Probability in the Engineering and Informational Sciences, **36**, 2022, 812–823.
doi:10.1017/S0269964821000097

ON DEGRADATION-BASED REMAINING LIFETIME

Maxim Finkelstein
Department of Mathematical Statistics, University of the Free State, Bloemfontein, South Africa

Ji Hwan Cha
Department of Statistics, Ewha Womans University, Seoul 120-750, Republic of Korea
E-mail: jhcha@ewha.ac.kr

The new reliability notion describing the remaining lifetime is introduced for items with monotonically increasing degradation. We consider the remaining lifetime of an item (to be called, the predicted remaining lifetime) after its degradation reaches the predetermined level. The prediction is executed at inception of an item into operation. For the nonhomogeneous stochastic processes of degradation, this characteristic depends on the random first passage time of a degradation process. Some properties of the predicted remaining lifetime and the corresponding stochastic comparisons are discussed for items from homogeneous and heterogeneous populations, and a generalization to the case of the n-component coherent system is outlined. The problem of regime switching is described, and the new notion of the corresponding virtual age after the switching is proposed.

Keywords: first passage time, gamma process, stochastic comparisons, virtual age

1. INTRODUCTION

1.1. Background and Setting

Let T be a lifetime of an item/system described by the Cdf $F(t)$: $\bar{F}(t) \equiv 1 - F(t)$. The conventional remaining lifetime for a system that was incepted into operation at $t = 0$ and did not fail in $[0, y)$, U_y is described by the following survival function:

$$P(U_y > t) = \overline{F}(t|y) = \frac{P(T - y > t)}{P(T > y)} = \frac{\bar{F}(t+y)}{\bar{F}(y)}. \quad (1)$$

This is one of the main classic reliability characteristics. Its properties in various settings and applications are widely studied in the literature. The remaining lifetime and the mean remaining lifetime play a pivotal role in demography, survival analysis and reliability studies. One of the recent meaningful extensions of this notion is due to considering the time y in (1) as a random variable Y, which happens in many applications. Then, obviously, (1) turns to

$$P(U_Y > t) = \frac{P(T - Y > t)}{P(T > Y)} = \frac{\int_0^\infty \bar{F}(y+t) f_Y(y)dy}{\int_0^\infty \bar{F}(y) f_Y(y)dy},$$

or

$$P(U_Y > t) = P(T - Y > t | T > Y) = \int_0^\infty P(T - Y > t | T > Y, Y = y) \frac{\bar{F}(y) f_Y(y)}{P(T > Y)} dy$$

$$= \int_0^\infty \frac{\bar{F}(t+y)}{\bar{F}(y)} \frac{\bar{F}(y) f_Y(y)}{\int_0^\infty \bar{F}(u) f_Y(u) du} dy, \quad (2)$$

where $f_Y(\cdot)$ is the pdf of Y. Stochastic properties of U_Y including the relevant stochastic comparisons were studied, for example, in Yue and Cao [19], Li and Zuo [12], Nanda and Kundu [15], and Cai and Zheng [2]. Other generalizations of the residual lifetime and the relevant analysis and comparisons can also be found in Li and Fang [10], Li and Lu [11], and Misra *et al.* [14]. Thus, in (2), by "averaging" $P(U_y > t)$ in (1) with respect to the conditional distribution of $(Y|T > Y)$, we make a *prediction* for the remaining lifetime of an item after Y on condition that it was operable at Y.

Another relevant model with numerous practical applications in reliability and demography has been also developed. This model describes the situation when an item that starts operating at $t = 0$ has already the initial age Y (see Finkelstein and Vaupel [7], Cha and Finkelstein [3,4], and Hazra *et al.* [8]). Stochastic analyses in these papers relied on the notion of an equilibrium distribution widely used in various branches of stochastics, for example, in describing limiting distributions of the excess and waiting times for renewal processes. The equilibrium distribution in the described context is defined as the distribution that is the same for the initial age and the remaining lifetime. Specifically, its pdf takes a very simple but meaningful form

$$f_e(t) = \frac{\bar{F}(t)}{\int_0^\infty \bar{F}(u) du}. \quad (3)$$

It can be also shown that the latter random age model is more general, as it reduces to (2) for the specifically chosen distribution of initial age [8].

The information on the dynamic performance of an item for defining the remaining lifetimes in the above models was binary: either an item is operating or it has failed. However, many real systems are characterized by the observed, monotonically increasing stochastic degradation, which results in the failure upon reaching a threshold. Thus, at time t, we observe the degradation of the *operating* item and are interested in the remaining lifetime. The first guess would be that for the given threshold level, we do not need the time of observation, as only the accumulated degradation matters. However, this is true only for the homogeneous processes of deterioration, whereas for the nonhomogeneous processes, the future (and, specifically, the remaining lifetime) will depend on time.

More specifically, at the time of inception of an item into operation, or even on the design stage, we are interested in predicting the remaining lifetime after reaching some predetermined level of degradation. This is, in fact, similar to the conventional remaining lifetime defined in (1), or the generalized one in (2). The difference is that we do not need the similar conditioning now, as the predetermined level of degradation is obviously smaller than the deterministic degradation threshold. Another crucial distinction, as mentioned, is that in our specific, degradation-wise setting, the remaining lifetime does not depend on Y for the homogeneous process of degradation. On the other hand, as it will be shown, for the nonhomogeneous degradation processes, this remaining lifetime will depend on the random time of accumulating the specified degradation only through the corresponding degradation rate (which is, obviously, constant for homogeneous processes). Thus, for the latter case, the random time to reach the specified level of degradation should be "integrated out" to end up in the *predicted remaining lifetime* (PRL), as we define this term. The more appropriate term would be probably the "expected remaining lifetime", however, this can create a confusion with the mean remaining lifetime defined as the expectation of U_y in (1).

A practically important application of the described setting is when we consider the remaining lifetime at inception into operation of a used item with the observed level of degradation. In this sense, the PRL can be loosely considered as the degradation-wise version of remaining lifetimes in Cha and Finkelstein [3,4] and Hazra *et al.* [8]. The setting with the used items will be especially relevant in practice for the n-component system considered in Section 4.

As degradation of items depends on the severity of an environment/regime, it is natural to model a change in the PRL after the switch of regimes. This is somehow related to the cumulative exposure model and similar principles used in accelerated life testing [16]. However while dealing with this problem, a new, specific notion of a random virtual age will be also introduced.

1.2. Preliminaries

Assume that the observable (continuously monitored or only at failure) internal deterioration process $\{W_t, t \geq 0\}$, $W_0 = 0$, describes the deterioration of an item/system. Assume that it has independent increments and is characterized by the monotonically increasing sample paths. A failure occurs when the process reaches the deterministic level w. Then, the lifetime of a system, T, can be described by the following survival function:

$$P(T > t) \equiv \bar{F}(t, w) = P(W_t \leq w). \tag{4}$$

Note that, for the fixed t, $\bar{F}(t, w)$ is, in fact, the Cdf (as a function of w) of a random variable W_t, whereas it is the survival function of a random variable T for the fixed w. We assume that $P(T > t)$ in (4) is an absolutely continuous distribution with respect to t for each fixed w. In accordance with (1), denote the Cdf $F(t, w) = 1 - \bar{F}(t, w) = P(W_t > w)$.

For degrading systems, it is natural to obtain the remaining lifetime after degradation of a system reaches a certain level as this information can be used for an efficient operation of the system. For example, based on this information, one can schedule a preventive maintenance. Denote this level of degradation for the described system by $\tilde{w}, 0 < \tilde{w} < w$. For the homogeneous $\{W_t, t \geq 0\}$, the remaining lifetime is, obviously, defined by the following survival function:

$$\bar{F}(t, w - \tilde{w}) = P(W_t \leq w - \tilde{w}), \tag{5}$$

whereas for the nonhomogeneous process, it already matters at what time degradation $\tilde{w}$ was accumulated.

Denote the random time to reach the degradation level $\tilde{w}$ by $V_{\tilde{w}}$. It follows from (4) that its Cdf for the fixed $\tilde{w}$ is given by$F(t, \tilde{w}) = 1 - \bar{F}(t, \tilde{w})$. Thus, the *predicted remaining lifetime* (PRL) $L_{\tilde{w}}$ in this case is defined by the following survival function:

$$P(L_{\tilde{w}} > t) \equiv \bar{F}^*(t, w, \tilde{w}) = \int_0^\infty P(W_{x+t} - W_x \leq w - \tilde{w}) f(x, \tilde{w}) dx, \tag{6}$$

where $f(x, \tilde{w}) = (\partial/\partial x) F(x, \tilde{w})$. For convenience, we will omit "predicted" where appropriate in what follows.

We will call $V_{\tilde{w}}$ the *virtual age* (to reach degradation $\tilde{w}$) as opposed to the "observed" calendar age which is a realization of $V_{\tilde{w}}$. Note that, this is a different notion than that employed in the imperfect repair modeling [5,9]. Moreover, as mentioned in the Introduction, this setting can be interpreted for a used item having a degradation level $\tilde{w}$ at inception into operation at $t = 0$. Then, its virtual age $V_{\tilde{w}}$ at $t=0$ is also defined as a random time to accumulate $\tilde{w}$ under the same static regime/stress it will operate in $t > 0$.

Let us consider now items from a heterogeneous population composed of homogeneous subpopulations defined via the continuous frailty variable Z with the corresponding pdf $\pi(z), z \in [0, \infty)$ (see, e.g., Finkelstein and Cha [6]). Thus, for given $Z = z$, the conditional distributions are indexed in the following way:

$$P(T > t|Z = z) \equiv \bar{F}_z(t, w) = P(W_t \leq w|Z = z), \tag{7}$$

which is assumed to be an absolutely continuous distribution with respect to t for each fixed w and given z. In accordance with this description, denote by W_t^z the degradation in a subpopulation with $Z = z$ is, that is,

$$W_t^z =_D (W_t|Z = z), \tag{8}$$

where "$=_D$" stands for the equality in distribution. For the homogeneous processes $\{W_t^z, t \geq 0\}$, the remaining population lifetime is, obviously, defined by the following mixed survival function:

$$\int_0^\infty \bar{F}_z(t, w - \tilde{w})\pi(z)dz = \int_0^\infty P(W_t \leq w - \tilde{w}|Z = z)\pi(z)dz,$$

whereas for the nonhomogeneous processes, taking into account (6), it is defined as

$$P(L_{\tilde{w}} > t) \equiv \bar{F}^*(t, w, \tilde{w}) = \int_0^\infty \int_0^\infty P(W_{x+t}^z - W_x^z \leq w - \tilde{w}) f(x, \tilde{w}|z)\pi(z)dxdz, \tag{9}$$

where $f(x, \tilde{w}|z) = (\partial/\partial x)F_z(x, \tilde{w})$ and $F_z(x, \tilde{w}) = 1 - \bar{F}_z(t, \tilde{w})$.

2. COMPARING LIFETIMES FOR TWO REGIMES

Consider now two lifetimes T_1 and T_2 that are defined by the corresponding deterioration processes $\{W_{1,t}, t \geq 0\}$ and $\{W_{2,t}, t \geq 0\}$ with the same failure threshold and assume that degradation in one process is more intensive than that in the other in the sense of the usual stochastic ordering, that is, $W_{1,t} \leq_{\mathrm{st}} W_{2,t}$ for all $t \geq 0$. This means that

$$\bar{F}_1(t, \tilde{w}) = P(W_{1,t} \leq \tilde{w}) \geq P(W_{2,t} \leq \tilde{w}) = \bar{F}_2(t, \tilde{w}) \tag{10}$$

for all t and $\tilde{w}$, $0 < \tilde{w} \leq w$. We assume that distributions in (10) are absolutely continuous with respect to t for each fixed $\tilde{w}$. When $\tilde{w} = w$, it follows from (4) and (10) that the corresponding lifetimes are ordered in the sense of the usual stochastic ordering, that is, $T_1 \geq_{\mathrm{st}} T_2$. Due to our definition of a virtual age, it means that the corresponding virtual ages are also ordered in the same sense

$$V_{1,\tilde{w}} \geq_{\mathrm{st}} V_{2,\tilde{w}}. \tag{11}$$

The following result for the predicted remaining lifetimes with the more specific assumptions can be also proved:

Proposition 1. *Let $W_{1,x+t} - W_{1,x} \leq_{st} W_{2,x+t} - W_{2,x}$ for all $x \geq 0, t > 0$, and $W_{2,x+t} - W_{2,x}$ is stochastically decreasing in x in the sense of the usual stochastic order for all fixed $t > 0$. Then,*

$$\bar{F}_1^*(t, w, \tilde{w}) \geq \bar{F}_2^*(t, w, \tilde{w}), \text{ for all } t > 0,$$

where $\bar{F}_1^*(t, w, \tilde{w})$ and $\bar{F}_2^*(t, w, \tilde{w})$, in accordance with the definition (4), are the survival functions of the expected remaining lifetimes for the first and the second items.

PROOF:

$$\begin{aligned}\bar{F}_1^*(t,w,\tilde{w}) &= \int_0^\infty P(W_{1,x+t}-W_{1,x}\le w-\tilde{w})f_1(x,\tilde{w})dx\\ &\ge \int_0^\infty P(W_{2,x+t}-W_{2,x}\le w-\tilde{w})f_1(x,\tilde{w})dx\\ &\ge \int_0^\infty P(W_{2,x+t}-W_{2,x}\le w-\tilde{w})f_2(x,\tilde{w})dx = \bar{F}_2^*(t,w,\tilde{w})\end{aligned}$$

The first inequality holds due to the assumption that $W_{1,x+t}-W_{1,x}\le_{\rm st}W_{2,x+t}-W_{2,x}$, for all $x\ge 0, t>0$, and the second inequality holds due to (10) and $W_{2,x+t}-W_{2,x}$ is stochastically decreasing in x in the usual stochastic order sense for all fixed $t>0$.

Indeed, integrating by parts:

$$\begin{aligned}&\int_0^\infty P(W_{2,x+t}-W_{2,x}\le w-\tilde{w})(f_1(x,\tilde{w})-f_2(x,\tilde{w}))dx\\ &\quad = -\int_0^\infty \frac{\partial}{\partial x}P(W_{2,x+t}-W_{2,x}\le w-\tilde{w})(F_1(x,\tilde{w})-F_2(x,\tilde{w}))dx \ge 0,\end{aligned}$$

where $f_1(x,\tilde{w})=\frac{\partial}{\partial x}F_1(x,\tilde{w})$, $f_2(x,\tilde{w})=\frac{\partial}{\partial x}F_2(x,\tilde{w})$. ■

Example 1: Consider two nonhomogeneous Gamma processes with $(\alpha_1(t),\lambda_1)$ and $(\alpha_2(t),\lambda_2)$ as the shape function (increasing and $\alpha_i(0)=0$) and the scale parameter, accordingly. It follows, for example, from Noortwijk [18] that the survival function for an increment of this process is given by

$$P(W_{i,x+t}-W_{i,x}\le w-\tilde{w}) = 1-\frac{\Gamma((\alpha_i(t+x)-\alpha_i(x)),\lambda_i(w-\tilde{w}))}{\Gamma(\alpha_i(t+x)-\alpha_i(x))},\quad i=1,2, \tag{12}$$

where $\Gamma(b)=\int_0^\infty z^{b-1}\exp\{-z\}dz$, $\Gamma(b,x)=\int_x^\infty z^{b-1}exp\{-z\}dz$, $b>0,\lambda>0$.

Specifically, when $x=0,\tilde{w}=0$,

$$\begin{aligned}P(W_{i,t}\le w)\equiv \bar{F}_i(t,w) &= 1-\frac{\Gamma((\alpha_i(t),\lambda_i w)}{\Gamma(\alpha_i(t))}\\ &= \textstyle\int_0^w \dfrac{1}{\Gamma(\alpha_i(t))}{\lambda_i}^{\alpha_i(t)}x^{\alpha_1(t)-1}exp(-\lambda_i x)\ \mathrm{d}x,\ i=1,2.\end{aligned} \tag{13}$$

Let $\alpha_1(x+t)-\alpha_1(x)\le\alpha_2(x+t)-\alpha_2(x)$ for all $x\ge 0, t>0$, and $\lambda_1\ge\lambda_2$, and $\alpha_2(t)$ is concave. Then, $\bar{F}_1(t,\tilde{w})\ge\bar{F}_2(t,\tilde{w})$, for all $t>0$. Indeed, consider the ratio of the pdfs of $W_{1,x+t}-W_{1,x}$ and $W_{2,x+t}-W_{2,x}$, that is,

$$\frac{\Gamma(\alpha_2(x+t)-\alpha_2(x))}{\Gamma(\alpha_1(x+t)-\alpha_1(x))}\frac{{\lambda_1}^{\alpha_1(x+t)-\alpha_1(x)}}{{\lambda_2}^{\alpha_2(x+t)-\alpha_2(x)}}u^{(\alpha_1(x+t)-\alpha_1(x))-(\alpha_2(x+t)-\alpha_2(x))}exp\{-(\lambda_1-\lambda_2)u\},$$

which is decreasing in u due to assumptions: $\alpha_1(x+t)-\alpha_1(x)\le\alpha_2(x+t)-\alpha_2(x)$, $x\ge 0, t>0$ and $\lambda_1\ge\lambda_2$. Thus, $W_{1,x+t}-W_{1,x}\le_{\rm lr}W_{2,x+t}-W_{2,x}$ and, accordingly, $W_{1,x+t}-W_{1,x}\le_{\rm st}W_{2,x+t}-W_{2,x}$, for all $x\ge 0,t>0$. Furthermore, as $\alpha_2(t)$ is concave, it can be shown that $W_{1,x_2+t}-W_{1,x_2}\le_{\rm lr}W_{2,x_1+t}-W_{2,x_1}$, for $x_1<x_2$, which also implies that $W_{2,x+t}-W_{2,x}$ is stochastically decreasing in x in the usual stochastic order for all fixed $t>0$.

https://doi.org/10.1017/S0269964821000097 Published online by Cambridge University Press

Following our discussion in the previous section, compare now predicted remaining lifetimes for items from two heterogeneous populations. Let these populations be described by the degradation processes $\{W_{1,t}, t \geq 0\}$ and $\{W_{2,t}, t \geq 0\}$ and the continuous frailty variables Z_1 and Z_2 with the corresponding pdfs $\pi_1(z)$ and$\pi_2(z)$, respectively. Then, the conditional lifetimes T_1 and T_2 for the same failure threshold w are specified, similar to (7), as

$$P(T_i > t|Z_i = z) \equiv \bar{F}_{i,z}(t, w) = P(W_{i,t} \leq w|Z_i = z),\ i = 1, 2.$$

We assume that the distributions $\bar{F}_{i,z}(t, w)$, $i = 1, 2$, are absolutely continuous with respect to t for each fixed w and given z.

Similar to (8), denote

$$W_{i,t}^z =_D (W_{i,t}|Z_i = z),\ i = 1, 2.$$

Proposition 2. *Let*

(i) $W_{1,x+t}^z - W_{1,x}^z \leq_{st} W_{2,x+t}^z - W_{2,x}^z$, *for all* $x \geq 0, t > 0$, *for each fixed* z,
(ii) $W_{1,x+t}^{z_1} - W_{1,x}^{z_1} \leq_{st} W_{1,x+t}^{z_2} - W_{1,x}^{z_2}$, *for* $z_1 < z_2$, *for all* $x \geq 0, t > 0$,
(iii) $W_{i,x+t}^z - W_{i,x}^z$ *is stochastically decreasing in* x *in the sense of the usual stochastic order for all fixed* $t > 0$, *for each fixed* z, $i = 1, 2$,
(iv) $Z_1 <_{st} Z_2$.

Then,

$$\bar{F}_1^*(t, w, \tilde{w}) \geq \bar{F}_2^*(t, w, \tilde{w}),\ \text{for all } t > 0.$$

where $\bar{F}_1^*(t, w, \tilde{w})$ *and* $\bar{F}_2^*(t, w, \tilde{w})$ *are the survival functions of the predicted remaining lifetimes for the items in the first and the second populations.*

Proof: Define the conditional survival functions of the expected remaining lifetimes $L_{1\tilde{w}}$ and $L_{2\tilde{w}}$ for the first and the second populations as

$$P(L_{i\tilde{w}} > t|Z_i = z) \equiv \bar{F}_i^*(t, w, \tilde{w}|z) = \int_0^\infty P(W_{i,x+t}^z - W_{i,x}^z \leq w - \tilde{w}) f_i(x, \tilde{w}|z) dx,\ i = 1, 2,$$

where $f_i(x, \tilde{w}|z) = (\partial/\partial x) F_{i,z}(x, \tilde{w})$ and $F_{i,z}(x, \tilde{w}) = 1 - \bar{F}_{i,z}(t, w)$. Denote by $V_{1,\tilde{w}}^z$ and $V_{2,\tilde{w}}^z$ the corresponding virtual ages for subpopulations with the following Cdfs

$$P(V_{i,\tilde{w}}^z \leq x) \equiv P(V_{i,\tilde{w}} \leq x|Z_i = z) = P(W_{i,x} > \tilde{w}|Z_i = z) = 1 - \bar{F}_{i,z}(x, \tilde{w}) = F_{i,z}(x, \tilde{w}),\ i = 1, 2.$$

Then, due to the assumption (i), $V_{1,\tilde{w}}^z \geq_{\text{st}} V_{2,\tilde{w}}^z$, and, by assumptions (i), (iii) and by the similar arguments stated in the proof of Proposition 1, it can be shown that

$$\bar{F}_1^*(t, w, \tilde{w}|z) \geq \bar{F}_2^*(t, w, \tilde{w}|z), \text{for all } t, z \geq 0.$$

Furthermore, for $z_1 < z_2$, due to assumption (ii), it obviously holds that $V_{1,\tilde{w}}^{z_1} \geq_{st} V_{1,\tilde{w}}^{z_2}$. By assumptions (ii), (iii) and again by the similar arguments stated in the proof of Proposition 1, it can be shown that $\bar{F}_1^*(t, w, \tilde{w}|z_1) \geq \bar{F}_1^*(t, w, \tilde{w}|z_2)$, for $z_1 < z_2$, which implies that $\bar{F}_1^*(t, w, \tilde{w}|z)$ is decreasing in z. Then,

$$P(L_{2\tilde{w}} > t) = \bar{F}_2^*(t, w, \tilde{w}) = \int_0^\infty \bar{F}_2^*(t, w, \tilde{w}|z) \pi_2(z) dz$$

$$\leq \int_0^\infty \bar{F}_1^*(t, w, \tilde{w}|z) \pi_2(z) dz \leq \int_0^\infty \bar{F}_1^*(t, w, \tilde{w}|z) \pi_1(z) dz = \bar{F}_1^*(t, w, \tilde{w}) = P(L_{1\tilde{w}} > t),$$

where, in the last inequality, similar to the proof of Proposition 1, the fact that $\bar{F}_1^*(t, w, \tilde{w}|z)$ is decreasing in z and assumption (iv) are applied. ■

Comparing the operation of two items characterized by different degradation processes can also help in dealing with the situation when an item, at first, operate in one regime and then is switched to another one, for instance, severer one. Thus, the virtual age of the first regime should be "recalculated" to the virtual age in the second one. This model will be considered in the next section. Note that, there are numerous publications related to this problem, for example, in the framework of accelerated life testing (Nelson [16,17] and Meeker and Escobar [13]), where this age correspondence is performed on the basis of the cumulative exposure principle, when the probabilities of failure in both regimes are equated. In contrast to the general modeling in these references, we are considering systems with more information at hand, that is, the observed monotone degradation, which allows to make this age correspondence in a different way.

3. SWITCHING THE REGIMES

Let, as previously, the system starts operating at $t=0$, under the first regime (lighter) described by the degradation process$\{W_{1,t}, t \geq 0\}$. If it does not fail in $[0, t_s)$, the regime is switched at t_s to the severer one characterized by the process$\{W_{2,t}, t \geq 0\}$. As before, define $\bar{F}_i(t, w) = P(W_{i,t} \leq w)$, $i = 1, 2$ and assume that these distributions are absolutely continuous. Thus, the accumulated degradation in $[0, t_s)$ becomes the initial degradation for the second phase. Assume first, that the processes $\{W_{i,t}, t \geq 0\}, i = 1, 2$ are homogeneous. Then, the combined (on the whole t-axis) process can be defined as

$$W(t) = \begin{cases} W_{1,t}, & 0 \leq t \leq t_s, \\ W_{1,t_s} + W_{2,t-t_s}, & t \geq t_s, \end{cases} \tag{14}$$

whereas the system survival function, for the same failure threshold w for both phases, in accordance with (4) and the definition of the virtual age, is given by

$$P(W(t) \leq w) = \begin{cases} \overline{F}_1(t, w), & 0 \leq t \leq t_s, \\ \int_0^w \int\limits_0 f_1(t_s, y)\overline{F}_2(t - t_s, w - y)dy, & t \geq t_s, \end{cases} \tag{15}$$

where $f_1(t_s, w) = (\partial/\partial w)\overline{F}_1(t_s, w)$. Thus, the second line in (15) can be considered as the PRL at t_s for a system that did not fail in $[0, t_s)$.

Modeling becomes more complex for nonhomogeneous processes, as after switching to the severer regime, the time t_s should be "recalculated" to some virtual time (age) that is defined as the time that a system had to operate in a severer regime to gain the same degradation $\tilde{w}$ as it was just before the switching. More specifically, we observe degradation $\tilde{w}$, $0 < \tilde{w} < w$ accumulated at time instant t_s by a system in Regime 1, which is then instantaneously switched to Regime 2.

What would be the time in this case for a system operating in Regime 2 to reach this degradation level? In accordance with our reasoning of the previous section, it is the random virtual age $V_{2,\tilde{w}}$ described by the Cdf $F_2(t, \tilde{w})$ defined in (10)–(11). It means that $V_{2,\tilde{w}}$ should be considered as the starting age after switching. Thus, the corresponding *predicted remaining lifetime* after switching is defined by the modified Equation (6) for the following survival function:

$$\bar{F}_2^*(t, w, \tilde{w}) = \int_0^\infty P(W_{2,x+t} - W_{2,x} \leq w - \tilde{w}) f_2(x, \tilde{w})dx, \tag{16}$$

where $f_2(x, \tilde{w}) = (\partial/\partial x)F_2(x, \tilde{w})$.

As the suggested approach presents the innovative, justified degradation-based method of recalculation of age when switching to another regime, we provide the formal definition.

Definition 1. *The degradation-based recalculated age after switching from the first regime to the second for the increasing nonhomogeneous stochastic processes with independent increments is defined as the virtual age* $V_{2,\tilde{w}}$ *(see* $F_2(t,\tilde{w})$ *in* (10)), *that is, the random time that is needed for the second process that starts at* $t=0$ *to accumulate the same degradation* $\tilde{w}$, $0<\tilde{w}\le w$ as *was accumulated by the first process before switching.*

Thus, everything is assessed/predicted at $t=0$, and we do not need to observe the time when reaching $\tilde{w}$.

Taking into account the above considerations, (15) should be modified for the nonhomogeneous processes as

$$P(W(t)\le w)=\begin{cases}\bar{F}_1(t,w), & 0\le t\le t_s,\\ \int_0^w f_1(t_s,y)\hat{F}_2(t,t_s,w-y)dy, & t\ge t_s,\end{cases} \tag{17}$$

where

$$\hat{F}_2(t,t_s,w-y)=\int_0^\infty \overline{F}_2(t+x-t_s,w-y)f_2(x,y)dx$$

and the variable x has a meaning of the realization of the virtual age $V_{2,\tilde{w}}$.

Remark 1: Note that, for the homogeneous case, the virtual age that a system should operate in Regime 2 to accumulate degradation $\tilde{w}$ that was accumulated in Regime 1, also, obviously, exists. However, it does not affect the PRL after switching.

Example 2: For the nonhomogeneous gamma processes (as in Example 1), the relevant functions in (17) take the form:

$$\bar{F}_1(t,w)=\int_0^w \frac{1}{\Gamma(\alpha_1(t))}{\lambda_1}^{\alpha_1(t)}x^{\alpha_1(t)-1}\exp(-\lambda_1 x)dx,$$

$$f_1(t_s,y)=\frac{1}{\Gamma(\alpha_1(t_s))}{\lambda_1}^{\alpha_1(t_s)}y^{\alpha_1(t_s)-1}\exp(-\lambda_1 y),$$

$$\begin{aligned}f_2(x,y)&=\frac{\partial}{\partial x}\left(1-\int_0^y \frac{1}{\Gamma(\alpha_2(x))}{\lambda_2}^{\alpha_2(x)}u^{\alpha_2(x)-1}\exp(-\lambda_2 u)du\right),\\ &=\frac{A(x,y)}{\left(\int_0^\infty s^{\alpha_2(x)-1}\exp(-s)ds\right)^2},\end{aligned}$$

where

$$A(x,y)=\int_0^y \Bigg[-(\ln\lambda_2+\ln u)\ {\alpha'}_2(x)\lambda_2^{\alpha_2(x)}u^{\alpha_2(x)-1}\exp(-\lambda_2 u)\ \left(\int_0^\infty s^{\alpha_2(x)-1}\exp(-s)ds\right)$$

$$+\lambda_2^{\alpha_2(x)}u^{\alpha_2(x)-1}\exp(-\lambda_2 u)\ \left(\int_0^\infty {\alpha'}_2(x)(\ln s)s^{\alpha_2(x)-1}\exp(-s)ds\right)\Bigg]\ du,$$

$$F_2(t+x-t_s,w-y)=\int_0^{w-y}\frac{1}{\Gamma(\alpha_2(t+x-t_s))}{\lambda_2}^{\alpha_2(t+x-t_s)}u^{\alpha_2(t+x-t_s)-1}\exp(-\lambda_2 u)du.$$

Note that, $f_1(t_s,y)$ is just the density of the corresponding gamma-distributed random variable for the fixed t_s, whereas $f_2(x,y)$ is already the pdf of the lifetime (the derivative of $F_2(x,y)$ is taken with respect to x, see (16)).

4. SYSTEM OF N COMPONENTS

In this section, we will briefly outline the meaningful generalization of the model described in Section 1.2 to the case of coherent, non-repairable systems with n independent components. In practical situations, it may happen that we are interested, for instance, in the predicted remaining lifetime of a system composed of the used components with observed levels of degradation. Then under some assumptions, the suggested approach can be extended to this important setting.

Denote the structure function of the described system [1] by $\phi(x_1, x_2, \ldots, x_n)$ and the corresponding reliability function by

$$r(p_1, p_2, \ldots, p_n) = E[\phi(X_1, X_2, \ldots, X_n)],$$

where X_i is the binary state variable of component i with $E[X_i] = p_i$, $i = 1, 2, \ldots, n$. Suppose that the deterioration of the components in the system is described by n statistically independent monotonically increasing degradation processes with independent increments, $\{W_{i,t}, t \geq 0\}$, $i = 1, 2, \ldots, n$. A failure of the ith component occurs if $W_{i,t}$ reaches its threshold w_i. In accordance with (4), we define

$$\bar{F}_i(t, w_i) \equiv P(W_{i,t} \leq w_i);\ F_i(t, w_i) = 1 - \bar{F}_i(t, w_i) = P(W_{i,t} > w_i),\ i = 1, 2, \ldots, n.$$

Then, the reliability function of the system's lifetime Tcan be written as

$$P(T > t) = r(\bar{F}_1(t, w_1), \bar{F}_2(t, w_1), \ldots, \bar{F}_n(t, w_n)).$$

Similar to the one-component case, at the time of inception of an item into operation (or even on the design stage), we are interested in the PRL of the system after its components' degradation levels reach the levels, that is, $\tilde{w}_i \leq w_i$ $i = 1, 2, \ldots, n$. Equivalently, as was stated above, at time $t = 0$, we want to define the remaining lifetime of a system with used components having initial degradation levels $\tilde{w}_i < w_i$, $i = 1, 2, \ldots, n$.

Denote by $\mathbf{V} = \{V_{1,\tilde{w}_1}, V_{2,\tilde{w}_2}, \ldots, V_{n,\tilde{w}_n}\}$ the n-variate vector of random virtual ages that corresponds to $\{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n\}$. However, note that all the n components in the system start to operate at $t = 0$ simultaneously and their degradation levels are also observed simultaneously at the same time point. This means that the operating times until the observation of degradation levels for all the components are the same, and thus, the virtual ages for all components should be the same, which is a simple but remarkable fact.

As before, define

$$f_i(v_i, \tilde{w}_i) = \frac{\partial}{\partial v_i} F_i(v_i, \tilde{w}_i),\ i = 1, 2, \ldots, n.$$

Then, we have to obtain the distribution of the common virtual age of the components $V_{(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)}$ given that $V_{1,\tilde{w}_1} = V_{2,\tilde{w}_2} = \cdots = V_{n,\tilde{w}_i}$. As the processes of degradation in the components are independent, the corresponding conditional pdf at time t should be proportional to $\prod_{i=1}^{n} f_i(t, \tilde{w}_i)$. Thus, by normalizing, it can be defined as

$$f_{(V_{(\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n)} | V_{1,\tilde{w}_1} = V_{2,\tilde{w}_2} = \cdots = V_{n,\tilde{w}_i})}(v) = \frac{\prod_{i=1}^{n} f_i(v, \tilde{w}_i)}{\int_0^{\infty} \prod_{i=1}^{n} f_i(u, \tilde{w}_i) du},\ 0 < t < \infty.$$

Then, the survival function of the PRL of the system, $L_{(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)}$, after reaching degradation levels $\{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n\}$ by the components (or "having" these levels at $t = 0$

by the used components) is given by

$$
\begin{aligned}
&P(L_{(\tilde{w}_1,\tilde{w}_2,\ldots,\tilde{w}_n)} > t) \\
&= \int_0^\infty r(P(W_{1,v+t} - W_{1,v} \le w_1 - \tilde{w}_1), P(W_{2,v+t} - W_{2,v} \le w_2 - \tilde{w}_2), \ldots, \\
&\quad P(W_{n,v+t} - W_{n,v} \le w_n - \tilde{w}_n)) \\
&\quad \times \frac{\prod_{i=1}^n f_i(v,\tilde{w}_i)}{\int_0^\infty \prod_{i=1}^n f_i(u,\tilde{w}_i)du} dv,
\end{aligned}
$$

where, if for some j, $w_j - \tilde{w}_j = 0$ (meaning a failure of a component), then $P(W_{j,v+t} - W_{j,v} \le 0) = 0$ in the reliability function on the integrand. The latter is not, obviously, relevant for the used components.

Example 3: Series and parallel system with three components.

As in Example 1, consider three nonhomogeneous gamma processes with parameters $(\alpha_i(t), \lambda_i)$ for $\{W_{i,t}, t \ge 0\}, i = 1, 2, 3$, respectively. Then,

(i) for a series system, the PRL is defined as

$$
\begin{aligned}
&P(L_{(\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)} > t) \\
&= \int_0^\infty P(W_{1,v+t} - W_{1,v} \le w_1 - \tilde{w}_1) P(W_{2,v+t} - W_{2,v} \le w_2 - \tilde{w}_2) \\
&\quad P(W_{3,v+t} - W_{3,v} \le w_3 - \tilde{w}_3) \\
&\quad \times \frac{f_1(v,\tilde{w}_1) f_2(v,\tilde{w}_2) f_3(v,\tilde{w}_3)}{\int_0^\infty f_1(u,\tilde{w}_1) f_2(u,\tilde{w}_2) f_3(u,\tilde{w}_3) du} dv,
\end{aligned}
$$

whereas

(a) for a parallel system,

$$
\begin{aligned}
&P(L_{(\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)} > t) \\
&= \int_0^\infty [1 - (1 - P(W_{1,v+t} - W_{1,v} \le w_1 - \tilde{w}_1))(1 - P(W_{2,v+t} - W_{2,v} \le w_2 - \tilde{w}_2)) \\
&\quad (1 - P(W_{3,v+t} - W_{3,v} \le w_3 - \tilde{w}_3))] \\
&\quad \times \frac{f_1(v,\tilde{w}_1) f_2(v,\tilde{w}_2) f_3(v,\tilde{w}_3)}{\int_0^\infty f_1(u,\tilde{w}_1) f_2(u,\tilde{w}_2) f_3(u,\tilde{w}_3) du} dv,
\end{aligned}
$$

where, similar to Example 2,

$$
\begin{aligned}
P(W_{i,v+t} - W_{i,v} \le w_i - \tilde{w}_i) &= \int_0^w \frac{1}{\Gamma(\alpha_i(v+t) - \alpha_i(t))} {\lambda_i}^{\alpha_i(v+t)-\alpha_i(t)} x^{\alpha_i(v+t)-\alpha_i(t)-1} \\
&\qquad \exp(-\lambda_i x) dx, \ i = 1, 2, 3, \\
f_i(v,\tilde{w}_i) &= \frac{A_i(v,\tilde{w}_i)}{\left(\int_0^\infty s^{\alpha_i(x)-1} \exp(-s) ds\right)^2}
\end{aligned}
$$

https://doi.org/10.1017/S0269964821000097 Published online by Cambridge University Press

and

$$A_i(v,\tilde{w}_i) = \int_0^{\tilde{w}_i} \Big[-(\ln\lambda_i + \ln u)\ \alpha_i{}'(v)\lambda_i^{\alpha_i(v)}u^{\alpha_i(v)-1}\exp(-\lambda_i u) \times \left(\int_0^\infty s^{\alpha_i(v)-1}\exp(-s)ds\right) + \lambda_i^{\alpha_i(x)}u^{\alpha_i(x)-1}\exp(-\lambda_i u)\left(\int_0^\infty \alpha'{}_i(x)(\ln s)\mathrm{s}^{\alpha_i(x)-1}\exp(-s)ds\right)\Big]\,du,$$
$$i=1,2,3.$$

5. CONCLUDING REMARKS

We consider a new notion of the remaining lifetime (called the predicted remaining lifetime) for degrading items with degradation modeled by the homogeneous and nonhomogeneous processes with independent increments. For this, we introduce the notion of the virtual age as a random time required to accumulate the specified degradation level. An important, novel application of the proposed setting is when the used item is incepted into operation with the observed level of degradation and we are interested in its remaining lifetime.

Some stochastic comparisons are obtained for homogeneous and heterogeneous populations of items operating in two regimes. Specifically, we show that under the formulated assumptions on the ordering of stochastic processes of deterioration in each regime, the corresponding predicted remaining lifetimes are ordered accordingly. Moreover, the problem of recalculation of the virtual age upon switching from one regime to another is considered and the corresponding PRL is derived using the suggested virtual age concept.

A generalization of the described approach to the case of coherent, non-repairable systems with n independent components is discussed. In practical situations, it may happen that we are interested, for instance, in the lifetime of a system composed of the used components with observed levels of degradation. Under certain assumptions, the suggested approach is extended to this important setting. Specifically, the general relationship for the distribution of the predicted remaining lifetime is derived and the specific cases of series and parallel systems of two independent components are considered.

We think that the developed approach can be extended in several directions. In the current paper, the failure threshold for a component was fixed. At some instances, it is reasonable to consider a random failure threshold. The n-component system in Section 4 was described under the assumption of independent components. This can be possibly lifted by considering the dependence structure in the form of the relevant copula.

Acknowledgments

The authors would like to thank the referees for helpful comments and advices, which have improved the presentation of this paper. The work of the second author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2019R1A2B5B02069500). The work of the second author was also supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant No. 2019R1A6A1A11051177).

References

1. Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing*. New York: Holt, Rinerhart & Winston.

2. Cai, N. & Zheng, Y. (2012). Preservation of generalized ageing class on the residual life at random time. *Journal of Statistical Planning and Inference* 142: 148–154.
3. Cha, J.H. & Finkelstein, M. (2018). On stochastic comparisons for population age and remaining lifetime. *Statistical Papers* 59: 199–213.
4. Cha, J.H. & Finkelstein, M. (2018). *Point processes for reliability analysis: shocks and repairable systems*. London: Springer.
5. Finkelstein, M. (2008). *Failure rate modelling for reliability and risk*. London: Springer.
6. Finkelstein, M. & Cha, J.H. (2013). *Stochastic modelling for reliability: shocks, burn-in and heterogeneous population*. London: Springer.
7. Finkelstein, M. & Vaupel, J.W. (2015). On random age and remaining lifetime for populations of items. *Applied stochastic Models in Business and Industry* 31: 681–689.
8. Hazra, N.K., Finkelstein, M., & Cha, J.H. (2018). Stochastic ordering for populations of manufactured items. *TEST* 27: 173–196.
9. Kijima, M. (1989). Some results for repairable systems with general repair. *Journal of Applied Probability* 26: 89–102.
10. Li, X. & Fang, R. (2018). Stochastic properties of two general versions of the residual lifetime at random times. *Applied Stochastic Models in Business and Industry* 34: 528–543.
11. Li, X. & Lu, J. (2003). Stochastic comparisons on residual life and inactivity time of series and parallel systems. *Probability in the Engineering and Informational Sciences* 17: 267–275.
12. Li, X. & Zuo, M.J. (2004). Stochastic comparison of residual life and inactivity time at a random time. *Stochastic Models* 20: 229–235.
13. Meeker, W.Q. & Escobar, L.A. (1998). *Statistical methods for reliability data*. New York: John Wiley.
14. Misra, N., Gupta, N., & Dhariyal, I.D. (2008). Stochastic properties of residual life and inactivity time at a random time. *Stochastic Models* 24: 89–102.
15. Nanda, A.K. & Kundu, A. (2009). On generalized stochastic orders of dispersion-type. *Calcutta Statistical Association Bulletin* 16: 241–244.
16. Nelson, W. (1982). *Applied life data analysis*. New York: Wiley.
17. Nelson, W. (1990). *Accelerated testing, statistical models, test plans and data analysis*. New York: Wiley.
18. van Noortwijk, J.M. (2009). A survey of the application of gamma processes in maintenance. *Reliability Engineering & System Safety* 94: 2–21.
19. Yue, D. & Cao, J. (2000). Some results on the residual life at random time. *Acta Mathematicae Applicatae Sinica* 16: 436–443.