



Explicit symmetric DGLA models of 3-cells

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Abstract. We give explicit formulae for differential graded Lie algebra (DGLA) models of 3-cells. In particular, for a cube and an n -faceted banana-shaped 3-cell with two vertices, n edges each joining those two vertices, and n bi-gon 2-cells, we construct a model symmetric under the geometric symmetries of the cell fixing two antipodal vertices. The cube model is to be used in forthcoming work for discrete analogues of differential geometry on cubulated manifolds.

1 Introduction

The discretization of differential equations is a necessity of life—most differential equations cannot be solved analytically, and a discrete numeric solution is the choice approach for many applications. However, by their nature, discrete differences lose “associativity” preserved in continuous derivatives [6]. A recent programme attempts to cure this problem by constructing a discrete analogue to differential geometry with C -infinity structure [8]. The starting point is to associate a differential graded Lie algebra (DGLA) with cell complexes.

For a regular CW complex X , it is possible to associate a DGLA model $A = A(X)$ over \mathbb{Q} satisfying the following conditions:

- as a completed graded Lie algebra, $A(X)$ is freely generated by a set of generators, one for each cell in X and whose grading is one less than the geometric degree of the cell;
- vertices (that is, 0-cells) in X give rise to generators a that satisfy the Maurer–Cartan equation $\partial a + \frac{1}{2}[a, a] = 0$ (a flatness condition);
- for a cell x in X , the part of ∂x without Lie brackets is the geometric boundary $\partial_0 x$ (where an orientation must be fixed on each cell);
- (locality) for a cell x in X , ∂x lies in the completed Lie algebra generated by the generators of $A(X)$ associated with cells of the closure \bar{x} .

The existence and general construction of such a model was demonstrated by Sullivan in the Appendix to [11]; however, the procedure given there is an iterative algorithm. By [1], there exist consistent (even symmetric) towers of models of simplices, and such towers are unique up to (exact) DGLA isomorphism. The explicit

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(unique) model associated with the interval was found in [10]. Explicit symmetric models of two-dimensional cells were demonstrated in previous work for bi-gons (see [5,6]) and arbitrary n -gons (see [7]), the main intermediate step being the construction of a ‘symmetric point’ in the model of the boundary of the cell, invariant under the full symmetries of the cell. In this paper we explicitly construct for the first time DGLA models of 3-cells, in particular for a banana-shaped cell (see Theorem 3.3) and for a cube (see Section 4), that are invariant under symmetries fixing a major diagonal.

In Section 2, we give a collection of general facts about DGLAs and models of cell complexes (some reproduced from [7, 10]), including defining the notion of a point (solution of MC), of a particular k -cell ($k > 1$) being localised at a point in a model, how to ‘twist’ a model so as to move the points of localisation of cells as well as the universal averages of [9]. In Section 3, we construct a symmetric model of the n -faceted banana in which the main cell is localised at a symmetric central point, by first constructing a model localised at one of the vertices of the banana and then twisting it. In Section 4, we derive a model of an arbitrary polyhedral 3-cell and give the example of the cube where the induced model is invariant under those symmetries of the cube fixing a main diagonal; this model will be used in [8].

2 General Properties of DGLAs and DGLA Models

2.1 General DGLAs

For simplicity, we will work over $k = \mathbb{Q}$, though the discussion also holds for any field of characteristic zero. Recall that a DGLA over k is a vector space A over k with \mathbb{Z} -grading $A = \bigoplus_{n \in \mathbb{Z}} A_n$ along with a bilinear map $[\cdot, \cdot]: A \times A \rightarrow A$ (bracket, respecting the grading) and a linear map $\partial: A \rightarrow A$ (differential, grading shift -1) for which $\partial^2 = 0$, while

$$\begin{aligned} \text{symmetry of bracket} & [b, a] = -(-1)^{|a||b|}[a, b]; \\ \text{Jacobi identity} & (-1)^{|a||b|}[[b, c], a] + (-1)^{|b||c|}[[c, a], b] + (-1)^{|c||a|}[[a, b], c] = 0; \\ \text{Leibniz rule} & \partial[a, b] = [\partial a, b] + (-1)^{|a|}[a, \partial b], \end{aligned}$$

for all homogeneous $a, b, c \in A$. Defining the *adjoint* action of A on itself by $\text{ad}_e(a) = [e, a]$, the operator $\text{ad}_e: A \rightarrow A$ has grading shift $|e|$, for homogeneous $e \in A$. The Jacobi identity and Leibniz rule can now be reformulated as operator equalities:

$$\begin{aligned} \text{Jacobi identity} & \text{ad}_{[a,b]} = [\text{ad}_a, \text{ad}_b]; \\ \text{Leibniz rule} & \text{ad}_{\partial a} = [\partial, \text{ad}_a]; \end{aligned}$$

in terms of the graded operator commutator, $[A, B] \equiv A \circ B - (-1)^{|A||B|} B \circ A$. Since the relations all preserve the number of brackets, it is meaningful to define an additional grading by the number of (Lie) brackets; in particular, for $x \in A$, let $x^{[m]}$ denote the part of x containing precisely m brackets.

2.2 Points and Localisation

An element $a \in A_{-1}$ is called a *point* (or is said to be *flat*) in the model if it satisfies the Maurer–Cartan equation $\partial a + \frac{1}{2}[a, a] = 0$. For any point $a \in A_{-1}$, define the *twisted*

differential ∂_a by $\partial_a \equiv \partial + \text{ad}_a$; the fact that $\partial_a^2 = 0$ is guaranteed by the Maurer–Cartan condition. By the *localisation* of A to a point a , denoted $A(a)$, we will mean the DGLA that as a graded Lie algebra is

$$(\ker \partial_a|_{A_0}) \oplus \bigoplus_{n>0} A_n$$

with the induced bracket from A and the differential ∂_a . This contains only non-negative gradings. Leibniz guarantees that $\ker \partial_a|_{A_0}$ is closed under Lie bracket.

2.3 Edges and Flows

Any element $e \in A_0$ defines a *flow* on A by

$$(2.1) \quad \frac{dx}{dt} = \partial e - \text{ad}_e(x) \quad \text{on } A_{-1}, \quad \frac{dx}{dt} = -\text{ad}_e(x) \quad \text{on } A_{\neq -1}.$$

This flow is called the *flow by e* and preserves the grading. (To define this rigorously, one can work in a space quotiented by all expressions involving $N + 1$ Lie brackets, as in [10], effectively truncating to the space of linear combinations of terms involving at most N Lie brackets, whose coefficients are polynomials in t with rational coefficients. Then one considers the tower of spaces as N increases. Equivalently, one can choose a basis for the finite-dimensional space of expressions involving exactly N Lie brackets, and then allowed expressions are formal combinations of these basis elements, over all N , with coefficients that are polynomials in t . While we talk about functions of t and their derivatives, these are well defined for rational t , with derivatives being well-defined, since all the coefficients are polynomial functions of t .)

Lemma 2.1 *For any $e \in A_0$, the flow by e in grading -1 preserves flatness. That is, if $x(t) \in A_{-1}$ satisfies (2.1) with initial condition $x(0)$ satisfying the Maurer–Cartan condition, then at any (rational) time t , $x(t)$ also satisfies Maurer–Cartan.*

Proof As in the proof of [10, Theorem 1], consider the curvature $f(t) \in A_{-2}$ defined by $f \equiv \partial x + \frac{1}{2}[x, x]$. It satisfies

$$\begin{aligned} \frac{df}{dt} &= \partial \frac{dx}{dt} + \left[x, \frac{dx}{dt} \right] = -\partial(\text{ad}_e x) + [x, \partial e] - [x, \text{ad}_e(x)] \\ &= -\partial \circ \text{ad}_e(x) + \text{ad}_{\partial e}(x) + (\text{ad}_x)^2 e = -\text{ad}_e \circ \partial(x) + \frac{1}{2} \text{ad}_{[x,x]} e = -\text{ad}_e f, \end{aligned}$$

a first order homogeneous linear ode for $f(t)$ with initial condition $f(0) = 0$, since $x(0)$ satisfies the Maurer–Cartan condition. Thus, $f(t) = 0$ for all t , as required. ■

Linearity of the differential equations (2.1) in e ensures that flowing by e for time t is equivalent to flowing by te for a unit time. Denote the result of flowing by e from $a \in A_{-1}$ for unit time, by $u_e(a)$, so that the solution of the first equation in (2.1) is $x(t) = u_{te}(x(0))$. Explicitly,

$$u_e(a) = e^{-\text{ad}_e} a + \frac{1 - e^{-\text{ad}_e}}{\text{ad}_e} \partial e = a + \partial e - \left[e, a + \frac{1}{2} \partial e \right] + (\geq 2 \text{ brackets}),$$

where the meaning of the operator quotient is the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\text{ad}_e)^n (\partial e)$.

Lemma 2.2 For a point a , the condition that $u_e(a) = a$ is equivalent to $\partial_a e = 0$, that is, $e \in A(a)_0$ (e is localised at a). This is a linear condition on e , and therefore, in this case, the flow by e fixes a at all time (not only after unit time).

Lemma 2.3 (see [6, Lemma 2.2]) If e flows from a point a to a point b in unit time, then $\partial_b \circ \exp(-ad_e) = \exp(-ad_e) \circ \partial_a$ so that $\exp(-ad_e)$ intertwines the localisation $A(a)$ to the localisation $A(b)$.

Example 2.4 The unique DGLA model, $A(I)$, of an interval has three generators; a, b of grading -1 (the endpoints), and e of grading 0 (the 1-cell). The differential is given by the condition $u_e(a) = b$ (see [10]). Explicitly,

$$\partial e = (\text{ad}_e)b + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\text{ad}_e)^i (b - a) = \frac{E}{1 - e^E} a + \frac{E}{1 - e^{-E}} b = b - a + \frac{E}{2}(a + b) + \dots,$$

where B_i denotes the i -th Bernoulli number defined as coefficients in the expansion $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$, $E \equiv \text{ad}_e$, and the expressions in E are considered as formal power series.

Example 2.5 In any DGLA model $A(X)$ of a regular cell complex X , for any 1-cell e in X with endpoints a, b , there is a natural DGLA homomorphism $A(I) \rightarrow A(X)$, while $u_e(a) = b$.

Denote by $\text{BCH}(x, y)$ the Baker–Campbell–Hausdorff formula for the element of the free Lie algebra (over \mathbb{Q}) in two variables x, y such that as formal series $\exp(x) \cdot \exp(y) = \exp \text{BCH}(x, y)$ (see [4] for a short proof of existence and [3] for a computational formula). Here, we collect some elementary properties that follow from the definition, Jacobi, and uniqueness of BCH as a free Lie algebra element.

Lemma 2.6 (i) The first few terms of $\text{BCH}(x, y)$ are

$$\text{BCH}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}(X^2 y + Y^2 x) - \frac{1}{24}XYXy + \dots,$$

where X, Y denote ad_x, ad_y .

- (ii) $\text{BCH}(\text{ad}_x, \text{ad}_y) = \text{ad}_{\text{BCH}(x, y)}$.
- (iii) BCH is associative; that is, $\text{BCH}(\text{BCH}(x, y), z) = \text{BCH}(x, \text{BCH}(y, z))$ for any symbols x, y, z . The iterated BCH of n symbols $x_1, \dots, x_n \in A$ will be written $\text{BCH}(x_1, \dots, x_n)$.

Lemma 2.7 There is a homomorphism from the group A_0 considered with operation BCH , to the group $\text{Aut}(A)$, defined by mapping $e \in A_0$ to the flow (in unit time) as defined on all gradings in A by equation (2.1).

Proof By [10, Lemma 3], and the explicit formula given for $u_e(a)$ above, it follows that

$$u_{e_2}(u_{e_1}(a)) = u_{\text{BCH}(e_1, e_2)}(a),$$

for any $a \in A_{-1}$. Thus, on elements of grading -1 , a flow by e_1 for unit time followed by a flow by e_2 for unit time is equivalent to a flow by $\text{BCH}(e_1, e_2)$ for unit time. Note that the flow for unit time by e acting on A_n for $n \neq -1$, is just the exponential operator $\exp(-\text{ad}_e)$ for which it is immediate that $\exp(-\text{ad}_{e_2}) \circ \exp(-\text{ad}_{e_1}) = \exp(-\text{ad}_{\text{BCH}(e_1, e_2)})$. ■

Definition 2.8 By a *piecewise linear path* γ in A , we mean a sequence of points $a_i \in A_{-1}$ ($0 \leq i \leq m$) along with elements $e_i \in A_0$ ($1 \leq i \leq m$), called *edges*, which are such that the edges define flows between the respective points, that is, $u_{e_i}(a_{i-1}) = a_i$ for all $1 \leq i \leq m$. For such a path, we denote by $\text{BCH}(\gamma) \in A_0$ the iterated BCH of the edges, $\text{BCH}(\gamma) \equiv \text{BCH}(e_1, \dots, e_m)$. A piecewise linear path in A is called a *loop* if its initial and final points agree, that is $a_0 = a_m$.

Lemma 2.9 (see [2]) *If X has c connected components and $\{a_1, \dots, a_c\}$ is a choice of basepoints, one in each connected component, then the set of points in $A(X)$ is*

$$\bigcup_{i=1}^c \{u_e(a_i) \mid e \in A_0\} \cup \{u_e(0) \mid e \in A_0\}.$$

For each i , the map $\pi_i: e \mapsto u_e(a_i)$ is a ‘fibration’, with fibre $\pi_i^{-1}(a_i)$ generated as a vector space by $\{\text{BCH}(\gamma) \mid \gamma \in \pi_1(X, a_i)\}$, while the map $\pi_0: e \mapsto u_e(0)$ is injective.

2.4 Localisation of Models

As noted in Section 1, $A(X)$ is not unique, but is well defined up to (exact) DGLA isomorphism.

Definition 2.10 A point $a \in A_{-1}$ in a model of a regular cell complex X is said to be *local to a cell f* in X if it lies in the submodel generated by the cells in the closure \bar{f} of f .

By Lemma 2.9, equivalently, a point is local to the cell f if it can be written as $u_e(a_0)$, where a_0 is a 0-cell in \bar{f} and e is a zero-graded element in the (free) Lie algebra generated by cells in \bar{f} .

Definition 2.11 In a DGLA model A of a regular cell complex X , a k -cell f (for $k > 1$) will be said to be *localised at the point $a \in A_{-1}$* if $\partial_a f = \partial f + [a, f]$ lies in the (free) Lie algebra generated by cells of the closure of the geometric boundary $\bar{\partial}_0 f$.

Here, by abuse of notation, we have used the same symbol for the geometric k -cell f and the element $f \in A_{k-1}$ in its model. By locality of the model and using freeness of the Lie algebra, we see that the point of localisation of a cell f in a model (if it exists) is unique and must be local to the cell (in the sense of Definition 2.10).

Remark The explicit constructions of models of the bi-gon in [6] and the triangle in [7], are localised (at their ‘centre’ points). Although not all models of X will be such

that all cells of dimension > 1 are localised (for example in the model of the bi-gon in [5], the main cell is not localised at any point), we will see in Section 4 that such models always exist in dimensions up to three. A similar construction should also work in higher dimensions.

Lemma 2.12 *If A is a model of a regular cell complex X in which the k -cell f ($k > 1$) is localised at the point $a \in A_{-1}$, then for any $e \in A_0$ in the subalgebra generated by the 1-skeleton of \bar{f} , there is a variation A' of the model in which the generator for f is replaced by*

$$f' = \exp(-ad_e) \cdot f$$

and the cell f is localised at the point $a' = u_e(a)$.

Proof This is immediate from Lemma 2.3, $\partial_{a'} f' = \exp(-ad_e) \cdot \partial_a f$. ■

The above lemma means that we can ‘twist’ any model in which a cell is localised, so that the cell is localised at any point we please (which is local to the cell). This technique was used in [6, 7] to generate symmetric models, by writing first a (non-symmetric) model of the relevant 2-cell localised at a point on its boundary and then twisting it so that it is localised at a symmetric point. All that remained was to verify that the model obtained was indeed symmetric.

Remark Since the 1-skeleton of \bar{f} is not simply connected, the construction given by Lemma 2.12 of a model A' in which f is localised at another point a' local to the face f , will not be unique. That is, there are different $e \in A_0$ for which $u_e(a) = a'$. In particular, a twist by $t\text{BCH}(\gamma)$ of a model in which f is localised at a_0 , for any $t \in \mathbb{Q}$ and any non-trivial loop γ based at a_0 in the 1-skeleton on \bar{f} , will yield another (distinct) such model.

2.5 Universal Averages

In [9], we were shown how to construct a universal expression $\mu_n(x_1, \dots, x_n)$ in the completed free Lie algebra generated by x_1, \dots, x_n such that

- (i) μ_n is totally symmetric in its arguments;
- (ii) in any DGLA model in which a, b are points with $u_{x_i}(a) = b$ for all $i = 1, \dots, n$, also $u_{\mu_n(x_1, \dots, x_n)}(a) = b$.

It was also shown that the expansion of μ_n up to three Lie brackets is

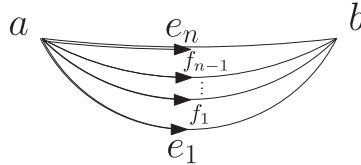
$$\mu_n(x_1, \dots, x_n) = \frac{1}{n} \sum_i x_i - \frac{1}{12n^2} \sum_{i,j, i \neq j} [x_i, [x_i, x_j]] + \dots,$$

and we call μ_n the universal average. There is a closed formula for μ_2 , found in [6],

$$\mu_2(x, y) = \text{BCH}\left(x, \frac{1}{2}\text{BCH}(-x, y)\right).$$

3 The Banana 3-cell

Let X_n be the n -faceted banana, with two 0-cells, $n-1$ 1-cells, n bi-gon 2-cells, and one 3-cell. The corresponding model will contain Lie algebra generators corresponding to each cell; denote them by a, b (grading -1), e_i ($1 \leq i \leq n$, grading 0), f_i ($1 \leq i \leq n$, grading 1), and h (grading 2), respectively. Here, the orientation on the 2-cell f_i is chosen so that its geometric boundary is $\partial_0 f_i = e_i - e_{i+1}$ where $e_{n+1} \equiv e_1$ (indices modulo n). The geometric boundary of h is $\partial_0 h = f_1 + \dots + f_n$.



3.1 Symmetries

The geometric symmetry group of the banana is $D_n \times \mathbb{Z}_2$. The dihedral group is generated by the rotation by $\frac{2\pi}{n}$ around the axis ab ,

$$\tau: e_i \mapsto e_{i+1}, \quad f_i \mapsto f_{i+1}, \quad a, b, h \text{ fixed,}$$

and the reflection in the plane containing e_n and the central axis of the banana,

$$\sigma: a, b \text{ fixed,} \quad e_i \mapsto e_{n-i}, \quad f_i \mapsto -f_{n-i-1}, \quad h \mapsto -h,$$

while the further \mathbb{Z}_2 factor is generated by reflection in the plane equidistant from a and b ,

$$\iota: a \leftrightarrow b, \quad e_i \mapsto -e_i, \quad f_i \mapsto -f_i, \quad h \mapsto -h,$$

which commutes with both σ and τ . For some purposes we will restrict to the subgroup $D_n = \langle \sigma, \tau \rangle$ of the symmetry group fixing the vertices.

3.2 0-, 1-, 2-cells

By the Leibniz rule, the differential ∂ is determined by its values on generators. On vertices, ∂ is fixed by the Maurer–Cartan condition, namely,

$$(3.1) \quad \partial a = -\frac{1}{2}[a, a], \quad \partial b = -\frac{1}{2}[b, b].$$

On 1-cells, ∂ is also unique (see Example 2.4):

$$(3.2) \quad \partial e_i = \frac{E_i}{1 - e^{E_i}} a + \frac{E_i}{1 - e^{-E_i}} b = b - a + \frac{1}{2}[a + b, e_i] + (\geq 2 \text{ brackets}),$$

where $E_i = \text{ad}_{e_i}$. The faces f_i are bi-gons, and we use the symmetric model of the bi-gon from [6] in which

$$(3.3) \quad \partial f_i = \text{BCH}(-\frac{1}{2}v_i, e_i, -e_{i+1}, \frac{1}{2}v_i) - [x_i, f_i] = e_i - e_{i+1} - \frac{1}{2}[a + b, f_i] + \dots$$

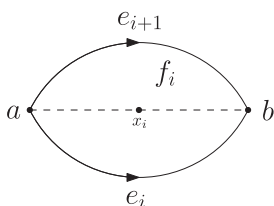
so that f_i is localised at its centre

$$x_i = u_{\frac{1}{2}v_i}(a) = \frac{1}{2}(a + b) + \frac{1}{16}[e_i + e_{i+1}, b - a] + (\geq 2 \text{ brackets}),$$

where v_i is the centreline from a to b given by the average

$$v_i = \mu_2(e_i, e_{i+1}) = \text{BCH}(e_i, \frac{1}{2}\text{BCH}(-e_i, e_{i+1})) = \frac{1}{2}(e_i + e_{i+1}) + (\geq 2 \text{ brackets})$$

in terms of the universal average μ_2 of [9].



3.3 Central Point of Banana

A totally symmetric point should be at the centre of the banana,

$$x = u_{\frac{1}{2}v}(a) = \frac{1}{2}(a + b) + \frac{1}{8n} \sum_{i=1}^n E_i(b - a) + \dots,$$

which is half way along a central diagonal of the banana from a to b , a path in the direction $v = \mu_n(e_1, \dots, e_n)$, the universal average of e_1, \dots, e_n . The fact that x is invariant under σ , τ , and ι follows from the total symmetry of μ_n along with the fact that ([9, Lemma 4.3])

$$\mu_n(-e_1, \dots, -e_n) = -\mu_n(e_1, \dots, e_n).$$

The only freedom remaining in the model is the boundary of the 3-cell, $\partial h \in A_1$, which, to give a valid model of X_n must be such that $\partial^2(h) = 0$, while $(\partial h)^{[0]}$ must coincide with the topological boundary $\partial_0 h = f_1 + \dots + f_n$. The purpose of this section is to give a formula for ∂h that is invariant under the D_n action of the symmetries of the banana fixing the vertices and in which h is localised at a central point. In order to do this, we will first construct a model in which all the 2-cells and the 3-cell are localised at a , and then twist using Lemma 2.12 to localise at a symmetric point.

3.4 Model Localised at a

Since f_i is localised at x_i , $f'_i = \exp(\frac{1}{2}\text{ad}_{v_i}) \cdot f_i$ is localised at a ; in particular,

$$(3.4) \quad \partial_a f'_i = \text{BCH}(e_i, -e_{i+1}).$$

Similarly, set $h' = \exp(\frac{1}{2}\text{ad}_v) \cdot h$; by Lemma 2.12, a model in which h is localised at x will have h' localised at a .

Lemma 3.1 A DGLA model of X_n is defined by generators a, b, e_i, f'_i, h' , with the differential defined by (3.1), (3.2), and (3.4) along with

$$(3.5) \quad \partial_a h' = \sum_{i=1}^n P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \cdot f'_i$$

for any polynomials P_i in $(n - 1)$ non-commuting variables whose initial term is 1 and satisfy the identity

$$(3.6) \quad \sum_{i=1}^{n-1} P_i(\text{ad}_{x_1}, \dots, \text{ad}_{x_{n-1}}) \cdot x_i = P_n(\text{ad}_{x_1}, \dots, \text{ad}_{x_{n-1}}) \cdot \text{BCH}(x_1, \dots, x_{n-1}).$$

In this model, the 3-cell and 2-cells are all localised at a .

Proof It is apparent that the initial term of $\partial_a h'$ as defined by (3.5) is $\sum_{i=1}^n f'_i$, as required. It remains only to verify that $\partial_a^2 h' = 0$, that is, that the RHS of (3.5) defines an element of $\ker \partial_a$. For this, observe that since $u_{e_i}(a) = b$, by Lemma 2.7, $u_{\text{BCH}(e_i, -e_{i+1})}(a) = a$, and so by Lemma 2.2, $\text{BCH}(e_i, -e_{i+1}) \in \ker \partial_a$. However, for $y_1, \dots, y_r \in \ker \partial_a$ of degree 0,

$$\partial_a [y_1, [y_2, \dots, [y_r, w] \dots]] = [y_1, [y_2, \dots, [y_r, \partial w] \dots]],$$

and so for any non-commuting polynomial $P, P(\text{ad}_{y_1}, \dots, \text{ad}_{y_r})$ commutes with ∂_a . By Lemma 2.6(ii), $\text{BCH}(\text{ad}_x, \text{ad}_y) = \text{ad}_{\text{BCH}(x,y)}$, and it follows that

$$\begin{aligned} \partial_a \sum_{i=1}^n P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \cdot f'_i &= \sum_{i=1}^n P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \cdot \partial_a f'_i \\ &= \sum_{i=1}^n P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \cdot \text{BCH}(e_i, -e_{i+1}) \\ &= 0, \end{aligned}$$

where the second step follows from (3.4) and the third from (3.6) applied to $x_i = \text{BCH}(e_i, -e_{i+1})$, since $-x_n = \text{BCH}(x_1, \dots, x_{n-1})$. ■

To see that the previous lemma's constructions actually lead to a model of the n -faceted banana, it remains only to show that P_1, \dots, P_n exist with initial term 1 and satisfying identity (3.6). This follows immediately from Lemma 2.6(iii), even with $P_n \equiv 1$; see the next subsection and (3.12). Indeed, there are many possible choices for P_1, \dots, P_n .

3.5 Model Localised at Central Point x

Twisting the model of Lemma 3.1 back so that the 2-cells are localised at their centres, and the 3-cell is localised at its centre x , we find the differential is given by (3.1), (3.2), and (3.3) along with

$$\partial_x h = e^{-\frac{1}{2}\text{ad}_v} \left(\sum_{i=1}^n P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \cdot e^{\frac{1}{2}\text{ad}_{v_i}} f_i \right).$$

The final requirement is that the model is symmetric under the action of D_n ; that is, it is invariant under σ , τ . By the geometry and total symmetry of the bi-gon model, (3.1), (3.2), and (3.3) will be invariant under σ , τ , and ι . Under τ , x, h, v are fixed, while $e_i \mapsto e_{i+1}, f_i \mapsto f_{i+1}, v_i \mapsto v_{i+1}$, so (3.7) is invariant under τ so long as

$$P_i(\text{BCH}(E_2, -E_3), \dots, \text{BCH}(E_n, -E_1)) = P_{i+1}(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)),$$

which is ensured by

$$P_{i+1}(x_1, \dots, x_{n-1}) = P_i(x_2, \dots, x_{n-1}, -\text{BCH}(x_1, \dots, x_{n-1})).$$

Under σ , v, x are fixed, $e_i \mapsto e_{n-i}, f_i \mapsto -f_{n-i-1}, v_i \mapsto v_{n-i-1}$, while h changes sign. In order for (3.7) to be invariant under σ , it is required that

$$P_i(\text{BCH}(E_{n-1}, -E_{n-2}), \dots, \text{BCH}(E_1, -E_n)) = P_{n-i-1}(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)),$$

which is ensured by (3.8) along with

$$P_{n-i}(x_1, \dots, x_{n-1}) = P_i(-x_{n-1}, \dots, -x_1).$$

Under ι , the quantities e_i, f_i, v_i, h , and v all change sign, while x is invariant, and so (3.7) is invariant under ι so long as

$$e^{\frac{1}{2}\text{ad}_v} P_i(\text{BCH}(-E_1, E_2), \dots, \text{BCH}(-E_{n-1}, E_n)) e^{-\frac{1}{2}\text{ad}_{v_i}} = e^{-\frac{1}{2}\text{ad}_v} P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) e^{\frac{1}{2}\text{ad}_{v_i}},$$

where $v_i = \mu_2(e_i, e_{i+1})$ and $v = \mu_n(e_1, \dots, e_n)$.

In conclusion, we have the following lemma.

Lemma 3.2 *The completed DGLA with (free) Lie algebra generators a, b, e_i, f_i, h and differential defined by (3.1), (3.2), (3.3), and (3.7) is a model for the n -faceted banana that is invariant under the geometric symmetries of the cell fixing the vertices, so long as the polynomials P_1, \dots, P_n in $(n - 1)$ non-commuting variables, with initial term 1, satisfy identities (3.6), (3.8), and (3.9). It will be completely invariant under the symmetries of the 3-cell if, in addition, (3.10) holds.*

By Lemma 2.6(i), we can write

$$\text{BCH}(x, y) = x + y + Q(X, Y)y,$$

where Q is a polynomial in two non-commuting variables whose lowest order terms are $\frac{1}{2}X + \frac{1}{12}(X^2 - YX) + \dots$. By Lemma 2.6(iii), iterating (3.11) gives

$$\text{BCH}(x_1, \dots, x_n) = \sum_{i=1}^n x_i + \sum_{i=2}^n Q(\text{BCH}(X_1, \dots, X_{i-1}), X_i)x_i$$

so that

$$(3.12) \quad P_1 = P_n = 1, \quad P_i = 1 + Q(\text{BCH}(X_1, \dots, X_{i-1}), X_i) \text{ for } 1 < i < n$$

is a solution of (3.6). Note that (3.6) is a linear condition on (P_1, \dots, P_n) that is invariant under the action of cyclic permutation of the P_i , while cyclically permuting the X_i (with $X_n = -\text{BCH}(X_1, \dots, X_{n-1})$), as well as reversing the order of the P_i while reversing the order and signs of the x_i . That is, (3.6) is invariant under a dihedral group action, and so there exists an invariant solution (that will satisfy (3.8) and (3.9)) given by averaging the orbit of the solution (3.12) under the just described dihedral action. The result is that

$$\begin{aligned} P_i &= 1 + \frac{1}{2n} \sum_{j=1+\delta_{in}}^{i-1} Q(\text{BCH}(X_j, \dots, X_{i-1}), X_i) \\ &\quad + \frac{1}{2n} \sum_{j=i+1}^{n-1} Q(-\text{BCH}(X_i, \dots, X_j), X_i) \\ &\quad + \frac{1}{2n} \sum_{j=1+\delta_{in}}^{i-1} Q(\text{BCH}(X_j, \dots, X_i), -X_i) \\ &\quad + \frac{1}{2n} \sum_{j=i+1}^{n-1} Q(-\text{BCH}(X_{i+1}, \dots, X_j), -X_i) \end{aligned}$$

satisfies (3.6), (3.8), and (3.9). The coefficient of f'_i in $\partial_a h'$ from (3.5) is now

$$\begin{aligned} &P_i(\text{BCH}(E_1, -E_2), \dots, \text{BCH}(E_{n-1}, -E_n)) \\ &= 1 + \frac{1}{2n} \sum_{\substack{j=1 \\ j \neq i, i+1}}^n \left(Q(\text{BCH}(E_j, -E_i), \text{BCH}(E_i, -E_{i+1})) \right. \\ &\quad \left. + Q(\text{BCH}(E_j, -E_{i+1}), \text{BCH}(E_{i+1}, -E_i)) \right). \end{aligned}$$

Theorem 3.3 *The completed DGLA with (free) Lie algebra generators a, b, e_i, f_i, h and differential defined by (3.1), (3.2), (3.3) and*

$$\begin{aligned} e^{\frac{1}{2}ad_v} \partial_x h &= \sum_{i=1}^n f'_i + \frac{1}{2n} \sum_{\substack{i, j=1 \\ j \neq i, i+1}}^n \left(Q(\text{BCH}(E_j, -E_i), \text{BCH}(E_i, -E_{i+1})) \right. \\ &\quad \left. + Q(\text{BCH}(E_j, -E_{i+1}), \text{BCH}(E_{i+1}, -E_i)) \right) f'_i \end{aligned}$$

defines a model for the n -faceted banana that is symmetric under the geometric symmetries of the cell fixing the vertices, where Q is defined by (3.11), $f'_i = e^{\frac{1}{2}ad_v} f_i$, $v_i = \mu_2(e_i, e_{i+1})$, $v = \mu_n(x_1, \dots, x_n)$, and $x = u_{\frac{1}{2}v}(a)$. The 3-cell is localised at x in this model and the bi-gon 2-cells are localised at their centres $x_i = u_{\frac{1}{2}v_i}(a)$.

Up to the first two non-trivial orders (in Lie brackets),

$$\begin{aligned} v_i &= \frac{1}{2}(e_i + e_{i+1}) - \frac{1}{48}(E_i^2 e_{i+1} + E_{i+1}^2 e_i) + \dots, \\ v &= \frac{1}{n} \sum_i e_i - \frac{1}{12n^2} \sum_{i \neq j} E_i^2 e_j + \dots, \\ x &= \frac{1}{2}(a + b) + \frac{1}{8n} \sum_i [e_i, b - a] + \dots, \\ x_i &= \frac{1}{2}(a + b) + \frac{1}{16}(E_i + E_{i+1})(b - a) + \dots. \end{aligned}$$

Meanwhile, up to second order (in Lie brackets) the differential is given by

$$\begin{aligned} \partial a &= -\frac{1}{2}[a, a], & \partial b &= -\frac{1}{2}[b, b], \\ \partial e_i &= b - a + \frac{1}{2}E_i(a + b) + \frac{1}{12}E_i^2(b - a) + \dots, \\ \partial f_i &= e_i - e_{i+1} - \frac{1}{2}[a + b, f_i] + \frac{1}{48}(E_{i+1}^2 e_i - E_i^2 e_{i+1}) + \frac{1}{16}[(E_i + E_{i+1})(a - b), f_i] + \dots, \\ \partial h &= \sum_i f_i - \frac{1}{2}[a + b, h] + \frac{1}{8n} \sum_i [E_i(a - b), h] + \sum_i (\text{quadratic in } E_j\text{'s})f_i + \dots. \end{aligned}$$

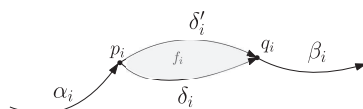
Remark The particular solution of (3.6), (3.8) and (3.9) constructed above does not satisfy (3.10) and so the model described in Theorem 3.3 is not symmetric under the full symmetry group $D_n \times \mathbb{Z}_2$, but only under the part fixing the vertices. In particular, it is not invariant under ι , although such a model does exist. However, this model and the model it induces in Section 4 on the cube have sufficient symmetry for the applications in [8], where cells will come with a preferred oriented main diagonal.

4 A Model for an Arbitrary Polyhedral 3-cell

Suppose that X is an arbitrary polyhedral 3-cell, with n faces. Choose two vertices a and b . Pick a shelling of the subdivision of the boundary, that is, a choice of n non-self intersecting paths $\gamma_1, \dots, \gamma_n$ each from a to b along edges of the polyhedron, in such a way that for each $i = 1, \dots, n$, the paths γ_i and γ_{i+1} (with $\gamma_{n+1} \equiv \gamma_1$) have common initial and final segments so that the intermediate segments together (one in reverse orientation) form the geometric boundary of the i -th face, g_i , of X . See the figure below; for each i ,

$$\gamma_i = \alpha_i \cup \delta_i \cup \beta_i, \quad \gamma_{i+1} = \alpha_i \cup \delta'_i \cup \beta_i,$$

while $\delta_i \cup -\delta'_i = \partial_0(i\text{-th face})$ with matching orientation (faces oriented outwards). Here, δ_i and δ'_i share common initial and final points, say p_i and q_i .



To construct a model $A(X)$ of X , first note that 0-cells and 1-cells have a unique description. For the 2-cells, pick a basepoint (say p_i). In a model in which the i -th cell f_i is localised at p_i :

$$\partial_{p_i} f_i = \text{BCH}(\delta_i, -\delta'_i).$$

To determine a suitable expression for the differential of the 3-cell h in X , we decide to localise it at a and then induce ∂h from the differential (3.5) on the 3-cell in the n -faceted banana X_n using the natural DGLA map $A(X_n) \rightarrow A(X)$ in which

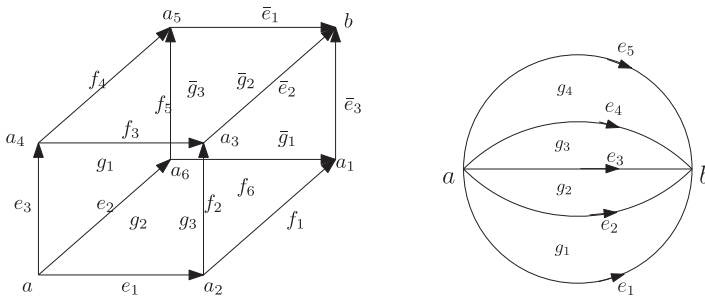
$$\begin{aligned} a, b &\mapsto a, b, \\ e_i &\mapsto \text{BCH}(\gamma_i), \\ f'_i &\mapsto \exp(\text{BCH}(\alpha_i))g_i, \\ h' &\mapsto h, \end{aligned}$$

using the notation of f'_i and h' cells in X_n localised at a as in (3.4) and (3.5) above. Thus, one can use in $A(X)$:

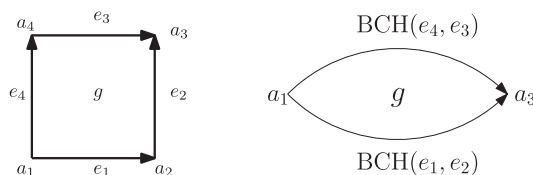
$$\partial_a h = \sum_{i=1}^n P_i(\text{BCH}(\text{ad}_{\gamma_i}, -\text{ad}_{\gamma_{i+1}}), \dots, \text{BCH}(\text{ad}_{\gamma_{n-1}}, -\text{ad}_{\gamma_n})) \cdot \exp(\text{BCH}(\alpha_i))g_i.$$

If it is desired to localise cells at symmetric central points in place of points on their boundary, then additional twists can be applied according to Lemma 2.12.

Example 4.1 We apply the above general argument to construct a model for the cube in which 2- and 3-cells are localised at their centres. Denote a pair of antipodal vertices by a, b . If these vertices and their adjacent edges are removed from the skeleton of the cube, then a hexagon remains joining the six remaining vertices; label the vertices of this hexagon a_1, \dots, a_6 and the edges f_1, \dots, f_6 (with f_i joining a_i and a_{i+1} modulo 6). Denote the edges adjacent to a by e_1, e_2, e_3 and their opposite edges (adjacent to b) by $\bar{e}_1, \bar{e}_2, \bar{e}_3$, respectively. Orient the edges so that those adjacent to a are oriented away from a and orient other edges so that parallel edges have matching orientation. The faces containing a are labelled g_1, g_2, g_3 and their opposite faces with a bar.



As usual, the differential on vertices x has $\partial x = -\frac{1}{2}[x, x]$, while the differential on all edges is also rigidly determined as in Example 2.4. The edge orientations give a partial ordering on the vertices, with a as minimal element and b as maximal element. Each face similarly has a minimal and a maximal element that are diagonally opposite. The differential on faces of the cube is chosen so that they will be localised at their centre.



For a square as depicted, the symmetric model of the bi-gon induces

$$\partial_x g = \exp(-\frac{1}{2} \text{ad}_v) \text{BCH}(e_1, e_2, -e_3, -e_4),$$

where $x = u_{v/2}(a_1)$ is the centre point defined in terms of the centre diagonal

$$v = \mu_2(\text{BCH}(e_1, e_2), \text{BCH}(e_4, e_3)).$$

The formula for the cube is induced from a symmetric model of a 6-faceted banana, by mapping the edges of X_6 to BCH of the 6 maximal chains from a to b on the cube, while mapping the faces f_i (based at their centres) to the conjugations of the corresponding faces of the cube, $\exp(-V_i)g_i$, where v_i is a zero graded element that flows the centre x_i of the i -th face of the cube to the centre of the corresponding bi-gon bounded by two maximal chains (thus changing the point at which it is localised). Here, by the centre of a square, we mean the centre point as in the formulae for ∂ of a square above. Up to terms with only one bracket (which are all that affect the result of $\exp(-V_i)g_i$ up to two brackets), the v 's for the squares $g_1, g_2, g_3, \bar{g}_1, \bar{g}_2, \bar{g}_3$ are $\frac{1}{2}\bar{e}_1, \frac{1}{2}\bar{e}_2, \frac{1}{2}\bar{e}_3, -\frac{1}{2}e_1, -\frac{1}{2}e_2, -\frac{1}{2}e_3$, respectively. That is, $A(X_6) \rightarrow A(\text{cube})$ is given on edges by

$$\begin{aligned} e_1 &\mapsto \text{BCH}(e_3, f_4, \bar{e}_1), \\ e_2 &\mapsto \text{BCH}(e_2, f_5, \bar{e}_1), \\ e_3 &\mapsto \text{BCH}(e_2, f_6, \bar{e}_3), \\ e_4 &\mapsto \text{BCH}(e_1, f_1, \bar{e}_3), \\ e_5 &\mapsto \text{BCH}(e_1, f_2, \bar{e}_2), \\ e_6 &\mapsto \text{BCH}(e_3, f_3, \bar{e}_2), \end{aligned}$$

and on faces by

$$\begin{aligned} g_1 &\mapsto -\exp(-\frac{1}{2}\bar{E}_1)g_1 + (\geq 2 \text{ brackets}), \\ g_2 &\mapsto -\exp(\frac{1}{2}E_2)\bar{g}_2 + (\geq 2 \text{ brackets}), \\ g_3 &\mapsto -\exp(-\frac{1}{2}\bar{E}_3)g_3 + (\geq 2 \text{ brackets}), \\ g_4 &\mapsto \exp(\frac{1}{2}E_1)\bar{g}_1 + (\geq 2 \text{ brackets}), \\ g_5 &\mapsto \exp(-\frac{1}{2}\bar{E}_2)g_2 + (\geq 2 \text{ brackets}), \\ g_6 &\mapsto \exp(\frac{1}{2}E_3)\bar{g}_3 + (\geq 2 \text{ brackets}). \end{aligned}$$

Up to one Lie bracket in $A(X_6)$,

$$\partial h = \sum_i g_i - \frac{1}{2}[a + b, h] + (\geq 2 \text{ brackets}),$$

which transforms on the cube to

$$\begin{aligned} \partial h = & \bar{g}_1 + g_2 + \bar{g}_3 - g_1 - \bar{g}_2 - g_3 - \frac{1}{2}[a + b, h] \\ & + \frac{1}{2}(E_3\bar{g}_3 - \bar{E}_2g_2 + E_1\bar{g}_1 + \bar{E}_3g_3 - E_2\bar{g}_2 + \bar{E}_1g_1) + (\geq 2 \text{ brackets}). \end{aligned}$$

The model of the cube so obtained will be symmetric under symmetries that fix the diagonal ab .

Remark Notice that the structure of the differential for 3-cells is essentially simpler than that for 2-cells, having a linear dependence on its codimension one boundary sub-cells. Thus, in (3.5), the dependence of $\partial_a h'$ upon f_i is linear. This is not true one dimension lower, where the dependence in (3.4) of $\partial_a f'_i$ on its boundary cells e_i and e_{i+1} is highly non-linear (through BCH).

Remark The functorial nature of the construction of $A(X)$ from X means that under a subdivision, there should be a corresponding DGLA map. The subdivision map for the interval (or 1-skeleta) is generated by BCH (see [10]), and for this reason, the differential on a 2-cell (in its simplest form) is the BCH of its boundary (see (3.4)). Similarly, the subdivision map for a 2-cell (say splitting a bi-gon into two bi-gons) will be linear, and the difference between the two sides gives an expression for the differential on a 3-cell as in (3.5). This same story should continue in higher dimensions (all dimensions > 2 will have a linear form).

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