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From homoclinics to quasi-periodic solutions for ordinary differential equations

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We consider the quasi-periodic solutions bifurcated from a degenerate homoclinic solution. Assume that the unperturbed system has a homoclinic solution and a hyperbolic fixed point. The bifurcation function for the existence of a quasi-periodic solution of the perturbed system is obtained by functional analysis methods. The zeros of the bifurcation function correspond to the existence of the quasi-periodic solution at the non-zero parameter values. Some solvable conditions of the bifurcation equations are investigated. Two examples are given to illustrate the results.

Keywords: degenerate homoclinic solution; quasi-periodic solution; Lyapunov–Schmidt reduction; Fredholm alternative; exponential dichotomy

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1. Introduction

Homoclinic bifurcations are interesting topics in dynamics because they are related to many important dynamical behaviours, such as subharmonic bifurcations and chaotic motions. The problem of determining the parameter values for which the perturbed system undergoes subharmonic or homoclinic bifurcations arises in a variety of applications. Some of these are predator–prey models [5, 7], climate systems [18], travelling waves in neurons and Nagumo equations [4,11,14] and chemical stirred tank reactors [1, 12].

Various techniques have been used to study homoclinic bifurcations. From a geometrical viewpoint, Melnikov [15] investigated the persistence of homoclinic solutions in $\mathbb{R}^2$. Mock [16] used the topological degree. By using functional analysis methods Chow *et al*. [6] considered the homoclinic and subharmonic bifurcations of Duffing's equation under damping and excitation. Palmer [17] generalized the methods in [6] to $\mathbb{R}^N$. Under the assumption that the unperturbed system had a homoclinic solution and a hyperbolic equilibrium, Palmer [17] obtained some conditions for which the perturbed system possessed a homoclinic solution. Note that the homoclinic solution for the unperturbed system is an orbit of the intersection of the stable and unstable manifolds for the hyperbolic equilibrium. When the dimension of the intersection is 1, the homoclinic solution is called non-degenerate. Otherwise, it is called degenerate. Generally, the homoclinic solution is non-degenerate in $\mathbb{R}^2$ or $\mathbb{R}^3$ since the dimension of the intersection is always 1. In § 4, we give an example in $\mathbb{R}^6$, which illustrates that the dimension of the intersection is 3. Hence, the dimension can be as much as $[N/2]$ in $\mathbb{R}^N$. From the late 1980s onwards, many authors have studied degenerate homoclinic bifurcations [3, 8–10, 21]. In [9], Gruendler studied the persistence of the degenerate homoclinic solution for the system in $\mathbb{R}^N$ with two parameters, μ_1 and μ_2. By the methods of functional analysis, he obtained some curves that passed through the origin in the (μ_1, μ_2)-plane. The perturbed system had a homoclinic solution when the parameters were on these curves.

The subharmonic bifurcations have been studied extensively in [2, 8, 13, 19–21]. In [2], Battelli and Fečkan investigated the subharmonic bifurcations for the singular system

$$\varepsilon\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \tag{1.1}$$

where $x \in \mathbb{R}^N$ and g is periodic in t. By assuming that the system $\dot{x} = f(x)$ had a non-degenerate homoclinic solution, Battelli and Fečkan obtained some conditions to ensure the persistence of the subharmonic orbit for system (1.1). The subharmonic bifurcations from a degenerate homoclinic orbit for system (1.1) were studied by Fečkan and Gruendler [8].

In [6, 8, 17, 19] and references therein the persistence of the homoclinics and subharmonics bifurcated from non-degenerate or degenerate homoclinic solutions were considered. In this paper, we investigate the existence of a quasi-periodic solution near a degenerate homoclinic solution. When the parameter values are zero, the system has a hyperbolic equilibrium and a known degenerate homoclinic solution. We obtain some criteria for which the quasi-periodic solution persists for small non-zero parameter values. Precisely, we consider the system

$$\dot{x}(t) = f(x(t)) + \varepsilon g(x(t), t, \varepsilon), \tag{1.2}$$

where $x \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$ and g is periodic in t. We make the following assumptions.

(H1) $f(0) = 0$, where the eigenvalues of $\mathrm{D}f(0)$ lie off the imaginary axis.

(H2) The unperturbed system

$$\dot{x}(t) = f(x(t)) \tag{1.3}$$

has a bounded solution γ that is homoclinic to 0. That is, there is a differentiable function $\gamma(t)$ satisfying $\dot{\gamma}(t) = f(\gamma(t))$ and $\lim_{|t|\to\infty} \gamma(t) = 0$.

(H3) $g(0, t, \varepsilon) = 0$ and $g(x, t + T, \varepsilon) = g(x, t, \varepsilon)$ for some $T > 0$.

Assumption (H1) implies that 0 is a hyperbolic equilibrium of (1.3). In [8, 9, 21], the homoclinics and subharmonics bifurcated from a known homoclinic are extensively studied. By methods of functional analysis, we investigate the appearance of quasi-periodic solutions near a homoclinic under the assumptions (H1)–(H3). Using a variant of the Lyapunov–Schmidt reduction, the bifurcation functions for the existence of quasi-periodic solutions are obtained. Some criteria for the solvability of the bifurcation function are given. When γ is a non-degenerate or degenerate homoclinic, some examples are provided to illustrate our results. Hence, the theorem for the existence of quasi-periodic solutions is valid for both non-degenerate and degenerate homoclinics.

2. Preliminaries and main result

Let $\mathrm{D}_i h$ denote the first derivative of h with respect to the ith variable, and let $\mathrm{D}_{ij} h$ denote the second derivative of h with respect to the ith and jth variables. By (H2), (1.3) has a homoclinic solution, γ. The linear variational equation of (1.3) along γ is

$$\dot{u}(t) = \mathrm{D}f(\gamma(t))u(t). \tag{2.1}$$

Let W^{s} and W^{u} denote the stable and unstable manifolds of the origin, and let d_{s} and d_{u} denote their corresponding dimensions, respectively. Since 0 is a hyperbolic fixed point and γ is homoclinic to 0, γ must approach origin exponentially along W^{s} as $t \to \infty$, and along W^{u} as $t \to -\infty$. Hence, $\gamma \subset W^{\mathrm{s}} \cap W^{\mathrm{u}}$. Note that $\mathrm{D}f(\gamma(t)) \to \mathrm{D}f(0)$ as $t \to \pm\infty$, giving a hyperbolic $N \times N$ matrix. By the roughness of exponential dichotomy, (2.1) has exponential dichotomies both in $\mathbb{R}^+$ and in $\mathbb{R}^-$. Fečkan and Gruendler [8] obtained two solutions for (2.1): one for $t \geqslant 0$ and one for $t \leqslant 0$. The two solutions match at $t = 0$. For the solutions of (2.1), the following lemma holds.

Lemma 2.1 (Fečkan and Gruendler [8]). *There exist a fundamental solution, U, of (2.1) along with a non-singular matrix C, constants $\alpha > 0$, $M > 0$ and four projections, P_{ss}, P_{us}, P_{su} and P_{uu}, such that $P_{\mathrm{ss}} + P_{\mathrm{us}} + P_{\mathrm{su}} + P_{\mathrm{uu}} = I$ and the following properties hold:*

(i) $|U(t)(P_{\mathrm{ss}} + P_{\mathrm{us}})U(s)^{-1}| \leqslant M\mathrm{e}^{2\alpha(s-t)}$ *for* $0 \leqslant s \leqslant t$;

(ii) $|U(t)(P_{\mathrm{su}} + P_{\mathrm{uu}})U(s)^{-1}| \leqslant M\mathrm{e}^{2\alpha(t-s)}$ *for* $0 \leqslant t \leqslant s$;

(iii) $|U(t)(P_{\mathrm{ss}} + P_{\mathrm{su}})U(s)^{-1}| \leqslant M\mathrm{e}^{2\alpha(t-s)}$ *for* $t \leqslant s \leqslant 0$;

(iv) $|U(t)(P_{\mathrm{us}} + P_{\mathrm{uu}})U(s)^{-1}| \leqslant M\mathrm{e}^{2\alpha(s-t)}$ *for* $s \leqslant t \leqslant 0$;

(v)
$$\sup_{t\geqslant 0}\left|U(t)(P_{\mathrm{ss}} + P_{\mathrm{us}})U(t)^{-1} - C\begin{pmatrix} I_{\mathrm{s}} & 0 \\ 0 & 0\end{pmatrix}C^{-1}\right|\mathrm{e}^{2\alpha t} \leqslant \infty;$$

(vi)
$$\sup_{t\geqslant 0}\left|U(t)(P_{\mathrm{su}} + P_{\mathrm{uu}})U(t)^{-1} - C\begin{pmatrix} 0 & 0 \\ 0 & I_{\mathrm{u}}\end{pmatrix}C^{-1}\right|\mathrm{e}^{2\alpha t} \leqslant \infty;$$

(vii)
$$\sup_{t\leqslant 0}\left|U(t)(P_{\mathrm{ss}} + P_{\mathrm{su}})U(t)^{-1} - C\begin{pmatrix} 0 & 0 \\ 0 & I_{\mathrm{u}}\end{pmatrix}C^{-1}\right|\mathrm{e}^{-2\alpha t} \leqslant \infty;$$

(viii)
$$\sup_{t\leqslant 0}\left|U(t)(P_{\mathrm{us}} + P_{\mathrm{uu}})U(t)^{-1} - C\begin{pmatrix} I_{\mathrm{s}} & 0 \\ 0 & 0\end{pmatrix}C^{-1}\right|\mathrm{e}^{-2\alpha t} \leqslant \infty,$$

where $I_{\rm s}$ and $I_{\rm u}$ are $d_{\rm s}\times d_{\rm s}$ and $d_{\rm u}\times d_{\rm u}$ identity matrices, respectively. In addition, $\operatorname{rank} P_{\rm ss} = \operatorname{rank} P_{\rm uu}$.

Let u_i denote the ith column of U. By (ii), (iv) and (i), (iii) of lemma 2.1, respectively, we know that

$$\lim_{|t|\to\infty} u_i(t) = \infty \quad \text{if } u_i \in P_{\rm uu}U, \qquad \lim_{|t|\to\infty} u_i(t) = 0 \quad \text{if } u_i \in P_{\rm ss}U.$$

Let $d := \operatorname{rank} P_{\rm ss}$. Renumbering if necessary, we can take $P_{\rm ss}$ and $P_{\rm uu}$ such that

$$P_{\rm uu} = \begin{pmatrix} I_d & 0 & 0 \\ 0 & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{\rm ss} = \begin{pmatrix} 0_d & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where I_d and 0_d are $d\times d$ identity and zero matrices. Hence, we have

$$\lim_{|t|\to\infty} u_i(t) = \infty, \quad \lim_{|t|\to\infty} u_{d+i}(t) = 0, \quad i = 1, 2, \dots, d.$$

Let $u_i^{\perp}$ be defined such that $(u_i^{\perp}, u_j) = \delta_{ij}$, $i, j = 1, \dots, d$, the Kronecker delta. The vectors $u_i^{\perp}$ can be computed as follows. Let $U^{\perp}$ be a matrix such that $u_i^{\perp}$ is its ith column. Then $U^{\perp t}U = I$, the identity matrix. Differentiating the equation, we get $\dot U^{\perp t}U + U^{\perp t}\dot U = 0$. Hence, $\dot U^{\perp t} = -U^{\perp t}\dot U U^{-1} = -U^{\perp t}\mathrm{D}f(\gamma)$. Then $\dot U^{\perp} = -\mathrm{D}f(\gamma)^t U^{\perp}$, which implies that $U^{\perp}$ is the adjoint of U.

We introduce some notation. For $i = 1, \dots, d$, let

$$M_i(\beta, \varepsilon) = \frac{1}{2}\sum_{j,k=1}^{d-1} \lambda_{ijk}\beta_j\beta_k + \varepsilon\eta_i, \tag{2.2}$$

where

$$\lambda_{ijk} = 2\int_{-\infty}^{\infty} (u_i^{\perp}(s), \mathrm{D}^2 f(\gamma(s))u_{d+j}(s)u_{d+k}(s))\,\mathrm{d}s,$$

$$\eta_i = 2\int_{-\infty}^{\infty} (u_i^{\perp}(s), g(\gamma(s), s, 0))\,\mathrm{d}s.$$

Define $M\colon \mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^d$ by

$$M(\beta, \varepsilon) = (M_1(\beta, \varepsilon), \dots, M_d(\beta, \varepsilon)). \tag{2.3}$$

The following is the main result of the paper.

Theorem 2.2. *Assume that (H1)–(H3) hold. If there exist some $(\beta^*, \varepsilon^*) \in \mathbb{R}^{d-1}\times\mathbb{R}$ such that $M(\beta^*, \varepsilon^*) = 0$ and $\mathrm{D}_{(\beta,\varepsilon)}M(\beta^*, \varepsilon^*)$ is non-singular $d\times d$ matrix, then there exist $s_0 > 0$, $p_0 > 0$ and a C^1-function $\varphi^*\colon (-s_0, s_0)\times(p_0, \infty)\to\mathbb{R}$ such that (1.2) with $\varepsilon = s^2(\varepsilon^* + \varphi^*(s, p))$ has a quasi-periodic solution for $(s, p) \in ((-s_0, s_0)/\{0\})\times(p_0, \infty)$.*

If $d = 1$, the homoclinic solution γ is non-degenerate. This is a special case of theorem 2.2. Then (2.1) has only one bounded solution. Hence, the adjoint equation of (2.1) also has only one bounded solution, say $u^{\perp}$. For the non-degenerate homoclinic bifurcation, there is no β in M. That is

$$M(\varepsilon) = \eta\varepsilon,$$

where

$$\eta = 2\int_{-\infty}^{\infty} (u^{\perp}(s), g(\gamma(s), s, 0))\,\mathrm{d}s.$$

Then we have the following corollary.

COROLLARY 2.3. *Assume that $d = 1$ and (H1)–(H3) hold. If $\eta \neq 0$, then there exist $s_0 > 0$, $p_0 > 0$ and a C^1-function $\varphi^*\colon (-s_0, s_0) \times (p_0, \infty) \to \mathbb{R}$ such that (1.2) with $\varepsilon = s^2\varphi^*(s,p)$ has a quasi-periodic solution for $(s,p) \in ((-s_0, s_0)/\{0\}) \times (p_0, \infty)$.*

At the end of the paper, we give some examples to illustrate our results. Example 4.1 is for the bifurcation of the non-degenerate homoclinic solution. This illustrates corollary 2.3. Example 4.2 considers the bifurcation of the degenerate homoclinic solution. It shows that the dimension of the intersection of stable and unstable manifolds can be as high as $[N/2]$. Theorem 2.2 is applied for the degenerate case.

3. The proof of theorem 2.2

To prove theorem 2.2, our strategy is to construct a quasi-periodic solution of (1.2), which is formed by two periodic functions with rationally independent periods. We define some Banach spaces. For each $p > 0$, let

$$X_p = C^0((-p,p), \mathbb{R}^N), \qquad \bar{X}_p = \{x \in X_p \mid x(-p) = x(p)\}.$$

For $x \in \bar{X}_p$, it is clear that x is a periodic function with period $2p$.

Consider a non-homogeneous equation

$$\dot{z} = \mathrm{D}f(\gamma)z + \mu, \tag{3.1}$$

where $\mu \in X_p$.

We define an operator $K_p\colon X_p \to C^0((-p,0)\cup(0,p), \mathbb{R}^N)$ by

$$(K_p\mu)(t) = \begin{cases} U(t)P_{\mathrm{su}}a_1 + U(t)(P_{\mathrm{us}} + P_{\mathrm{uu}})U^{-1}(-p)b_1 \\ \quad + U(t)\displaystyle\int_0^t ((P_{\mathrm{ss}} + P_{\mathrm{su}})U^{-1}(s), \mu(s))\,\mathrm{d}s \\ \quad + U(t)\displaystyle\int_{-p}^t ((P_{\mathrm{us}} + P_{\mathrm{uu}})U^{-1}(s), \mu(s))\,\mathrm{d}s, \quad t \in (-p,0), \\ U(t)P_{\mathrm{us}}a_2 + U(t)(P_{\mathrm{su}} + P_{\mathrm{uu}})U^{-1}(p)b_2 \\ \quad + U(t)\displaystyle\int_0^t ((P_{\mathrm{ss}} + P_{\mathrm{us}})U^{-1}(s), \mu(s))\,\mathrm{d}s \\ \quad - U(t)\displaystyle\int_t^p ((P_{\mathrm{su}} + P_{\mathrm{uu}})U^{-1}(s), \mu(s))\,\mathrm{d}s, \quad t \in (0,p), \end{cases} \tag{3.2}$$

where $a_i, b_i \in \mathbb{R}^N$, $i = 1, 2$, are unknowns.

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It is easy to check that $(K_p\mu)$ is a solution of (3.1) on $(-p,0)\cup(0,p)$. If there exist some a_i and b_i such that $(K_p\mu)(0^-)=(K_p\mu)(0^+)$ and $(K_p\mu)(-p)=(K_p\mu)(p)$, then $(K_p\mu)\in\bar{X}_p$ is a periodic solution of (3.1). As in [8], if $(K_p\mu)(0^-)=(K_p\mu)(0^+)$ and $(K_p\mu)(-p)=(K_p\mu)(p)$, then there exist $b_i^*(p,\mu)$ satisfying $|b_i^*(p,\mu)|=|\mu|O(\mathrm{e}^{-\alpha p})$ such that

$$\int_{-p}^{p}(P_{\mathrm{uu}}U^{-1}(s),\mu(s))\,\mathrm{d}s+P_{\mathrm{uu}}U^{-1}(-p)b_1^*(p,\mu)-P_{\mathrm{uu}}U^{-1}(p)b_2^*(p,\mu)=0$$

for $\mu\in X_p$. Define the operator $\mathcal{L}_p\colon X_p\to\mathbb{R}^N$ by

$$\mathcal{L}_p(\mu)=U^{-1}(-p)b_1^*(p,\mu)-U^{-1}(p)b_2^*(p,\mu).$$

Clearly, $|\mathcal{L}_p(\mu)|=|\mu|O(\mathrm{e}^{-\alpha p})$.

Using $\mathcal{L}_p$, we define a subset of X_p by

$$\tilde{X}_p=\left\{z\in X_p\;\middle|\;\int_{-p}^{p}(P_{\mathrm{uu}}U^{-1}(s),z(s))\,\mathrm{d}s+P_{\mathrm{uu}}\mathcal{L}_p(z)=0\right\}.$$

From the above discussions, we know that (3.1) has a periodic solution $(K_p\mu)$ with period $2p$ if $\mu\in\tilde{X}_p$. Thus, $K_p\colon\tilde{X}_p\to\bar{X}_p$.

Let $b\colon\mathbb{R}\to[0,\infty)$ be a smooth function with compact support and $\operatorname{supp}(b)\subset(-2,2)$. For each $a>2$,

$$\int_{-a}^{a}b(s)\,\mathrm{d}s=\int_{-a}^{-2}b(s)\,\mathrm{d}s+\int_{-2}^{2}b(s)\,\mathrm{d}s+\int_{2}^{a}b(s)\,\mathrm{d}s=\int_{-2}^{2}b(s)\,\mathrm{d}s.$$

Let

$$\zeta(t)=\frac{b(t)}{\int_{-2}^{2}b(s)\,\mathrm{d}s}.$$

For $a>2$, it is clear that

$$\int_{-a}^{a}\zeta(s)\,\mathrm{d}s=1.$$

Define a matrix

$$A_p=(a_{ij}(p))_{d\times d}\,,$$

where $a_{ij}(p)=(\mathcal{L}_p\zeta u_j)_i$, $i,j=1,\dots,d$. Clearly, $a_{ij}(p)=O(\mathrm{e}^{-\alpha p})$. Then $|A_p|=O(\mathrm{e}^{-\alpha p})$. Hence, there exists $p_1>0$ such that the matrix $(I+A_p)$ is invertible for $p\in(p_1,\infty)$. Define a vector $V\colon X_p\to\mathbb{R}^N$ by

$$V(z)=((I+A_p)^{-1}v(z),0,\dots,0)^{\mathrm{T}}:=(V_1(z),\dots,V_d(z),0,\dots,0)^{\mathrm{T}},\tag{3.3}$$

where $v(z)=(v_1(z),\dots,v_d(z))^{\mathrm{T}}$ and

$$v_i(z)=\int_{-p}^{p}(u_i^{\perp}(s),z(s))\,\mathrm{d}s+(\mathcal{L}_pz)_i.\tag{3.4}$$

Note that $(I+A_p)^{-1}=I-A_p+A_p^2-\cdots$ and $|A_p|=O(\mathrm{e}^{-\alpha p})$. We get $V_i(z)=v_i(z)+O(\mathrm{e}^{-\alpha p})$. Define a map $\varPi_p\colon X_p\to X_p$ by

$$(\varPi_pz)(t)=\zeta(t)P_{\mathrm{uu}}U(t)V(z)=\sum_{i=1}^{d}\zeta(t)u_i(t)V_i(z).$$

With a proof similar to that in [8], we get that Π_p is a projection and $\mathrm{Im}(I-\Pi_p) \subset \tilde{X}_p$ for $p > 2$.

Since $K_p\colon \tilde{X}_p \to \bar{X}_p$ and $\mathrm{Im}(I-\Pi_p) \subset \tilde{X}_p$, $K_p(I-\Pi_p)(\mu)$ is a periodic solution of period $2p$. We want to construct another periodic solution with period $2\sqrt{2}p$. For each $p > 0$, let

$$X_{\sqrt{2}p} = C^0((-\sqrt{2}p, \sqrt{2}p), \mathbb{R}^N), \qquad \bar{X}_{\sqrt{2}p} = \{x \in X_{\sqrt{2}p} \mid x(-\sqrt{2}p) = x(\sqrt{2}p)\}.$$

For $x \in \bar{X}_{\sqrt{2}p}$, it is obvious that x is a periodic function with period $2\sqrt{2}p$. Now we return to considering the solutions of (3.1) with $\mu \in X_{\sqrt{2}p}$.

As in (3.2), we define an operator $K_{\sqrt{2}p}\colon X_{\sqrt{2}p} \to ((-\sqrt{2}p, 0) \cup (0, \sqrt{2}p), \mathbb{R}^N)$ by

$$(K_{\sqrt{2}p}\mu)(t) = \begin{cases} U(t)P_{\mathrm{su}}\tilde{a}_1 + U(t)(P_{\mathrm{us}} + P_{\mathrm{uu}})U^{-1}(-\sqrt{2}p)\tilde{b}_1 \\ \quad + U(t)\displaystyle\int_0^t ((P_{\mathrm{ss}} + P_{\mathrm{su}})U^{-1}(s), \mu(s))\,\mathrm{d}s \\ \quad + U(t)\displaystyle\int_{-\sqrt{2}p}^t ((P_{\mathrm{us}} + P_{\mathrm{uu}})U^{-1}(s), \mu(s))\,\mathrm{d}s, \quad t \in (-\sqrt{2}p, 0), \\ U(t)P_{\mathrm{us}}\tilde{a}_2 + U(t)(P_{\mathrm{su}} + P_{\mathrm{uu}})U^{-1}(\sqrt{2}p)\tilde{b}_2 \\ \quad + U(t)\displaystyle\int_0^t ((P_{\mathrm{ss}} + P_{\mathrm{us}})U^{-1}(s), \mu(s))\,\mathrm{d}s \\ \quad - U(t)\displaystyle\int_t^{\sqrt{2}p} ((P_{\mathrm{su}} + P_{\mathrm{uu}})U^{-1}(s), \mu(s))\,\mathrm{d}s, \quad t \in (0, \sqrt{2}p), \end{cases} \tag{3.5}$$

where $\tilde{a}_i, \tilde{b}_i \in \mathbb{R}^N$, $i = 1, 2$, are unknowns.

It is clear that $(K_{\sqrt{2}p}\mu)$ is a solution of (3.1) with $\mu \in X_{\sqrt{2}p}$ on $(-\sqrt{2}p, 0) \cup (0, \sqrt{2}p)$. If

$$(K_{\sqrt{2}p}\mu)(0^-) = (K_{\sqrt{2}p}\mu)(0^+) \quad \text{and} \quad (K_{\sqrt{2}p}\mu)(-\sqrt{2}p) = (K_{\sqrt{2}p}\mu)(\sqrt{2}p)$$

for some $\tilde{a}_i$ and $\tilde{b}_i$, then $(K_{\sqrt{2}p}\mu)$ is a periodic solution of (3.1) with period $2\sqrt{2}p$.

Assume that

$$(K_{\sqrt{2}p}\mu)(0^-) = (K_{\sqrt{2}p}\mu)(0^+) \quad \text{and} \quad (K_{\sqrt{2}p}\mu)(-\sqrt{2}p) = (K_{\sqrt{2}p}\mu)(\sqrt{2}p).$$

Following the proofs in [8], we find that there exist $\tilde{b}_i = \tilde{b}_i^*(p, \mu)$ satisfying

$$|\tilde{b}_i^*(p, \mu)| = |\mu| O(\mathrm{e}^{-\alpha p})$$

and

$$\int_{-\sqrt{2}p}^{\sqrt{2}p} (P_{\mathrm{uu}}U^{-1}(s), \mu(s))\,\mathrm{d}s + P_{\mathrm{uu}}U^{-1}(-\sqrt{2}p)\tilde{b}_1^*(p, \mu) - P_{\mathrm{uu}}U^{-1}(\sqrt{2}p)\tilde{b}_2^*(p, \mu) = 0$$

for $\mu \in X_{\sqrt{2}p}$. Define a map $\mathcal{L}_{\sqrt{2}p}\colon X_{\sqrt{2}p} \to \mathbb{R}^N$ by

$$\mathcal{L}_{\sqrt{2}p}(\mu) = U^{-1}(-\sqrt{2}p)\tilde{b}_1^*(p, \mu) - U^{-1}(\sqrt{2}p)\tilde{b}_2^*(p, \mu).$$

Then $|\mathcal{L}_{\sqrt{2}p}(\mu)| = |\mu| O(\mathrm{e}^{-\alpha p})$. Let

$$\tilde{X}_{\sqrt{2}p} = \left\{ z \in X_{\sqrt{2}p} \;\middle|\; \int_{-\sqrt{2}p}^{\sqrt{2}p} (P_{\mathrm{uu}} U^{-1}(s), z(s))\,\mathrm{d}s + P_{\mathrm{uu}} \mathcal{L}_{\sqrt{2}p}(z) = 0 \right\}.$$

It is clear that $(K_{\sqrt{2}p}\mu)$ is a periodic solution of (3.1) with period $2\sqrt{2}p$ for $\mu \in \tilde{X}_{\sqrt{2}p}$. This implies that the map $K_{\sqrt{2}p} \colon \tilde{X}_{\sqrt{2}p} \to \tilde{X}_{\sqrt{2}p}$.

Using $\mathcal{L}_{\sqrt{2}p}$, we define a matrix

$$A_{\sqrt{2}p} = (\tilde{a}_{ij}(\sqrt{2}p))_{d \times d}\,,$$

where $\tilde{a}_{ij}(\sqrt{2}p) = (\mathcal{L}_{\sqrt{2}p} \zeta u_j)_i$, $i, j = 1, \dots, d$. It is obvious that

$$|\tilde{a}_{ij}(\sqrt{2}p)| = O(\mathrm{e}^{-\alpha p}),$$

and hence $|A_{\sqrt{2}p}| = O(\mathrm{e}^{-\alpha p})$. Then, there exists $p_2 > 0$ such that the matrix $(I + A_{\sqrt{2}p})$ is invertible for $p \in (p_2, \infty)$. Define a vector $W \colon X_{\sqrt{2}p} \to \mathbb{R}^N$ by

$$W(z) = ((I + A_{\sqrt{2}p})^{-1} w(z), 0, \dots, 0)^{\mathrm{T}} := (W_1(z), \dots, W_d(z), 0, \dots, 0)^{\mathrm{T}}, \qquad (3.6)$$

where $w(z) = (w_1(z), \dots, w_d(z))^{\mathrm{T}}$ and

$$w_i(z) = \int_{-\sqrt{2}p}^{\sqrt{2}p} (u_i^{\perp}(s), z(s))\,\mathrm{d}s + (\mathcal{L}_{\sqrt{2}p} z)_i.$$

Since $(I + A_{\sqrt{2}p})^{-1} = I - A_{\sqrt{2}p} + A^2_{\sqrt{2}p} + \cdots$, we know that

$$W_i(z) = w_i(z) + O(\mathrm{e}^{-\alpha p})$$

for $z \in X_{\sqrt{2}p}$ and $p \in (p_2, \infty)$. Define a map $\mathit{\Pi}_{\sqrt{2}p} \colon X_{\sqrt{2}p} \to X_{\sqrt{2}p}$ by

$$(\mathit{\Pi}_{\sqrt{2}p} z)(t) = \zeta(t) P_{\mathrm{uu}} U(t) W(z) = \sum_{i=1}^{d} \zeta(t) u_i(t) W_i(z).$$

Similarly to [8], we get that $\mathit{\Pi}_{\sqrt{2}p}$ is a projection, and

$$\mathrm{Im}(I - \mathit{\Pi}_{\sqrt{2}p}) \subset \tilde{X}_{\sqrt{2}p} \quad \text{for } p > 2.$$

Hence, $K_{\sqrt{2}p}(I - \mathit{\Pi}_{\sqrt{2}p})(\mu)$ is a periodic solution of period $2\sqrt{2}p$.

Let $\mathit{\Pi}_{p,\sqrt{2}p} = \frac{1}{2}(\mathit{\Pi}_p + \mathit{\Pi}_{\sqrt{2}p})$ be defined by $(\mathit{\Pi}_{p,\sqrt{2}p})(z) = \frac{1}{2}(\mathit{\Pi}_p(z) + \mathit{\Pi}_{\sqrt{2}p}(z))$ for $z \in X_{\sqrt{2}p}$.

Lemma 3.1. *$\mathit{\Pi}_{p,\sqrt{2}p}$ is a projection.*

Proof. Let $\bar{W}(z) = (W_1(z), \dots, W_d(z))$ for $z \in X_{2\sqrt{2}p}$, where the $W_i(z)$ are defined in (3.6). For $z \in X_{\sqrt{2}p}$, we get from (3.4) and the definition of $\mathit{\Pi}_{\sqrt{2}p}$ that

$$\begin{aligned} v_i(\mathit{\Pi}_{\sqrt{2}p} z) &= \int_{-p}^{p} (u_i^{\perp}(s), (\mathit{\Pi}_{\sqrt{2}p} z)(s))\,\mathrm{d}s + (\mathcal{L}_p(\mathit{\Pi}_{\sqrt{2}p} z))_i \\ &= \int_{-p}^{p} (u_i^{\perp}(s), \zeta(s) P_{\mathrm{uu}} U(s) W(z))\,\mathrm{d}s + (\mathcal{L}_p \zeta(s) P_{\mathrm{uu}} U(s) W(z))_i \\ &= (\bar{W}(z))_i + (A_p \bar{W}(z))_i = ((I + A_p)\bar{W}(z))_i \quad \text{for } i = 1, \dots, d. \end{aligned}$$

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Hence, we obtain that

$$v(\Pi_{\sqrt{2}p}z) = (v_1(\Pi_{\sqrt{2}p}z), \dots, v_d(\Pi_{\sqrt{2}p}z)) = (I + A_p)\bar{W}(z). \tag{3.7}$$

By (3.7) and the definition of Π_p, we have

$$\begin{aligned}
\Pi_p(\Pi_{\sqrt{2}p}z)(t) &= \zeta(t)P_{\mathrm{uu}}U(t)V(\Pi_{\sqrt{2}p}z) \\
&= \zeta(t)P_{\mathrm{uu}}U(t)((I + A_p)^{-1}v(\Pi_{\sqrt{2}p}z), 0\dots, 0)^{\mathrm{T}} \\
&= \zeta(t)P_{\mathrm{uu}}U(t)((I + A_p)^{-1}((I + A(p)\bar{W}(z)), 0\dots, 0))^{\mathrm{T}} \\
&= \zeta(t)P_{\mathrm{uu}}U(t)(W_1(z), \dots, W_d(z), 0\dots, 0)^{\mathrm{T}} \\
&= \zeta(t)P_{\mathrm{uu}}U(t)W(z) = (\Pi_{\sqrt{2}p}z)(t),
\end{aligned}$$

which implies that $\Pi_p(\Pi_{\sqrt{2}p}z) = \Pi_{\sqrt{2}p}(z)$ for $z \in X_{\sqrt{2}p}$.

Similarly, we can obtain $\Pi_{\sqrt{2}p}(\Pi_p z) = \Pi_p(z)$ for $z \in X_{\sqrt{2}p}$.

For $z \in X_{\sqrt{2}p}$, we obtain that

$$\begin{aligned}
\Pi^2_{p,\sqrt{2}p}(z) &= \Pi_{p,\sqrt{2}p}(\Pi_{p,\sqrt{2}p}(z)) \\
&= \tfrac{1}{2}(\Pi_p + \Pi_{\sqrt{2}p})(\Pi_{p,\sqrt{2}p}(z)) \\
&= \tfrac{1}{2}[\Pi_p(\Pi_{p,\sqrt{2}p}(z)) + \Pi_{\sqrt{2}p}(\Pi_{p,\sqrt{2}p}(z))] \\
&= \tfrac{1}{2}[\Pi_p(\tfrac{1}{2}(\Pi_p(z) + \Pi_{\sqrt{2}p}(z))) + \Pi_{\sqrt{2}p}(\tfrac{1}{2}(\Pi_p(z) + \Pi_{\sqrt{2}p}(z)))] \\
&= \tfrac{1}{4}[\Pi^2_p(z) + \Pi_p\Pi_{\sqrt{2}p}(z) + \Pi_{\sqrt{2}p}\Pi_p(z) + \Pi^2_{\sqrt{2}p}(z)] \\
&= \tfrac{1}{4}[\Pi_p(z) + \Pi_{\sqrt{2}p}(z) + \Pi_p(z) + \Pi_{\sqrt{2}p}(z)] \\
&= \tfrac{1}{2}(\Pi_p(z) + \Pi_{\sqrt{2}p}(z)) = \Pi_{p,\sqrt{2}p}(z),
\end{aligned}$$

which implies that $\Pi_{p,\sqrt{2}p}$ is a projection. □

Remark 3.2. By the proof of lemma 3.1, we know that $\Pi_p(\Pi_{\sqrt{2}p}) = \Pi_{\sqrt{2}p}$ and $\Pi_{\sqrt{2}p}(\Pi_p) = \Pi_p$. Hence, we obtain that

$$\begin{aligned}
(I - \Pi_p)\Pi_{p,\sqrt{2}p} &= \Pi_{p,\sqrt{2}p} - \Pi_p\Pi_{p,\sqrt{2}p} \\
&= \Pi_{p,\sqrt{2}p} - \tfrac{1}{2}\Pi_p(\Pi_p + \Pi_{\sqrt{2}p}) \\
&= \Pi_{p,\sqrt{2}p} - \tfrac{1}{2}(\Pi^2_p + \Pi_p\Pi_{\sqrt{2}p}) \\
&= \Pi_{p,\sqrt{2}p} - \tfrac{1}{2}(\Pi_p + \Pi_{\sqrt{2}p}) \\
&= \Pi_{p,\sqrt{2}p} - \Pi_{p,\sqrt{2}p} = 0.
\end{aligned}$$

Similarly, we have $(I - \Pi_{\sqrt{2}p})\Pi_{p,\sqrt{2}p} = 0$.

Let $K_{p,\sqrt{2}p}\colon X_{\sqrt{2}p} \to X_{\sqrt{2}p}$ be defined by

$$K_{p,\sqrt{2}p}(z) = K_p(I - \Pi_p)(z) + K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p})(z),$$

where $z \in X_{\sqrt{2}p}$. Note that $K_p(I - \Pi_p)(z)$ and $K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p})(z)$ are periodic solutions with period $2p$ and $2\sqrt{2}p$, respectively. Then, $K_{p,\sqrt{2}p}(z)$ is a quasi-periodic solution.

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For each $a > 0$, let

$$\phi(\beta, a) = \gamma(-a) - \gamma(a) + \sum_{j=1}^{d-1} \beta_j (u_{d+j}(-a) - u_{d+j}(a)),$$

where $\beta = (\beta_1, \dots, \beta_{d-1})$ and $\beta_j \in \mathbb{R}$. Since γ is a homoclinic solution and u_{d+j}, $j = 1, \dots, d-1$, are bounded solutions, we know that $|\phi(\beta, a)| = O(\mathrm{e}^{-\alpha a})$ for large a.

Introduce a transformation

$$x(t) = \gamma(t) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(t) + z(t) + \frac{\phi(\beta, p)}{4p} t + \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} t, \tag{3.8}$$

where $z \in X_{\sqrt{2}p}$.

By substituting (3.8) into (1.2), we obtain

$$\dot{z}(t) = \mathrm{D}f(\gamma(t))z(t) + \tilde{h}_p(z, \beta, \varepsilon, t), \tag{3.9}$$

where $z \in X_{\sqrt{2}p}$ and

$$\begin{aligned} \tilde{h}_p(z, \beta, \varepsilon, t) = f\biggl(\gamma(t) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(t) + z(t) + \frac{\phi(\beta, p)}{4p} t + \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} t\biggr) \\ - f(\gamma(t)) - \mathrm{D}f(\gamma(t))z(t) - \frac{\phi(\beta, p)}{4p} - \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} \\ - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(t)) u_{d+j}(t) \\ + \varepsilon g\biggl(\gamma(t) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(t) + z(t) + \frac{\phi(\beta, p)}{4p} t + \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} t, t, \varepsilon\biggr). \end{aligned}$$

Define a map $h_p \colon X_{\sqrt{2}p} \times \mathbb{R}^{d-1} \times \mathbb{R} \to X_{\sqrt{2}p}$ by $h_p(z, \beta, \varepsilon)(t) = \tilde{h}_p(z, \beta, \varepsilon, t)$. Then (3.9) is

$$\dot{z} = \mathrm{D}f(\gamma)z + h_p(z, \beta, \varepsilon). \tag{3.10}$$

Since $\phi(\beta, a) = O(\mathrm{e}^{-\alpha a})$, there exists $p_4 > 0$ such that $\phi(\beta, p) = O(\mathrm{e}^{-\alpha p})$ and $\phi(\beta, \sqrt{2}p) = O(\mathrm{e}^{-\alpha p})$ for $p \in (p_4, \infty)$. Hence, we get that

$$\begin{aligned} h_p(z, \beta, \varepsilon) = f\biggl(\gamma + \sum_{j=1}^{d-1} \beta_j u_{d+j} + z\biggr) - f(\gamma) - \mathrm{D}f(\gamma)z - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma) u_{d+j} \\ + \varepsilon g\biggl(\gamma + \sum_{j=1}^{d-1} \beta_j u_{d+j} + z, t, \varepsilon\biggr) + O(\mathrm{e}^{-\alpha p}). \end{aligned}$$

Through direct computations, the function h_p has the following properties.

$$\left.\begin{aligned} |h_p(0,0,0)| = O(\mathrm{e}^{-\alpha p}), \qquad |h_p(0,0,\varepsilon)| = O(|\varepsilon|) + O(\mathrm{e}^{-\alpha p}), \\ |D_1 h_p(0,0,\varepsilon)| = O(|\varepsilon|) + O(\mathrm{e}^{-\alpha p}), \\ \left|\frac{\partial h_p}{\partial \beta_j}(0,0,0)\right| = O(\mathrm{e}^{-\alpha p}), \qquad \left|\frac{\partial h_p}{\partial \beta_j}(0,0,\varepsilon)\right| = O(|\varepsilon|) + O(\mathrm{e}^{-\alpha p}), \\ \left|\frac{\partial^2 h_p}{\partial \beta_j \partial \beta_k}(0,0,0)\right| = \mathrm{D}^2 f(\gamma) u_{d+j} u_{d+k} + O(\mathrm{e}^{-\alpha p}), \\ \left|\frac{\partial h_p}{\partial \varepsilon}(0,0,0)\right| = g(\gamma, t, 0) + O(\mathrm{e}^{-\alpha p}). \end{aligned}\right\} \tag{3.11}$$

Our goal is to solve (3.10) for $z = z^*(\beta, \varepsilon, p)$ in $X_{\sqrt{2}p}$. Note that $\Pi_{p,\sqrt{2}p}$ and $(I - \Pi_{p,\sqrt{2}p})$ are projections. By (3.10), if we can solve

$$\dot{z} = \mathrm{D} f(\gamma) z + (I - \Pi_{p,\sqrt{2}p}) h_p(z, \beta, \varepsilon) \tag{3.12}$$

for $z = z^*(\beta, \varepsilon, p)$, then we can get the bifurcation equation

$$0 = \Pi_{p,\sqrt{2}p} h_p(z^*(\beta, \varepsilon, p), \beta, \varepsilon). \tag{3.13}$$

If there are some parameters $(\beta, \varepsilon, p) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ satisfying (3.13), then $z^*(\beta, \varepsilon, p)$ is a solution of (3.10).

Lemma 3.3. *There exist a neighbourhood $\Omega \subset \mathbb{R}^{d-1}$ and constants $\varepsilon_1 > 0$ and $p_5 > 0$ such that (3.12) has a solution, $z = z^*(\beta, \varepsilon, p)$, for $(\beta, \varepsilon, p) \in \Omega \times (-\varepsilon_1, \varepsilon_1) \times (p_5, \infty)$ satisfying $|z^*(0, 0, p)| = O(\mathrm{e}^{-\alpha p})$.*

Proof. We define a map $F_{p,\sqrt{2}p} \colon X_{\sqrt{2}p} \times \mathbb{R}^{d-1} \times \mathbb{R} \to X_{\sqrt{2}p}$ by

$$F_{p,\sqrt{2}p}(z, \beta, \varepsilon) = K_{p,\sqrt{2}p}(I - \Pi_{p,\sqrt{2}p}) h_p(z, \beta, \varepsilon). \tag{3.14}$$

Clearly, the fixed points of $F_{p,\sqrt{2}p}(\cdot, \beta, \varepsilon)$ are solutions of (3.12). By the definition of $K_{p,\sqrt{2}p}$ and remark 3.2, we have

$$\begin{aligned} K_{p,\sqrt{2}p}&(I - \Pi_{p,\sqrt{2}p}) h_p(z, \beta, \varepsilon) \\ &= K_p(I - \Pi_p)(I - \Pi_{p,\sqrt{2}p}) h_p(z, \beta, \varepsilon) \\ &\quad + K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p})(I - \Pi_{p,\sqrt{2}p}) h_p(z, \beta, \varepsilon) \\ &= K_p[(I - \Pi_p) - (I - \Pi_p)\Pi_{p,\sqrt{2}p}] h_p(z, \beta, \varepsilon) \\ &\quad + K_{\sqrt{2}p}[(I - \Pi_{\sqrt{2}p})(I - \Pi_{p,\sqrt{2}p})] h_p(z, \beta, \varepsilon) \\ &= K_p(I - \Pi_p) h_p(z, \beta, \varepsilon) + K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p}) h_p(z, \beta, \varepsilon). \end{aligned}$$

Then, we have

$$F_{p,\sqrt{2}p}(z, \beta, \varepsilon) = K_p(I - \Pi_p) h_p(z, \beta, \varepsilon) + K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p}) h_p(z, \beta, \varepsilon).$$

Hence, the fixed points of $F_{p,\sqrt{2}p}(\cdot, \beta, \varepsilon)$ are quasi-periodic solutions.

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Note that $K_{p,\sqrt{2}p}$ and $(I-\Pi_{p,\sqrt{2}p})$ are uniformly bounded in p. By (3.11) and (3.14), we have

$$\left.\begin{aligned} |F_{p,\sqrt{2}p}(0,0,0)| &= O(\mathrm{e}^{-\alpha p}),\\ |F_{p,\sqrt{2}p}(0,0,\varepsilon)| &= O(|\varepsilon|)+O(\mathrm{e}^{-\alpha p}),\\ |D_1F_{p,\sqrt{2}p}(0,0,\varepsilon)| &= O(|\varepsilon|)+O(\mathrm{e}^{-\alpha p}),\\ \left|\frac{\partial F_{p,\sqrt{2}p}}{\partial\beta_j}(0,0,\varepsilon)\right| &= O(|\varepsilon|)+O(\mathrm{e}^{-\alpha p}). \end{aligned}\right\} \tag{3.15}$$

Let $B_1(0,r_1)\subset X_{\sqrt{2}p}$ and $B_2(0,r_2)\subset\mathbb{R}^{d-1}$ be balls centred at the origin with radius $r_1>0$ and $r_2>0$, respectively. By (3.15), there exist sufficiently small $r_1>0$, $r_2>0$ $(r_1>r_2)$, $\varepsilon_{11}>0$ and large $p_{51}>0$ such that

$$|F_{p,\sqrt{2}p}(0,0,\varepsilon)|<\tfrac{1}{3}r_1,\qquad |D_1F_{p,\sqrt{2}p}(z,\beta,\varepsilon)|<\tfrac{1}{3},\qquad |D_2F_{p,\sqrt{2}p}(z,\beta,\varepsilon)|<\tfrac{1}{3} \tag{3.16}$$

for $(z,\beta,\varepsilon,p)\in B_1(0,r_1)\times B_2(0,r_2)\times(-\varepsilon_{11},\varepsilon_{11})\times(p_{51},\infty)$.

For $(z,\beta,\varepsilon,p)\in B_1(0,r_1)\times B_2(0,r_2)\times(-\varepsilon_{11},\varepsilon_{11})\times(p_{51},\infty)$, let $\rho_1\colon[0,1]\to X_{\sqrt{2}p}$ be defined by $\rho_1(\tau)=F_{p,\sqrt{2}p}(\tau z,\tau\beta,\varepsilon)$. By the smoothness of $F_{p,\sqrt{2}p}$ in (β,ε), we see that ρ_1 is C^1 in (β,ε). Hence, we have

$$\begin{aligned} F_{p,\sqrt{2}p}(z,\beta,\varepsilon) &= \rho_1(0)+\int_0^1\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_1(\tau)\,\mathrm{d}\tau\\ &= F_{p,\sqrt{2}p}(0,0,\varepsilon)+\int_0^1[D_1F_{p,\sqrt{2}p}(\tau z,\tau\beta,\varepsilon)z+D_2F_{p,\sqrt{2}p}(\tau z,\tau\beta,\varepsilon)\beta]\,\mathrm{d}\tau. \end{aligned}$$

Taking norms, we have

$$\begin{aligned} |F_{p,\sqrt{2}p}(z,\beta,\varepsilon)| &\leqslant |F_{p,\sqrt{2}p}(0,0,\varepsilon)|+\int_0^1[|D_1F_{p,\sqrt{2}p}(\tau z,\tau\beta,\varepsilon)||z|+|D_2F_{p,\sqrt{2}p}(\tau z,\tau\beta,\varepsilon)||\beta|]\,\mathrm{d}\tau\\ &\leqslant \tfrac{1}{3}r_1+\int_0^1[\tfrac{1}{3}r_1+\tfrac{1}{3}r_2]\,\mathrm{d}\tau<r_1, \end{aligned}$$

where we have used (3.16). Thus, $F_{p,\sqrt{2}p}(\cdot,\beta,\varepsilon)\colon B_1(0,r_1)\to B_1(0,r_1)$ for $(\beta,\varepsilon,p)\in B_2(0,r_2)\times(-\varepsilon_{11},\varepsilon_{11})\times(p_{51},\infty)$.

For $z_1,z_2\in B_1(0,r_1)$, $(\beta,\varepsilon,p)\in B_2(0,r_2)\times(-\varepsilon_{11},\varepsilon_{11})\times(p_{51},\infty)$, define

$$\rho_2\colon[0,1]\to X_{\sqrt{2}p}$$

by $\rho_2(\tau)=F_{p,\sqrt{2}p}(\tau z_1+(1-\tau)z_2,\beta,\varepsilon)$. Clearly, ρ_2 is C^1. Then there exists $\tau_0\in(0,1)$ such that

$$\begin{aligned} F_{p,\sqrt{2}p}(z_1,\beta,\varepsilon)-F_{p,\sqrt{2}p}(z_2,\beta,\varepsilon) &= \rho_2(1)-\rho_2(0)=\rho_2'(\tau_0)\\ &= D_1F_{p,\sqrt{2}p}(\tau_0z_1+(1-\tau_0)z_2,\tau\beta,\varepsilon)(z_1-z_2), \end{aligned}$$

which implies that

$$
\begin{aligned}
|F_{p,\sqrt{2}p}(z_1,\beta,\varepsilon) - F_{p,\sqrt{2}p}(z_2,\beta,\varepsilon)| &= |D_1F_{p,\sqrt{2}p}(\tau_0 z_1 + (1-\tau_0)z_2, \tau\beta, \varepsilon)||z_1 - z_2| \\
&\leqslant \tfrac{1}{3}|z_1 - z_2|,
\end{aligned}
$$

where (3.16) is used. Hence, $F_{p,\sqrt{2}p}(\cdot,\beta,\varepsilon)$ is a uniformly contractive map. By the uniformly contractive principle, there exist a neighbourhood $0 \in \Omega \subset \mathbb{R}^{d-1}$, constants $\varepsilon_1 > 0$ ($\varepsilon_1 < \varepsilon_{11}$) and $p_5 > 0$ ($p_5 > p_{51}$) such that $F_{p,\sqrt{2}p}(\cdot,\beta,\varepsilon)\colon B_1(0,r_1) \to B_1(0,r_1)$ has a unique fixed point $z = z^*(\beta,\varepsilon,p)$ satisfying

$$
z^*(\beta,\varepsilon,p) = F_{p,\sqrt{2}p}(z^*(\beta,\varepsilon,p),\beta,\varepsilon) \quad \text{for } (\beta,\varepsilon,p) \in \Omega \times (-\varepsilon_1,\varepsilon_1) \times (p_5,\infty). \tag{3.17}
$$

By the definition of $F_{p,\sqrt{2}p}$, (3.14) and (3.17), we have

$$
\begin{aligned}
z^*(\beta,\varepsilon,p) &= K_{p,\sqrt{2}p}(I - \Pi_{p,\sqrt{2}p})h_p(z^*,\beta,\varepsilon) \\
&= K_p(I-\Pi_p)h_p(z^*,\beta,\varepsilon) + K_{\sqrt{2}p}(I - \Pi_{\sqrt{2}p})h_p(z^*,\beta,\varepsilon).
\end{aligned} \tag{3.18}
$$

From (3.17), we have

$$
z^*(0,0,p) = F_{p,\sqrt{2}p}(z^*(0,0,p),0,0). \tag{3.19}
$$

Define a map $\rho_3\colon [0,1] \to X_{\sqrt{2}p}$ by $\rho_3(\tau) = F_{p,\sqrt{2}p}(\tau z^*(0,0,p),0,0)$. Then we obtain

$$
\begin{aligned}
&F_{p,\sqrt{2}p}(z^*(0,0,p),0,0) \\
&\quad = \rho_3(0) + \int_0^1 \rho_3'(\tau)\,\mathrm{d}\tau \\
&\quad = F_{p,\sqrt{2}p}(0,0,0) + \int_0^1 D_1F_{p,\sqrt{2}p}(\tau z^*(0,0,p),0,0)z^*(0,0,p)\,\mathrm{d}\tau.
\end{aligned}
$$

Taking norms, we get from (3.16) and (3.19) that

$$
\begin{aligned}
|z^*(0,0,p)| &= |F_{p,\sqrt{2}p}(z^*(0,0,p),0,0)| \\
&\leqslant |F_{p,\sqrt{2}p}(0,0,0)| + \int_0^1 |D_1F_{p,\sqrt{2}p}(\tau z^*(0,0,p),0,0)|\,|z^*(0,0,p)|\,\mathrm{d}\tau \\
&\leqslant |F_{p,\sqrt{2}p}(0,0,0)| + \int_0^1 \tfrac{1}{3}|z^*(0,0,p)|\,\mathrm{d}\tau \\
&= |F_{p,\sqrt{2}p}(0,0,0)| + \tfrac{1}{3}|z^*(0,0,p)|.
\end{aligned} \tag{3.20}
$$

By (3.15), $|F_{p,\sqrt{2}p}(0,0,0)| = O(\mathrm{e}^{-\alpha p})$. Hence, (3.20) implies that $|z^*(0,0,p)| = O(\mathrm{e}^{-\alpha p})$. □

From lemma 3.3, we see that $z = z^*(\beta,\varepsilon,p)$ is a solution of (3.12). By substituting $z = z^*$ into (3.13), we get the bifurcation equation,

$$
\Pi_{p,\sqrt{2}p}h_p(z^*,\beta,\varepsilon) = \tfrac{1}{2}\zeta(t)\sum_{i=1}^{d} u_i(t)[V_i(h_p(z^*,\beta,\varepsilon)) + W_i(h_p(z^*,\beta,\varepsilon))] = 0.
$$

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By the linear independence of $u_1, \dots, u_d$, the bifurcation function is equivalent to the system

$$H_i(\beta,\varepsilon,p) := V_i(h_p(z^*,\beta,\varepsilon)) + W_i(h_p(z^*,\beta,\varepsilon)) = 0, \quad i = 1,\dots,d.$$

It is obvious that $z = z^*$ is a solution of (3.10) if $H_i(\beta,\varepsilon,p) = 0$, $i = 1,\dots,d$, for some (β,ε,p). Since $V_i(z) = v_i(z) + O(\mathrm{e}^{-\alpha p})$ and $W_i(z) = w_i(z) + O(\mathrm{e}^{-\alpha p})$ by (3.3) and (3.6), the bifurcation equations are

$$\begin{aligned}
&H_i(\beta,\varepsilon,p)\\
&\quad = v_i(h_p(z^*,\beta,\varepsilon)) + w_i(h_p(z^*,\beta,\varepsilon)) + O(\mathrm{e}^{-\alpha p})\\
&\quad = \int_{-p}^{p} (u_i^{\perp}(s), h_p(z^*,\beta,\varepsilon)(s))\,\mathrm{d}s + (\mathcal{L}_p h_p(z^*,\beta,\varepsilon))_i\\
&\qquad\quad + \int_{-\sqrt{2}p}^{\sqrt{2}p} (u_i^{\perp}(s), h_p(z^*,\beta,\varepsilon)(s))\,\mathrm{d}s + (\mathcal{L}_{\sqrt{2}p} h_p(z^*,\beta,\varepsilon))_i + O(\mathrm{e}^{-\alpha p})\\
&\quad = \int_{-p}^{p} (u_i^{\perp}(s), h_p(z^*,\beta,\varepsilon)(s))\,\mathrm{d}s + \int_{-\sqrt{2}p}^{\sqrt{2}p} (u_i^{\perp}(s), h_p(z^*,\beta,\varepsilon)(s))\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p}).
\end{aligned}$$

Since $h_p(z,\beta,\varepsilon)(t) = \tilde{h}_p(z,\beta,\varepsilon,t)$, by the definition of $\tilde{h}_p$ in (3.9), we have

$$\begin{aligned}
&H_i(\beta,\varepsilon,p)\\
&= \int_{-p}^{p} (u_i^{\perp}(s), \tilde{h}_p(z^*,\beta,\varepsilon,s))\,\mathrm{d}s + \int_{-\sqrt{2}p}^{\sqrt{2}p} (u_i^{\perp}(s), \tilde{h}_p(z^*,\beta,\varepsilon,s))\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p})\\
&= \int_{-p}^{p} \biggl(u_i^{\perp}(s), f\biggl(\gamma(s) + \sum_{j=1}^{d-1}\beta_j u_{d+j}(s) + z^*(s) + \frac{\phi(\beta,p)}{4p}s + \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p}s\biggr)\\
&\qquad\quad - f(\gamma(s)) - \mathrm{D}f(\gamma(s))z^*(s) - \frac{\phi(\beta,p)}{4p} - \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p}\\
&\qquad\quad - \sum_{j=1}^{d-1}\beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s)\\
&\qquad\quad + \varepsilon g\biggl(\gamma(s) + \sum_{j=1}^{d-1}\beta_j u_{d+j}(s) + z^*(s) + \frac{\phi(\beta,p)}{4p}s + \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p}s, s, \varepsilon\biggr)\biggr)\,\mathrm{d}s\\
&\quad + \int_{-\sqrt{2}p}^{\sqrt{2}p} \biggl(u_i^{\perp}(s), f\biggl(\gamma(s) + \sum_{j=1}^{d-1}\beta_j u_{d+j}(s) + z^*(s) + \frac{\phi(\beta,p)}{4p}s + \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p}s\biggr)\\
&\quad - f(\gamma(s)) - \mathrm{D}f(\gamma(s))z^*(s) - \frac{\phi(\beta,p)}{4p} - \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p} - \sum_{j=1}^{d-1}\beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s)\\
&\quad + \varepsilon g\biggl(\gamma(s) + \sum_{j=1}^{d-1}\beta_j u_{d+j}(s) + z^*(s)\\
&\qquad\qquad + \frac{\phi(\beta,p)}{4p}s + \frac{\phi(\beta,\sqrt{2}p)}{4\sqrt{2}p}s, s, \varepsilon\biggr)\biggr)\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-p}^{p} \Bigg(u_i^{\perp}(s), f\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s)\Bigg) - f(\gamma(s)) - \mathrm{D}f(\gamma(s))z^*(s) \\
&\qquad - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s) + \varepsilon g\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s), s, \varepsilon\Bigg)\Bigg)\,\mathrm{d}s \\
&\quad + \int_{-\sqrt{2}p}^{\sqrt{2}p} \Bigg(u_i^{\perp}(s), f\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s)\Bigg) - f(\gamma(s)) \\
&\qquad\qquad - \mathrm{D}f(\gamma(s))z^*(s) - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s) \\
&\qquad\qquad + \varepsilon g\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s), s, \varepsilon\Bigg)\Bigg)\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p}) \\
&= \int_{-\infty}^{\infty} \Bigg(u_i^{\perp}(s), f\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s)\Bigg) - f(\gamma(s)) - \mathrm{D}f(\gamma(s))z^*(s) \\
&\qquad - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s) + \varepsilon g\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s), s, \varepsilon\Bigg)\Bigg)\,\mathrm{d}s \\
&\quad + \int_{-\infty}^{\infty} \Bigg(u_i^{\perp}(s), f\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s)\Bigg) \\
&\qquad\qquad - f(\gamma(s)) - \mathrm{D}f(\gamma(s))z^*(s) - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s) \\
&\qquad\qquad + \varepsilon g\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s), s, \varepsilon\Bigg)\Bigg)\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p}).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
H_i(\beta, \varepsilon, p) &= 2\int_{-\infty}^{\infty} \Bigg(u_i^{\perp}(s), f\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s)\Bigg) f(\gamma(s))z^*(s) \\
&\qquad\qquad - f(\gamma(s)) - \mathrm{D} - \sum_{j=1}^{d-1} \beta_j \mathrm{D}f(\gamma(s))u_{d+j}(s) \\
&\qquad\qquad + \varepsilon g\Bigg(\gamma(s) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(s) + z^*(s), s, \varepsilon\Bigg)\Bigg)\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p}) \\
&= 2\int_{-\infty}^{\infty} (u_i^{\perp}(s), h_p(z^*, \beta, \varepsilon)(s))\,\mathrm{d}s + O(\mathrm{e}^{-\alpha p}). \qquad (3.21)
\end{aligned}
$$

Let $H(\beta, \varepsilon, p) = (H_1(\beta, \varepsilon, p), \dots, H_d(\beta, \varepsilon, p))$. Through direct calculations, by (3.11) and lemma 3.3, we get the following lemma.

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LEMMA 3.4. *For $i = 1, \dots, d$ and $j, k = 1, \dots, d-1$, $H_i(\beta, \varepsilon, p)$ satisfy the following properties:*

(1) *$z = z^*(\beta, \varepsilon, p)$ is a solution of (3.10) if $H(\beta, \varepsilon, p) = 0$ for some (β, ε, p);*

(2) *$H_i(0, 0, p) = O(\mathrm{e}^{-\alpha p})$ and*

$$\frac{\partial H_i}{\partial \beta_j}(0, 0, p) = O(\mathrm{e}^{-\alpha p});$$

(3)

$$\begin{aligned}\frac{\partial^2 H_i}{\partial \beta_j \partial \beta_k}(0, 0, p) &= 2\int_{-\infty}^{\infty} (u_i^{\perp}(s), \mathrm{D}^2 f(\gamma(s)) u_{d+j}(s) u_{d+k}(s))\, \mathrm{d}s + O(\mathrm{e}^{-\alpha p}) \\ &= \lambda_{ijk} + O(\mathrm{e}^{-\alpha p});\end{aligned}$$

(4)

$$\frac{\partial H_i}{\partial \varepsilon}(0, 0, p) = 2\int_{-\infty}^{\infty} (u_i^{\perp}(s), g(\gamma(s), s, 0))\, \mathrm{d}s + O(\mathrm{e}^{-\alpha p}) = \eta_i + O(\mathrm{e}^{-\alpha p});$$

here λ_{ijk} and η_i are defined in (2.2).

By (3.21), lemma 3.4 and the notation of $M(\beta, \varepsilon)$ defined in (2.3), we see that $M(\beta, \varepsilon)$ is the main part of $H(\beta, \varepsilon, p)$. Thus,

$$H(\beta, \varepsilon, p) = M(\beta, \varepsilon) + \text{higher-order terms}.$$

LEMMA 3.5. *If there exist some $(\beta^*, \varepsilon^*) \in \mathbb{R}^{d-1} \times \mathbb{R}$ such that $M(\beta^*, \varepsilon^*) = 0$, and $\mathrm{D}_{(\beta,\varepsilon)} M(\beta^*, \varepsilon^*)$ is a non-singular $d \times d$ matrix, then there exist $s_0 > 0$, $p_6 > 0$ and the C^1-functions $\psi^*\colon (-s_0, s_0) \times (p_6, \infty) \to \mathbb{R}^{d-1}$, $\varphi^*\colon (-s_0, s_0) \times (p_6, \infty) \to \mathbb{R}$ such that $z = z^*(\beta, \varepsilon, p)$ is a solution of (3.10), where $\beta = s(\beta^* + \psi^*(s, p))$ and $\varepsilon = s^2(\varepsilon^* + \varphi^*(s, p))$, for $(s, p) \in (-s_0, s_0) \times (p_6, \infty)$.*

Proof. Define a map $\boldsymbol{H}\colon (\mathbb{R}^{d-1} \times \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$ by

$$\boldsymbol{H}((\psi, \varphi), p, s) = \begin{cases} \dfrac{1}{s^2} H(s(\beta^* + \psi), s^2(\varepsilon^* + \varphi), p), & s \neq 0, \\ M(\beta^* + \psi, \varepsilon^* + \varphi), & s = 0. \end{cases}$$

For $s \neq 0$, it is clear that $\boldsymbol{H}((\psi, \varphi), p, s) = 0$ iff $H(s(\beta^* + \psi), s^2(\varepsilon^* + \varphi), p) = 0$.

Define a function $G\colon \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$ by

$$G(\beta, \varepsilon) = (\tilde{H}_1(\beta, \varepsilon), \dots, \tilde{H}_d(\beta, \varepsilon)),$$

where $\tilde{H}_i(\beta, \varepsilon) = \lim_{p\to\infty} H_i(\beta, \varepsilon, p)$. We define a map $\boldsymbol{G}\colon (\mathbb{R}^{d-1} \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}^d$ by

$$\boldsymbol{G}((\psi, \varphi), s) = \begin{cases} \dfrac{1}{s^2} G(s(\beta^* + \psi), s^2(\varepsilon^* + \varphi)), & s \neq 0, \\ M(\beta^* + \psi, \varepsilon^* + \varphi), & s = 0. \end{cases}$$

It is easy to check that

$$\boldsymbol{G}((0,0),0)=0, \quad D_{(\psi,\varphi)}\boldsymbol{G}((0,0),0)=D_{(\psi,\varphi)}M(\beta^*,\varepsilon^*).$$

Since $\mathcal{M}:=D_{(\psi,\varphi)}M(\beta^*,\varepsilon^*)$ is non-singular, $\mathcal{M}^{-1}$ is bounded. Let

$$\begin{gathered}\mathcal{H}((\psi,\varphi),s,p)=(\psi,\varphi)-\mathcal{M}^{-1}\boldsymbol{H}((\psi,\varphi),s,p),\\ \mathcal{G}((\psi,\varphi),s)=(\psi,\varphi)-\mathcal{M}^{-1}\boldsymbol{G}((\psi,\varphi),s).\end{gathered}$$

It is clear that the fixed points of $\mathcal{H}(\cdot,s,p)$ are zeros of $\boldsymbol{H}(\cdot,s,p)$.

By the formula for $\mathcal{G}((\psi,\varphi),s)$, we obtain that

$$\mathcal{G}((0,0),0)=0, \quad D_{(\psi,\varphi)}\mathcal{G}((0,0),0)=0. \tag{3.22}$$

Let $r>0$, and let $B(0,r)\subset\mathbb{R}^{d-1}\times\mathbb{R}$ be a ball with radius r centred at the origin. By (3.22), there exist $r_1>0$ and $s_1>0$ such that

$$|\mathcal{G}((\psi,\varphi),s)|<\tfrac{1}{4}, \quad D_{(\psi,\varphi)}\mathcal{G}((\psi,\varphi),s)<\tfrac{1}{4} \tag{3.23}$$

for $(\psi,\varphi)\in B(0,r_1)$ and $s\in(-s_1,s_1)$.

Note that

$$\begin{aligned}&\mathcal{H}((\psi,\varphi),p,s)-\mathcal{G}((\psi,\varphi),s)\\ &\quad=\mathcal{M}^{-1}\begin{cases}\dfrac{1}{s^2}[H(s(\beta^*+\psi),s^2(\varepsilon^*+\varphi),p)-G(s(\beta^*+\psi),s^2(\varepsilon^*+\varphi))], & s\neq0,\\ 0, & s=0\end{cases}\end{aligned}$$

and

$$\begin{aligned}&D_{(\psi,\varphi)}\mathcal{H}((\psi,\varphi),s,p)-D_{(\psi,\varphi)}\mathcal{G}((\psi,\varphi),s)\\ &\qquad=\mathcal{M}^{-1}\begin{cases}\dfrac{1}{s^2}[D_{(\psi,\varphi)}H(s(\beta^*+\psi),s^2(\varepsilon^*+\varphi),p)\\ \qquad\qquad -D_{(\psi,\varphi)}G(s(\beta^*+\psi),s^2(\varepsilon^*+\varphi))], & s\neq0,\\ 0, & s=0.\end{cases}\end{aligned}$$

Then we have

$$|\mathcal{H}((0,0),s,p)-\mathcal{G}((0,0),s)|=\begin{cases}O(|s|)+O(\mathrm{e}^{-\alpha p}), & s\neq0,\\ 0, & s=0,\end{cases} \tag{3.24}$$

and

$$|D_{(\psi,\varphi)}\mathcal{H}((0,0),s,p)-D_{(\psi,\varphi)}\mathcal{G}((0,0),s)|=\begin{cases}O(|s|)+O(\mathrm{e}^{-\alpha p}), & s\neq0,\\ 0, & s=0.\end{cases} \tag{3.25}$$

By (3.24) and (3.25), there exist small $r_2,s_2>0$ and large $p_{61}>0$ such that

$$\left.\begin{aligned}|\mathcal{H}((\psi,\varphi),s,p)-\mathcal{G}((\psi,\varphi),s)|&<\tfrac{1}{4},\\ |D_{(\psi,\varphi)}\mathcal{H}((\psi,\varphi),s,p)-D_{(\psi,\varphi)}\mathcal{G}((\psi,\varphi),s)|&<\tfrac{1}{4}\end{aligned}\right\} \tag{3.26}$$

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for $((\psi,\varphi),s,p)\in B(0,r_2)\times(-s_2,s_2)\times(p_{61},\infty)$. By (3.23) and (3.26), we have

$$\begin{aligned}|D_{(\psi,\varphi)}\mathcal{H}((\psi,\varphi),s,p)| &\leqslant |D_{(\psi,\varphi)}\mathcal{H}((\psi,\varphi),s,p)-D_{(\psi,\varphi)}\mathcal{G}((\psi,\varphi),s)|\\ &\quad+|D_{(\psi,\varphi)}\mathcal{G}((\psi,\varphi),s)|\\ &\leqslant \tfrac14+\tfrac14=\tfrac12. \end{aligned}\tag{3.27}$$

Let $r_0=\min\{r_1,r_2\}$. Then, (3.22) and (3.24) imply that there exist small $s_3>0$ and large $p_{62}>0$ such that

$$\left.\begin{aligned}|\mathcal{G}((0,0),s)|&<\tfrac14 r_0\\ |\mathcal{H}((0,0),s,p)-\mathcal{G}((0,0),s)|&<\tfrac14\end{aligned}\right\}\quad\text{for }(s,p)\in(-s_3,s_3)\times(p_{62},\infty).$$

Thus, we get

$$|\mathcal{H}((0,0),s,p)|\leqslant|\mathcal{H}((0,0),s,p)-\mathcal{G}((0,0),s)|+|\mathcal{G}((0,0),s)|<\tfrac12 r_0.\tag{3.28}$$

Let $\tilde{s}_0=\min\{s_1,s_2,s_3\},\tilde{p}_6=\max\{p_{61},p_{62}\}$. By (3.28) and (3.27), we have

$$|\mathcal{H}((0,0),s,p)|\leqslant\tfrac12 r_0,\qquad |D_{(\psi,\varphi)}\mathcal{H}((\psi,\varphi),s,p)|\leqslant\tfrac12\tag{3.29}$$

for $((\psi,\varphi),s,p)\in B(0,r_0)\times(-\tilde{s}_0,\tilde{s}_0)\times(\tilde{p}_6,\infty)$.

For $((\psi,\varphi),s,p)\in B(0,r_0)\times(-\tilde{s}_0,\tilde{s}_0)\times(\tilde{p}_6,\infty)$, define $\xi_1\colon[0,1]\to\mathbb{R}^d$ by $\xi_1(\tau)=\mathcal{H}(\tau(\psi,\varphi),s,p)$. Then

$$\begin{aligned}\mathcal{H}((\psi,\varphi),s,p)&=\xi_1(0)+\int_0^1\xi_1'(\tau)\,\mathrm{d}\tau\\ &=\mathcal{H}((0,0),s,p)+\int_0^1 D_{(\psi,\varphi)}\mathcal{H}(\tau(\psi,\varphi),s,p)(\psi,\varphi)\,\mathrm{d}\tau.\end{aligned}$$

Hence,

$$|\mathcal{H}((\psi,\varphi),s,p)|\leqslant\tfrac12 r_0+\int_0^1\tfrac12 r_0\,\mathrm{d}\tau=r_0,\tag{3.30}$$

where we have used (3.29).

For $(\psi_1,\varphi_1),(\psi_2,\varphi_2)\in B(0,r_0)$, $(s,p)\in(-\tilde{s}_0,\tilde{s}_0)\times(\tilde{p}_6,\infty)$, define $\xi_2\colon[0,1]\to\mathbb{R}^d$ by $\xi_2(\tau)=\mathcal{H}(\tau(\psi_1,\varphi_1)+(1-\tau)(\psi_2,\varphi_2),s,p)$. Then ξ_2 is C^1. Hence, there exists $\tau_0\in(0,1)$ such that

$$\begin{aligned}&\mathcal{H}((\psi_1,\varphi_1),s,p)-\mathcal{H}((\psi_2,\varphi_2),s,p)\\ &\qquad=\xi_2(1)-\xi_2(0)=\xi_2'(\tau_0)\\ &\qquad=D_{(\psi,\varphi)}\mathcal{H}(\tau_0(\psi_1,\varphi_1)+(1-\tau_0)(\psi_2,\varphi_2),s,p)((\psi_1,\varphi_1)-(\psi_2,\varphi_2)).\end{aligned}$$

Taking norms, we get from (3.29) that

$$|\mathcal{H}((\psi_1,\varphi_1),s,p)-\mathcal{H}((\psi_2,\varphi_2),s,p)|\leqslant\tfrac12|(\psi_1,\varphi_1)-(\psi_2,\varphi_2)|.\tag{3.31}$$

By (3.30) and (3.31), $\mathcal{H}(\cdot,s,p)\colon B(0,r_0)\to B(0,r_0)$ is a uniformly contractive map on $(-\tilde{s}_0,\tilde{s}_0)\times(\tilde{p}_6,\infty)$. This implies that there exist $s_4>0$, $p_{63}>0$ and the

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functions $\psi^*\colon (-s_4, s_4) \times (p_{63}, \infty) \to \mathbb{R}^{d-1}$, $\varphi^*\colon (-s_4, s_4) \times (p_{63}, \infty) \to \mathbb{R}$ such that $\mathcal{H}(\cdot, s, p)$ has a fixed point,

$$(\psi, \varphi) = (\psi^*(s,p), \varphi^*(s,p)) \quad \text{for } (s,p) \in (-s_4, s_4) \times (p_{63}, \infty).$$

Let $s_0 = \min\{s_1, s_2, s_3, s_4\}$ and $p_6 = \min\{p_{61}, p_{62}, p_{63}\}$. By the definitions of H, $\boldsymbol{H}$ and $\mathcal{H}$, we obtain

$$H(s(\beta^* + \psi^*(s,p)), s^2(\varepsilon^* + \varphi^*(s,p)), p) = 0$$

for $(s,p) \in ((-s_0, 0) \cup (0, s_0)) \times (p_6, \infty)$. Since the bifurcation functions H vanish, we know from property (1) of lemma 3.4 that $z = z^*(s(\beta^* + \psi^*(s,p)), s^2(\varepsilon^* + \varphi^*(s,p)), p)$ is a solution of (3.10) for $(s,p) \in ((-s_0, 0) \cup (0, s_0)) \times (p_6, \infty)$. □

Let $p_0 = \max\{2, p_1, \dots, p_6\}$. For $(s,p) \in ((-s_0, 0) \cup (0, s_0)) \times (p_0, \infty)$, from transformation (3.8), system (1.2) with $\varepsilon = s^2(\varepsilon^* + \varphi^*(s,p))$ has a solution given by

$$x^*(t) = \gamma(t) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(t) + z^*(\beta, \varepsilon, p)(t) + \frac{\phi(\beta, p)}{4p} t + \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} t, \tag{3.32}$$

where $\beta = (s(\beta_1^* + \psi_1^*(s,p)), \dots, s(\beta_{d-1}^* + \psi_{d-1}^*(s,p))), \varepsilon = s^2(\varepsilon^* + \varphi^*(s,p))$, $\beta^* = (\beta_1^*, \dots, \beta_{d-1}^*)$, ψ^*, φ^* are given in lemma 3.5, z^* is given in lemma 3.3. By (3.18), (3.32) can be written as

$$\begin{aligned} x^*(t) &= \gamma(t) + \sum_{j=1}^{d-1} \beta_j u_{d+j}(t) + K_{p,\sqrt{2}p}(I - \mathit{\Pi}_{p,\sqrt{2}p}) h_p(z^*, \beta, \varepsilon)(t) \\ &\quad + \frac{\phi(\beta, p)}{4p} t + \frac{\phi(\beta, \sqrt{2}p)}{4\sqrt{2}p} t \\ &:= x_1^*(t) + x_2^*(t), \end{aligned}$$

where

$$\begin{aligned} x_1^*(t) &= \tfrac{1}{2}\gamma(t) + \sum_{j=1}^{d-1} \frac{s(\beta_j^* + \psi_j^*(s,p))}{2} u_{d+j}(t) \\ &\quad + K_p(I - \mathit{\Pi}_p) h_p(z^*, \beta, \varepsilon)(t) + \frac{\phi(s(\beta^* + \psi^*(s,p)), p)}{4p} t, \\ x_2^*(t) &= \tfrac{1}{2}\gamma(t) + \sum_{j=1}^{d-1} \frac{s(\beta_j^* + \psi_j^*(s,p))}{2} u_{d+j}(t) \\ &\quad + K_{\sqrt{2}p}(I - \mathit{\Pi}_{\sqrt{2}p}) h_p(z^*, \beta, \varepsilon)(t) + \frac{\phi(s(\beta^* + \psi^*(s,p)), \sqrt{2}p)}{4\sqrt{2}p} t. \end{aligned}$$

Since $(I - \mathit{\Pi}_p)\colon X_p \to \tilde{X}_p$ and $K_p\colon \tilde{X}_p \to \bar{X}_p$, we have $K_p(I - \mathit{\Pi}_p)\colon X_p \to \bar{X}_p$. Thus, $K_p(I - \mathit{\Pi}_p) h_p(z^*, \beta, \varepsilon) \in \bar{X}_p$, and hence $K_p(I - \mathit{\Pi}_p) h_p(z^*, \beta, \varepsilon)$ is a periodic

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function with period $2p$. From the formula for $\phi(\beta, p)$ we obtain that

$$\begin{aligned}
x_1^*(-p) &= \tfrac{1}{2}\gamma(-p) + \sum_{j=1}^{d-1} \tfrac{1}{2} s(\beta_j^* + \psi_j^*(s,p)) u_{d+j}(-p) + K_p(I - \Pi_p) h_p(z^*, \beta, \varepsilon)(-p) \\
&\quad - \tfrac{1}{4}\phi(s(\beta^* + \psi_j^*(s,p)), p) \\
&= \tfrac{1}{2}\gamma(-p) + \sum_{j=1}^{d-1} \tfrac{1}{2} s(\beta_j^* + \psi_j^*(s,p)) u_{d+j}(-p) + K_p(I - \Pi_p) h_p(z^*, \beta, \varepsilon)(-p) \\
&\quad - \frac{1}{4}\biggl(\gamma(-p) - \gamma(p) + \sum_{j=1}^{d-1} s(\beta_j^* + \psi_j^*(s,p))(u_{d+j}(-p) - u_{d+j}(p))\biggr) \\
&= \frac{1}{4}\biggl\{\gamma(-p) + \gamma(p) + \sum_{j=1}^{d-1} s(\beta_j^* + \psi_j^*(s,p))(u_{d+j}(-p) - u_{d+j}(p))\biggr\} \\
&\quad + K_p(I - \Pi_p) h_p(z^*, \beta, \varepsilon)(-p) \\
&= x_1^*(p).
\end{aligned}$$

Thus, x_1^* is a periodic function with period $2p$. Similarly, we have that x_2^* is a periodic function with period $2\sqrt{2}p$. Hence, $x^* = x_1^* + x_2^*$ is a quasi-periodic function and hence a quasi-periodic solution of (1.2) with $\varepsilon = s^2(\varepsilon^* + \varphi^*(s,p))$ for $(s,p) \in ((-s_0, s_0)/\{0\}) \times (p_0, \infty)$.

4. Examples

We now give some examples to conclude the paper.

Example 4.1. Consider the system

$$\left.\begin{aligned}
\dot{x}_1 &= x_2 + \varepsilon(x_1^3 + x_3), \\
\dot{x}_2 &= x_1 - 2x_1^3 + \varepsilon(x_1 + x_2), \\
\dot{x}_3 &= 2x_3 + \varepsilon(x_1 x_2 + x_2 \cos 2\pi t).
\end{aligned}\right\} \tag{4.1}$$

Let $r(t) = \operatorname{sech} t$. The unperturbed system

$$\left.\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_1 - 2x_1^3, \\
\dot{x}_3 &= 2x_3
\end{aligned}\right\} \tag{4.2}$$

has a homoclinic solution $\gamma = (r, \dot{r}, 0)$. The linear variational equation of (4.2) along γ is

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 - 6r & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{4.3}$$

Let P be a differentiable function satisfying $\dot{P}\dot{r}^2 = 1$. Through direct calculation, we see that (4.3) has the following fundamental solution:

$$u_1 = (P\dot{r}, (P\dot{r})^{\cdot}, 0), \qquad u_2 = (\dot{r}, \ddot{r}, 0), \qquad u_3 = (0, 0, \mathrm{e}^{2t}).$$

Clearly,

$$\lim_{|t|\to\infty}|u_1(t)|=\infty,$$
$$\lim_{|t|\to\infty}|u_2(t)|=0,$$
$$\lim_{t\to\infty}|u_3(t)|=\infty,$$
$$\lim_{t\to-\infty}|u_3(t)|=0.$$

Hence, $d=1$. Let $u_1^{\perp}=(-\ddot{r},\dot{r},0)$. $u_1^{\perp}$ is the bounded solution of the adjoint equation of (4.3). Note that $d=1$. There is no β in M. We compute η as follows:

$$\begin{aligned}\eta&=\int_{-\infty}^{\infty}(u_1(s)^{\perp},g(\gamma(s),s,0))\,\mathrm{d}s\\&=\int_{-\infty}^{\infty}((-\ddot{r}(s),\dot{r}(s),0),\mathrm{col}(r(s)^3,r(s)+\dot{r}(s),r(s)\dot{r}(s)+\dot{r}(s)\cos(2\pi s)))\,\mathrm{d}s\\&=-\int_{-\infty}^{\infty}\ddot{r}(s)r(s)^3-\dot{r}(s)r(s)-\dot{r}(s)^2\,\mathrm{d}s\\&=\int_{-\infty}^{\infty}3\dot{r}(s)^2r(s)^2+\dot{r}(s)^2\,\mathrm{d}s=\tfrac{22}{15}\neq 0.\end{aligned}$$

By corollary 2.2, there exist $s_0>0$, $p_0>0$ and a function $\varphi^*\colon(-s_0,s_0)\times(p_0,\infty)\to\mathbb{R}$ such that system (4.1) with $\varepsilon=s^2\varphi^*(s,p)$ has a quasi-periodic solution for $(s,p)\in((-s_0,s_0)/\{0\})\times(p_0,\infty)$.

In example 4.1, the homoclinic solution of the unperturbed system is non-degenerate. We use an example with degenerate homoclinic solution to illustrate theorem 2.2.

Example 4.2. Consider the system

$$\left.\begin{aligned}\dot{x}_1&=x_2+\varepsilon(x_1+x_3+x_5\sin t),\\\dot{x}_2&=x_1-2x_1x_5^2+x_2^2+\varepsilon(x_1+x_2),\\\dot{x}_3&=x_4-\varepsilon x_5\sin t,\\\dot{x}_4&=x_3-2x_3x_5^2+x_2x_4+\varepsilon(x_1x_2+\varepsilon x_2\cos t),\\\dot{x}_5&=x_6+3\varepsilon^2x_4\cos t,\\\dot{x}_6&=x_5-2x_5^2+x_3x_4+\varepsilon(x_1x_2-\tfrac{1}{4}x_5\sin t).\end{aligned}\right\}\tag{4.4}$$

Let

$$\begin{aligned}x&=(x_1,x_2,\dots,x_6),\\f(x)&=(x_2,x_1-2x_1x_5^2+x_2^2,x_4,x_3-2x_3x_5^2+x_2x_4,x_6,x_5-2x_5^2+x_3x_4)\end{aligned}$$

and

$$\begin{aligned}g(x,t,\varepsilon)=(&x_1+x_3+x_5\sin t,x_1+x_2,-x_5\sin t,\\&x_1x_2+\varepsilon x_2\cos t,3\varepsilon x_4\cos t,x_1x_2-\tfrac{1}{4}x_5\sin t).\end{aligned}$$

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Then (4.4) falls in the form (1.2). Clearly, $f(0) = 0$ and $\mathrm{D}f(0)$ has six eigenvalues: $-1, -1, -1, 1, 1, 1$. Hence, (H1) holds. Through direct calculations, we see that $g(0, t, \varepsilon) = 0$ and $g(x, t + 2\pi, \varepsilon) = g(x, t, \varepsilon)$, which imply (H3). The unperturbed system of (4.4),

$$\dot{x} = f(x), \tag{4.5}$$

has a homoclinic solution $\gamma = (0, 0, 0, 0, r, \dot{r})$. Thus, (H2) holds.

The linear variational equation of (4.5) along γ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 - 2r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - 2r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 - 6r & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}. \tag{4.6}$$

Let Q be a differentiable function satisfying $\dot{Q}r^2 = 1$. It is easy to obtain the dichotomous fundamental solution of (4.6):

$$\begin{aligned} &u_1 = (Qr, (Qr)^{\cdot}, 0, 0, 0, 0), && u_2 = (0, 0, Qr, (Qr)^{\cdot}, 0, 0), && u_3 = (0, 0, 0, 0, Pr, (P\dot{r})^{\cdot}), \\ &u_4 = (r, \dot{r}, 0, 0, 0, 0), && u_5 = (0, 0, r, \dot{r}, 0, 0), && u_6 = (0, 0, 0, 0, \dot{r}, \ddot{r}). \end{aligned}$$

It is clear that u_1, u_2, u_3 are unbounded solutions and u_4, u_5, u_6 are bounded ones. Correspondingly, the adjoint equation of (4.6) has bounded solutions

$$u_1^{\perp} = (-\dot{r}, r, 0, 0, 0, 0), \qquad u_2^{\perp} = (0, 0, -\dot{r}, r, 0, 0), \qquad u_3^{\perp} = (0, 0, 0, 0, -\ddot{r}, \dot{r}).$$

In the notation of theorem 2.2, we obtain

$$\begin{aligned} \lambda_{111} &= 2 \int_{-\infty}^{\infty} (u_1^{\perp}(s), \mathrm{D}^2 f(\gamma(s)) u_4(s) u_4(s))\,\mathrm{d}s \\ &= 2 \int_{-\infty}^{\infty} (u_1^{\perp}(s), \mathrm{col}(0, 2\dot{r}^2(s), 0, 0, 0, 0))\,\mathrm{d}s \\ &= 4 \int_{-\infty}^{\infty} r(s)\dot{r}^2(s)\,\mathrm{d}s = \tfrac{1}{2}\pi. \end{aligned}$$

Similarly, we have $\lambda_{212} = \lambda_{221} = \lambda_{322} = \frac{1}{2}\pi$ and $\lambda_{112} = \lambda_{121} = \lambda_{122} = \lambda_{211} = \lambda_{222} = \lambda_{311} = \lambda_{312} = \lambda_{321} = 0$.

We compute η_i. By the formulae for η_i, we get

$$\begin{aligned} \eta_1 &= 2 \int_{-\infty}^{\infty} (u_1^{\perp}(s), g(\gamma(s), s, 0))\,\mathrm{d}s \\ &= 2 \int_{-\infty}^{\infty} (u_1^{\perp}(s), \mathrm{col}(r(s)\sin s, 0, -r(s)\sin s, 0, 0, -\tfrac{1}{4} r(s)\sin s))\,\mathrm{d}s \\ &= -2 \int_{-\infty}^{\infty} \dot{r}(s) r(s) \sin s\,\mathrm{d}s = -2\pi\,\mathrm{cosech}\,\tfrac{1}{2}\pi. \end{aligned}$$

Similarly, we have $\eta_2 = 2\pi$ and $\mathrm{cosech}\,\frac{1}{2}\pi$, $\eta_3 = -\frac{1}{2}\pi\,\mathrm{cosech}\,\frac{1}{2}\pi$. Hence, we obtain that

$$M(\beta, \varepsilon) = (M_1(\beta, \varepsilon), M_2(\beta, \varepsilon), M_3(\beta, \varepsilon)),$$

where

$$M_1(\beta,\varepsilon) = \tfrac{1}{4}\pi\beta_1^2 - 2\pi \operatorname{cosech} \tfrac{1}{2}\pi\varepsilon,$$
$$M_2(\beta,\varepsilon) = \tfrac{1}{2}\pi\beta_1\beta_2 + 2\pi \operatorname{cosech} \tfrac{1}{2}\pi\varepsilon,$$
$$M_3(\beta,\varepsilon) = \tfrac{1}{4}\pi\beta_2^2 - \tfrac{1}{2}\pi \operatorname{cosech} \tfrac{1}{2}\pi\varepsilon.$$

Take $\beta_1^* = 2$, $\beta_2^* = -1$ and $\varepsilon^* = \frac{1}{2}\sinh\frac{1}{2}\pi$. Through direct computation, we obtain $M(\beta^*, \varepsilon^*) = 0$ and $|D_{(\beta,\varepsilon)}M|(\beta^*, \varepsilon^*) = \pi^3 \operatorname{cosech}\frac{1}{2}\pi \neq 0$. Note that (H1)–(H3) hold. Hence, theorem 2.2 can be applied. There exist $s_0 > 0$, $p_0 > 0$ and a function $\varphi^* \colon (-s_0, s_0) \times (p_0, \infty) \to \mathbb{R}$ such that system (4.4) with $\varepsilon = s^2(\varepsilon^* + \varphi^*(s,p))$ has a quasi-periodic solution for $(s,p) \in ((-s_0, s_0)/\{0\}) \times (p_0, \infty)$.

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