$$S_R^{\alpha_0}(f)(x) - f(x) = o(R^{-\gamma}) \quad \text{a.e. as } R \to \infty.$$

If $f \in B_q^2(G)$, *then*

$$S_R^{\alpha_0}(f)(x) - f(x) = O(R^{-2}) \quad \text{a.e. as } R \to \infty.$$

As a corollary of Theorem 2.1, we have the following result.

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Corollary 2.3. *Let* $1 < q < \infty$ *and*

$$\alpha > (n-1)|1/q - 1/2|.$$

If $f \in \dot{L}^q_\gamma(G)$ *for* $0 \le \gamma < 2$, *then*

$$S^\alpha_R(f)(x) - f(x) = o(R^{-\gamma}) \quad \text{a.e. as } R \to \infty.$$

If $f \in \dot{L}^q_2(G)$, *then*

$$S^\alpha_R(f)(x) - f(x) = O(R^{-2}) \quad \text{a.e. as } R \to \infty.$$

Also, the saturation of S^α_R *is* 2.

3. Proof of main theorems

Note that for $f \in C^\infty(G)$, the function $B^\alpha_R * f(x) - f(x)$ has the Fourier series

$$B^\alpha_R * f(x) - f(x) \sim \sum_{\lambda\in\Lambda}\bigg(\bigg(1 - \frac{\|\lambda+\delta\|^2}{R^2}\bigg)^\alpha_+ - 1\bigg) d_\lambda \chi_\lambda * f(x), \quad R > 1.$$

We can view that $B^\alpha_R * f(x) - f(x) = (S^\alpha_R - \mathbf{Id})f(x)$, where $\mathbf{Id}$ is the identity operator. This says that $(S^\delta_R - \mathbf{Id})$ is a convolution operator with the symbol $m((\lambda + \delta)/R)$, where

$$m(\lambda+\delta) = (1 - \|\lambda+\delta\|^2)^\alpha_+ - 1, \quad \lambda \in \Lambda.$$

We decompose m as a sum of three subsymbols centralizing at $0, 1$ and near ∞, respectively. To this end, let Φ_0, Φ_1 and Φ_∞ be three $C^\infty(\mathbb{R}^m)$ radial nonnegative-valued functions satisfying

(i) $\Phi_0(\xi) \equiv 1$ on the set $\{\xi : |\xi| \le 1/4\}$, $\mathrm{supp}(\Phi_0) \subset \{\xi : |\xi| \le 1/2\}$,
(ii) $\Phi_1(\xi) \equiv 1$ on the set $\{\xi : 1/2 \le |\xi| \le 3/2\}$, $\mathrm{supp}(\Phi_1) \subset \{\xi : 1/4 \le |\xi| \le 2\}$,
(iii) $\Phi_\infty(\xi) \equiv 1$ on the set $\{\xi : |\xi| \ge 2\}$, $\mathrm{supp}(\Phi_\infty) \subset \{\xi : |\xi| \ge 3/2\}$,
(iv) $\Phi_0(\xi) + \Phi_1(\xi) + \Phi_\infty(\xi) \equiv 1$.

Write $\mu = \lambda + \delta$. For $0 \le \gamma \le 2$, we decompose

$$\frac{(1 - \|\mu\|^2)^\alpha_+ - 1}{\|\mu\|^\gamma} = \frac{((1 - \|\mu\|^2)^\alpha_+ - 1)\Phi_0(\|\mu\|)}{\|\mu\|^\gamma} + \frac{\Phi_1(\|\mu\|)(1 - \|\mu\|^2)^\alpha_+}{\|\mu\|^\gamma} - \frac{\Phi_1(\|\mu\|) + \Phi_\infty(\|\mu\|)}{\|\mu\|^\gamma},$$

where if $s = 0$, then we define the value of $((1 - s^2)^\alpha - 1)s^{-\gamma}$ as the limit

$$\lim_{s\to 0^+} \frac{(1 - s^2)^\alpha - 1}{s^\gamma}.$$

For simplicity, we write $\Psi_\infty = -\Phi_1 - \Phi_\infty$. Then Ψ_∞ is a C^∞ function supported on the set $\{\xi : |\xi| \ge 1/4\}$ and $\Psi_\infty(\xi) \equiv 1$ on the set $\{\xi : |\xi| \ge 1/2\}$. Now we write the Fourier series of $R^\gamma(B^\delta_R * f - f)$ as

$$T^\alpha_{R,0}(g) + T^\alpha_{R,1}(g) + T^\alpha_{R,\infty}(g),$$

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where $g = I_{-\gamma}(f)$ and

$$T^{\alpha}_{R,0}(g) = \sum_{\lambda\in\Lambda} \frac{((1-\|\mu\|^2/R^2)^{\alpha}_{+}-1)\Phi_0(\|\mu\|/R)}{(\|\mu\|/R)^{\gamma}} d_\lambda\chi_\lambda * g,$$

$$T^{\alpha}_{R,1}(g) = \sum_{\lambda\in\Lambda} \frac{\Phi_1(\|\mu\|/R)(1-\|\mu\|^2/R^2)^{\alpha}_{+}}{(\|\mu\|/R)^{\gamma}} d_\lambda\chi_\lambda * g,$$

$$T^{\alpha}_{R,\infty}(g) = \sum_{\lambda\in\Lambda} \frac{\Psi_\infty(\|\mu\|/R)}{(\|\mu\|/R)^{\gamma}} d_\lambda\chi_\lambda * g.$$

We denote, for $j = 0, 1, \infty$,

$$T^{\alpha}_{*,j}(g) = \sup_{R>0} |T^{\alpha}_{R,j}(g)|.$$

Proof of Theorem 2.1. The result for $\gamma = 0$ is known in [5]. Thus we only need to show the theorem for the case $0 < \gamma \le 2$. To prove Theorem 2.1, it suffices to show that under the condition of Theorem 2.1,

$$\|T^{\alpha}_{*,j}(g)\|_{L^q(G)} \le C\|g\|_{L^q(G)}, \quad j = 0, 1, \infty,$$

where C is a constant independent of g.

Firstly, we bring the following two lemmas.

Lemma 3.1 [5, Theorem 1]. *Suppose that φ is a function on $[0,\infty)$ satisfying $|\varphi(r)| \le r^{-n-\varepsilon}$ for any $\varepsilon > 0$. Let*

$$\phi(s) = 2\pi \int_0^\infty \varphi(r) V_{(m-2)/2}(rs) r^{m-1}\, dr,$$

where and in what follows $V_\gamma(t) = J_\gamma(t)/t^\gamma$ and $J_\gamma(t)$ is the Bessel function of order γ. Assume that for any $\ell, 0 \le \ell \le k = (n-m)/2$, ϕ satisfies the following condition:

$$\left|\left(\frac{1}{s}\frac{d}{ds}\right)^{\ell} \phi(s)\right| \le s^{-\ell-m-\varepsilon}.$$

Then

$$\varphi_R(\exp\xi) = \sum_{\lambda\in\Lambda} \varphi\Big(\frac{\|\lambda+\delta\|}{R}\Big) d_\lambda\chi_\lambda(\exp\xi)$$
$$= \frac{C}{D(\exp(\xi))} R^n \sum_{\eta\in\Gamma} \Big(\prod_{a\in\triangle}\langle a, \xi+\eta\rangle\Big)\Big(\Big(\frac{1}{s}\frac{d}{ds}\Big)^k \phi\Big)(R|\xi+\eta|).$$

Lemma 3.2 [16, page 155]. *Suppose that f is a radial function in $L^1(\mathbb{R}^m)$, $m \ge 2$; thus, $f(x) = f_0(|x|)$ for a.e. $x \in \mathbb{R}^m$. Then*

$$\widehat{f}(\xi) = (2\pi)^{m/2} \int_0^\infty f_0(s) s^{m-1} V_{(m-2)/2}(2\pi s|\xi|)\, ds.$$

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Let φ be the function in Lemma 3.1. Since

$$\phi(s) = 2\pi \int_0^\infty \varphi(r) V_{(m-2)/2}(rs) r^{m-1}\, dr,$$

using the formula (see [15, page 338])

$$\frac{d}{dt}\Big(\frac{J_\gamma(t)}{t^\gamma}\Big) = -\frac{J_{\gamma+1}(t)}{t^\gamma},$$

then

$$\begin{aligned}\Big(\frac{1}{s}\frac{d}{ds}\Big)^\ell \phi(s) &= (-1)^\ell (2\pi) \int_0^\infty \varphi(r) V_{(m-2)/2+\ell}(rs) r^{m-1+2\ell}\, dr \\ &= (-1)^\ell (2\pi)^{1-(m+2\ell)/2} \widehat{\varphi_{(\ell)}}((2\pi)^{-1} s),\end{aligned}$$

where $\widehat{\varphi_{(\ell)}}(s)$ is the Fourier transform of $\varphi(r)$ in $\mathbb{R}^{m+2\ell}$ at $r = |y|$ for $y \in \mathbb{R}^{m+2\ell}$.

Therefore, combining Lemmas 3.1 and 3.2, we obtain the following handy lemma that will be used in our estimates.

Lemma 3.3. *Suppose that φ is a function on $[0,\infty)$ satisfying $|\varphi(r)| \le r^{-n-\varepsilon}$ and $|\widehat{\varphi_{(\ell)}}(s)| \le s^{-\ell-m-\varepsilon}$ for any ℓ, $0 \le \ell \le k = (n-m)/2$, where $\varepsilon > 0$ and $\widehat{\varphi_{(\ell)}}(s)$ is the Fourier transform of $\varphi(r)$ in $\mathbb{R}^{m+2\ell}$ at $r = |y|$ for $y \in \mathbb{R}^{m+2\ell}$. Then*

$$\begin{aligned}\varphi_R(\exp\xi) &= \sum_{\lambda\in\Lambda} \varphi\Big(\frac{\|\lambda+\delta\|}{R}\Big) d_\lambda \chi_\lambda(\exp\xi) \\ &= \frac{C}{D(\exp(\xi))} R^n \sum_{\eta\in\Gamma} \Big(\prod_{a\in\triangle} \langle a, \xi+\eta\rangle\Big) \widehat{\varphi_{((n-m)/2)}}((2\pi)^{-1} R|\xi+\eta|).\end{aligned}$$

Note that $|\triangle| = (n-m)/2$, $\Gamma \subset \mathbb{Z}^m$ and $n - |\triangle| = (n+m)/2 > m$. By Lemma 3.3, it is easy to see that if φ is a radial Schwartz function on $\mathbb{R}^m$,

$$\begin{aligned}&\Big| R^n \sum_{\eta\in\Gamma\setminus\{0\}} \Big(\prod_{a\in\triangle} \langle a, \xi+\eta\rangle\Big) \Big(\Big(\frac{1}{s}\frac{d}{ds}\Big)^k \phi\Big)(R|\xi+\eta|)\Big| \\ &\quad \le \sum_{\eta\in\Gamma\setminus\{0\}} \frac{\prod_{a\in\triangle} |\langle a, \xi+\eta\rangle|}{|\eta|^n} \le \sum_{\eta\in\Gamma\setminus\{0\}} \frac{1}{|\eta|^{n-|\triangle|}} = O(1),\end{aligned}$$

uniformly for $R > 0$ and $\xi \in Q$. It yields that

$$\varphi_R(\exp\xi) = \frac{C}{D(\exp(\xi))} R^n \prod_{a\in\triangle} \langle a, \xi\rangle \widehat{\varphi_{(\frac{n-m}{2})}}((2\pi)^{-1} R|\xi|) + O\Big(\frac{1}{|D(\exp(\xi))|}\Big)$$

uniformly for $R > 0$ and $\xi \in Q$. According to the definition of Weyl denominator $D(\exp(\xi))$, it is easy to see that

$$\frac{\prod_{a\in\triangle} \langle a, \xi\rangle}{D(\exp(\xi))} = O(1)$$

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uniformly for $\xi \in Q$. Hence we know that

$$\begin{aligned} ||\varphi_R| * f(x)| &= \int_G |\varphi_R(z)| f(xz^{-1})| \, dz \\ &\leq \int_G |\mathfrak{I}_R(z)| |f(xz^{-1})| \, dz + \frac{1}{|D|} * |f|(x), \end{aligned}$$

where $\mathfrak{I}_R(z)$ is a central kernel satisfying

$$|\mathfrak{I}_R(z)| \leq R^n |\widehat{\varphi_{((n-m)/2)}}((2\pi)^{-1} R|\xi|)|, \quad z \sim \exp \xi.$$

Now since $\widehat{\varphi_{((n-m)/2)}}$ is a Schwartz function and $|\xi| \simeq d(z, e)$ when $|\xi| \leq \sigma_0$ for a fixed positive σ_0, by a standard estimate,

$$\sup_{R>0} ||\varphi_R| * f(x)| \leq M(f)(x) + \frac{1}{|D|} * |f|(x),$$

where $M(f)$ is the Hardy–Littlewood maximal function of f. For more details we refer the reader to [1]. This observation gives the following estimate for $T^\alpha_{*,1}$.

Proposition 3.4. *Let* $1 < q < \infty$ *and* $\alpha > (n-1)|1/q - 1/2|$. *For any* $\gamma \geq 0$,

$$\|T^\alpha_{*,1}(g)\|_{L^q(G)} \leq C\|g\|_{L^q(G)}.$$

Proof. Using the Hardy–Littlewood maximal function and the known result for Bochner–Riesz means (see [1, 5]),

$$\|T^\alpha_{*,1}(g)\|_{L^q(G)} \leq \|M(S^\alpha_*(g)\|_{L^q(G)} + \left\| \frac{1}{|D|} * (S^\alpha_*(g)) \right\|_{L^q(G)} \leq \|g\|_{L^q(G)}.$$

Let dx (respectively dt) be the normalized Haar measure on G (respectively $\mathbb{T}^m$). For any central function f on G, by the Weyl integration formula,

$$\int_G f(x) \, dx = \frac{1}{|W|} \int_{\mathbb{T}^m} f(t) |D(t)|^2 \, dt.$$

With this and (2-1), one can easily obtain $1/|D| \in L^1(G)$. It follows from Young's inequality that

$$\left\| \frac{1}{|D|} * (S^\alpha_*(g)) \right\|_{L^q(G)} \leq \|S^\alpha_*(g)\|_{L^q(G)}. \qquad \square$$

To estimate $T^\alpha_{*,0}$, we recall that

$$T^\alpha_{R,0}(g) = \psi_R * g,$$

where

$$\psi_R(\exp(\xi)) = \sum_{\lambda \in \Lambda} \frac{((1 - \|\mu\|^2/R^2)^\alpha_+ - 1)\Phi_0(\|\mu\|/R)}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(\exp(\xi)).$$

For

$$\psi(y) = |y|^{-\gamma}((1 - |y|^2)^\alpha_+ - 1)\Phi_0(y),$$

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we in [7, Lemma 2.2] proved that for any ℓ,

$$0 \le \ell \le k = (n-m)/2, |\widehat{\psi_{(\ell)}}(s)| \le (1+|s|)^{-m-2\ell-2+\gamma}$$

if $\gamma < 2$ and $|\widehat{\psi_{(\ell)}}(s)| \le (1+|s|)^{-m-2\ell-1}$ if $\gamma = 2$. Thus, $\psi(y)$ satisfies all conditions in Lemma 3.3 when $\gamma \le 2$. Following the same argument used in Proposition 3.4, we have the following.

Proposition 3.5. *Let* $1 < q < \infty$. *For any* $0 \le \gamma \le 2$,

$$\|T^{\alpha}_{*,0}(g)\|_{L^q(G)} \le C\|g\|_{L^q(G)}.$$

Finally, we turn to estimate $T^{\alpha}_{*,\infty}(g)$. Recall that

$$T^{\alpha}_{R,\infty}(g) = \omega_R * g,$$

where

$$\omega_R(\exp(\xi)) = \sum_{\lambda\in\Lambda} \frac{\Psi_\infty(\|\mu\|/R)}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(\exp(\xi)).$$

Since we cannot use Lemma 3.3 directly on ω_R, we choose a C^∞ function $Z(t)$ on the interval $(0,\infty)$ with support in the interval $[1,2]$ and

$$\sum_{k=-\infty}^{\infty} Z_k(t) = 1$$

for all $t \in (0,\infty)$, where $Z_k(t) = Z(t/2^k)$. By the support condition, we may write

$$\omega_R(\exp(\xi)) = \sum_{\lambda\in\Lambda}\sum_{k=-1}^{\infty} \frac{\Psi_\infty(\|\mu\|/R)Z_k(\|\mu\|/R)}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(\exp(\xi)).$$

Note that the function $\Psi_\infty(|y|)Z_k(|y|)/|y|^\gamma$ is a radial Schwartz function if $k=-1$. When $k \ge 0$, $\Psi_\infty(|y|) \equiv 1$ on the support of $Z_k(|y|)$. Hence, we may write

$$\omega_R(\exp(\xi)) = \sum_{k=0}^{\infty}\Big(\sum_{\lambda\in\Lambda} \frac{Z_k(\|\mu\|/R)}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(\exp(\xi))\Big) + \Omega_R(\exp(\xi)),$$

where

$$\Omega_R(\exp(\xi)) = \sum_{\lambda\in\Lambda} \Omega(\|\mu\|/R) d_\lambda \chi_\lambda(\exp(\xi))$$

and Ω is a radial Schwartz function. Denote

$$\omega_{R,k}(\exp(\xi)) = \sum_{\lambda\in\Lambda} \frac{Z_k(\|\mu\|/R)}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(\exp(\xi)).$$

We notice that

$$\omega_{R,k}(\exp(\xi)) = 2^{-k\gamma}\omega_{2^kR,0}(\exp(\xi)).$$

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For $t > 0$, since $Z(t)/t^{\gamma}$ is a Schwartz function, by Lemma 3.3 and the same proof of Proposition 3.4,

$$\Big\|\sup_{R>0}|\omega_{R,k} * g|\Big\|_{L^q(G)} \lesssim 2^{-k\gamma}\Big\|\sup_{R>0}|\omega_{R,0} * g|\Big\|_{L^q(G)} \lesssim 2^{-k\gamma}\|g\|_{L^q(G)}$$

and

$$\Big\|\sup_{R>0}|\Omega_R * g|\Big\|_{L^q(G)} \lesssim \|g\|_{L^q(G)}.$$

By the Minkowski inequality and recalling that we assume $\gamma > 0$, we conclude that

$$\|T^{\alpha}_{*,\infty}(g)\|_{L^q(G)} \leq \sum_{k=0}^{\infty}\Big\|\sup_{R>0}|\omega_{R,k} * g|\Big\|_{L^q(G)} + \Big\|\sup_{R>0}|\Omega_R * g|\Big\|_{L^q(G)} \lesssim \|g\|_{L^q(G)}.$$

The proof of Theorem 2.1 is completed. □

The proof of Corollary 2.3 follows a standard argument, we omit the details. To show the rate $O(R^{-2})$ is sharp, we consider a 'polynomial' $\mathcal{P}$, which means

$$\mathcal{P}(x) = \sum_{\lambda\in\Lambda_N} d_\lambda\chi_\lambda * \mathcal{P}(x),$$

where $\Lambda_N = \{\lambda \in \Lambda : \|\lambda + \delta\| < N\}$ for some fixed integer N. Clearly, $\mathcal{P} \in C^{\infty}(G)$. Now, if R is sufficiently large (for instance $R > 2N$), with the help of Taylor's formula

$$(1 - t)^{\alpha} - 1 = -\alpha t + O(t^2) \quad \text{as } t \to 0,$$

then

$$\begin{aligned} R^2(S^{\alpha}_R(\mathcal{P})(x) - \mathcal{P}(x)) &= R^2\sum_{\lambda\in\Lambda_N}\Big(-\frac{\alpha\|\lambda+\delta\|^2}{R^2}d_\lambda\chi_\lambda * \mathcal{P}(x) + O\Big(\Big(\frac{\|\lambda+\delta\|^2}{R^2}\Big)^2\Big)\Big) \\ &= C\Delta\mathcal{P}(x) + O(R^{-2}), \end{aligned}$$

where Δ is the Laplacian on G. Hence we can always find a function $\mathcal{P} \in C^{\infty}(G)$ such that it fails to have

$$\lim_{R\to\infty} R^2(S^{\alpha}_R(\mathcal{P})(x) - \mathcal{P}(x)) = 0 \quad \text{a.e. } x \in G.$$

Proof of Theorem 2.2. For $f \in B^{\gamma}_q(G)$, we need to show that for any $\varepsilon > 0$,

$$\Big|\Big\{x \in G : \limsup_{R\to\infty}|R^{\gamma}(S^{(n-1)/2}_R f(x) - f(x))| > \varepsilon\Big\}\Big| = 0.$$

As in the proof of Theorem 2.1,

$$\begin{aligned} &\Big|\Big\{x \in G : \limsup_{R\to\infty}|R^{\gamma}(S^{(n-1)/2}_R f(x) - f(x))| > \varepsilon\Big\}\Big| \\ &\quad = \Big|\Big\{x \in G : \limsup_{R\to\infty}|(T^{(n-1)/2}_{R,0} + T^{(n-1)/2}_{R,1} + T^{(n-1)/2}_{R,\infty})(g)(x)| > \varepsilon\Big\}\Big|, \end{aligned}$$

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where $g = I_{-\gamma}(f) \in B_q(G)$. So we have the block decomposition

$$g(x) = \sum_{k=1}^{\infty} c_k b_k(x),$$

where each b_k is a q-block and $N(\{c_k\}) < \infty$. Let $\varepsilon > 0$, and choose a sufficiently large $N = N(\varepsilon)$ such that

$$\sum_{k=N+1}^{\infty} |c_k|\Big(1 + \log \frac{\sum_j |c_j|}{|c_k|}\Big) < \varepsilon^2.$$

Rewrite g as

$$g(x) = \sum_{k=1}^{\infty} c_k b_k(x) = \sum_{k=1}^{N} c_k b_k(x) + \sum_{k=N+1}^{\infty} c_k b_k(x) = g_1(x) + g_2(x).$$

For any fixed N, it is easy to check that g_1 is an L^q function. Hence for any fixed $\beta > 0$, Corollary 2.3 implies

$$\Big|\Big\{x \in G : \limsup_{R\to\infty} |(T_{R,0}^{(n-1)/2} + T_{R,1}^{(n-1)/2} + T_{R,\infty}^{(n-1)/2})(g_1)(x)| > \beta/2\Big\}\Big| = 0.$$

If we can prove there is a constant C independent of any q-block b such that

$$\Big|\Big\{x \in G : \sup_{R>0} |(T_{R,0}^{(n-1)/2} + T_{R,1}^{(n-1)/2} + T_{R,\infty}^{(n-1)/2})(b)(x)| > \beta/2\Big\}\Big| \le C/\beta$$

then for any $\varepsilon > 0$, by the Stein–Weiss formula [17]

$$\Big|\Big\{x \in G : \limsup_{R\to\infty} |(T_{R,0}^{(n-1)/2} + T_{R,1}^{(n-1)/2} + T_{R,\infty}^{(n-1)/2})(g_2)(x)| > \varepsilon/2\Big\}\Big|$$
$$\le \varepsilon^{-1} \sum_{k=N+1}^{\infty} |c_k|\Big(1 + \log \frac{\sum_j |c_j|}{|c_k|}\Big) < \varepsilon.$$

Thus it remains to show that for any $\beta > 0$, the following the weak type inequality

$$\Big|\Big\{x \in G : \sup_{R>0} |T_{R,j}^{(n-1)/2}(b)(x)| > \beta\Big\}\Big| \le C/\beta \quad \text{for } j = 0, 1, \infty,$$

holds for any q-block b.

Without loss of generality, assume that b is supported in a ball B centered at I, the identity of G. Let $|B|$ be the volume of B. If $\beta > 1/|B|$, by Theorem 2.1,

$$\Big|\Big\{x \in G : \sup_{R>0} |(T_{R,0}^{(n-1)/2} + T_{R,1}^{(n-1)/2} + T_{R,\infty}^{(n-1)/2})(b)(x)| > \frac{\beta}{2}\Big\}\Big| \le \Big(\frac{\|b\|_{L^q(G)}}{\beta}\Big)^q \le \frac{1}{\beta}.$$

Let $B^* = 2^n B$, where $2^n B$ denotes the ball concentric with B whose radius is 2^n times the radius of B. If $\beta \le 1/|B|$, then it suffices to show that for $j = 0, 1, \infty$,

$$\Big|\Big\{x \notin B^* : \sup_{R>0} |T_{R,j}^{(n-1)/2}(b)(x)| > \beta\Big\}\Big| \le C/\beta.$$

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Recalling the proof of Theorem 2.1, we know that for $j = 0, \infty$,

$$\sup_{R>0} |T_{R,j}^{(n-1)/2}(b)(x)| \le M(b)(x) + \frac{1}{|D|} * |b|(x),$$

and we obtain the weak $(1, 1)$ inequality

$$\left|\left\{x \notin B^* : \sup_{R>0} |T_{R,j}^{(n-1)/2}(b)(x)| > \lambda\right\}\right| \lesssim \|b\|_{L^1}/\beta \lesssim |B|^{1-1/q}\|b\|_q/\beta \lesssim 1/\beta,$$

for $j = 0, \infty$, where we have used Hölder's inequality and the size estimate of the atom b.

Our final step is to show

$$\left|\left\{x \notin B^* : \sup_{R>0} |T_{R,1}^{(n-1)/2}(b)(x)| > \beta\right\}\right| \lesssim \|b\|_{L^1}/\beta.$$

Recall that

$$\begin{aligned} T_{R,1}^{(n-1)/2}(b)(x) &= \sum_{\lambda\in\Lambda} \frac{\Phi_1(\|\mu\|/R)(1 - \| \mu \|^2/R^2)_+^{(n-1)/2}}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda * b(x) \\ &= \mathfrak{I}_R * b(x), \end{aligned}$$

where

$$\mathfrak{I}_R(x) = \sum_{\lambda\in\Lambda} \frac{\Phi_1(\|\mu\|/R)(1 - \| \mu \|^2/R^2)_+^{(n-1)/2}}{(\|\mu\|/R)^\gamma} d_\lambda \chi_\lambda(x).$$

In order to apply Lemma 3.3, we need to check the Fourier transforms of $\widehat{\mathfrak{I}_{(\ell)}}, 0 \le \ell \le (n - m)/2$, for the radial function

$$\mathfrak{I}(|y|) = |y|^{-\gamma}\Phi_1(|y|)(1 - |y|^2)_+^{(n-1)/2}.$$

Next we will prove that the estimate

$$|\widehat{\mathfrak{I}_{(\ell)}}(s)| \lesssim (1 + |s|)^{-m-2\ell} \tag{3-1}$$

holds for $s \in \mathbb{R}$ and $0 \le \ell \le k = (n - m)/2$.

Note that the Fourier transform of $|y|^{-\gamma}\Phi_1(|y|)$ is a Schwartz function, and it will be denoted by $\Theta(2\pi\xi)$. Let $u(y) = (1 - |y|^2)_+^{(n-1)/2}$, $y \in \mathbb{R}^{m+2\ell}$. Using the result in [16, Theorem 4.15, page 171], we see that

$$\widehat{u}(\xi) = C_{n,m,\ell} V_{(m+2\ell+n-1)/2}(2\pi|\xi|),$$

where $C_{n,m,\ell} = 2^{(m+2\ell+n-1)/2}\pi^{(m+2\ell)/2}\Gamma((n + 1)/2)$ and $\Gamma(\cdot)$ is the gamma function. Thus we may write

$$\begin{aligned} \widehat{\mathfrak{I}_{(\ell)}}(s) &= C_{n,m,\ell}(V_{(m+2\ell+n-1)/2} * \Theta)(2\pi x) \\ &= C_{n,m,\ell} \int_{\mathbb{R}^{m+2\ell}} V_{(m+2\ell+n-1)/2}(|2\pi x - z|)\Theta(z)\, dz \\ &= (2\pi)^n C_{n,m,\ell} \int_{\mathbb{R}^{m+2\ell}} V_{(m+2\ell+n-1)/2}(2\pi|x - y|)\Theta(2\pi y)\, dy, \end{aligned}$$

where $s = |x|$.

Applying the property of the Bessel function (see [10, Appendix B]), we have $V_{(m+2\ell+n-1)/2}(z) = O(1)$ for all $z \in \mathbb{C}$. Then

$$\Big|\int_{\mathbb{R}^{m+2\ell}} V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy\Big| \lesssim \int_{\mathbb{R}^{m+2\ell}} |\Theta(y)|\,dy \lesssim 1$$

for $|x| \leq 10$.

For $|x| \geq 10$, we write

$$\begin{aligned}\int_{\mathbb{R}^{m+2\ell}} & V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy \\ &= \Big(\int_{|y|/2\leq|x|\leq2|y|} + \int_{|x|<|y|/2} + \int_{|x|>2|y|}\Big)V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy.\end{aligned}$$

Since ψ is a Schwartz function,

$$\begin{aligned}\Big|\int_{|y|/2\leq|x|\leq2|y|} V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy\Big| &\lesssim \int_{|x|/2\leq|y|\leq2|x|} |\Theta(2\pi y)|\,dy \\ &\lesssim |x|^{-N}\end{aligned}$$

for any positive integer N.

By the asymptotic behavior of the Bessel function $J_\nu(r)$ as $r \to \infty$ (see [16, Lemma 3.11, page 158] or [8, Proposition 5.1, page 93]), we know that

$$|V_{(m+2\ell+n-1)/2}(|y|)| \lesssim |y|^{-(m+2\ell+n)/2}.$$

Noting that $|x| < |y|/2$, we have $|x-y| \simeq |y|$. Since $n \geq 2\ell + m$,

$$\begin{aligned}\Big|\int_{|x|<|y|/2} & V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy\Big| \\ &\lesssim \int_{|y|>2|x|} |x-y|^{-(m+2\ell+n)/2}|\Theta(2\pi y)|\,dy \\ &\lesssim \int_{|y|>2|x|} |y|^{-(m+2\ell+n)/2}|\Theta(2\pi y)|\,dy \\ &\lesssim \int_{|y|>2|x|} |y|^{-m-2\ell}|\Theta(2\pi y)|\,dy \\ &\lesssim |2x|^{-m-2\ell}\int_{|y|>2|x|} |\Theta(2\pi y)|\,dy \\ &\lesssim |x|^{-m-2\ell}.\end{aligned}$$

Similarly,

$$\Big|\int_{|x|>2|y|} V_{(m+2\ell+n-1)/2}(2\pi|x-y|)\Theta(2\pi y)\,dy\Big| \lesssim |x|^{-m-2\ell}.$$

Combining all the estimates, we prove that (3-1) holds.

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Hence it follows from Lemma 3.3 that for $xz^{-1} \sim \exp \xi$,

$$\begin{aligned}\mathfrak{I}_R(xz^{-1}) &= O(R^n \widehat{\varphi_{(n)}}((2\pi)^{-1}R|\xi|)) + \frac{1}{|D(\exp(\xi))|}\\ &= O(|\xi|^{-n}) + \frac{1}{|D(\exp(\xi))|}.\end{aligned}$$

For $x \notin B^*$ and $z \in B$, we have $|\xi| \simeq d(xz^{-1}, I) = d(x, z) \geq d(x, I)/2$. Thus for $x \notin B^*$,

$$\begin{aligned}|T_{R,1}^{(n-1)/2}(b)(x)| = \left|\int_B \mathfrak{I}_R(xz^{-1})b(y)\,dy\right| &\leq \left|\frac{1}{|D|} * b(x)\right| + d(x, I)^{-n}\int_B |b(z)|\,dz\\ &\leq \left|\frac{1}{|D|} * b(x)\right| + d(x, I)^{-n}\|b\|_{L^1}.\end{aligned}$$

It then easily yields the desired inequality

$$\left|\left\{x \notin B^* : \sup_{R>0} |T_{R,1}^{(n-1)/2}(b)(x)| > \beta\right\}\right| \leq 1/\beta.$$

Theorem 2.2 is proved. □

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