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Range-renewal structure in continued fractions

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Abstract. Let $\omega = [a_1, a_2, \ldots]$ be the infinite expansion of a continued fraction for an irrational number $\omega \in (0, 1)$, and let $R_n(\omega)$ (respectively, $R_{n,k}(\omega)$, $R_{n,k+}(\omega)$) be the number of distinct partial quotients, each of which appears at least once (respectively, exactly k times, at least k times) in the sequence $a_1, \ldots, a_n$. In this paper, it is proved that, for Lebesgue almost all $\omega \in (0, 1)$ and all $k \geq 1$,

$$\lim_{n\to\infty} \frac{R_n(\omega)}{\sqrt{n}} = \sqrt{\frac{\pi}{\log 2}}, \quad \lim_{n\to\infty} \frac{R_{n,k}(\omega)}{R_n(\omega)} = \frac{C_{2k}^k}{(2k-1)\cdot 4^k}, \quad \lim_{n\to\infty} \frac{R_{n,k}(\omega)}{R_{n,k+}(\omega)} = \frac{1}{2k}.$$

The Hausdorff dimensions of certain level sets about R_n are discussed.

1. *Introduction*

In early 2011, a beautiful range-renewal structure in independent and identically distributed models was found by Chen *et al* [**2**]. Among others, the typical main results in [**2**] say that, given n samples of a heavy-tailed *regular* (see [**2**] for the definition) discrete distribution π with an intrinsic index $\gamma = \gamma(\pi) \in (0, 1)$,

$$\lim_{n\to\infty} \frac{R_n}{\mathbb{E}R_n} = 1, \tag{1.1}$$

$$\lim_{n\to\infty} \frac{R_{n,k}}{R_n} = r_k(\gamma) := \frac{\gamma \cdot \Gamma(k-\gamma)}{k! \cdot \Gamma(1-\gamma)}, \tag{1.2}$$

$$\lim_{n\to\infty} \frac{R_{n,k}}{R_{n,k+}} = \frac{\gamma}{k}, \tag{1.3}$$

where R_n (respectively, $R_{n,k}$, $R_{n,k+}$) stands for the number of distinct sample values, each of which appears at least once (respectively, exactly k times, at least k times). Also, the so-called range-renewal speed $\mathbb{E}R_n$ can be calculated explicitly in n: for instance, if

$$\pi_x = \frac{C}{x^\alpha} \cdot [1 + o(1)], \quad x \in \mathbb{N}$$

with $1 < \alpha < +\infty$, then

$$\gamma = \gamma(\pi) := 1/\alpha \quad \text{and} \quad \mathbb{E}R_n = \Gamma(1-\gamma)\cdot(C\cdot n)^{\gamma}\cdot[1+o(1)].$$

Soon after that, it was realized by the second author that the above results should be somehow universal. In particular, it should be true in the continued fractions system (equipped with the well-known Gauss measure μ) since this system, in a certain sense, is stationary and strongly mixing, which means that the system is very nearly an independent and identically distributed model. Confidence in this was strengthened, in the spring of 2012, after a numerical simulation carried out with the help of Mr. Peng Liu.

Given an irrational number $\omega \in (0, 1)$, let $R_n(\omega)$ (respectively, $R_{n,k}(\omega)$, $R_{n,k+}(\omega)$) be the number of distinct partial quotients, each of which appears at least once (respectively, exactly k times, at least k times) in the first n partial quotients of ω. In this paper, we shall prove the following interesting result: for Lebesgue almost all $\omega \in (0, 1)$ and all $k \geq 1$

$$\frac{R_n(\omega)}{\sqrt{n}} = \sqrt{\frac{\pi}{\log 2}} + o(1), \quad \frac{R_{n,k}(\omega)}{R_n(\omega)} = \frac{C_{2k}^k}{(2k-1)\cdot 4^k} + o(1) \quad \text{and}$$
$$\frac{R_{n,k}(\omega)}{R_{n,k+}(\omega)} = \frac{1}{2k} + o(1)$$

as $n \to +\infty$ (see Theorem 1 for the explicit statement and its proof in §3). Moreover, we will discuss the Hausdorff dimension of certain level sets (see Theorem 2 and its proof in §4). As pointed out in Remark 1, although the continued fraction system (equipped with the Gauss measure μ) is in fact a positive recurrent system, we can observe a certain kind of escape phenomenon: for any $k \in \mathbb{N}$ and Lebesgue almost all $\omega \in (0, 1)$

$$\lim_{n\to+\infty} \frac{R_{n,k}(\omega)}{R_{n,k+}(\omega)} = \frac{1}{2k}.$$

In the simple symmetrical random walk model in $\mathbb{Z}^d$ (with $d \geq 3$), the above limited ratio is always the escape rate γ_d [**4**, **5**, **25**], where R_n (respectively, $R_{n,k}$, $R_{n,k+}$) is interpreted as the number of distinct sites visited at least once (respectively, exactly k times, at least k times) up to time n.

This article contains a combination of pure probability theory, ergodic theory (the proof of Theorem 1) and fractal theory (the proof of Theorem 2). It is worthwhile to point out that the ideas in this paper are applicable to other systems to obtain results that are similar to Theorem 1. In view of the techniques developed in this article, it is also possible to obtain further results (for example, those in [**2**]) for the current continued fractions model.

2. *Main settings and results*

Throughout this paper, the notation $y = O(z)$ implies that there exists some universal constant $C > 0$ such that $C^{-1} \leq |y/z| \leq C$; the notation $y = \overline{O}(z)$ implies that there exists some universal constant $C > 0$ such that $|y/z| \leq C$. The notation $y = o(z)$ is understood in the usual way. We shall use C_0 to denote universal constants which may change from line to line. For two sets A, B, we will write $AB := A \cap B$, for simplicity.

Let $\mathbb{X} = (0, 1)\backslash\mathbb{Q}$ be the set of irrational numbers in the interval (0, 1). For any $\omega \in \mathbb{X}$, let $\{a_n = a_n(\omega)\}_{n=1}^{\infty}$ be the partial quotients of ω in continued fraction form: that is,

$$\omega = [a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Therefore, for any $\omega \in \mathbb{X}$, there is a unique (natural) coding $(a_1, a_2, \ldots)$ (still writing ω for simplicity) in $\Omega = \mathbb{N}^{\mathbb{N}}$. Also, the Gauss map $T : \mathbb{X} \to \mathbb{X}$

$$T(\omega) := \frac{1}{\omega} \ (\mathrm{mod}\ 1) = [a_2, a_3, \ldots]$$

induces the natural left-shift map $\sigma : \Omega \to \Omega$. The Gauss measure μ (which satisfies $d\mu(\omega) = d\omega/((\log 2) \cdot (1 + \omega))$ and which is invariant under T) on (0, 1) naturally induces a probability measure $\mathbb{P}$ on Ω. For any $x \in \mathbb{N}$ we know

$$\pi_x := \mathbb{P}(a_1 = x) = -\log_2\left[1 - \frac{1}{(x+1)^2}\right] = \frac{1}{(\log 2) \cdot (x+1)^2} + O\left(\frac{1}{x^4}\right) \tag{2.1}$$

as $x \to +\infty$. There is a probability measure $\pi = (\pi_x : x \in \mathbb{N})$ on $\mathbb{N}$ which also naturally induces an infinitely independent product measure $\widetilde{\mathbb{P}} := \pi^{\infty}$ on Ω. The expectation operator of the probability measure $\mathbb{P}$ (respectively, $\widetilde{\mathbb{P}}$) will be denoted by $\mathbb{E}$ (respectively, $\widetilde{\mathbb{E}}$). Also, we have the following commuting graph

$$\begin{array}{ccc} (\Omega, \mathbb{P}) & \xrightarrow{\sigma} & (\Omega, \mathbb{P}) \\ \boldsymbol{p}\downarrow & & \downarrow\boldsymbol{p} \\ (\mathbb{X}, \mu) & \xrightarrow{T} & (\mathbb{X}, \mu) \end{array}$$

with $\boldsymbol{p}$ being the natural projection

$$\boldsymbol{p}(a_1, a_2, \ldots) := [a_1, a_2, \ldots].$$

Due to this obvious identification, we shall *not* distinguish the spaces Ω and $\mathbb{X} = (0, 1)\backslash\mathbb{Q}$ from here on.

Given $\omega \in \mathbb{X}$. For any fixed $x \in \mathbb{N}$, write

$$N_n(x) = N_n(x, \omega) := \sum_{k=1}^{n} 1_{\{a_k(\omega)=x\}}, \tag{2.2}$$

which is the visiting number of the state x by the partial quotients $a_1(\omega), \ldots, a_n(\omega)$. Define

$$R_n(\omega) := \sum_{x\in\mathbb{N}} 1_{\{N_n(x,\omega)\geq 1\}}. \tag{2.3}$$

This is the number of distinct values of $a_1(\omega), \ldots, a_n(\omega)$: i.e., $R_n(\omega) = \sharp\{a_1(\omega), \ldots, a_n(\omega)\}$. Also, define, for any $k \in \mathbb{N}$,

$$R_{n,k}(\omega) := \sum_{x\in\mathbb{N}} 1_{\{N_n(x,\omega)=k\}}. \tag{2.4}$$

This is the number of distinct partial quotients, each of which appears exactly k times in the finite sequence $a_1(\omega), \ldots, a_n(\omega)$.

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Our main result is the following theorem.

THEOREM 1. *For Lebesgue almost all* $\omega \in (0, 1)$,

$$\lim_{n\to\infty} \frac{R_n(\omega)}{\sqrt{n}} = \sqrt{\frac{\pi}{\log 2}}. \tag{2.5}$$

Furthermore, for any $k \geq 1$,

$$\lim_{n\to\infty} \frac{R_{n,k}(\omega)}{R_n(\omega)} = \frac{C_{2k}^k}{(2k-1)\cdot 4^k} =: r_k, \tag{2.6}$$

where $C_n^m := n!/(m!(n-m)!)$ *and* $\sum_{k=1}^{\infty} r_k = 1$.

Remark 1. The above theorem leads to the following.

(1) As $k \to \infty$, $r_k = 1/(2\sqrt{\pi}) \cdot k^{-3/2} + O(k^{-5/2})$, which is a power law with index 3/2.

(2) Let

$$R_{n,k+}(\omega) := \sum_{\ell=k}^{n} R_{n,\ell}(\omega). \tag{2.7}$$

Then for Lebesgue almost all $\omega \in (0, 1)$

$$\lim_{n\to+\infty} \frac{R_{n,k}(\omega)}{R_{n,k+}(\omega)} = \frac{1}{2k}, \tag{2.8}$$

which can be interpreted as an average escape rate at the level $k \geq 1$, although the model itself is in fact positive recurrent. This result can be seen as the following. Put, for $k \geq 2$,

$$r_{k+} = 2k \cdot r_k = \frac{2k}{2k-1} \cdot \frac{C_{2k}^k}{4^k} = \prod_{j=1}^{k-1}\left(1 - \frac{1}{2j}\right)$$

and $r_{1+} = 1$. Obviously,

$$r_{k+} - r_{(k+1)+} = r_{k+} - r_{k+} \cdot \left(1 - \frac{1}{2k}\right) = \frac{1}{2k} \cdot r_{k+} = r_k, \quad k \geq 1.$$

Thus $r_{k+} = \sum_{\ell \geq k} r_\ell$. Then the result follows from Theorem 1.

(3) We could recall a standard result in the field of the simple symmetrical random walk (*SSRW*) on $\mathbb{Z}^d$ (with $d \geq 3$), although it was not expressed explicitly in [**5**] (but nearly explicitly in [**25**, p. 220, Theorem 20.11]). It says that the limited ratio in (2.8) is always γ_d, the usual escape rate of the random walk. The proof of this result has only several lines using a subadditive ergodic theorem [**13–15**] (essentially the argument of Derriennic [**3**]) which we would like to list as the following. In a *SSRW* on $\mathbb{Z}^d$ (with $d \geq 3$), R_n, $R_{n,k-} := R_n - R_{n,(k+1)+}$ are all subadditive. Hence

$$\lim_{n\to+\infty} \frac{R_n}{n} = \lim_{n\to+\infty} \frac{\mathbb{E}R_n}{n} = \gamma_d,$$

$$\lim_{n\to+\infty} \frac{R_{n,k-}}{n} = \lim_{n\to+\infty} \frac{\mathbb{E}R_{n,k-}}{n} = \gamma_d \cdot [1 - (1-\gamma_d)^k].$$

(Cf. [**31**] for more detailed calculations.) The result follows. This is a different approach compared with Dvoretzky and Erdös' result [**4**] for $d \geq 3$; the variation estimations are not needed in this proof.

Remark 2. In view of Corollary 4, the result in the above theorem can be strengthened as follows. For any fixed $k \geq 1$,

$$\frac{R_n(\omega)}{\sqrt{n}} = \sqrt{\frac{\pi}{\log 2}} + o\bigg(\frac{1}{n^{0.0698}}\bigg), \tag{2.9}$$

$$\frac{R_{n,k}(\omega)}{R_n(\omega)} = \frac{C_{2k}^k}{(2k-1)\cdot 4^k} + o\bigg(\frac{1}{n^{0.0698}}\bigg), \tag{2.10}$$

$$\frac{R_{n,k}(\omega)}{R_{n,k+}(\omega)} = \frac{1}{2k} + o\bigg(\frac{1}{n^{0.0698}}\bigg) \tag{2.11}$$

almost surely, since $1/(6(1+2\log 2)) = 0.069843\ldots$.

Put $c > 0$. Then, for any $\beta \geq 0$,

$$E(\beta, c) := \bigg\{\omega \in (0,1)\backslash\mathbb{Q} : \lim_{n\to+\infty} \frac{R_n(\omega)}{n^\beta} = c\bigg\}.$$

The above theorem implies that

$$\dim_H E(\beta, c) = 1 \quad \text{for } \beta = \frac{1}{2} \text{ and } c = \sqrt{\frac{\pi}{\log 2}}.$$

One may wonder what happens for other choices of (β, c).

Let

$$d_H(\beta) := \dim_H \bigg\{\omega \in (0,1)\backslash\mathbb{Q} : 0 < \lim_{n\to+\infty} \frac{R_n(\omega)}{n^\beta} < +\infty\bigg\}.$$

The above theorem proves that $d_H(\frac{1}{2}) = 1$. The classical result of Jarník (see [**10**]) implies that $d_H(0) = 1$. Therefore a natural conjecture seems to be

$$d_H(\beta) = 1 \quad \text{for all } \beta \in [0, 1).$$

In fact we have the following result.

THEOREM 2. *Let $E(\beta, c)$ be defined as above and put $F(c) = E(1, c)$: that is,*

$$E(\beta, c) := \bigg\{\omega \in [0,1) : \lim_{n\to\infty} \frac{R_n(\omega)}{n^\beta} = c\bigg\},$$

$$F(c) := \bigg\{\omega \in [0,1) : \lim_{n\to\infty} \frac{R_n(\omega)}{n} = c\bigg\}.$$

Then:

(1) *for any $\beta \in (0, 1)$ and $c > 0$,*

$$\dim_H E(\beta, c) = 1; \textit{ and} \tag{2.12}$$

(2) *for any $c \in (0, 1]$,*

$$\dim_H F(c) = \tfrac{1}{2}. \tag{2.13}$$

Remark 3. Actually, the proof of Theorem 2 in §4 can be modified to prove the following, more general, result. For any smooth function ψ with $\psi(n) \nearrow \infty$, $\psi(n)/n \searrow 0$ as $n \to \infty$, the set

$$E_\psi := \bigg\{\omega \in [0,1) : \lim_{n\to+\infty} \frac{R_n(\omega)}{\psi(n)} = 1\bigg\}$$

is always of Hausdorff dimension one.

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It is interesting to point out that $e - 2 = 0.71828\ldots \in F(\frac{1}{3})$, due to Euler (cf. [**9**, p. 12]). The authors guess that $\pi - 3 = 0.14159\ldots$ satisfies the conclusions of Theorem 1.

The above results in Theorem 2 and in Remark 3 fall into the so-called *fractional dimensional theory*. This theory has attracted much attention in the study of the exceptional sets arising from the metrical theory of continued fractions. It seems that the first published work in this area was a paper by Jarník [**10**]. Later on, Good [**6**] gave an overall investigation of sets, with some restrictions on their partial quotients. For more results in this area, one can refer to the work of Hirst [**7**], Lúczak [**20**], Mauldin and Urbánski [**21**, **22**], Pollicott and Weiss [**23**], Wang and Wu [**27**], Li *et al* [**19**] and references therein.

3. *Proof of Theorem 1*

The proof essentially follows the main strategy of [**4**]. The main tool used is the one developed in [**2**] (also, in a sense, traced back to [**4**]). As it is a straightforward application of Borel–Cantelli's lemma, we restate it here, without proof.

LEMMA 3. *Let $\{Y_n\}_1^\infty$ be a sequence of non-negative random variables in a probability space $(\Omega, \mathbb{P})$. Let $S_n := \sum_{k=1}^n Y_n$. Suppose $\lim_{n\to+\infty} \mathbb{E}S_n = +\infty$, $\sup\{\mathbb{E}Y_n : n \geq 1\} < +\infty$ and*

$$\mathrm{Var}(S_n) \leq C \cdot (\mathbb{E}S_n)^{2-\delta} \tag{3.1}$$

for some $\delta > 0$, $C > 0$ and all sufficiently large n. Then

$$\lim_{n\to\infty} \frac{S_n}{\mathbb{E}S_n} = 1 \tag{3.2}$$

holds true almost surely. The condition (3.1) can even be weakened to

$$\mathrm{Var}(S_n) \leq C \cdot (\mathbb{E}S_n)^2/(\log \mathbb{E}S_n)^{1+\delta}. \tag{3.3}$$

COROLLARY 4. *The conclusion of the above lemma can be strengthened as follows. For any fixed $\beta \in (0, \delta/3)$,*

$$\frac{S_n}{\mathbb{E}S_n} = 1 + \bar{O}\left(\frac{1}{(\mathbb{E}S_n)^\beta}\right) \tag{3.4}$$

holds true almost surely if condition (3.1) holds. If condition (3.3) holds instead of (3.1), then

$$\frac{S_n}{\mathbb{E}S_n} = 1 + \bar{O}\left(\frac{1}{(\log \mathbb{E}S_n)^\beta}\right). \tag{3.5}$$

For the convenience of the reader, we translate the main results in [**2**] for the probability measure $\widetilde{\mathbb{P}}$ induced by independent and identical distribution $(\pi_x, x \in \mathbb{N})$ (see (2.1) for the definition) as shown below. This is, in fact, also the motivation of the current paper. Such a result is obtained by careful calculations in view of (2.1). We omit the proof here, since the calculations involved are somewhat tedious but routine; interested readers can follow the ideas in [**2**] for the calculations.

LEMMA 5. *We have the following estimations for the measure $\widetilde{\mathbb{P}}$ as $n \to +\infty$.*

(1) $\widetilde{\mathbb{E}}R_n = \sum_{x=1}^{\infty}[1 - (1 - \pi_x)^n] = \sqrt{\frac{\pi n}{\log 2}} + \overline{O}(1), \quad \mathrm{Var}_{\widetilde{\mathbb{P}}}(R_n) \leq \widetilde{\mathbb{E}}R_n.$

(2) *For any fixed $k \geq 1$,*

$$\widetilde{\mathbb{E}} R_{n,k} = \sum_{x=1}^{\infty} C_n^k \cdot \pi_x^k \cdot (1-\pi_x)^{n-k} = \sqrt{\frac{\pi n}{\log 2}} \cdot r_k + \overline{O}(1),$$

$$\mathrm{Var}_{\widetilde{\mathbb{P}}}(R_{n,k}) \leq [1+o(1)] \cdot \widetilde{\mathbb{E}} R_{n,k},$$

where r_k is defined in Theorem 1. Let $r_{k+} := \sum_{\ell=k}^{\infty} r_\ell = 1 - \sum_{\ell=1}^{k-1} r_\ell$, then

$$\widetilde{\mathbb{E}} R_{n,k+} = \sqrt{\frac{\pi n}{\log 2}} \cdot r_{k+} + \overline{O}(1).$$

(3) *For $\widetilde{\mathbb{P}}$-almost every ω and all $k \geq 1$,* $\lim_{n\to+\infty} \frac{R_n(\omega)}{\widetilde{\mathbb{E}} R_n} = 1, \quad \lim_{n\to+\infty} \frac{R_{n,k}(\omega)}{R_n(\omega)} = r_k.$

In order to prove Theorem 1, in view of Lemma 3, we should try to obtain suitable estimations for $\mathbb{E}R_n$ and $\mathrm{Var}(R_n)$ (respectively, for $\mathbb{E}R_{n,k+}$ and $\mathrm{Var}(R_{n,k+})$), which are presented in the following subsections.

3.1. *Estimating $\mathbb{E}R_n$.* In this part, we shall estimate $\mathbb{E}R_n$ by proving the following lemma.

LEMMA 6. *For sufficiently large n, $|\mathbb{E}R_n - \widetilde{\mathbb{E}}R_n| \leq O(n^{0.2905})$. Also, by Lemma 5,*

$$\mathbb{E}R_n = \sqrt{\frac{\pi n}{\log 2}} + \overline{O}(n^{0.2905}).$$

For this purpose, we need to estimate the probability

$$\mathbb{P}(N_n(x) \geq 1) = \mathbb{P}(a_k(\omega) = x \text{ for some } 1 \leq k \leq n)$$

for any $x \in \mathbb{N}$, in view of (2.3): this could be done by comparing $\mathbb{P}(N_n(x) \geq 1)$ with $\widetilde{\mathbb{P}}(N_n(x) \geq 1)$. Actually, we will start by doing such estimations for large enough x: that is, for

$$x > x_n^* := \left\lfloor \sqrt{\frac{Cn}{\log n}} \right\rfloor, \tag{3.6}$$

with $C > 0$ to be determined later (here $\lfloor a \rfloor$ denotes the integer part of a real number a).

From here on, we will always write

$$N_A = N_A(\omega) := \sum_{k=1}^{n} 1_{A_k}(\omega), \quad A_{i_1,\ldots,i_k} := \bigcap_{r=1}^{k} A_{i_r} \tag{3.7}$$

for a sequence $A = \{A_k\}_1^n$ of measurable sets. The basic idea for the estimation of probability $\mathbb{P}(N_n(x) \geq 1)$ is to exploit the following standard result in probability theory.

LEMMA 7. *Let $\{A_k\}_1^n$ be a sequence of measurable sets in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then*

$$\mathbb{P}(N_A \geq 1) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}(A_{i_1,\ldots,i_k}). \tag{3.8}$$

Equation (3.8) is just a direct application of the following standard fact about indicator functions: that is

$$1_{\{N_A\geq 1\}} = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} 1_{A_{i_1,\ldots,i_k}}. \tag{3.9}$$

Obviously, in order to estimate $\mathbb{P}(N_n(x)\geq 1)$ by exploiting the formula (3.8), one has to take into account the strong mixing property of the model under investigation. Therefore we will introduce this property into the subsequent work.

For any $n\geq 1$ and $(a_1, \ldots, a_n)\in\mathbb{N}^n$, call

$$I(a_1, \ldots, a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{when } n \text{ is even,} \\ \left(\dfrac{p_n+p_{n-1}}{q_n+q_{n-1}}, \dfrac{p_n}{q_n}\right] & \text{when } n \text{ is odd,} \end{cases}$$

an nth order cylinder, where $\{p_k, \ q_k\}_{k=1}^n$ are determined by the following recursive relations

$$p_k = a_k\cdot p_{k-1} + p_{k-2}, \quad q_k = a_k\cdot q_{k-1} + q_{k-2}, \quad 1\leq k\leq n, \tag{3.10}$$

with the conventions that $p_{-1}=1$, $p_0=0$, $q_{-1}=0$, $q_0=1$. It is well known (see [**11**]), that $I(a_1, \ldots, a_n)$ just represents the set of points in $[0, 1)$ which have a continued fraction expansion beginning with $a_1, \ldots, a_n$.

PROPOSITION 8. [**11**] *For any $n\geq 1$ and $(a_1, \ldots, a_n)\in\mathbb{N}^n$,*

$$|I(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n+q_{n-1})}, \tag{3.11}$$

where $|I(a_1, \ldots, a_n)|$ denotes the length of $I(a_1, \ldots, a_n)$.

The following lemma is a standard result in the continued fraction model, and is our desired mixing property.

PROPOSITION 9. (See [**1**] or [**9**, Corollary 1.3.15]) *In the continued fraction model,*

$$\left|\frac{\mu(I(\tilde{x})\bigcap T^{-(m+L)}I(\tilde{y}))}{\mu(I(\tilde{x}))\cdot\mu(I(\tilde{y}))} - 1\right| \leq O(q^L) \tag{3.12}$$

for some $q\in(0, 1)$, all (fixed) $\tilde{x}=(x_1, \ldots, x_m)$, $\tilde{y}=(y_1, \ldots, y_n)$ and sufficiently large L.

The first contributor to the mixing property of continued fractions was Kuzmin [**16**] (see also [**11**, **12**]), who proved a sub-exponential decay rate when solving Gauss's conjecture on continued fractions. Lévy [**17**] (see also [**18**, Ch. IX]) independently proved the exponential decay rate with $q=3.5-2\sqrt{2}=0.67157\ldots$, also solving Gauss's conjecture. Using Kuzmin's approach, Szűsz [**26**] claimed to have lowered the Lévy estimate for q to 0.4. However, his argument yields $q=0.485$ rather than $q=0.4$. The optimal value of $q=0.303663002898732658 59\ldots$ was determined by Wirsing [**29**].

The above equations alone are not sufficient. We need the following observation, the proof of which is omitted since it is somewhat routine.

LEMMA 10. *For any* $m, n \geq 1$ *and* $\tilde{x} = (x_1, \ldots, x_m) \in \mathbb{N}^m$, $\tilde{y} = (y_1, \ldots, y_n) \in \mathbb{N}^n$

$$K' := \log 2 \leq \frac{\mu(I(\tilde{x}, \tilde{y}))}{\mu(I(\tilde{x})) \cdot \mu(I(\tilde{y}))} \leq K := 2 \log 2. \tag{3.13}$$

The bounds in the above lemma are optimal; we will only use the bound K later. One can compare (3.13) with the standard result that

$$\frac{1}{2} \leq \frac{|I(\tilde{x}, \tilde{y})|}{|I(\tilde{x})| \cdot |I(\tilde{y})|} \leq 2, \tag{3.14}$$

both bounds of which are also optimal. This can easily be proved in view of Proposition 8.

Now, for a fixed integer $x > x_n^*$, we put, for $i = 1, \ldots, n$,

$$A_i = A_i^x := \{\omega : a_i(\omega) = x\}, \tag{3.15}$$

in equation (3.8). We also put

$$A_{i_1,\ldots,i_k}^x := \bigcap_{r=1}^{k} A_{i_r}^x = \{\omega : a_{i_r}(\omega) = x, r = 1, \ldots, k\}. \tag{3.16}$$

Noting that $\mathbb{P}(A_i^x) = \widetilde{\mathbb{P}}(A_i^x) = \pi_x$,

$$\begin{aligned} \Delta(x) &:= |\mathbb{P}(N_n(x) \geq 1) - \widetilde{\mathbb{P}}(N_n(x) \geq 1)| \\ &\leq \sum_{k=2}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |\mathbb{P}(A_{i_1,\ldots,i_k}^x) - \widetilde{\mathbb{P}}(A_{i_1,\ldots,i_k}^x)|. \end{aligned}$$

We choose an integer

$$K_n := \lfloor C_1 \cdot \log n \rfloor$$

for some sufficiently large C_1 (to be determined later). Consider the sum

$$\Delta_1(x) := \sum_{k=K_n+1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |\mathbb{P}(A_{i_1,\ldots,i_k}^x) - \widetilde{\mathbb{P}}(A_{i_1,\ldots,i_k}^x)|. \tag{3.17}$$

Clearly, $\widetilde{\mathbb{P}}(A_{i_1,\ldots,i_k}^x) = \pi_x^k$ and, in view of Lemma 10,

$$\Delta_1(x) \leq \sum_{k=K_n+1}^{n} C_n^k \cdot 2 \cdot K^{k-1} \cdot \pi_x^k.$$

Since $\pi_x \leq 1/(\log 2) \cdot x^{-2}$,

$$\begin{aligned} \sum_{x > x_n^*} \Delta_1(x) &\leq \sum_{k=K_n+1}^{n} C_n^k \cdot 2 \cdot K^{k-1} \sum_{x > x_n^*} \pi_x^k \\ &\leq \sum_{k=K_n+1}^{n} C_n^k \cdot 2 \cdot K^{k-1} \cdot C_0 \cdot \left(\frac{1}{\log 2}\right)^k \cdot \frac{1}{2k-1} \cdot (x_n^*)^{-(2k-1)} \\ &\leq \sum_{k=K_n+1}^{n} \frac{n^k}{k!} \cdot \left(\frac{K}{\log 2}\right)^k \cdot C_0 \cdot \frac{1}{k} \left(\frac{\log n}{Cn}\right)^{k-1/2} \\ &\leq \sqrt{\frac{n}{\log n}} \cdot C_0 \cdot \sum_{k=K_n+1}^{n} \frac{1}{(k+1)!} \cdot \left(\frac{K}{C \log 2} \cdot \log n\right)^k, \end{aligned}$$

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where the C_0 are universal constants which may change from line to line, as explained at the beginning of §2. Let $\alpha := K/(C \log 2) = 2/C$. By Stirling's formula, we see that if $C_1 \cdot C > 4e^2 \approx 29.55 \ldots$, then

$$\begin{aligned}\log(K_n!) - K_n \log(\alpha \log n) &= \log \sqrt{2\pi K_n} + K_n \log K_n - K_n + o(1) - K_n \log(\alpha \log n)\\ &= \log \sqrt{2\pi K_n} + K_n \log \frac{K_n}{e \cdot \alpha \log n} + o(1)\\ &\geq C_1 \log n\end{aligned}$$

for sufficiently large n. Also,

$$\frac{\alpha \log n}{k} < \frac{1}{2} \quad \text{for all } k > K_n.$$

Therefore, if $C_1 \geq 5$ and $C \geq 6$,

$$\begin{aligned}\sum_{x > x_n^*} \Delta_1(x) &\leq \sqrt{\frac{n}{\log n}} \cdot C_0 \cdot \sum_{\ell=1}^{n-K_n} \frac{(\alpha \cdot \log n)^{K_n}}{K_n!} \cdot \left(\frac{1}{2}\right)^{\ell}\\ &\leq C_0 \cdot n^{-(C_1 - 1/2)} \leq O(n^{-4}).\end{aligned}$$

Now choose a number $C_2 \geq 4$ and put

$$\lambda_0 := -\log q > 0, \quad L_n := \left\lfloor \frac{C_2}{\lambda_0} \log n \right\rfloor + 1. \tag{3.18}$$

We want to estimate

$$\Delta_2(x) := \sum_{k=2}^{K_n} \sum_{*} |\mathbb{P}(A^x_{i_1,\ldots,i_k}) - \widetilde{\mathbb{P}}(A^x_{i_1,\ldots,i_k})| \tag{3.19}$$

and

$$\Delta_3(x) := \sum_{k=2}^{K_n} \sum_{**} |\mathbb{P}(A^x_{i_1,\ldots,i_k}) - \widetilde{\mathbb{P}}(A^x_{i_1,\ldots,i_k})|, \tag{3.20}$$

where the sum $\sum_*$ is over all $1 \leq i_1 < \cdots < i_k \leq n$ with $i_{u+1} - i_u > L_n$, $u = 1, \ldots, k-1$, and the sum $\sum_{**}$ is over the rest $1 \leq i_1 < \cdots < i_k \leq n$.

In view of Proposition 9,

$$\prod_{\ell=1}^{k-1} (1 - C_0 q^{i_{\ell+1} - i_\ell}) \leq \frac{\mathbb{P}(A^x_{i_1,\ldots,i_k})}{\widetilde{\mathbb{P}}(A^x_{i_1,\ldots,i_k})} \leq \prod_{\ell=1}^{k-1} (1 + C_0 q^{i_{\ell+1} - i_\ell}).$$

Hence, for sufficiently large n (noting that $C_2 \geq 4$ and $i_{\ell+1} - i_\ell \geq L_n$, $q^{L_n} \leq n^{-C_2}$),

$$\begin{aligned}\Delta_2(x) &\leq \sum_{k=2}^{K_n} C^k_{n-(k-1)L_n} \cdot \max[(1 + C_0 q^{L_n})^{k-1} - 1, 1 - (1 - C_0 q^{L_n})^{k-1}] \cdot \pi_x^k\\ &\leq \sum_{k=2}^{K_n} \frac{n^k}{k!} \cdot \frac{C_0}{n^3} \cdot \pi_x^k.\end{aligned}$$

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Thus, if $C \geq 6 > 2/(\log 2) = 2.88539\ldots$, then

$$\sum_{x>x_n^*} \Delta_2(x) \leq C_0 \cdot \sum_{k=2}^{K_n} \frac{n^{k-3}}{k!} \cdot \sum_{x>x_n^*} \pi_x^k$$
$$\leq C_0 \cdot \sum_{k=2}^{K_n} \frac{n^{k-3}}{k!} \cdot \left(\frac{1}{\log 2}\right)^k \cdot \frac{1}{2k-1} \cdot \left(\frac{\log n}{Cn}\right)^{k-1/2}$$
$$\leq C_0 \cdot n^{-5/2} \sum_{k=2}^{K_n} \frac{1}{(k+1)!} \cdot \left(\frac{\log n}{C \log 2}\right)^k$$
$$\leq C_0 \cdot n^{-5/2} \cdot n^{1/(C \log 2)} \leq C_0 \cdot n^{-2}.$$

Now we estimate $\Delta_3(x)$. Clearly,

$$C_n^k - C_{n-(k-1)L_n}^k \leq C_0 \cdot \frac{n^{k-1}}{k!} \cdot k^2 \cdot L_n.$$

Therefore

$$\Delta_3(x) \leq C_0 \cdot \sum_{k=2}^{K_n} \frac{n^{k-1}}{k!} \cdot k^2 \cdot L_n \cdot 2K^{k-1} \cdot \pi_x^k.$$

From here on we take

$$C := 4 + \frac{2}{\log 2} = 6.88539\ldots \tag{3.21}$$

Then $\alpha := K/(C \log 2) = (\log 2)/(1 + 2\log 2) = 0.290470\ldots$. For sufficiently large n,

$$\sum_{x>x_n^*} \Delta_3(x) \leq \sum_{k=2}^{K_n} C_0 \cdot \frac{n^{k-1}}{k!} \cdot k^2 \cdot L_n \cdot \left(\frac{K}{\log 2}\right)^k \cdot \frac{1}{2k-1} \cdot \left(\frac{\log n}{Cn}\right)^{k-1/2}$$
$$\leq C_0 \cdot \frac{L_n}{\sqrt{n \cdot \log n}} \cdot \sum_{k=2}^{K_n} \frac{1}{(k-1)!} \cdot \left(\frac{K}{C \log 2} \cdot \log n\right)^k$$
$$\leq C_0 \cdot \frac{(\log n)^{1.5}}{\sqrt{n}} \cdot n^{\alpha}$$
$$= O\left(\frac{(\log n)^{1.5}}{n^{\varepsilon_0}}\right) \quad \text{with } \varepsilon_0 := \frac{1}{2(1 + 2\log 2)} = 0.209529\ldots.$$

A direct, but similar, calculation of $\Delta_1(x_n^*)$, $\Delta_2(x_n^*)$, $\Delta_2(x_n^*)$ reveals

$$|\mathbb{P}(N_n(x_n^*) \geq 1) - [1 - (1 - \pi_{x_n^*})^n]| \leq O\left(\frac{(\log n)^3}{n^{0.5+\varepsilon_0}}\right).$$

By combining the above results and noting that $\widetilde{\mathbb{P}}(N_n(x) \geq 1) = 1 - (1 - \pi_x)^n$, we have proved the following lemma.

LEMMA 11. *Let* $x_n^* = \lfloor\sqrt{(Cn)/(\log n)}\rfloor$ *with* $C = 4 + 2/(\log 2)$. *Then*

$$\sum_{x>x_n^*} |\mathbb{P}(N_n(x) \geq 1) - [1 - (1 - \pi_x)^n]| \leq O(n^{-0.2095}),$$
$$|\mathbb{P}(N_n(x_n^*) \geq 1) - [1 - (1 - \pi_{x_n^*})^n]| \leq O(n^{-0.7095}).$$

Now it is natural to see what happens for $|\mathbb{P}(N_n(x) \geq 1) - [1 - (1 - \pi_x)^n]|$ with $x \leq x_n^*$. The inequality $0 \leq \mathbb{P}(N_n(x) \geq 1) \leq 1$ is surely not sufficient for our purpose, in view of Lemma 3. Intuitively, we should have the following result.

LEMMA 12. *$f(x) := \mathbb{P}(N_n(x) \geq 1)$ is decreasing in x.*

But a mathematically rigorous proof is not so obvious. We will postpone it until later. From Lemma 12,

$$f(x_n^*) \leq f(x) \leq 1 \quad \text{for } x = 1, 2, \ldots, x_n^*.$$

We already know, from Lemma 11, that

$$f(x_n^*) \geq 1 - (1 - \pi_{x_n^*})^n - O(n^{-0.7095}).$$

Since $1/(C \log 2) = 1/(2 + 4 \log 2) = \varepsilon_0 = 0.209529\ldots$, for sufficiently large n,

$$f(x_n^*) \geq 1 - C_0 \cdot n^{-0.2095},$$

which implies the following lemma.

LEMMA 13. *Let x_n^* be defined as above. Then for sufficiently large n,*

$$|\mathbb{P}(N_n(x) \geq 1) - [1 - (1 - \pi_x)^n]| \leq C_0 \cdot n^{-0.2095}, \quad x = 1, 2, \ldots, x_n^*.$$

In order to prove Lemma 12, we observe the following important fact.

LEMMA 14. (Comparison lemma for continued fractions) *Given two sequences of natural numbers $\tilde{x} = (x_k : 1 \leq k \leq n)$ and $\tilde{y} = (y_k : 1 \leq k \leq n)$. Suppose $x_k \geq y_k$ for $k = 1, \ldots, n$. Then $\mu(I(\tilde{x})) \leq \mu(I(\tilde{y}))$.*

Proof. Let $p_k/q_k = [x_1, \ldots, x_k]$, $\bar{p}_k/\bar{q}_k = [y_1, \ldots, y_k]$ be irreducible fractions. Then

$$\mu(I(\tilde{x})) = \int_0^1 \frac{d\omega}{(\log 2) \cdot (q_n + \omega q_{n-1}) \cdot [p_n + q_n + \omega(p_{n-1} + q_{n-1})]} =: \int_0^1 \rho(\omega; \tilde{x})\, d\omega$$

and a similar equation holds for $\mu(I(\tilde{y}))$. Then the condition in the lemma implies

$$p_{n-1} \geq \bar{p}_{n-1}, \quad q_{n-1} \geq \bar{q}_{n-1}, \quad p_n \geq \bar{p}_n, \quad q_n \geq \bar{q}_n.$$

Therefore the densities satisfy $\rho(\omega; \tilde{x}) \leq \rho(\omega; \tilde{y})$, which implies $\mu(I(\tilde{x})) \leq \mu(I(\tilde{y}))$. □

Remark 4. The proof of Lemma 14 says more. Let $\tilde{x} = (x_1, \ldots, x_m)$, $\tilde{y} = (y_1, \ldots, y_n)$ be two natural number tuples which may be of different length. Let

$$\frac{p_m}{q_m} := [x_1, \ldots, x_m], \quad \frac{\bar{p}_n}{\bar{q}_n} := [y_1, \ldots, y_m]$$

be irreducible fractions. If

$$\begin{bmatrix} p_{m-1} & p_m \\ q_{m-1} & q_m \end{bmatrix} \geq \begin{bmatrix} \bar{p}_{n-1} & \bar{p}_n \\ \bar{q}_{n-1} & \bar{q}_n \end{bmatrix} \quad \text{(in element-by-element sense)},$$

then $\mu(I(\tilde{x})) \leq \mu(I(\tilde{y}))$.

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Proof of Lemma 12. For any $x < y$, we would prove that $f(x) \geq f(y)$. Noting that

$$f(x) := \mathbb{P}(N_n(x) \geq 1) = \mathbb{P}(N_n(x) \geq 1,\, N_n(y) = 0) + \mathbb{P}(N_n(x) \geq 1,\, N_n(y) \geq 1),$$

we only need to prove that, for any $k = 1, 2, \ldots, n$,

$$\mathbb{P}(N_n(x) = k,\, N_n(y) = 0) \geq \mathbb{P}(N_n(y) = k,\, N_n(x) = 0).$$

But this is obvious, in view of Lemma 14, since

$$\mathbb{P}(N_n(x) = k,\, N_n(y) = 0) = \sum_{***} \mu(I(x_1, \ldots, x_n)),$$

where $\sum_{***}$ is over all tuples $(x_1, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq y$, for all i and $\sum_{i=1}^{n} 1_{\{x_i = x\}} = k$. □

Summing the above estimations in Lemmas 11–13 together, we have proved Lemma 6.

3.2. *Estimating* $\mathrm{Var}(R_n)$. The main result of this subsection is the following lemma.

LEMMA 15. *For sufficiently large n,* $\mathrm{Var}(R_n) \leq O(n^{0.7905})$*, and hence*

$$\mathrm{Var}(R_n) \leq C_0 \cdot (\mathbb{E}R_n)^{2-\delta} \quad \text{with } \delta = 0.4190 \text{ and large constant } C_0.$$

In order to prove the above lemma, noting that

$$\mathbb{E}(R_n^2 - R_n) = \sum_{x \neq y} \mathbb{P}(N_n(x) \geq 1,\, N_n(y) \geq 1),$$

we would need to estimate the probability

$$\mathbb{P}(N_n(x) \geq 1,\, N_n(y) \geq 1) \quad \text{for all } x \neq y.$$

As in the above subsection, we need the following elemental fact.

LEMMA 16. *Let* $\{A_k\}_1^n$, $\{B_k\}_1^n$ *be two sequences of measurable sets in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *which satisfies* $A_k \cap B_k = \emptyset$, $k = 1, \ldots, n$*. Then*

$$\mathbb{P}(N_A \geq 1,\, N_B \geq 1) = \sum_{k=2}^{n} (-1)^k \sum_{\substack{r+s=k \\ 1 \leq r < k}} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} \mathbb{P}(A_{i_1,\ldots,i_r} B_{j_1,\ldots,j_s}), \tag{3.22}$$

where, in the above summation, the indices $i_1, \ldots, i_r$ *and* $j_1, \ldots, j_s$ *are all distinct and both groups of indices are in increasing order.*

One can prove equation (3.22) directly using equation (3.9).

Then we immediately derive (see (3.15) and (3.16) for the involved notation)

$$\begin{aligned}\Delta(x, y) &:= |\mathbb{P}(N_n(x) \geq 1,\, N_n(y) \geq 1) - \widetilde{\mathbb{P}}(N_n(x) \geq 1,\, N_n(y) \geq 1)| \\ &\leq \sum_{k=2}^{n} \sum_{\substack{r+s=k \\ 1 \leq r < k}} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} |\mathbb{P}(A^x_{i_1,\ldots,i_r} A^y_{j_1,\ldots,j_s}) - \widetilde{\mathbb{P}}(A^x_{i_1,\ldots,i_r} A^y_{j_1,\ldots,j_s})|. \end{aligned} \tag{3.23}$$

First, we make the estimation for $x > y \geq x_n^*$. A similar calculation yields the following lemma.

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LEMMA 17. *For sufficiently large n,* $\sum_{x>y\geq x_n^*}\Delta(x,y)\leq O(\sqrt{n})$.

For $x<y\leq x_n^*$,

$$\begin{aligned}
&\mathbb{P}(N_n(x)\geq 1,\,N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(x)\geq 1,\,N_n(y)\geq 1)\\
&\quad\leq|\mathbb{P}(N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(y)\geq 1)|+\widetilde{\mathbb{P}}(N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(x)\geq 1,\,N_n(y)\geq 1)\\
&\quad=|\mathbb{P}(N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(y)\geq 1)|+\widetilde{\mathbb{P}}(N_n(x)=0,\,N_n(y)\geq 1)\\
&\quad\leq O(n^{-0.2095})+\widetilde{\mathbb{P}}(N_n(x)=0)=O(n^{-0.2095})+(1-\pi_x)^n\\
&\quad\leq O(n^{-0.2095})+(1-\pi_{x_n^*})^n=O(n^{-0.2095}).
\end{aligned}\tag{3.24}$$

Hence we have the following lemma.

LEMMA 18. *For sufficiently large n,*

$$\sum_{x<y\leq x_n^*}[\mathbb{P}(N_n(x)\geq 1,\,N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(x)\geq 1,\,N_n(y)\geq 1)]\leq O(n^{0.7905}).$$

Similarly, for $x<x_n^*\leq y$, noting equation (3.24),

$$\begin{aligned}
&\mathbb{P}(N_n(x)\geq 1,\,N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(x)\geq 1,\,N_n(y)\geq 1)\\
&\quad\leq|\mathbb{P}(N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(y)\geq 1)|+\widetilde{\mathbb{P}}(N_n(x)=0,\,N_n(y)\geq 1)\\
&\quad\leq|\mathbb{P}(N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(y)\geq 1)|+[(1-\pi_x)^n-(1-\pi_x-\pi_y)^n]
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{x<x_n^*\leq y}[\mathbb{P}(N_n(x)\geq 1,\,N_n(y)\geq 1)-\widetilde{\mathbb{P}}(N_n(x)\geq 1,\,N_n(y)\geq 1)]\\
&\quad\leq O\bigg(\sqrt{\frac{n}{\log n}}\bigg)\cdot O(n^{-0.2095})+\sum_{x<x_n^*\leq y}[(1-\pi_x)^n-(1-\pi_x-\pi_y)^n]\\
&\quad\leq O(n^{0.2905})+\sum_{x<x_n^*}(1-\pi_x)^n\cdot\sum_{y\geq x_n^*}\bigg[1-\bigg(1-\frac{\pi_y}{1-\pi_x}\bigg)^n\bigg]\\
&\quad\leq O(n^{0.2905})+\sum_{x<x_n^*}(1-\pi_x)^n\cdot\sum_{y\geq x_n^*}[1-(1-2\cdot\pi_y)^n]\\
&\qquad\qquad(\text{noting }\pi_x\leq\pi_1=0.4150\ldots)\\
&\quad=O(n^{0.2905})+\sum_{x<x_n^*}(1-\pi_x)^n\cdot O(\sqrt{n})\quad(*)\\
&\quad\leq O(n^{0.2905})+O(\sqrt{n})\cdot O\bigg(\sqrt{\frac{n}{\log n}}\bigg)\cdot O(n^{-0.2095\ldots})\quad(**)\\
&\quad\leq O(n^{0.7905}).
\end{aligned}$$

We note here that, in the step $(*)$, we have exploited the estimating technique developed in [**2**]. Alternatively, one can compute directly that

$$\sum_{y\geq x_n^*}[1-(1-2\cdot\pi_y)^n]\leq C_0\cdot\sqrt{n}.$$

In the step $(**)$, we should note that $(1-\pi_x)^n\leq(1-\pi_{x_n^*})^n=O(n^{-0.2095})$ and $x_n^*=O(\sqrt{n})$.

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From the above, we have proved the following lemma.

LEMMA 19. *For sufficiently large n,*

$$\sum_{x<x_n^*\le y}[\mathbb{P}(N_n(x)\ge 1,\, N_n(y)\ge 1)-\widetilde{\mathbb{P}}(N_n(x)\ge 1,\, N_n(y)\ge 1)]\le O(n^{0.7905}).$$

Note that, by Lemma 5,

$$\mathrm{Var}_{\widetilde{\mathbb{P}}}(R_n)\le \widetilde{\mathbb{E}}R_n=O(\sqrt{n})$$

and, in view of the above estimations,

$$\begin{aligned}\mathrm{Var}(R_n)-\mathrm{Var}_{\widetilde{\mathbb{P}}}(R_n)&=\mathbb{E}R_n^2-\widetilde{\mathbb{E}}R_n^2+(\widetilde{\mathbb{E}}R_n)^2-(\mathbb{E}R_n)^2\\&=[\mathbb{E}R_n-\widetilde{\mathbb{E}}R_n]+[(\widetilde{\mathbb{E}}R_n)^2-(\mathbb{E}R_n)^2]\\&\quad+2\sum_{x<y}[\mathbb{P}(N_n(x)\ge 1,\, N_n(y)\ge 1)-\widetilde{\mathbb{P}}(N_n(x)\ge 1,\, N_n(y)\ge 1)]\\&\le O(n^{0.2905})+O(n^{0.2905}\cdot\sqrt{n})+O(n^{0.7905})=O(n^{0.7905}),\end{aligned}$$

which proves Lemma 15.

Now, by Lemma 3, for μ-almost every (and hence for Lebesgue almost all) $\omega\in(0,1)$ $\lim_{n\to+\infty}(R_n(\omega))/(\mathbb{E}R_n)=1$: that is,

$$\lim_{n\to+\infty}\frac{R_n(\omega)}{\sqrt{n}}=\sqrt{\frac{\pi}{\log 2}}.$$

3.3. *More estimations for $R_{n,k+}$ with $k\ge 2$.* The estimation of $\mathbb{E}R_{n,k+}$ and $\mathrm{Var}(R_{n,k+})$ follows almost the same line as in the previous subsections except that we need some additional treatments.

First, we would need new equations in place of equations (3.8) and (3.22), which we state as the following lemmas without proof.

LEMMA 20. *Let $\{A_i\}_1^n$ be a sequence of measurable sets in a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then, for any $1\le k\le n$,*

$$\mathbb{P}(N_A\ge k)=\sum_{r=k}^{n}(-1)^{r-k}\cdot C_{r-1}^{k-1}\cdot\sum_{1\le i_1<\cdots<i_r\le n}\mathbb{P}(A_{i_1,\ldots,i_r}).$$

LEMMA 21. *Let $\{A_i\}_1^n$, $\{B_i\}_1^n$ be two sequences of measurable sets in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that $A_i\cap B_i=\emptyset$ for all i. Then, for any $1\le k\le n$,*

$$\mathbb{P}(N_A\ge k,\, N_B\ge k)=\sum_{r=2k}^{n}(-1)^r\cdot\sum_{\substack{a+b=r\\a,b\ge k}}C_{a-1}^{k-1}\cdot C_{b-1}^{k-1}\cdot\sum_{\substack{i_1,\ldots,i_a\\j_1,\ldots,j_b}}\mathbb{P}(A_{i_1,\ldots,i_a}\cap B_{j_1,\ldots,j_b}),$$

where, in the above summation, the two group of indices $i_1,\ldots,i_a$ and $j_1,\ldots,j_b$ are all distinct and are both in increasing order.

We believe that there is monotonicity in the function

$$f_k(x):=\mathbb{P}(N_n(x)\ge k) \tag{3.25}$$

for all (fixed) $n\ge k\ge 2$ as Lemma 12 states, but we cannot give a rigorous and relatively easy proof for such result. Therefore we present an alternative treatment here.

LEMMA 22. *For all* $1 \le x \le x_n^*$ *and sufficiently large* n, $\mathbb{P}(N_n(x) \ge k) \ge 1 + O(n^{-0.2095})$. *Hence*

$$|\mathbb{P}(N_n(x) \ge k) - \widetilde{\mathbb{P}}(N_n(x) \ge k)| \le O(n^{-0.2095}).$$

Proof. Assume $k \ge 2$. Let L_n be defined as above and let $s_1 := 0$. Put

$$A_n := \left\lfloor \frac{n - (k-1) \cdot L_n}{k} \right\rfloor,$$

$$t_j := (j-1) \cdot L_n + j \cdot A_n, \quad s_{j+1} = j \cdot (L_n + A_n), \quad j = 1, \ldots, k.$$

Clearly, $t_k \le n$ and $0 = s_1 < t_1 < s_2 < t_2 < \cdots < s_k < t_k \le n$. We will write

$$N_x(\Delta) := \sum_{i \in \Delta} 1_{\{a_i(\omega) = x\}}$$

for any interval $\Delta \subset \mathbb{N}$, which is the visiting number at x with times $n \in \Delta$ such that $a_n(\omega) = x$. Then, obviously,

$$\mathbb{P}(N_n(x) \ge k) \ge \mathbb{P}(N_x((s_j, t_j]) \ge 1 \text{ for } j = 1, \ldots, k).$$

Hence

$$\begin{aligned}
\mathbb{P}(N_n(x) \ge k) &\ge (1 - C_0 q^{L_n})^{k-1} \cdot \prod_{j=1}^{k} \mathbb{P}(N_x((s_j, t_j]) \ge 1) \\
&= (1 - C_0 q^{L_n})^{k-1} \cdot [\mathbb{P}(N_x((0, A_n]) \ge 1)]^k \\
&\ge [1 - C_0 \cdot n^{-4}]^{k-1} \cdot [1 - C_0 \cdot n^{-0.2095}]^k \\
&\ge 1 - C_0 \cdot n^{-0.2095}.
\end{aligned}$$

We have the same bound for $\widetilde{\mathbb{P}}(N_n(x) \ge k)$. □

With the help of the above lemmas, one can prove that

$$\lim_{n \to +\infty} \frac{R_{n,k+}(\omega)}{\mathbb{E} R_{n,k+}} = 1,$$

following the same method indicated above; the details are omitted here. By Lemma 5,

$$\lim_{n \to +\infty} \frac{\mathbb{E} R_{n,k+}}{\mathbb{E} R_n} = r_{k+} \quad \text{for all } k \ge 1.$$

Therefore our main theorem follows.

4. *Proof of Theorem 2*

4.1. *Some elementary facts on continued fractions.*

In this subsection, we collect some elementary properties shared by continued fractions that will be used later.

For any $n \ge 1$ and $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, let $q_n(a_1, a_2, \ldots, a_n) = q_n$ be defined by (3.10). Then we have the following proposition.

PROPOSITION 23. [**30**] *For any* $n \ge 1$ *and* $1 \le k \le n$,

$$\frac{a_k + 1}{2} \le \frac{q_n(a_1, a_2, \ldots, a_n)}{q_{n-1}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)} \le a_k + 1. \tag{4.1}$$

For any positive integer $B \geq 2$, let E_B be the set of continued fractions with partial quotients between 1 and B: that is,

$$E_B = \{\omega \in [0, 1) : 1 \leq a_n(\omega) \leq B, \forall n \geq 1\}.$$

Good [**6**] proved the following result.

PROPOSITION 24. [**6**] *For any $n \geq 1$, let σ_n be the unique root of*

$$\sum_{1 \leq a_1, a_2, \ldots, a_n \leq B} \frac{1}{q_n(a_1, a_2, \ldots, a_n)^{2s}} = 1.$$

Then

$$\dim_H E_B = \lim_{n \to \infty} \sigma_n.$$

Moreover, $\lim_{B \to \infty} \dim_H E_B = 1$.

4.2. *Non-autonomous conformal iterated function systems.* In this part, we present the construction and some basic properties of a non-autonomous conformal *iterated function system* (*IFS*) which was introduced quite recently by Rempe-Gillen and Urbanski in [**24**]. It is a variant of the construction studied in [**8**, **28**].

Fix a compact set $X \subset \mathbb{R}^d$ with $\overline{\text{int}(X)} = X$ such that ∂X is smooth or X is convex. Given a conformal map $\varphi : X \to X$, we denote the derivative of φ at x by $D\varphi(x)$ and the operator norm of the differential by $|D\varphi(x)|$. Put

$$\|D\varphi\| = \sup\{|D\varphi(x)| : x \in X\}, \quad |||D\varphi||| = \inf\{|D\varphi(x)| : x \in X\}.$$

For any $n \geq 1$, let $I^{(n)}$ be a (finite or countable infinite) index set. For each $i \in I^{(n)}$, there is a conformal map $\varphi_i^{(n)} : X \to X$, and we can write $\Phi^{(n)} = \{\varphi_i^{(n)} : i \in I^{(n)}\}$.

Definition 4.1. We call $\Phi = \{\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)} \ldots\}$ a non-autonomous conformal IFS on the set X if the following conditions hold.

(a) Open set condition: we have

$$\varphi_i^{(n)}(\text{int}(X)) \cap \varphi_j^{(n)}(\text{int}(X)) = \emptyset$$

for all $n \in \mathbb{N}$ and all distinct indices $i, j \in I^{(n)}$.

(b) Conformality: there exists an open connected set $V \supset X$ such that for each $n \geq 1$ and $i \in I^{(n)}$, $\varphi_i^{(n)}$ can be extended to a C^1 conformal diffeomorphism of V into V.

(c) Bounded distortion: there exists a constant $K \geq 1$ such that, for any $k \leq l$ and any $i_k, i_{k+1}, \ldots, i_l$ with $i_j \in I^{(j)}$, the map $\varphi = \varphi_{i_k}^{(k)} \circ \cdots \circ \varphi_{i_l}^{(l)}$ satisfies

$$|D\varphi(x)| \leq K|D\varphi(y)|$$

for all $x, y \in V$.

(d) Uniform contraction: there exists a constant $\eta < 1$ such that

$$|D\varphi(x)| \leq \eta^m$$

for all sufficiently large m, all $x \in X$ and all $\varphi = \varphi_{i_j}^{(j)} \circ \cdots \circ \varphi_{i_{j+m}}^{(j+m)}$, where $j \geq 1$ and $i_k \in I^{(k)}$.

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For any $0 < m \le n < \infty$, write

$$I^n := \prod_{j=1}^{n} I^{(j)}, \quad I^\infty := \prod_{j=1}^{\infty} I^{(j)}, \quad I^{m,n} := \prod_{j=m}^{n} I^{(j)} \quad \text{and} \quad I^{m,\infty} := \prod_{j=m}^{\infty} I^{(j)}.$$

If $\tilde{i} = i_m i_{m+1} \cdots i_n \in I^{m,n}$, write $\varphi_{\tilde{i}}^{m,n} = \varphi_{i_1}^{(m)} \circ \cdots \circ \varphi_{i_n}^{(n)}$. When $m = 1$, we also abbreviate $\varphi_{\tilde{i}} := \varphi_{\tilde{i}}^{n} := \varphi_{\tilde{i}}^{1,n}$.

For any $n \ge 1$ and $i \in I^n$, let $X_i = \varphi_i(X)$. The limit set (or attractor) of Φ is defined as

$$J := J(\Phi) := \bigcap_{n=1}^{\infty} \bigcup_{i \in I^n} X_i. \tag{4.2}$$

Definition 4.2. For any $t \ge 0$ and $n \in \mathbb{N}$, we define

$$Z_n(t) = \sum_{i \in I^n} \|D\varphi_i\|^t,$$

and the upper and lower pressure functions are defined as

$$\underline{P}(t) = \varliminf_{n\to\infty} \frac{1}{n} \log Z_n(t), \quad \overline{P}(t) = \varlimsup_{n\to\infty} \frac{1}{n} \log Z_n(t).$$

Rempe-Gillen and Urbanski [**24**] proved the following results.

PROPOSITION 25. *If* $\lim_{n\to\infty} (1/n) \log \sharp I^{(n)} = 0$, *then*

$$\begin{aligned} \dim_H J &= \sup\{t \ge 0 : \underline{P}(t) > 0\} = \inf\{t \ge 0 : \underline{P}(t) < 0\} \\ &= \sup\{t \ge 0 : Z_n(t) \to \infty\}. \end{aligned}$$

PROPOSITION 26. *Suppose that both limits*

$$a =: \lim_{n\to\infty} \frac{1}{n} \log \sharp I^{(n)}$$

and

$$b =: \lim_{n\to\infty, j \in I^{(n)}} \frac{1}{n} \log\left(\frac{1}{\|D\varphi_j^{(n)}\|}\right)$$

exist and are positive finite. Then $\dim_H J = a/b$.

4.3. *Proof of equations (2.12) and (2.13).*

Proof of Equation (2.12). Put $\gamma := 1/\beta > 1$. We only prove that, for $c = 1$, $E(\beta) := E(\beta, c)$ is of Hausdorff dimension one; results for other values of c can be proved in the same way (the only modifications would be changing terms like k^γ into terms like $(k/c)^\gamma$ and taking such changes into account in related calculations).

For any $\varepsilon > 0$, by Proposition 24, we can find a positive integer B with $\dim E_B > 1 - \varepsilon/4$, and an integer $N_0 > 0$ such that $\sigma_n > 1 - \varepsilon/2$ for any $n \ge N_0$.

Define a subset $E_B(\beta)$ of $E(\beta)$ as

$$E_B(\beta) = \{\omega \in [0, 1) : a_{\lfloor k^\gamma \rfloor}(\omega) = k \ \forall k \ge 1, \text{ and } 1 \le a_n(\omega) \le B, \forall \text{ other } n \ge 1\}.$$

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It is direct to check that

$$E_B(\beta) \subset E(\beta).$$

Define $\Phi = \{\Phi^{(1)}, \ \Phi^{(2)}, \ldots\}$ as follows. For any $n \geq 1$, choose $k \geq 1$ such that $\lfloor k^\gamma \rfloor \leq n < \lfloor (k+1)^\gamma \rfloor$. If $n = \lfloor k^\gamma \rfloor$ for some $k \geq 1$, let $\Phi^{(n)} = \{\varphi_k^{(n)}(x) := 1/(x+k)\}$; for other n let $\Phi^{(n)} = \{\varphi_j^{(n)} := 1/(x+j) : j = 1, \ldots, B\}$. Functions $\{f_j(x) := 1/(x+j) : j \in \mathbb{N}\}$ on interval $(0, 1)$ are C^1 conformal contractions and $f_i(0, 1) \cap f_j(0, 1) = \emptyset$ for all $i \neq j$. Also,

$$|Df_i(x)| = \frac{1}{(x+j)^2} \in \left(\frac{1}{(j+1)^2}, \frac{1}{j^2}\right),$$

which implies

$$\|Df_j\| \leq \frac{1}{j^2} \leq 1 \quad \text{and} \quad \frac{\|Df_j\|}{|||Df_j|||} \leq \frac{(j+1)^2}{j^2} \leq 4.$$

Therefore Φ is a non-autonomous conformal IFS, and $E_B(\beta)$ is the associated limit set. $I^{(n)} = \{k\}$, if $n = \lfloor k^\gamma \rfloor$ for some $k \geq 1$, and $I^{(n)} = \{1, \ 2, \ldots, B\}$ otherwise. Thus $\sharp I^{(n)} = 1$, if $n = \lfloor k^\gamma \rfloor$ for some $k \geq 1$, and $\sharp I^{(n)} = B$ otherwise. By Proposition 25,

$$\dim E_B(\beta) = \sup\{t \geq 0 : Z_n(t) \to \infty\}.$$

We will compute $Z_n(t)$ below.

Now for any $n \geq 1$, assume $k \geq 1$ is such that $\lfloor k^\gamma \rfloor \leq n < \lfloor (k+1)^\gamma \rfloor$. We know that, for any conformal interval maps f, g, $\|D(f \circ g)\| = \|Df \cdot Dg\| \geq |||Df||| \cdot |||Dg|||$. Therefore

$$\begin{aligned} Z_n(1-\varepsilon) &= \sum_{\tilde{i} \in I^n} \|D\varphi_{\tilde{i}}\|^{(1-\varepsilon)} \\ &\geq \prod_{j=1}^{k-1} \bigg(\sum_{\tilde{i} \in I^{\lfloor j^\gamma \rfloor, \lfloor (j+1)^\gamma \rfloor - 1}} |||D\varphi_{\tilde{i}}^{\lfloor j^\gamma \rfloor, \lfloor (j+1)^\gamma \rfloor - 1}|||^{1-\varepsilon} \bigg) \cdot \sum_{\tilde{i} \in I^{\lfloor k^\gamma \rfloor, n}} |||D\varphi_{\tilde{i}}^{\lfloor k^\gamma \rfloor, n}|||^{1-\varepsilon}. \end{aligned}$$

For any $j \geq 1$, write $l(j) = \lfloor (j+1)^\gamma \rfloor - \lfloor j^\gamma \rfloor$. Notice that when j is large enough such that $l(j) - 1 \geq N_0$,

$$\begin{aligned} &\sum_{\tilde{i} \in I^{\lfloor j^\gamma \rfloor, \lfloor (j+1)^\gamma \rfloor - 1}} |||D\varphi_{\tilde{i}}^{\lfloor j^\gamma \rfloor, \lfloor (j+1)^\gamma \rfloor - 1}|||^{1-\varepsilon} \\ &= \sum_{1 \leq a_1, a_2, \ldots, a_{l(j)-1} \leq B} \left(\frac{1}{q_{l(j)}(j, a_1, a_2, \ldots, a_{l(j)-1}) + q_{l(j)-1}(j, a_1, a_2, \ldots, a_{l(j)-2})} \right)^{2(1-\varepsilon)} \\ &\geq \frac{1}{2^{2(1-\varepsilon)}} \sum_{1 \leq a_1, a_2, \ldots, a_{l(j)-1} \leq B} \left(\frac{1}{q_{l(j)}(j, a_1, a_2, \ldots, a_{l(j)-1})} \right)^{2(1-\varepsilon)} \\ &\geq \frac{1}{(2(j+1))^{2(1-\varepsilon)}} \sum_{1 \leq a_1, a_2, \ldots, a_{l(j)-1} \leq B} \left(\frac{1}{q_{l(j)-1}(a_1, a_2, \ldots, a_{l(j)-1})} \right)^{2(1-\varepsilon)} \quad \text{(noting (4.1))} \\ &\geq \frac{1}{(2(j+1))^{2(1-\varepsilon)}} \sum_{1 \leq a_1, a_2, \ldots, a_{l(j)-1} \leq B} \left(\frac{1}{q_{l(j)-1}(a_1, a_2, \ldots, a_{l(j)-1})} \right)^{2\sigma_{l(j)-1} - \varepsilon} \\ &\geq \frac{1}{(2(j+1))^{2(1-\varepsilon)}} \cdot 2^{((l(j)-1)/2)\varepsilon} \geq \frac{1}{(2(j+1))^2} \cdot 2^{(((j+1)^\gamma - j^\gamma - 3)/2)\varepsilon}. \end{aligned} \tag{4.3}$$

In a similar way (taking $j = k$ and changing the term $\lfloor (j+1)^\gamma \rfloor - 1$ into n in the above equations), if $n - \lfloor k^\gamma \rfloor - 1 \geq N_0$,

$$\sum_{\tilde{i} \in I^{\lfloor k^\gamma \rfloor, n}} |||D\varphi_{\tilde{i}}^{\lfloor k^\gamma \rfloor, n}|||^{1-\varepsilon} \geq \frac{1}{(2(k+1))^2} \cdot 2^{((n-k^\gamma-2)/2)\varepsilon}. \tag{4.4}$$

If $n - [k^\gamma] - 1 < N_0$,

$$\sum_{\tilde{i} \in I^{[k^\gamma], n}} |||D\varphi_{\tilde{i}}^{[k^\gamma], n}|||^{1-\varepsilon} \geq \frac{1}{(2(k+1))^2} \cdot \frac{1}{(B+1)^{N_0}}. \tag{4.5}$$

Combining (4.3)–(4.5), we have $\lim_{n\to\infty} Z_n(1-\varepsilon) = \infty$. This implies $\dim E_B(\beta) \geq 1 - \varepsilon$. Since ε is arbitrary, we finish the proof of equation (2.12). □

Proof of equation (2.13). We divide the proof into two parts.

Upper bound: for any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, let $R_n(a_1, a_2, \ldots, a_n)$ be the number of distinct ones among $a_1, a_2, \ldots, a_n$.

For any $0 < \varepsilon < c$, let $t = \frac{1}{2} + \varepsilon$ and $s = \frac{1}{2} + \varepsilon/2$. For any $n \geq 1$, let

$$\Lambda_n = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : R_n(a_1, a_2, \ldots, a_n) \geq (c - \varepsilon)n\}.$$

Then

$$F(c) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{(a_1,\ldots,a_n)\in\Lambda_n} I(a_1, \ldots, a_n).$$

For any $N \geq 1$,

$$\begin{aligned}
&H^t\bigg(\bigcap_{n=N}^{\infty} \bigcup_{(a_1,\ldots,a_n)\in\Lambda_n} I(a_1, \ldots, a_n)\bigg) \\
&\leq \varliminf_{n\to\infty} \sum_{(a_1,\ldots,a_n)\in\Lambda_n} |I(a_1, \ldots, a_n)|^t \\
&\leq \varliminf_{n\to\infty} \sum_{(a_1,\ldots,a_n)\in\Lambda_n} (a_1 \cdot a_2 \cdots a_n)^{-2t} \\
&\leq \varliminf_{n\to\infty} \frac{1}{(\lfloor (c-\varepsilon)n \rfloor !)^\varepsilon} \sum_{(a_1,\ldots,a_n)\in\Lambda_n} (a_1 \cdot a_2 \cdots a_n)^{-2s} \\
&\leq \varliminf_{n\to\infty} \frac{1}{(\lfloor (c-\varepsilon)n \rfloor !)^\varepsilon} \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} (a_1 \cdot a_2 \cdots a_n)^{-2s} \\
&= \varliminf_{n\to\infty} \frac{1}{(\lfloor (c-\varepsilon)n \rfloor !)^\varepsilon} \cdot [\zeta(1+\varepsilon)]^n \quad \text{(where } \zeta(\cdot) \text{ is Riemann's zeta function)} \\
&= 0.
\end{aligned}$$

This finishes the proof of the *upper bound*.

Lower bound: for any given $c \in (0, 1]$, let

$$\begin{aligned}
G := \{\omega \in [0, 1) : 2^n \leq a_k(\omega) < 2^{n+1} &\text{ if } k = \lfloor n/c \rfloor \text{ for some } n \geq 1, \\
&\text{and for other } k,\ a_k(\omega) = 1\}.
\end{aligned}$$

It is direct to check that

$$G \subset F(c).$$

Define $\Phi = \{\Phi^{(1)}, \Phi^{(2)}, \ldots\}$ as follows. For any $n \geq 1$, let $K_n = \lfloor n/c \rfloor$, $K'_n = \lfloor (n+1)/c \rfloor$ and put

$$\Phi^{(n)} = \{\varphi^{(n)}_{1\cdots 1k}(x) := [1, \ldots, 1, k + x] : 2^n \leq k < 2^{n+1}\},$$

where the 1s in the continued fractional function formula $[1, \ldots, 1, k + x]$ appear exactly $K'_n - K_n - 1$ (possibly $K'_n - K_n - 1 = 0$) times. Then Φ is a non-autonomous conformal IFS, and G is the associated limit set. It is easy to check that

$$\lim_{n\to\infty} \frac{1}{n} \log \sharp I^{(n)} = \log 2$$

and

$$\lim_{n\to\infty, j\in I^{(n)}} \frac{1}{n} \log\left(\frac{1}{\|D\varphi^{(n)}_j\|}\right) = 2 \log 2.$$

By Proposition 26, we have

$$\dim_H G = \tfrac{1}{2},$$

and this finishes the proof of the lower bound. □

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