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NON-CLASSICAL FOUNDATIONS OF SET THEORY

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Abstract. In this paper, we use algebra-valued models to study cardinal numbers in a class of non-classical set theories. The algebra-valued models of these non-classical set theories validate the Axiom of Choice, if the ground model validates it. Though the models are non-classical, the foundations of cardinal numbers in these models are similar to those in classical set theory. For example, we show that mathematical induction, Cantor's theorem, and the Schröder–Bernstein theorem hold in these models. We also study a few basic properties of cardinal arithmetic. In addition, the generalized continuum hypothesis is proved to be independent of these non-classical set theories.

§1. Introduction. The study of the cardinal numbers and the related properties are one of the corner stones in set theory. To explore different branches of mathematics, the use of cardinal numbers is invincible. In particular, the study of the transfinite cardinal numbers in classical set theory, initiated by George Cantor at the second half of the nineteenth century, has drawn special attention from the set theorists. On the contrary, cardinal numbers in different non-classical set theories are not investigated in a large scale, except the intuitionistic Zermelo–Fraenkel set theory (IZF) and the constructive Zermelo–Fraenkel set theory (CZF; cf. [1, 2, 5]). In this paper, we develop a foundation of mathematics by exploring the cardinal numbers in the algebra-valued models of certain non-classical set theories, extending the study of ordinals in [11]: the validity or independence of the set-theoretic sentences like Cantor's theorem, the Schröder–Bernstein theorem, mathematical induction, the Axiom of Choice, the generalized continuum hypothesis, etc. is the same as their status in the classical set theory.

The set theories, which are studied in this paper, are said to be non-classical as their underlying logics are non-classical. To compare it to the well-known non-classical set theory IZF, we notice that the underlying logic of IZF is the *intuitionistic logic*, where *tertium non datur* fails: there exists a formula φ such that $\varphi \vee \neg\varphi$ is not valid. But, the set-theoretic axioms are the same as the classical Zermelo–Fraenkel (ZF) axioms. As one of the examples of the non-classical set theories developed in this paper, we may consider paraconsistent set theories, whose underlying logics are *paraconsistent*: there are formulas φ and ψ such that the formula $(\varphi \wedge \neg\varphi) \rightarrow \psi$ is invalid. But the set-theoretic axiom system is a fragment of the classical ZF axioms, viz. the *negation-free fragment* of ZF, formally defined in Section 2. However, if the

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underlying logic is considered to be the classical logic or the intuitionistic logic, then the negation-free fragment of ZF is proved to be equivalent with the full ZF axiom system.

Although, the theory of cardinal numbers is well-developed in the literature of IZF, one can find some basic differences between the properties of cardinal numbers in IZF and the classical set theory. On the contrary, the properties of cardinal numbers in the non-classical set theories, developed in this paper, are similar to those in the classical set theory. A comparative study in this perspective between these non-classical set theories, IZF, and the classical set theory can be found in Section 7.

We can argue similarly when it turns about most of the paraconsistent set theories already developed. Syntactic developments of different paraconsistent logics can be found in the literature (cf. [3, 10]), but semantic developments are harder to find [4, 14]. The study of the paraconsistent set theories is also very rare in the literature. One of the fundamental problems in studying paraconsistent set theories is a lack of models, although some investigation has been done (cf. [7–9, 11, 13]). As a result, it is also hard to find the paraconsistent mathematical realms. This paper produces a class of paraconsistent set theories besides the other non-classical set theories, more precisely, the algebra-valued models of these theories. We prove that the basic foundations of cardinal numbers in these paraconsistent set theories are similar to those of classical mathematics.

In Section 2, we discuss the background of generalized algebra-valued models of set theories, following the construction of [9]. From Section 3 onwards, we develop algebra-valued models of the non-classical set theories mentioned above. As a first step, we define a class of *well-ordered deductive reasonable implication algebras*, generalizing [7], then we discuss their corresponding algebra-valued models. This section ends with a discussion of the ordinal numbers, which generalizes the study of ordinal numbers found in [11]. The natural numbers in these models are studied in Section 4. It is proved that the smallest *inductive set* is the set of all natural numbers. Finally, it is shown that all these models validate the principle of *mathematical induction*. At the beginning of Section 5 we prove that the Axiom of Choice is valid in these non-classical algebra-valued models if it is valid in the ground model. This is used to define one of the most important notions of this paper, cardinality. Cantor's theorem on cardinal numbers and the Schröder–Bernstein theorem are proved to be valid. We then move on to the generalized continuum hypothesis and prove that, as in classical set theory, it remains independent of the set theories we are discussing. In Section 6, we discuss *cardinal arithmetic*: cardinal addition, cardinal multiplication, and cardinal exponentiation and their properties. As a conclusion, in Section 7, we compare the non-classical set theories developed in this paper with IZF and a paraconsistent set theory established in [14]. Finally, as a summary, we give an overview of the similarities of these non-classical set theories with the classical set theory.

§2. Background: generalized algebra-valued models. Boolean-valued models of the classical set theory ZFC were introduced by Dana Scott, Robert M. Solovay, and Petr Vopěnka in the 1960s to study Paul Cohen's *forcing* in a different way. See [2] for the details of these models' construction. This method of construction was used to construct generalized algebra-valued models of set theories in [9].

2.1. Generalized algebra-valued models. Following [9], we give an overview of generalized algebra-valued models of set theory. Consider the language $\mathcal{L}$ of ZF having the binary predicate symbol $\in$, connectives $\wedge, \vee, \rightarrow, \neg, \bot$, and quantifiers $\forall, \exists$. Let $\mathbf{V}$ be a model of ZF and ORD be the class of all ordinal numbers in $\mathbf{V}$. Let us now assume that $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ is a complete distributive lattice, augmented with one binary operator $\Rightarrow$ and one unary operator *. Then a universe of *names* is defined by transfinite recursion as follows. For any $\alpha \in$ ORD,

$$\mathbf{V}_\alpha^{(\mathbb{A})} = \{x \,:\, x \text{ is a function where } \mathrm{ran}(x) \subseteq \mathbf{A} \text{ and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{(\mathbb{A})})\}, \text{ and}$$
$$\mathbf{V}^{(\mathbb{A})} = \{x \,:\, \exists \alpha (x \in \mathbf{V}_\alpha^{(\mathbb{A})})\}.$$

After obtaining $\mathbf{V}^{(\mathbb{A})}$, extend the language $\mathcal{L}$ by adding a constant symbol for every element in $\mathbf{V}^{(\mathbb{A})}$. The new language is denoted by $\mathcal{L}_\mathbb{A}$.

As usual, define an assignment function $[\![\cdot]\!]$ from the class of all formulas in $\mathcal{L}_\mathbb{A}$ to the set $\mathbf{A}$ of *truth values* as follows. If $u, v \in \mathbf{V}^{(\mathbb{A})}$ and φ, ψ are any two closed well-formed formulas in $\mathcal{L}_\mathbb{A}$, then

$$[\![\bot]\!] = \mathbf{0},$$
$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!]),$$
$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]),$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!],$$
$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!],$$
$$[\![\varphi \rightarrow \psi]\!] = [\![\varphi]\!] \Rightarrow [\![\psi]\!],$$
$$[\![\neg \varphi]\!] = [\![\varphi]\!]^*,$$
$$[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in \mathbf{V}^{(\mathbb{A})}} [\![\varphi(u)]\!], \text{ and}$$
$$[\![\exists x \varphi(x)]\!] = \bigvee_{u \in \mathbf{V}^{(\mathbb{A})}} [\![\varphi(u)]\!].$$

A *filter*[1] $D \subseteq \mathbf{A}$ will be called a *designated set* of $\mathbb{A}$. A formula φ of $\mathcal{L}_\mathbb{A}$ is said to be D-valid in $\mathbf{V}^{(\mathbb{A})}$, denoted by $\mathbf{V}^{(\mathbb{A})} \models_D \varphi$ or simply $\mathbf{V}^{(\mathbb{A})} \models \varphi$ (when the corresponding designated set is clear from the context), if $[\![\varphi]\!] \in D$.

It is well known that if $\mathbb{A}$ is a complete Boolean algebra then $\mathbf{V}^{(\mathbb{A})}$ becomes an algebra-valued model of ZF, i.e., $\mathbf{V}^{(\mathbb{A})} \models$ ZF. This method of constructing a

[1] A set $D \subseteq \mathbf{A}$ is called a filter in $\mathbb{A}$ if the following conditions hold:

(i) $\mathbf{1} \in D$,
(ii) $\mathbf{0} \notin D$,
(iii) if $x \in D$ and $x \leq y$ then $y \in D$, and
(iv) for $x, y \in D$, we have $x \wedge y \in D$.

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Boolean-valued model was first used by R. J. Grayson to produce an algebra-valued model of a non-classical set theory: if $\mathbb{A}$ is a complete Heyting algebra then $\mathbf{V}^{(\mathbb{A})}$ becomes an algebra-valued model of IZF, i.e., $\mathbf{V}^{(\mathbb{A})} \models$ IZF [6].

Consider the following properties, viz. *bounded quantification properties* BQ_φ, for every formula φ in $\mathcal{L}$:

$$[\![\forall x(x \in u \rightarrow \varphi(x))]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]). \qquad (\mathrm{BQ}_\varphi)$$

If $\mathbb{A}$ is a Boolean algebra or Heyting algebra, then BQ_φ holds for every formula φ in $\mathbf{V}^{(\mathbb{A})}$. However, it was proved in [9] that there are algebras $\mathbb{A}$ such that BQ_φ does not hold in general in the algebra-valued models $\mathbf{V}^{(\mathbb{A})}$.

2.2. Negation-free fragment. By NFF, the negation-free fragment of $\mathcal{L}$, we mean the closure of the atomic formulas of $\mathcal{L}$ under $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\exists$, and $\forall$. The elements of NFF are called negation-free formulas. By NFF-ZF and NFF-ZF$^-$ we mean the negation-free fragment of ZF and the negation-free fragment of ZF excluding the Axiom of Foundation,[2] respectively. The Axiom of Infinity is taken as follows:

$$\exists x\big(\exists y(\forall z(z \in y \rightarrow \perp) \wedge y \in x) \wedge \forall w(w \in x \rightarrow \exists u(u \in x \wedge w \in u))\big),$$

which makes it a negation-free formula. The NFF-bounded quantification property (NFF-BQ_φ) stands for the bounded quantification property BQ_φ for all negation-free formulas φ. Classically and intuitionistically, every formula is equivalent to a negation-free formula, as $\neg\varphi$ is equivalent to $\varphi \rightarrow \perp$. Hence, in these cases, we can conclude that NFF-ZF and NFF-IZF are equivalent to ZF and IZF, respectively.

2.3. The logic corresponding to an algebra $\mathbb{A}$ and a designated set D. Let us consider an algebra $\mathbb{A}$ and a designated set D of it. Let Prop be the collection of all propositional formulas of the language having the same signature as $\mathbb{A}$. We know that a *valuation* is a homomorphism from Prop into the algebra $\mathbb{A}$. The logic of $(\mathbb{A}, D)$, denoted by $\mathbf{L}(\mathbb{A}, D)$, is defined as:

$$\mathbf{L}(\mathbb{A}, D) = \{\varphi \in \mathsf{Prop} : \text{ for all valuation function } v, v(\varphi) \in D\}.$$

2.4. Reasonable implication algebra. [9, p. 194] An algebra $\mathbb{A} := \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0}\rangle$ is called a *deductive reasonable implication algebra* if $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0}\rangle$ is a complete distributive lattice and the following properties hold: for any $x, y, z \in \mathbf{A}$

P1: $(x \wedge y) \leq z$ implies $x \leq (y \Rightarrow z)$,
P2: $y \leq z$ implies $(x \Rightarrow y) \leq (x \Rightarrow z)$,
P3: $y \leq z$ implies $(z \Rightarrow x) \leq (y \Rightarrow x)$, and
P4: $((x \wedge y) \Rightarrow z) = (x \Rightarrow (y \Rightarrow z))$.

The following theorem gave rise to generalized algebra-valued models of set theories.

[2] Following [2], we interpret the Axiom of Foundation as the following scheme, which is also referred as Set Induction: $\forall x\big(\forall y(y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x)\big) \rightarrow \forall z\varphi(z)$.

THEOREM 1 [9, Theorems 3.3 and 3.4]. *If* $\mathbb{A}$ *is a deductive reasonable implication algebra such that* NFF-BQ$_\varphi$ *holds in* $\mathbf{V}^{(\mathbb{A})}$, *then for any choice of the designated set* D, *we have* $\mathbf{V}^{(\mathbb{A})} \models_D$ NFF-ZF$^-$.

§3. Well-ordered algebra-valued models and ordinal-like elements. It was shown in [9] that any complete Boolean algebra or complete Heyting algebra is a deductive reasonable implication algebra. However, there are reasonable implication algebras which are neither Boolean nor Heyting.

3.1. Well-ordered deductive reasonable implication algebra. Consider a lattice $\langle \mathbf{A}, \wedge, \vee \rangle$. A binary relation $\leq$ is uniquely defined on the lattice as follows: $a \leq b$ iff $a \wedge b = a$, for all $a, b \in \mathbf{A}$. The relation $\leq$ is said to be the lattice order of the lattice considered. For any two elements $a, b \in \mathbf{A}$, we define $a < b$ iff $a \leq b$ and $a \neq b$. The lattice order $\leq$ is said to be well-ordered on $\mathbf{A}$ if

(i) it is totally-ordered: for any $a, b \in \mathbf{A}$ exactly one of $a < b$, $a = b$, and $b < a$ holds, and
(ii) for any non-empty $X \subseteq \mathbf{A}$, there exists a least element with respect to the lattice order $\leq$, i.e., there exists an element $a \in X$ such that for any $b \in X$, $a \leq b$.

DEFINITION 2. An algebra $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is called a well-ordered deductive reasonable implication algebra (WoDRIA) if the following conditions hold:

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a bounded lattice having top and bottom elements $\mathbf{1}$ and $\mathbf{0}$, respectively,
(ii) the lattice order $\leq$ is a well-ordered relation on $\mathbf{A}$, and
(iii) the operator $\Rightarrow$ is defined by

$$a \Rightarrow b = \begin{cases} \mathbf{0}, & \text{if } a \neq \mathbf{0} \text{ and } b = \mathbf{0}, \\ \mathbf{1}, & \text{otherwise}, \end{cases}$$

for any two elements $a, b \in \mathbf{A}$.

LEMMA 3. *Any* WoDRIA *is a deductive reasonable implication algebra.*

PROOF. Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ be a WoDRIA. Then, $\mathbb{A}$ is a complete distributive lattice. We shall prove that all the properties **P1**, **P2**, **P3**, and **P4** will be satisfied by $\mathbb{A}$.

For **P1**: Let $a, b, c \in \mathbf{A}$ be such that $a \wedge b \leq c$. We have to prove that $a \leq b \Rightarrow c$. If $b \Rightarrow c = \mathbf{1}$, then we are done. Let $b \Rightarrow c = \mathbf{0}$, which implies that $b \neq \mathbf{0}$ and $c = 0$. Therefore, by our assumption, $a = \mathbf{0}$. Hence, in any case, we conclude that $a \leq b \Rightarrow c$.

For **P2**: Let $a \leq b$ holds in $\mathbf{A}$. We have to prove that $c \Rightarrow a \leq c \Rightarrow b$, for any $c \in \mathbf{A}$. If $c \Rightarrow b = \mathbf{1}$, the proof is done. Suppose $c \Rightarrow b = \mathbf{0}$. Then, by the definition of $\Rightarrow$, $c \neq \mathbf{0}$ and $b = \mathbf{0}$. Since $a \leq b$, we have $a = \mathbf{0}$ also. Hence, it can be concluded that $c \Rightarrow a = \mathbf{0}$. Therefore, $c \Rightarrow a \leq c \Rightarrow b$, for any $c \in \mathbf{A}$.

For **P3**: Let us consider any $a, b \in \mathbf{A}$ which satisfy the property $a \leq b$. We shall prove that for any $c \in \mathbf{A}$, $b \Rightarrow c \leq a \Rightarrow c$ holds. Similar to the previous cases, there is nothing to prove if $a \Rightarrow c = \mathbf{1}$. Hence, assume that $a \Rightarrow c = \mathbf{0}$. This implies that

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$a \neq \mathbf{0}$ and $c = \mathbf{0}$. Since, by our assumption, $a \leq b$, we get that $b \neq \mathbf{0}$ as well. Hence, $b \Rightarrow c = \mathbf{0}$. So, in any case, $b \Rightarrow c \leq a \Rightarrow c$ holds for any $c \in \mathbf{A}$.

For **P4**: Let us consider any three elements $a, b, c \in \mathbf{A}$. If $(a \wedge b) \Rightarrow c = \mathbf{0}$, then $a, b \neq \mathbf{0}$ and $c = \mathbf{0}$, which entails that $a \Rightarrow (b \Rightarrow c) = \mathbf{0}$. Conversely, if $a \Rightarrow (b \Rightarrow c) = \mathbf{0}$, we have $a, b \neq \mathbf{0}$ and $c = \mathbf{0}$. Since $\leq$ is a well-order (and hence linear-order) relation in $\mathbf{A}$, either $a \wedge b = a$ or $a \wedge b = b$. In either case, $a \wedge b \neq \mathbf{0}$. Therefore, $(a \wedge b) \Rightarrow c = \mathbf{0}$. Hence, we can conclude that, $(a \wedge b) \Rightarrow c = \mathbf{0}$ iff $a \Rightarrow (b \Rightarrow c) = \mathbf{0}$. Since the range of the binary operator $\Rightarrow$ is defined to be $\{\mathbf{1}, \mathbf{0}\}$, from the previous conclusion, we get that $(a \wedge b) \Rightarrow c = \mathbf{1}$ iff $a \Rightarrow (b \Rightarrow c) = \mathbf{1}$. Combining both the results we finally have $(a \wedge b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$. $\dashv$

If a WoDRIA contains more than two elements then it can be neither a Boolean algebra nor a Heyting algebra. In [7], the authors introduced *totally ordered complete distributive lattices*, $\mathbb{L}$. The domain of $\mathbb{L}$ is a complete distributive lattice totally ordered by the relation $\leq$, and the lattices are equipped with the operator $\Rightarrow$ as defined in Definition 2. Hence, every WoDRIA is a totally ordered complete distributive lattice.

At this stage it is worth to observe that for any WoDRIA $\mathbb{A}$ and a pair of elements $u, v \in \mathbf{V}^{(\mathbb{A})}$, the value of $[\![u = v]\!]$ is either $\mathbf{1}$ or $\mathbf{0}$, due to the definitions of $[\![\cdot = \cdot]\!]$ and the operator $\Rightarrow$ in a WoDRIA. On the other hand, for any element a of $\mathbb{A}$ there exist $u, v \in \mathbf{V}^{(\mathbb{A})}$ such that $[\![u \in v]\!] = a$, which can be witnessed by taking any $u \in \mathbf{V}^{(\mathbb{A})}$ and then by fixing an element $v = \{\langle u, a\rangle\}$ of $\mathbf{V}^{(\mathbb{A})}$.

THEOREM 4 [7]. *If $\mathbb{A}$ is a* WoDRIA *then* BQ_φ *holds in* $\mathbf{V}^{(\mathbb{A})}$ *for every negation-free formula* φ.

As an application of Theorem 1, Lemma 3, and Theorem 4, if $\mathbb{A}$ is a WoDRIA then $\mathbf{V}^{(\mathbb{A})} \models_D$ NFF-ZF$^-$, for any designated set D. In addition, it can also be proved that $\mathbf{V}^{(\mathbb{A})} \models_D$ NFF-Foundation. Combining all these results we have the following theorem.

THEOREM 5 [7]. *If $\mathbb{A}$ is a* WoDRIA *and D is any designated set of $\mathbb{A}$, then* $\mathbf{V}^{(\mathbb{A})} \models_D$ NFF-ZF.

3.2. Negation introduction in WoDRIA. Until now we have discussed the validity of sentences in the negation-free fragment of ZF. To explore full ZF we need to introduce a unary operator * in WoDRIA which will be the algebraic interpretation of the connective $\neg$ in the language of set theory, $\mathcal{L}$.

DEFINITION 6. Let $\mathcal{PS}$ be the collection of all algebras $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ satisfying the following conditions:

(i) $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0}\rangle$ is a WoDRIA, and
(ii) * is a unary operator on $\mathbf{A}$ which satisfies $\mathbf{1}^* = \mathbf{0}$, $\mathbf{0}^* = \mathbf{1}$.

Any algebra $\mathbb{A} \in \mathcal{PS}$ will be called a $\mathcal{PS}$-algebra.

The logic $\mathbf{L}(\mathbb{A}, D)$ of a $\mathcal{PS}$-algebra $\mathbb{A}$ will vary depending on the unary operator * and the choice of the designated set D. The following are few such examples.

EXAMPLE 1 (The logic is classical). Note that, the $\mathcal{PS}$-algebra $\mathbb{A}$ having two elements is the two-valued Boolean algebra. The only possible designated set in $\mathbb{A}$ is $D = \{\mathbf{1}\}$. Hence, $\mathbf{L}(\mathbb{A}, D)$ is the classical propositional logic.

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EXAMPLE 2 (The logic is an existing paraconsistent logic). Consider the three-valued $\mathcal{PS}$-algebra $\mathrm{PS}_3 = \langle\{\mathbf{1}, {}^1\!/_2, \mathbf{0}\}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$, where ${}^1\!/_2{}^* = {}^1\!/_2$. The designated set is taken as $D_{\mathrm{PS}_3} = \{\mathbf{1}, {}^1\!/_2\}$. This algebra was introduced as a deductive reasonable implication algebra in [9, p. 194] to produce a non-classical algebra-valued model of NFF-ZF. The logic $\mathbf{L}(\mathrm{PS}_3, D_{\mathrm{PS}_3})$ was studied in [12], which was named as LPS_3. It was proved that the logic LPS_3 is a paraconsistent logic, which is sound and (weak) complete with respect to PS_3 [12].

EXAMPLE 3 (The logic is both paraconsistent and paracomplete). Consider any $\mathcal{PS}$-algebra $\mathbb{A}$ and a designated set D such that the algebra contains more than two elements. If there exists a non-zero element $a \in \mathbf{A} \setminus D$ such that $a^* = a$, then clearly the logic $\mathbf{L}(\mathbb{A}, D)$ becomes *paracomplete* (as tertium non datur fails) and paraconsistent both. To have a concrete case, consider the three-valued algebra PS_3 itself. But, fix the designated set as $D = \{\mathbf{1}\}$, instead of the set $\{\mathbf{1}, {}^1\!/_2\}$. In this case, ${}^1\!/_2 \notin D$ and since ${}^1\!/_2{}^* = {}^1\!/_2$, we have that ${}^1\!/_2 \vee {}^1\!/_2{}^* \notin D$. This implies that, if φ is a propositional formula assigned to the value ${}^1\!/_2$ through a certain valuation function $v : \mathsf{Prop} \to \mathrm{PS}_3$, then $v(\varphi \vee \neg\varphi) = {}^1\!/_2$, which makes the formula $\varphi \vee \neg\varphi$ invalid in the logic of PS_3 with respect to the designated set D, i.e., $\varphi \vee \neg\varphi \notin \mathbf{L}(\mathrm{PS}_3, D)$. On the other hand, if ψ be a propositional formula assigned to the element $\mathbf{0}$ of PS_3 by the same valuation function v, then $v((\varphi \wedge \neg\varphi) \to \psi) = \mathbf{0}$. Hence, $(\varphi \wedge \neg\varphi) \to \psi \notin \mathbf{L}(\mathrm{PS}_3, D)$, which entails that the logic $\mathbf{L}(\mathrm{PS}_3, D)$ is paraconsistent.

EXAMPLE 4 (The logic is paraconsistent but not paracomplete). Consider any $\mathcal{PS}$-algebra $\mathbb{A}$ and a designated set D containing more than one element such that the unary operator * is defined as follows:

$$a^* = \begin{cases} \mathbf{0}, & \text{if } a = \mathbf{1}, \\ a, & \text{if } a \in D \setminus \{\mathbf{1}\}, \\ \mathbf{1}, & \text{otherwise.} \end{cases}$$

Let φ and ψ be two propositional formulas and v be a valuation from Prop to $\mathbb{A}$ such that $v(\varphi) = a_0$, where $a_0 \in D \setminus \{\mathbf{1}\}$, and $v(\psi) = \mathbf{0}$. Then, $v((\varphi \wedge \neg\varphi) \to \psi) = \mathbf{0}$, which shows that $(\varphi \wedge \neg\varphi) \to \psi \notin \mathbf{L}(\mathbb{A}, D)$. Hence, the logic $\mathbf{L}(\mathbb{A}, D)$ is paraconsistent. On the other hand, it is clear that for any $b \in \mathbb{A}$, $b \vee b^* \in D$. Hence, for any $\varphi \in \mathsf{Prop}$, $\varphi \vee \neg\varphi \in \mathbf{L}(\mathbb{A}, D)$, which proves that $\mathbf{L}(\mathbb{A}, D)$ is not paracomplete. The algebra PS_3 with the designated set D_{PS_3}, explained in Example 2, is a particular case for this kind of example.

EXAMPLE 5 (The logic is paracomplete but not paraconsistent). Let $\mathbb{A}$ be any $\mathcal{PS}$-algebra and D be its designated set such that $\mathbf{A} \setminus D$ contains more than one element and the unary operator * is defined as follows:

$$a^* = \begin{cases} \mathbf{1}, & \text{if } a = \mathbf{0}, \\ \mathbf{0}, & \text{if } a \neq \mathbf{0}. \end{cases}$$

Since for every $a \in \mathbf{A}$, $a \wedge a^* = \mathbf{0}$, for any $\varphi, \psi \in \mathsf{Prop}$ and any valuation $v : \mathsf{Prop} \to \mathbf{A}$, we have $v((\varphi \wedge \neg\varphi) \to \psi) = \mathbf{1}$. Hence, $(\varphi \wedge \neg\varphi) \to \psi \in \mathbf{L}(\mathbb{A}, D)$, for all $\varphi, \psi \in \mathsf{Prop}$, which implies that $\mathbf{L}(\mathbb{A}, D)$ is not a paraconsistent logic. To prove that it is a paracomplete logic, let us consider a non-zero element $a_0 \in \mathbf{A} \setminus D$. Suppose $\varphi \in \mathsf{Prop}$ is such that with respect to a certain valuation v, $v(\varphi) = a_0$.

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Then, $v(\varphi \vee \neg\varphi) = a_0 \notin D$. Hence, $\varphi \vee \neg\varphi \notin \mathbf{L}(\mathbb{A}, D)$, which shows that $\mathbf{L}(\mathbb{A}, D)$ is paracomplete. To have a specific case, consider the three-valued $\mathcal{PS}$-algebra $\mathbb{A}_3 = \langle\{\mathbf{1}, 1/2, \mathbf{0}\}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$, where $1/2^* = \mathbf{0}$. Let the designated set be $D = \{\mathbf{1}\}$. Then, by the above argument, $\mathbf{L}(\mathbb{A}_3, D)$ is paracomplete but not paraconsistent.

Observation 7. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and any designated set D, $\mathbf{V}^{(\mathbb{A})} \models$ NFF-ZF, by Theorem 5, since any $\mathcal{PS}$-algebra is by definition a* WoDRIA. *Moreover, if the logic $\mathbf{L}(\mathbb{A}, D)$ is non-classical then $\mathbf{V}^{(\mathbb{A})}$ becomes an algebra-valued model of a non-classical set theory.*

3.3. Ordinal numbers in $\mathcal{PS}$-algebra-valued models. In [11], ordinal numbers in the algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$ were developed by transfinite induction. More precisely, ordinal-like elements were defined. We shall now generalize the definition of ordinal-like elements in the $\mathcal{PS}$-algebra-valued models.

Definition 8. Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra. An element $x \in \mathbf{V}^{(\mathbb{A})}$ is said to be

(i) 0-like if $x(y) = \mathbf{0}$ for every $y \in \mathrm{dom}(x)$,
(ii) α-like for some $\alpha \in \mathrm{ORD}$ if for each $\beta \in \alpha$ there exists a β-like $y \in \mathrm{dom}(x)$ such that $x(y) \in D$, and for any $z \in \mathrm{dom}(x)$ if it is not β-like for any $\beta \in \alpha$ then $x(z) = \mathbf{0}$, and
(iii) ordinal-like if it is α-like for some $\alpha \in \mathrm{ORD}$.

Note that $\varnothing$ is also a 0-like element and for every $\alpha \in \mathrm{ORD}$ there are *class many* α-like elements. Following the proofs of [11, Theorem 9] and [11, Theorem 10] the following theorem can be derived.

Theorem 9. *Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra and $u \in \mathbf{V}^{(\mathbb{A})}$ be α-like for some $\alpha \in$* ORD. *Then, independent of the choice of the designated set,*

(i) *for any $v \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models u = v$ if and only if v is α-like, and*
(ii) *for any $v \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models v \in u$ if and only if v is β-like for some $\beta \in \alpha$.*

We shall prove in Theorem 13 that ordinal-like elements are characterized by the first-order formula $\mathsf{Ord}(x)$, defined as follows:

$$\begin{aligned}
\mathrm{Trans}(x) &:= \forall y \forall z (z \in y \wedge y \in x \to z \in x),\\
\mathrm{LO}(x) &:= \forall y \forall z ((y \in x \wedge z \in x) \to (y \in z \vee y = z \vee z \in y)),\\
\mathrm{WO}_{\in}(x) &:= \mathrm{LO}(x) \wedge \forall y \big((y \subseteq x \wedge (y \neq \varnothing)) \to \exists z (z \in y \wedge z \cap y = \varnothing)\big),\\
\mathsf{Ord}(x) &:= \mathrm{Trans}(x) \wedge \mathrm{WO}_{\in}(x),
\end{aligned}$$

where the abbreviations used in $\mathrm{WO}_{\in}(x)$ are

$$\begin{aligned}
y \subseteq x &:= \forall t (t \in y \to t \in x),\\
(y \neq \varnothing) &:= \exists z (z \in y),\\
(z \cap y = \varnothing) &:= \exists w (w \in z \wedge w \in y) \to \bot.
\end{aligned}$$

The formulas $\mathrm{Trans}(x)$, $\mathrm{LO}(x)$, $\mathrm{WO}_{\in}(x)$, and $\mathsf{Ord}(x)$ naively state that 'x is a *transitive set*', 'x is a *linear ordered set*', 'x is a *well-ordered set* with respect to $\in$', and 'x is an ordinal number', respectively. In [11, Theorem 13], it was proved that if

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u is an ordinal-like element, then $\mathbf{V}^{(\mathrm{PS}_3)} \models_{D_{\mathrm{PS}_3}} \mathsf{Ord}(u)$. In this paper, we shall also prove the converse in the general set-up.

Definition 10. Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra, D be a designated set of $\mathbb{A}$, and $u \in \mathbf{V}^{(\mathbb{A})}$ be an arbitrary element. Then a sub-collection $\mathrm{dom}_D(u)$ of $\mathrm{dom}(u)$ is defined as $\mathrm{dom}_D(u) = \{x \in \mathrm{dom}(u) : u(x) \in D\}$.

Theorem 11. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and designated set D, the class relation $\sim$ on $\mathbf{V}^{(\mathbb{A})}$, defined as $u \sim v$ if and only if $\mathbf{V}^{(\mathbb{A})} \models u = v$, is an equivalence relation.*

Proof. That the relation $\sim$ is reflexive and symmetric follows immediately from the definition of $[\![\cdot = \cdot]\!]$. We shall now prove that $\sim$ is transitive as well. Let us consider three elements $u, v, w \in \mathbf{V}^{(\mathbb{A})}$ such that $u \sim v$ and $v \sim w$ hold, i.e., $[\![u = v]\!], [\![v = w]\!] \in D$. Hence, we get that $[\![u = v]\!] = \mathbf{1} = [\![v = w]\!]$, as the only possible values of $[\![u = v]\!]$ and $[\![v = w]\!]$ are $\mathbf{1}$ and $\mathbf{0}$. Then, it is enough to prove that $[\![u = w]\!] = \mathbf{1}$ as well. If not, then without loss of generality, let there exists $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \Rightarrow [\![x_0 \in w]\!] = \mathbf{0}$. Hence, $u(x_0) \neq \mathbf{0}$ and $[\![x_0 \in w]\!] = \mathbf{0}$. Since, by our assumption, $[\![u = v]\!] = \mathbf{1}$ and $u(x_0) \neq \mathbf{0}$, we have $[\![x_0 \in v]\!] \neq \mathbf{0}$. Therefore, there exists $y_0 \in \mathrm{dom}(v)$ such that $v(y_0) \neq \mathbf{0}$ and $[\![x_0 = y_0]\!] = \mathbf{1}$. Hence, we can derive that $[\![y_0 \in w]\!] = \mathbf{0}$, as $[\![x_0 \in w]\!] = \mathbf{0}$. This implies that $v(y_0) \Rightarrow [\![y_0 \in w]\!] = \mathbf{0}$. Hence, we get that $[\![v = w]\!] = \mathbf{0}$, which contradicts our assumption. Finally we can conclude that $[\![u = w]\!] = \mathbf{1}$. Hence, $u \sim w$ holds. ⊣

Definition 12. Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra and D be a designated set of $\mathbb{A}$. For every element $u \in \mathbf{V}^{(\mathbb{A})}$ the set $\mathrm{Part}(\mathrm{dom}_D(u))$ is defined to be the quotient (partition) of $\mathrm{dom}_D(u)$ by the equivalence relation $\sim$, i.e., $\mathrm{Part}(\mathrm{dom}_D(u)) = \mathrm{dom}_D(u)/{\sim}$.

Theorem 13. *Let $\mathbb{A}$ be an arbitrary $\mathcal{PS}$-algebra and D be any designated set. For an element $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Ord}(u)$ if and only if u is an α-like element for some $\alpha \in$ ORD.*

Proof. Following the proof of Theorem 13 of [11], one can prove that if $u \in \mathbf{V}^{(\mathbb{A})}$ is an α-like element for some $\alpha \in \mathrm{ORD}$, then $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Ord}(u)$.

Conversely, let $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Ord}(u)$ for some $u \in \mathbf{V}^{(\mathbb{A})}$. We shall prove that u is an α-like element for some $\alpha \in \mathrm{ORD}$. If $\mathrm{dom}_D(u)$ is empty then u is a 0-like element and the proof is complete. Hence, we assume that $\mathrm{dom}_D(u)$ is non-empty. Since $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Trans}(u)$, if $v \in \mathrm{dom}_D(u)$ and $w \in \mathrm{dom}_D(v)$ then $w \in \mathrm{dom}_D(u)$. This leads to the fact that there exists a 0-like element in $\mathrm{dom}_D(u)$. Also, if there exists a β-like element in $\mathrm{dom}_D(u)$ for some $\beta \in \mathrm{ORD}$, then for every $\gamma \in \beta$ there exists a γ-like element in $\mathrm{dom}_D(u)$. Using this fact and the case that every element of $\mathbf{V}^{(\mathbb{A})}$ is sets (not proper classes) in $\mathbf{V}$, we can say that there exists $\alpha \in \mathrm{ORD}$ such that for every $\beta \in \alpha$, there exists at least one β-like element in $\mathrm{dom}_D(u)$, but there does not exist any γ-like element in $\mathrm{dom}_D(u)$ where $\gamma = \alpha$ or $\alpha \in \gamma$. If $\mathrm{dom}_D(u)$ does not contain any other element then u is an α-like element and hence the proof is complete. If possible, let $\mathrm{dom}_D(u)$ contain an element x_i which is not a β-like element for any $\beta \in \alpha$. Since $\mathbf{V}^{(\mathbb{A})} \models \mathrm{LO}(u)$, $\mathrm{dom}_D(x_i)$ contains a β-like element for each $\beta \in \alpha$. But, by a similar argument, it contains an element x_j which is not β-like for any $\beta \in \alpha$. But $x_j \in \mathrm{dom}_D(u)$. Continuing the process, we get a subset $\{x_i : i \in \Lambda\}$ of $\mathrm{dom}_D(u)$, where Λ is an index set such that

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(i) for every $i \in \Lambda$, x_i is not β-like for any $\beta \in \alpha$, and
(ii) for every $i \in \Lambda$, x_i contains a β-like element for each $\beta \in \alpha$.

Now consider an element $y \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(y) = \{x_i : i \in \Lambda\}$ and $\mathrm{ran}(y) = \{\mathbf{1}\}$. Then $\mathbf{V}^{(\mathbb{A})} \models y \subseteq u \ \wedge \ (y \neq \varnothing)$. Since $\mathbf{V}^{(\mathbb{A})} \models \mathrm{WO}_{\in}(u)$, the second conjunct of $\mathrm{WO}_{\in}(u)$ is also valid in $\mathbf{V}^{(\mathbb{A})}$. So, there exists an element $z \in \mathbf{V}^{(\mathbb{A})}$ such that for some $k \in \Lambda$, $[\![z = x_k]\!] \in D$ and $\mathrm{dom}_D(z)$ does not contain any element which is equal to x_i for any $i \in \Lambda$. This fact and the construction of x_k show that $\mathrm{dom}_D(x_k)$ contains at least one β-like element for each $\beta \in \alpha$ and nothing else. Hence, from the definition, we get that x_k is an α-like element. Moreover, by our construction, $x_k \in \mathrm{dom}_D(u)$. This contradicts the fact that $\mathrm{dom}_D(u)$ does not contain any α-like element. Hence, the proof is complete. ⊣

§4. Natural numbers and mathematical induction. In this section, we explore the first-order formulas corresponding to the natural numbers and prove that for each natural number n, the domain of definition of the formula representing the natural number n contains precisely the n-like elements. We show that *mathematical induction* holds in $\mathbf{V}^{(\mathbb{A})}$ for any $\mathcal{PS}$-algebra $\mathbb{A}$ and designated set D.

4.1. Natural number-like elements. The formula $\mathsf{Empty}_{\exists}(x) := \neg\exists y(y \in x)$ represents the *empty set* in classical set theory. But there exists a $\mathcal{PS}$-algebra $\mathbb{A}$ and a designated set D such that the domain of $\mathsf{Empty}_{\exists}(x)$ does not contain only 0-like elements in the corresponding $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. For example, consider a $\mathcal{PS}$-algebra $\mathbb{A}$ and an element $u \in \mathbf{V}^{(\mathbb{A})}$ so that $\mathrm{ran}(u) = \{a\}$, where both $a, a^* \in D$. Clearly, u is not a 0-like element but it validates the formula $\mathsf{Empty}_{\exists}(x)$ in $\mathbf{V}^{(\mathbb{A})}$. Moreover, u validates $\neg\mathsf{Empty}_{\exists}(x)$ as well. From now on, we shall consider the formula $\mathsf{Empty}(x) := \exists y(y \in x) \to \perp$ as the first-order representation of the empty set in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. Observe that if classical set theory or intuitionistic set theory is concerned, $\mathsf{Empty}(x)$ and $\mathsf{Empty}_{\exists}(x)$ are equivalent.

Let $\mathsf{Nat}(0, x)$ be the abbreviation for the formula $\mathsf{Empty}(x)$. Then for a $\mathcal{PS}$-algebra $\mathbb{A}$ and $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(0, u)$ if and only if u is a 0-like element, corresponding to any designated set D. For each positive natural number n, we shall recursively define the formula $\mathsf{Nat}(n, x)$, which uniquely represents the natural number n in classical set theory. For each $n \in \omega$,

$$\begin{aligned}\mathsf{Nat}(n+1, x) := \ & \exists x_0\, \exists x_1 \ldots \exists x_n \big((\mathsf{Nat}(0, x_0) \wedge \cdots \wedge \mathsf{Nat}(n, x_n)) \wedge \\ & (x_0 \in x \wedge \cdots \wedge x_n \in x) \wedge \forall y(y \in x \to (y = x_0 \vee \cdots \vee y = x_n))\big).\end{aligned}$$

THEOREM 14. *Let $n \in \omega$ be arbitrarily chosen. For any $\mathcal{PS}$-algebra $\mathbb{A}$, designated set D, and $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(n+1, u)$ if and only if u is an $(n+1)$-like element in $\mathbf{V}^{(\mathbb{A})}$.*

PROOF. The theorem will be proved by meta-induction. Let us arbitrarily fix a $\mathcal{PS}$-algebra $\mathbb{A}$ with a designated set D.

Base step: Consider the case for $n = 0$. We need to prove that the 1-like elements are the only instances of the formula

$$\mathsf{Nat}(1, x) := \exists x_0 \ \big(\mathsf{Nat}(0, x_0) \ \wedge \ x_0 \in x \ \wedge \ \forall y \ (y \in x \to y = x_0)\big).$$

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So $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(1, x)$ if and only if there exists $u \in \mathbf{V}^{(\mathbb{A})}$ such that $[\![\mathsf{Nat}(0, u)]\!]$, $[\![u \in x]\!]$, $[\![\forall y\ (y \in x \to y = u)]\!] \in D$.

(i) $[\![\mathsf{Nat}(0, u)]\!] \in D$ if and only if u is 0-like.

(ii) $[\![u \in x]\!] \in D$, i.e., $\bigvee_{t \in \mathrm{dom}(x)} (x(t) \wedge [\![t = u]\!]) \in D$, if and only if there exists $t \in \mathrm{dom}_D(x)$ such that $[\![t = u]\!] \in D$, i.e., t is 0-like, using Theorem 9(i).

(iii) $[\![\forall y\ (y \in x \to y = u)]\!] \in D$, i.e., $\bigwedge_{t \in \mathrm{dom}(x)} (x(t) \Rightarrow [\![t = u]\!]) \in D$, if and only if for each $t \in \mathrm{dom}_D(x)$, $[\![t = u]\!] \in D$, i.e., t is 0-like.

Hence, combining (i), (ii), and (iii), it can be concluded that $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(1, u)$ if and only if u is a 1-like element.

Induction hypothesis: Let the proposition be true for all natural numbers less than $m \in \omega \setminus \{0\}$, i.e., for $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(k, u)$ if and only if u is a k-like element for each $k \in \{1, 2, \ldots, m\}$.

Induction step: We shall prove the proposition for the natural number m. Hence, we have to prove $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(m+1, u)$ if and only if u is an $(m+1)$-like element. Since

$$\begin{aligned}\mathsf{Nat}(m+1, u) := \ &\exists x_0\, \exists x_1 \ldots \exists x_m \big((\mathsf{Nat}(0, x_0) \wedge\ \cdots\ \wedge \mathsf{Nat}(m, x_m)) \wedge \\ &(x_0 \in u\ \wedge \cdots \wedge\ x_m \in u)\ \wedge\ \forall y (y \in u \to (y = x_0\ \vee \cdots \vee\ y = x_m))\big),\end{aligned}$$

$[\![\mathsf{Nat}(m+1, u)]\!] \in D$ if and only if there exist $u_0, u_1, \ldots, u_m \in \mathbf{V}^{(\mathbb{A})}$ such that

(i) $[\![\mathsf{Nat}(0, u_0)]\!]$, $[\![\mathsf{Nat}(1, u_1)]\!]$, $\ldots$, $[\![\mathsf{Nat}(m, u_m)]\!] \in D$,

(ii) $[\![u_0 \in u]\!]$, $[\![u_1 \in u]\!]$, $\ldots$, $[\![u_m \in u]\!] \in D$, and

(iii) $[\![\forall y (y \in u \to (y = u_0\ \vee\ \cdots\ \vee\ y = u_m))]\!] \in D$.

By the induction hypothesis, (i) holds if and only if each u_i is an i-like element where $i \in \{0, 1, \ldots, m\}$. Condition (ii) holds if and only if for each $i \in \{0, 1, 2, \ldots, m\}$ there exists an i-like element, in $\mathrm{dom}_D(u)$. Condition (iii) holds if and only if every $y \in \mathrm{dom}_D(u)$ is i-like for some $i \in \{0, 1, 2, \ldots, m\}$. Hence, $[\![\mathsf{Nat}(m+1, u)]\!] \in D$ if and only if u is an $(m+1)$-like element, which entails that the proposition also holds for m. Therefore, by the principle of mathematical induction, the proposition holds for all $n \in \omega$. ⊣

We next prove that in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, the successor of an n-like element will be an $(n+1)$-like element, for any $n \in \omega$.

PROPOSITION 15. *For any $\mathcal{PS}$-algebra $\mathbb{A}$, any designated set D, and a natural number n,*

$$\mathbf{V}^{(\mathbb{A})} \models \forall x (\mathsf{Nat}(n, x) \wedge \forall y \forall z (z \in y \leftrightarrow z \in x \vee z = x) \to \mathsf{Nat}(n+1, y)).$$

PROOF. Let us take an n-like element x for some $n \in \omega$. Now let

$$[\![\forall y \forall z (z \in y \leftrightarrow z \in x \vee z = x)]\!] \in D.$$

From the first conjunct we get $[\![\forall y \forall z (z \in y \to z \in x \vee z = x)]\!] \in D$, which implies that for any $y \in \mathbf{V}^{(\mathbb{A})}$, $[\![\forall z (z \in y \to z \in x \vee z = x)]\!] \in D$, i.e.,

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$$\bigwedge_{z \in \mathrm{dom}(y)} (y(z) \Rightarrow [\![z \in x]\!] \vee [\![z = x]\!]) \in D,$$

i.e., any $z \in \mathrm{dom}_D(y)$ is either an m-like element, where $m < n$, or an n-like element.

The second conjunct gives $[\![\forall y \forall z(z \in x \vee z = x \rightarrow z \in y)]\!] \in D$. This implies that for any $y \in \mathbf{V}^{(\mathbb{A})}$, $[\![\forall z(z \in x \vee z = x \rightarrow z \in y)]\!] \in D$, i.e.,

$$\bigwedge_{z \in \mathbf{V}^{(\mathbb{A})}} ([\![z \in x]\!] \vee [\![z = x]\!] \Rightarrow [\![z \in y]\!]) \in D,$$

i.e., for any $z \in \mathbf{V}^{(\mathbb{A})}$ which is either m-like for some $m < n$ or n-like, there exists $t \in \mathrm{dom}_D(y)$ which is either m-like or n-like.

Combining the above two derivations, we can say that if x is any n-like element and $[\![\forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x)]\!] \in D$ holds, then by definition, y is an $(n+1)$-like element. This completes the proof. ⊣

4.2. Mathematical induction. In classical set theory we know that the smallest *inductive set* is the collection of all natural numbers, ω. By Definition 8, we know that in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, if u is an ω-like element, then $\mathrm{dom}_D(u)$ contains at least one natural-number-like element for every natural number in ω and nothing else. However, from this fact it is not immediate that an ω-like element is the smallest inductive set in $\mathbf{V}^{(\mathbb{A})}$. Theorem 19 confirms that this is the case, which also leads to the validity of mathematical induction in $\mathbf{V}^{(\mathbb{A})}$.

Consider the following three formulas:

(i) $\mathrm{Ind}(I) := \exists e(\mathrm{Nat}(0, e) \wedge e \in I) \wedge \forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)$,
(ii) $\mathsf{Nat}(x) := \forall I(\mathrm{Ind}(I) \rightarrow x \in I)$, and
(iii) $\mathrm{SetNat}(w) := \forall x(x \in w \leftrightarrow \mathsf{Nat}(x))$.

Intuitively, the formula $\mathrm{Ind}(I)$ is naively interpreted as 'I is *inductive*', $\mathsf{Nat}(x)$ is interpreted as 'x belongs to every inductive set', i.e., in the sense of classical set theory we can think of the formula $\mathsf{Nat}(x)$ as expressing that x is a natural number, and the formula $\mathrm{SetNat}(w)$ is interpreted as 'w consists of all natural numbers'.

Intuitively, Lemma 16 portrays that if $I \in \mathbf{V}^{(\mathbb{A})}$ is an inductive set and there does not exist any m-like element (where $m \in \omega \setminus \{0\}$) in I, then I does not contain any $(m-1)$-like element.

LEMMA 16. *Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra and D be any designated set of $\mathbb{A}$. Suppose $I \in \mathbf{V}^{(\mathbb{A})}$ is an element which satisfies the following two conditions*:

(i) $[\![\forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)]\!] \in D$, *and*
(ii) *for an arbitrary $m \in \omega \setminus \{0\}$ there does not exist any m-like element $v \in \mathrm{dom}_D(I)$.*

Then there does not exist any $(m-1)$-like element $u \in \mathrm{dom}_D(I)$.

PROOF. If possible, let u be an $(m-1)$-like element such that $u \in \mathrm{dom}_D(I)$. We have

$$[\![\forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)]\!] \leq I(u) \Rightarrow [\![\exists s \forall y(y \in s \leftrightarrow y \in u \vee y = u) \wedge s \in I]\!].$$

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Condition (i) and $I(u) \in D$ together imply

$$[\![\exists s \forall y(y \in s \leftrightarrow y \in u \vee y = u) \wedge s \in I]\!] \in D,$$

i.e., there exists $v \in \mathbf{V}^{(\mathbb{A})}$ such that

$$[\![\forall y(y \in v \leftrightarrow y \in u \vee y = u) \wedge v \in I]\!] \in D,$$

which implies $[\![\forall y(y \in v \leftrightarrow y \in u \vee y = u)]\!] \wedge [\![v \in I]\!] \in D$. Using Theorem 9 the first conjunct assures that v is m-like. The second conjunct implies $I(v) \in D$. This contradicts condition (ii). Hence, the proof is complete. ⊣

LEMMA 17. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and any designated set D, if $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Ind}(I)$ for some $I \in \mathbf{V}^{(\mathbb{A})}$ then for each natural number n there exists an n-like element $u \in \mathrm{dom}_D(I)$.*

PROOF. Let us consider an $I \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Ind}(I)$, i.e., both the conjuncts of $\mathrm{Ind}(I)$ are valid in $\mathbf{V}^{(\mathbb{A})}$. Hence, from the first conjunct there exists a 0-like element in $\mathrm{dom}_D(I)$. Since the second conjunct is also valid, by Lemma 16 it can be concluded that for a natural number m in $\mathbf{V}$, if there exists an m-like element $u \in \mathrm{dom}_D(I)$ then there exists an $(m+1)$-like element $v \in \mathrm{dom}_D(I)$. Hence, by meta-induction on natural numbers in $\mathbf{V}$, the proof is complete. ⊣

Lemma 17 shows that every inductive set contains all the natural numbers in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. Hence, in $\mathbf{V}^{(\mathbb{A})}$, the intersection of all inductive sets should be the collection of all natural numbers. This is confirmed by Theorem 18.

THEOREM 18. *For any $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Nat}(u)$ if and only if u is n-like for some natural number n, corresponding to any designated set D of $\mathbb{A}$.*

PROOF. Let u be an n-like element for some $n \in \omega$. We get

$$\begin{aligned} [\![\mathsf{Nat}(u)]\!] &= [\![\forall I(\mathrm{Ind}(I) \to u \in I)]\!] \\ &= \bigwedge_{I \in \mathbf{V}^{(\mathbb{A})}} ([\![\mathrm{Ind}(I)]\!] \Rightarrow [\![u \in I]\!]) \\ &\in D, \end{aligned}$$

since for each $I \in \mathbf{V}^{(\mathbb{A})}$, if $[\![\mathrm{Ind}(I)]\!] \in D$ then by Lemma 17 we also get $[\![u \in I]\!] \in D$.

Conversely, let $[\![\mathsf{Nat}(x)]\!] \in D$. Hence, for each $I \in \mathbf{V}^{(\mathbb{A})}$, if $[\![\mathrm{Ind}(I)]\!] \in D$ then $[\![x \in I]\!] \in D$. We shall show that x is a natural-number-like element. Consider an ω-like element $u \in \mathbf{V}^{(\mathbb{A})}$. Then $[\![\mathrm{Ind}(u)]\!] \in D$ is immediate. But if x is not a natural-number-like element then using Lemma 17 we get $[\![x \in I]\!] \notin D$. This implies that x is a natural-number-like element and the theorem is proved. ⊣

As an application of Theorem 18 we get the following theorem, which ensures that the smallest inductive set is an ω-like element in any $\mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A} \in \mathcal{PS}$.

THEOREM 19. *For any $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, $\mathbf{V}^{(\mathbb{A})} \models \mathrm{SetNat}(u)$ if and only if u is an ω-like element, with respect to any designated set D of $\mathbb{A}$.*

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Following the principle of mathematical induction in classical set theory, we can intuitively think about the same principle in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ as follows: for any two names $x, y \in \mathbf{V}^{(\mathbb{A})}$, if x is an ω-like element, y is a subset of x in $\mathbf{V}^{(\mathbb{A})}$, and y is inductive, then $x = y$ holds in $\mathbf{V}^{(\mathbb{A})}$. Let us now consider the formula

$$\forall x \forall y(\mathrm{SetNat}(x) \wedge y \subseteq x \wedge \mathrm{Ind}(y) \rightarrow x = y). \tag{MI}$$

In the following theorem we shall prove that mathematical induction holds in every $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ even if the logic of $(\mathbb{A}, D)$ is non-classical.

THEOREM 20. *The formula* MI *is valid in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, i.e., $\mathbf{V}^{(\mathbb{A})} \models$* MI, *corresponding to any designated set D of $\mathbb{A}$.*

PROOF. Consider any $x, y \in \mathbf{V}^{(\mathbb{A})}$ such that $[\![\mathrm{SetNat}(x) \wedge y \subseteq x \wedge \mathrm{Ind}(y)]\!] \in D$. We shall prove $[\![x = y]\!] \in D$, i.e., y is an ω-like element. Using Theorem 19, $[\![\mathrm{SetNat}(x)]\!] \in D$ implies that x is an ω-like element. From the second conjunct we have $[\![y \subseteq x]\!] \in D$, which implies

$$[\![\forall t(t \in y \rightarrow t \in x)]\!] = \bigwedge_{t \in \mathrm{dom}(y)} (y(t) \Rightarrow [\![t \in x]\!]) \in D.$$

Since x is ω-like, if there exists $t \in \mathrm{dom}(y)$ which is not a natural-number-like element then $y(t) \notin D$. The third conjunct of our assumption gives us that $[\![\mathrm{Ind}(y)]\!] \in D$. Then, using Lemma 17, we get that for each natural number n there exists an n-like element in $\mathrm{dom}_D(y)$. These results together show that y is an ω-like element. ⊣

§5. Axiom of Choice, cardinality, and GCH in $\mathcal{PS}$-algebra-valued models. In this section, we explore the cardinality of an element in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ following the construction of cardinality in classical set theory. It will be shown that the properties of cardinality in $\mathbf{V}^{(\mathbb{A})}$ mostly depend on the properties in $\mathbf{V}$.

To avoid repetition, from now on by saying 'for any $\mathcal{PS}$-algebra $\mathbb{A}$ (any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$)' we shall mean that 'any $\mathcal{PS}$-algebra $\mathbb{A}$, associated with any designated set D of it (any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ where $\mathbb{A}$ is associated with any designated set D)'.

5.1. The Axiom of Choice and the Well-Ordering Theorem in $\mathbf{V}^{(\mathbb{A})}$. It is already discussed that all the axioms of NFF-ZF are valid in $\mathbf{V}^{(\mathbb{A})}$, for every $\mathcal{PS}$-algebra $\mathbb{A}$. In this section, we shall prove that the Axiom of Choice is also valid in $\mathbf{V}^{(\mathbb{A})}$ if it is valid in the ground model $\mathbf{V}$. The validity of the Axiom of Choice will be needed in defining cardinality in the $\mathcal{PS}$-algebra-valued models $\mathbf{V}^{(\mathbb{A})}$.

To proceed further, we shall use the following abbreviations:[3]

(i) $\mathbf{z} = \{\mathbf{x}\} := \exists y\ (y \in z) \wedge \forall y\ (y \in z \rightarrow y = x)$,

(ii) $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\} := \exists s\ (s \in z \wedge s = x) \wedge \exists t\ (t \in z \wedge t = y) \wedge \forall w\ (w \in z \rightarrow w = x \vee w = y)$,

[3] Note that to increase readability, in the abbreviations $\mathbf{z} = \{\mathbf{x}\}$ and $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$, we have used '=' as an arbitrary symbol. It does not represent the predicate symbol '=' of the language of set theory.

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(iii) $\text{Pair}(z; x, y) := \exists s\ (s \in z \wedge (\mathbf{s} = \{\mathbf{x}\})) \wedge \exists t\ (t \in z \wedge (\mathbf{t} = \{\mathbf{x}, \mathbf{y}\})) \wedge$
$\forall w\ (w \in z \rightarrow (\mathbf{w} = \{\mathbf{x}\}) \vee (\mathbf{w} = \{\mathbf{x}, \mathbf{y}\}))$,
(iv) $\text{Func}(f) := \forall x\ (x \in f \rightarrow \exists s\ \exists t\ \text{Pair}(x; s, t)) \wedge \forall x\ \forall y\ \forall s\ \forall t\ \forall w\ \forall w'((x \in f \wedge y \in f \wedge \text{Pair}(x; w, s) \wedge \text{Pair}(y; w', t) \wedge w = w') \rightarrow s = t)$,
(v) $\text{Dom}(f; x) := \forall y\ (y \in x \rightarrow \exists w\ \exists z\ (w \in f \wedge \text{Pair}(w; y, z))) \wedge$
$\forall w\ (w \in f \rightarrow \exists y\ \exists z\ \text{Pair}(w; y, z) \wedge y \in x)$,
(vi) $\text{Codom}(f; x) := \forall y\ (y \in x \rightarrow \exists w\ \exists z\ (w \in f \wedge \text{Pair}(w; y, z))) \wedge$
$\forall w\ (w \in f \rightarrow \exists y\ \exists z\ \text{Pair}(w; y, z) \wedge z \in x)$,
(vii) $\text{InjFunc}(f; x, y) := \text{Func}(f) \wedge \text{Dom}(f; x) \wedge \text{Codom}(f; y) \wedge \forall x\ \forall y\ \forall s\ \forall t$
$\forall p\ \forall q((x \in f \wedge y \in f \wedge \text{Pair}(x; s, t) \wedge \text{Pair}(y; p, q) \wedge (t = q)) \rightarrow (s = p))$,
(viii) $\text{SurFunc}(f; x, y) := \text{Func}(f) \wedge \text{Dom}(f; x) \wedge \text{Codom}(f; y) \wedge$
$\forall z\ (z \in y \rightarrow \exists s\ \exists w\ (w \in f \wedge s \in x \wedge \text{Pair}(w; s, z)))$, and
(ix) $\text{BijFunc}(f; x, y) := \text{InjFunc}(f; x, y) \wedge \text{SurFunc}(f; x, y)$.

The formulas $\mathbf{z} = \{\mathbf{x}\}$ and $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$ are naively interpreted as 'z is a singleton set $\{x\}$' and 'z is the set $\{x, y\}$', respectively. Observe that the possibility of the sets x and y being equal is alive in the formula $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$. Naive interpretations of the formulas $\text{Func}(f)$, $\text{Dom}(f; x)$, $\text{Codom}(f; x)$, $\text{InjFunc}(f; x, y)$, $\text{SurFunc}(f; x, y)$, and $\text{BijFunc}(f; x, y)$ are 'f is a function', 'x is the domain of f', 'x is a co-domain of f', 'f is an injective function from x into y', 'f is a surjective function from x onto y', and 'f is a bijective function from x to y', respectively.

Definition 21. An element $w \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra and D is an arbitrarily fixed designated set of $\mathbb{A}$, is said to be (a, b)-like for some $a, b \in \mathbf{V}^{(\mathbb{A})}$ if

(i) there exists $p \in \text{dom}_D(w)$ such that any element in $\text{dom}_D(p)$ is equivalent to a with respect to $\sim$,
(ii) there exists $p \in \text{dom}_D(w)$ such that $\text{dom}_D(p)$ contains two elements u and v such that $u \sim a$ and $v \sim b$; moreover, any element in $\text{dom}_D(p)$ is either equivalent to a or equivalent to b,
(iii) $\text{dom}_D(w)$ does not contain any element other than those described in (i) and (ii).

The first-order formula defining the Axiom of Choice is given below:

$$\forall u(u \neq \varnothing \rightarrow \exists f(\text{Func}(f) \wedge \text{Dom}(f; u) \wedge \forall x(x \in u \wedge x \neq \varnothing \rightarrow \exists z \exists y(\text{Pair}(z; x, y) \wedge z \in f \wedge y \in x)))), \tag{AC}$$

where, as earlier, $u \neq \varnothing$ and $x \neq \varnothing$ are the abbreviations for the formulas $\exists z(z \in u)$ and $\exists z(z \in x)$, respectively.

Theorem 22. *For any $\mathcal{PS}$-algebra $\mathbb{A}$, if $\mathbf{V} \models \text{AC}$ then $\mathbf{V}^{(\mathbb{A})} \models \text{AC}$.*

Proof. Let us consider a $\mathcal{PS}$-algebra $\mathbb{A}$ with a designated set D and a model of set theory $\mathbf{V}$ such that $\mathbf{V} \models \text{AC}$. Suppose $u \in \mathbf{V}^{(\mathbb{A})}$ be an arbitrary element such that $\mathbf{V}^{(\mathbb{A})} \models u \neq \varnothing$. Then, we have $\text{dom}_D(u) \neq \varnothing$ in $\mathbf{V}$. Let us take an element $\bar{x} \in \text{Part}(\text{dom}_D(u))$.

Case I. Suppose $\bar{x}$ does not contain 0-like elements. Arbitrarily fix an element $a_{\bar{x}} \in \bar{x}$. By our assumption $\text{dom}_D(a_{\bar{x}}) \neq \varnothing$ and hence choose an element $b_{\bar{x}} \in \text{dom}_D(a_{\bar{x}})$. Construct two elements $p_{\bar{x}}$ and $q_{\bar{x}}$ in $\mathbf{V}^{(\mathbb{A})}$ such that $\text{dom}(p_{\bar{x}}) = \{a_{\bar{x}}\}$

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and $\mathrm{dom}(q_{\bar{x}}) = \{a_{\bar{x}}, b_{\bar{x}}\}$, where $\mathrm{ran}(p_{\bar{x}}) = \{\mathbf{1}\} = \mathrm{ran}(q_{\bar{x}})$. Using these elements $p_{\bar{x}}$ and $q_{\bar{x}}$ we define another element $w_{\bar{x}} \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(w_{\bar{x}}) = \{p_{\bar{x}}, q_{\bar{x}}\}$ and $\mathrm{ran}(w_{\bar{x}}) = \{\mathbf{1}\}$. Hence by Definition 21, it can be said that $w_{\bar{x}}$ is an $(a_{\bar{x}}, b_{\bar{x}})$-like element.

Case II. Suppose $\bar{x}$ contains a 0-like element. Then by Theorem 9(i), all the elements of $\bar{x}$ are 0-like elements. Hence, $\bar{x}$ is the class of all 0-like elements of $\mathrm{dom}_D(u)$. Let us arbitrarily fix any two 0-like elements $s, t \in \mathbf{V}^{(\mathbb{A})}$, not necessarily different in $\mathbf{V}$. Following the same construction as in Case I, construct an element $w_{\bar{x}} \in \mathbf{V}^{(\mathbb{A})}$ such that it becomes an (s, t)-like element.

Let us now consider an element f such that $\mathrm{dom}(f) = \{w_{\bar{x}} : \bar{x} \in \mathrm{Part}(\mathrm{dom}_D(u))\}$ and $\mathrm{ran}(f) = \{\mathbf{1}\}$. The existence of f in $\mathbf{V}$ is assured by the fact that $\mathbf{V} \models \mathrm{AC}$. Then by the construction $f \in \mathbf{V}^{(\mathbb{A})}$. It can be checked that $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Func}(f) \wedge \mathrm{Dom}(f; u)$. We shall now prove that

$$\mathbf{V}^{(\mathbb{A})} \models \forall x(x \in u \wedge x \neq \varnothing \rightarrow \exists z \exists y(\mathrm{Pair}(z; x, y) \wedge z \in f \wedge y \in x))).$$

Consider an element $v \in \mathbf{V}^{(\mathbb{A})}$ such that $[\![v \in u]\!] \wedge [\![v \neq \varnothing]\!] \in D$. Then, there exists an element $x \in \mathrm{dom}_D(u)$ such that $\mathbf{V}^{(\mathbb{A})} \models v = x$, and x is not 0-like. Consider the equivalence class $\bar{x}$ containing x in $\mathrm{Part}(\mathrm{dom}_D(u))$. By the construction of f, there exists $w_{\bar{x}} \in \mathrm{dom}(f)$ which is an $(a_{\bar{x}}, b_{\bar{x}})$-like element, where $a_{\bar{x}} \in \bar{x}$ and $b_{\bar{x}} \in \mathrm{dom}_D(a_{\bar{x}})$. Since $a_{\bar{x}} \in \bar{x}$, we get that $\mathbf{V}^{(\mathbb{A})} \models a_{\bar{x}} = x$, which implies $\mathbf{V}^{(\mathbb{A})} \models a_{\bar{x}} = v$. Hence, we can derive that $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Pair}(w_{\bar{x}}; v, b_{\bar{x}}) \wedge w_{\bar{x}} \in f \wedge b_{\bar{x}} \in v$. Hence combining all the above results we can finally conclude that $\mathbf{V}^{(\mathbb{A})} \models \mathrm{AC}$. $\dashv$

Unless otherwise stated, from now on we shall assume that the ground model $\mathbf{V}$ is a model of ZFC. In this paper, the results on cardinality heavily depend on Theorem 23, which shows how the bijection between two elements u and v in $\mathbf{V}^{(\mathbb{A})}$ depends on the bijection between $\mathrm{Part}(\mathrm{dom}_D(u))$ and $\mathrm{Part}(\mathrm{dom}_D(v))$ in $\mathbf{V}$, and similarly for the injection and surjection.

Theorem 23. *For any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ and a pair of elements $u, v \in \mathbf{V}^{(\mathbb{A})}$,*

(i) *there exists an injection from* $\mathrm{Part}(\mathrm{dom}_D(u))$ *into* $\mathrm{Part}(\mathrm{dom}_D(v))$ *in* $\mathbf{V}$ *if and only if* $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{InjFunc}(f; u, v)$,

(ii) *there exists a surjection from* $\mathrm{Part}(\mathrm{dom}_D(u))$ *onto* $\mathrm{Part}(\mathrm{dom}_D(v))$ *in* $\mathbf{V}$ *if and only if* $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{SurjFunc}(f; u, v)$,

(iii) *there exists a bijection between* $\mathrm{Part}(\mathrm{dom}_D(u))$ *and* $\mathrm{Part}(\mathrm{dom}_D(v))$ *in* $\mathbf{V}$ *if and only if* $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{BijFunc}(f; u, v)$.

Proof. We shall prove (iii) only; statements (i) and (ii) can be proved similarly. Let $g : \mathrm{Part}(\mathrm{dom}_D(u)) \rightarrow \mathrm{Part}(\mathrm{dom}_D(v))$ be a bijection in $\mathbf{V}$. Let $\bar{x} \in \mathrm{Part}(\mathrm{dom}_D(u))$ be arbitrarily chosen and $a \in \bar{x}$ and $b \in g(\bar{x})$ be arbitrarily fixed elements. Then, by definition, $a, b \in \mathbf{V}^{(\mathbb{A})}$. Consider two elements $p, q \in \mathbf{V}^{(\mathbb{A})}$, defined by $\mathrm{dom}(p) = \{a\}$, $\mathrm{dom}(q) = \{a, b\}$, and $\mathrm{ran}(p) = \{\mathbf{1}\} = \mathrm{ran}(q)$. Let us now take the element $w_{\bar{x}}$ of $\mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(w_{\bar{x}}) = \{p, q\}$ and $\mathrm{ran}(w_{\bar{x}}) = \{\mathbf{1}\}$. It can be proved that $w_{\bar{x}}$ is (a, b)-like. Clearly, for each $\bar{x} \in \mathrm{Part}(\mathrm{dom}_D(u))$, we have one element $w_{\bar{x}}$ which is (a, b)-like for some $a \in \bar{x}$ and $b \in g(\bar{x})$. Now consider

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$f \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(f) = \{w_{\bar{x}} : \bar{x} \in \mathrm{Part}(\mathrm{dom}_D(u))\}$ and $\mathrm{ran}(f) = \{\mathbf{1}\}$; such a function f exists in $\mathbf{V}$, and hence in $\mathbf{V}^{(\mathbb{A})}$, since $\mathbf{V} \models \mathrm{AC}$. By the definition, it is immediate that $[\![\mathrm{BijFunc}(f; u, v)]\!] \in D$.

Conversely, let $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{BijFunc}(f; u, v)$. We shall prove that there exists a bijection between $\mathrm{Part}(\mathrm{dom}_D(u))$ and $\mathrm{Part}(\mathrm{dom}_D(v))$. By the assumption, there exists $f \in \mathbf{V}^{(\mathbb{A})}$ such that

$$[\![\mathrm{Func}(f)]\!],\ [\![\mathrm{Dom}(f; u)]\!],\ [\![\mathrm{Codom}(f; v)]\!],\ [\![\mathrm{InjFunc}(f; u, v)]\!],\\ [\![\mathrm{SurFunc}(f; u, v)]\!] \in D.$$

By analysing all these, we conclude the following.

(i) If $w \in \mathrm{dom}_D(f)$ then w is (a, b)-like for some $a, b \in \mathbf{V}^{(\mathbb{A})}$.
(ii) For any $w, w' \in \mathrm{dom}_D(f)$, if w is (a, b)-like and w' is (c, d)-like for some $a, b, c, d \in \mathbf{V}^{(\mathbb{A})}$, where $a \sim c$ holds, then $b \sim d$ also holds.
(iii) For every $a \in \mathrm{dom}_D(u)$ there exist $w \in \mathrm{dom}_D(f)$ and $b \in \mathbf{V}^{(\mathbb{A})}$ such that w is (a, b)-like.
(iv) For every $w \in \mathrm{dom}_D(f)$ there exist $a \in \mathrm{dom}_D(u)$ and $b \in \mathbf{V}^{(\mathbb{A})}$ such that w is (a, b)-like.
(v) For every $w \in \mathrm{dom}_D(f)$ there exist $b \in \mathrm{dom}_D(v)$ and $a \in \mathbf{V}^{(\mathbb{A})}$ such that w is (a, b)-like.
(vi) For any $w, w' \in \mathrm{dom}_D(f)$ and $a, b, c, d \in \mathbf{V}^{(\mathbb{A})}$ if w is (a, b)-like, w' is (c, d)-like, and $b \sim d$, then $a \sim c$.
(vii) If $b \in \mathrm{dom}_D(v)$ then there exist $a \in \mathrm{dom}_D(u)$ and $w \in \mathrm{dom}_D(f)$ such that w is (a, b)-like.

Combining all these results, we get a bijection between $\mathrm{Part}(\mathrm{dom}_D(u))$ and $\mathrm{Part}(\mathrm{dom}_D(v))$ in $\mathbf{V}$. ⊣

Corollary 24. *Let $u \in \mathbf{V}^{(\mathbb{A})}$ be arbitrarily chosen and the cardinality of* $\mathrm{Part}(\mathrm{dom}_D(u))$ *in* $\mathbf{V}$, *denoted by* $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}}$, *be* κ. *Then in* $\mathbf{V}^{(\mathbb{A})}$, *there exist bijections between u and any κ-like element, but there does not exist any bijection between u and any α-like element, where $\alpha < \kappa$ in* $\mathbf{V}$.

A set together with a well-order relation is said to be a well-ordered set. In classical set theory, it is known that every well-ordered set is order isomorphic to an ordinal number. This implies that, a set can be well ordered by a relation if there exists a bijection between the set and an ordinal number. Using this fact, we get the following definition.

Definition 25. Let $\mathbf{V}^{(\mathbb{A})}$ be a $\mathcal{PS}$-algebra-valued model. An element $u \in \mathbf{V}^{(\mathbb{A})}$ is called well-ordered (or it can be well-ordered) if $\mathbf{V}^{(\mathbb{A})} \models \exists x(\mathsf{Ord}(x) \wedge \exists f\ \mathrm{BijFunc}(f; u, x))$ holds.

In set theory, the Well-Ordering Theorem states that 'every set can be well-ordered'. We shall now prove that the validity of the Well-Ordering Theorem in any $\mathcal{PS}$-algebra-valued model depends on its validity in the ground model, where the following abbreviation will be used to represent the Well-Ordering Theorem:

$$\forall y(\exists x(\mathsf{Ord}(x) \wedge \exists f\ \mathrm{BijFunc}(f; y, x)). \qquad \text{(WOT)}$$

THEOREM 26 (Well-Ordering Theorem in $\mathbf{V}^{(\mathbb{A})}$). *Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra and $\mathbf{V}$ be a model of ZF. If every element of $\mathbf{V}$ can be well-ordered, i.e., for every $y \in \mathbf{V}$ there exists a bijective function between y and an element $x \in \mathrm{ORD}$ in $\mathbf{V}$, then any element of $\mathbf{V}^{(\mathbb{A})}$ can also be well-ordered, i.e., $\mathbf{V}^{(\mathbb{A})} \models \mathrm{WOT}$.*

PROOF. Let the Well-Ordering Theorem hold in $\mathbf{V}$ and hence $\mathbf{V} \models \mathrm{AC}$. Suppose $u \in \mathbf{V}^{(\mathbb{A})}$ is any arbitrary element. Then, $\mathrm{Part}(\mathrm{dom}_D(u))$ is an element of $\mathbf{V}$. By our assumption, there exists $\alpha \in \mathrm{ORD}$ and a bijection between $\mathrm{Part}(\mathrm{dom}_D(u))$ and α in $\mathbf{V}$. Let us now consider any α-like element v. Then, by using Theorems 13 and 23(iii) we get that $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Ord}(v) \wedge \exists f\ \mathrm{BijFunc}(f; u, v))$. This completes the proof. $\dashv$

It is also well-known that in classical set theory, the Well-Ordering Theorem is equivalent to the Axiom of Choice. If $\mathbb{A}$ is a $\mathcal{PS}$-algebra and $\mathbf{V}$ is a model of ZF, then Theorems 22 and 26 are not sufficient to conclude the equivalence of these two statements. However, we have the following conjecture.

CONJECTURE . The Axiom of Choice and the Well-Ordering Theorem are equivalent in $\mathbf{V}^{(\mathbb{A})}$:

$$\mathbf{V}^{(\mathbb{A})} \models \mathrm{AC} \leftrightarrow \mathrm{WOT},$$

for any $\mathcal{PS}$-algebra $\mathbb{A}$ and any model $\mathbf{V}$ of ZF.

5.2. Cardinal numbers in $\mathcal{PS}$-algebra-valued models. The first-order formula expressing that x is a cardinal number is as follows:

$$\mathsf{Card}(x) := \mathsf{Ord}(x) \wedge \forall y (y \in x \rightarrow \neg \exists f\ \mathrm{BijFunc}(f; x, y)).$$

DEFINITION 27. For any $\mathcal{PS}$-algebra $\mathbb{A}$, an element $u \in \mathbf{V}^{(\mathbb{A})}$ is called a cardinal number in $\mathbf{V}^{(\mathbb{A})}$ if $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Card}(u)$ holds.

Using Theorem 13 and Corollary 24 we have the following theorem, which explicitly gives the cardinal numbers in $\mathbf{V}^{(\mathbb{A})}$.

THEOREM 28. *For an element $u \in \mathbf{V}^{(\mathbb{A})}$, a $\mathcal{PS}$-algebra-valued model, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Card}(u)$ if and only if u is a κ-like element, where κ is a cardinal number in $\mathbf{V}$.*

The first-order formula which naively expresses that 'y is the cardinality of x' is the following:

$$\mathsf{Card}(x, y) := \mathsf{Card}(y) \wedge \exists f\ \mathrm{BijFunc}(f; x, y).$$

Let us arbitrarily fix a $\mathcal{PS}$-algebra $\mathbb{A}$. For any $u \in \mathbf{V}^{(\mathbb{A})}$, consider the class

$$\mathsf{Cardinal}_u := \{v \in \mathbf{V}^{(\mathbb{A})} : \mathbf{V}^{(\mathbb{A})} \models \mathsf{Card}(u, v)\}.$$

By our assumption, $\mathbf{V}$ is a model of ZFC. Hence, for any $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathsf{Cardinal}_u \neq \varnothing$, by Theorem 26.

DEFINITION 29. Consider an element $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra. Any element of the class $\mathsf{Cardinal}_u$ is called a name of the cardinal number of u or simply a cardinal number of u in $\mathbf{V}^{(\mathbb{A})}$.

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For each $x \in \mathbf{V}$, let us define an element $\hat{x}$ of $\mathbf{V}^{(\mathbb{A})}$ recursively as: $\hat{\varnothing} = \varnothing$ and $\hat{x} = \{\langle \hat{y}, \mathbf{1}\rangle : y \in x\}$.

Observation 30. *For any $u \in \mathbf{V}^{(\mathbb{A})}$, if $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$, then using Corollary 24 and Theorem 28 it can be concluded that $\mathsf{Cardinal}_u$ consists of all κ-like elements. This implies that $\mathbf{V}^{(\mathbb{A})}$ cannot separate the elements in $\mathsf{Cardinal}_u$, i.e., for any two cardinal numbers v and w of u, $\mathbf{V}^{(\mathbb{A})} \models v = w$. In particular, for any $v \in \mathsf{Cardinal}_u$, $\mathbf{V}^{(\mathbb{A})} \models v = \hat{\kappa}$.*

Theorem 31. *Let $\mathbb{A}$ be any $\mathcal{PS}$-algebra and $u, v \in \mathbf{V}^{(\mathbb{A})}$ be such that $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$ and $|\mathrm{Part}(\mathrm{dom}_D(v))|_{\mathbf{V}} = \eta$. Then the following are equivalent.*

(i) $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{BijFunc}(f; u, v)$.
(ii) $\mathsf{Cardinal}_u = \mathsf{Cardinal}_v$.
(iii) $\mathbf{V}^{(\mathbb{A})} \models \hat{\kappa} = \hat{\eta}$.

Proof. The proof can be done by using Observation 30 and the fact that, for any $u, v, w \in \mathbf{V}^{(\mathbb{A})}$, if both $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{BijFunc}(f; u, v)$ and $\mathbf{V}^{(\mathbb{A})} \models \exists g\ \mathrm{BijFunc}(g; v, w)$ hold then $\mathbf{V}^{(\mathbb{A})} \models \exists h\ \mathrm{BijFunc}(h; u, w)$ holds as well. ⊣

5.3. Ordering in cardinal numbers. In classical set theory we know that for any two elements $u, v \in \mathbf{V}$, $|u|_{\mathbf{V}} < |v|_{\mathbf{V}}$ if and only if

$$\mathbf{V} \models \exists f\ \mathrm{InjFunc}(f; u, v) \text{ but } \mathbf{V} \not\models \exists f\ \mathrm{InjFunc}(f; v, u).$$

We shall use the same ontology to define ordering in cardinal numbers in any $\mathcal{PS}$-algebra-valued model.

Definition 32. Let $\mathbf{V}^{(\mathbb{A})}$ be a $\mathcal{PS}$-algebra-valued model. For any two elements $u, v \in \mathbf{V}^{(\mathbb{A})}$ a cardinal number of u is said to be less than a cardinal number of v if

$$\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{InjFunc}(f; u, v) \text{ but } \mathbf{V}^{(\mathbb{A})} \not\models \exists f\ \mathrm{InjFunc}(f; v, u).$$

Theorem 33 shows that this definition is unambiguous in the sense that if a cardinal number of u is less than a cardinal number of v then we always get an injective function from any element of $\mathsf{Cardinal}_u$ into any element of $\mathsf{Cardinal}_v$ but not the other way around.

Theorem 33. *Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra, and let u and v be as defined in Theorem 31. Then the following are equivalent.*

(i) *A cardinal number of u is less than a cardinal number of v.*
(ii) *For any $p \in \mathsf{Cardinal}_u$ and $q \in \mathsf{Cardinal}_v$, $\mathbf{V}^{(\mathbb{A})} \models \exists f\ \mathrm{InjFunc}(f; p, q)$ but $\mathbf{V}^{(\mathbb{A})} \not\models \exists f\ \mathrm{InjFunc}(f; q, p)$.*
(iii) *For any $p \in \mathsf{Cardinal}_u$ and $q \in \mathsf{Cardinal}_v$, $\mathbf{V}^{(\mathbb{A})} \models p \in q$.*
(iv) $\mathbf{V}^{(\mathbb{A})} \models \hat{\kappa} \in \hat{\eta}$.

Proof. A cardinal number of u is less than a cardinal number of v $\iff$ for any $p \in \mathsf{Cardinal}_u$ and $q \in \mathsf{Cardinal}_v$ there is an injection from $\mathrm{Part}(\mathrm{dom}_D(p))$ into $\mathrm{Part}(\mathrm{dom}_D(q))$ but no injection from $\mathrm{Part}(\mathrm{dom}_D(q))$ into $\mathrm{Part}(\mathrm{dom}_D(p))$ in $\mathbf{V}$; this is because, applying Theorem 23, we get bijections between $\mathrm{Part}(\mathrm{dom}_D(u))$

and $\text{Part}(\text{dom}_D(p))$ and also between $\text{Part}(\text{dom}_D(v))$ and $\text{Part}(\text{dom}_D(q))$ in $\mathbf{V}$, $\iff$ $\mathbf{V}^{(\mathbb{A})} \models p \in q$, since, by Observation 30, p is κ-like and q is η-like, $\iff$ $\mathbf{V}^{(\mathbb{A})} \models \hat{\kappa} \in \hat{\eta}$. ⊣

Let us fix a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. An order relation $<$ between any two ordinal-like elements u and v in $\mathbf{V}^{(\mathbb{A})}$ is defined as follows:[4]

$$u < v \text{ if and only if } \mathbf{V}^{(\mathbb{A})} \models u \in v.$$

Definition 34. For two elements $u, v \in \mathbf{V}^{(\mathbb{A})}$, it will be said that the cardinality of u is less than the cardinality of v, denoted by $|u|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$, if for any $p \in \mathsf{Cardinal}_u$ and $q \in \mathsf{Cardinal}_v$, $p < q$ holds in $\mathbf{V}^{(\mathbb{A})}$.

Using Theorem 33, it can be proved that Definition 34 is well defined. It can also be proved by Theorem 33 that, for two elements $u, v \in \mathbf{V}^{(\mathbb{A})}$, where $|\text{Part}(\text{dom}_D(u))|_{\mathbf{V}} = \kappa$ and $|\text{Part}(\text{dom}_D(v))|_{\mathbf{V}} = \eta$, $|u|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$ if and only if $\hat{\kappa} < \hat{\eta}$ in $\mathbf{V}^{(\mathbb{A})}$. We shall use the abbreviation $|u|_{\mathbf{V}^{(\mathbb{A})}} \leq |v|_{\mathbf{V}^{(\mathbb{A})}}$ if either $|u|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$ or $\mathsf{Cardinal}_u = \mathsf{Cardinal}_v$ holds in $\mathbf{V}^{(\mathbb{A})}$.

5.4. Cantor's theorem and the Schröder–Bernstein theorem. We shall prove that two basic demands on cardinality, Cantor's theorem and the Schröder–Bernstein theorem, hold in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$.

Theorem 35 (Schröder–Bernstein theorem). *For any $\mathcal{PS}$-algebra $\mathbb{A}$, if $u, v \in \mathbf{V}^{(\mathbb{A})}$ are such that*

$$\mathbf{V}^{(\mathbb{A})} \models \exists g \text{ InjFunc}(g; u, v) \wedge \exists h \text{ InjFunc}(h; v, u)$$

holds then $\mathbf{V}^{(\mathbb{A})} \models \exists f \text{ BijFunc}(f; u, v)$ also holds.

Proof. As an application of Theorem 23, there exist injections from $\text{Part}(\text{dom}_D(u))$ into $\text{Part}(\text{dom}_D(v))$ and from $\text{Part}(\text{dom}_D(v))$ into $\text{Part}(\text{dom}_D(u))$ in $\mathbf{V}$. Then, by the Schröder–Bernstein theorem in $\mathbf{V}$, there exists a bijection between $\text{Part}(\text{dom}_D(u))$ and $\text{Part}(\text{dom}_D(v))$. Hence, one more application of Theorem 23 completes the proof. ⊣

To prove Cantor's theorem in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ we have to first recognize the names corresponding to the power set of a given set in $\mathbf{V}^{(\mathbb{A})}$. Since the Power Set Axiom is valid in $\mathbf{V}^{(\mathbb{A})}$, for any $x \in \mathbf{V}^{(\mathbb{A})}$ there exists $y \in \mathbf{V}^{(\mathbb{A})}$ such that

$$[\![\forall t(t \subseteq x \leftrightarrow t \in y)]\!] \in D.$$

Let $\text{Pow}(x, y) := \forall t(t \subseteq x \leftrightarrow t \in y)$. Then, $\text{Pow}(x, y)$ is the first-order formula, having two free variables x and y, which naively interprets that 'y is the power set of x'.

Definition 36. Let $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, be any arbitrary element. An element $v \in \mathbf{V}^{(\mathbb{A})}$ is called a name of the power set of u or simply a power set of u in $\mathbf{V}^{(\mathbb{A})}$ if $\mathbf{V}^{(\mathbb{A})} \models \text{Pow}(u, v)$.

[4]To keep the expression simple, we shall use the same symbol $<$ for the order relation between two ordinal numbers in both $\mathbf{V}$ and $\mathbf{V}^{(\mathbb{A})}$, unless the domain of definition of $<$ is not clear from the context.

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It is not hard to check that for any $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, if v and w are two names of the power sets of u in $\mathbf{V}^{(\mathbb{A})}$, then v and w are identical in $\mathbf{V}^{(\mathbb{A})}$. So we have the following theorem.

THEOREM 37. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and $u \in \mathbf{V}^{(\mathbb{A})}$, if two elements $v, w \in \mathbf{V}^{(\mathbb{A})}$ are such that $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Pow}(u, v)$ and $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Pow}(u, w)$ both hold, then $\mathbf{V}^{(\mathbb{A})} \models v = w$.*

To distinguish activities in the ground model $\mathbf{V}$ and the model $\mathbf{V}^{(\mathbb{A})}$, 'subset in $\mathbf{V}$' will be expressed by the symbols $\subseteq_{\mathbf{V}}$, where it is needed.

LEMMA 38. *Let $v \in \mathbf{V}^{(\mathbb{A})}$ be a power set of an element $u \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra. If $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$ then*

(i) $\mathrm{dom}_D(v)$ *consists of elements x_A corresponding to each $A \subseteq_{\mathbf{V}} \kappa$, and*
(ii) *if $x_A, x_B \in \mathrm{dom}_D(v)$ where $A, B \subseteq_{\mathbf{V}} \kappa$ and $A \neq B$ in $\mathbf{V}$, then $[\![x_A = x_B]\!] = \mathbf{0}$.*

PROOF. It is given that $[\![\forall x(x \subseteq u \leftrightarrow x \in v)]\!] \in D$. So, both $[\![\forall x(x \subseteq u \rightarrow x \in v)]\!]$ and $[\![\forall x(x \in v \rightarrow x \subseteq u]\!]$ are in D, i.e.,

$$[\![\forall x(\forall t(t \in x \rightarrow t \in u) \rightarrow x \in v)]\!] \in D,$$

$$[\![\forall x(x \in v \rightarrow \forall t(t \in x \rightarrow t \in u))]\!] \in D.$$

From the first condition, we get that if for any $x \in \mathbf{V}^{(\mathbb{A})}$, $[\![\forall t(t \in x \rightarrow t \in u)]\!] \in D$, i.e., for any $t \in \mathrm{dom}_D(x)$ there exists some $t' \in \mathrm{dom}_D(u)$ such that $[\![t = t']\!] \in D$, then $[\![x \in v]\!] \in D$ as well, i.e., there exists $x' \in \mathrm{dom}_D(v)$ such that $[\![x = x']\!] \in D$. From the second condition, it can be said that

$$\bigwedge_{x \in \mathrm{dom}(v)} (v(x) \Rightarrow [\![x \subseteq u]\!]) \in D,$$

i.e., for any $x \in \mathrm{dom}_D(v)$ and $t \in \mathrm{dom}_D(x)$, there exists some $t' \in \mathrm{dom}_D(u)$ such that $[\![t = t']\!] \in D$.

By the given condition, $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$. Hence, there exists a bijection f between $\mathrm{Part}(\mathrm{dom}_D(u))$ and κ in $\mathbf{V}$. Let $A \subseteq_{\mathbf{V}} \kappa$ be arbitrarily chosen. Then consider $\bigcup f^{-1}(A)$ in $\mathbf{V}$. Clearly, $\bigcup f^{-1}(A) \subseteq_{\mathbf{V}} \mathrm{dom}_D(u)$. Construct an element $x \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(x) = \bigcup f^{-1}(A)$ and $\mathrm{ran}(x) = \{\mathbf{1}\}$. Therefore, we have $[\![\forall t(t \in x \rightarrow t \in u)]\!] \in D$. Hence, using the first condition, we can say that there exists an element $x_A \in \mathrm{dom}_D(v)$ such that $[\![x = x_A]\!] \in D$.

Now suppose $x \in \mathrm{dom}_D(v)$ is an arbitrary element. We shall prove that there exists $A \subseteq_{\mathbf{V}} \kappa$ such that $[\![x = x_A]\!] \in D$. By the second condition, for any $t \in \mathrm{dom}_D(x)$ there exists an element $t' \in \mathrm{dom}_D(u)$ such that $[\![t = t']\!] \in D$. Now consider the class $[t']$ in $\mathrm{Part}(\mathrm{dom}_D(u))$ and the image of it under the bijective function f. Let $A = \{f([t']) : t' \in \mathrm{dom}_D(u)$ and there exists some $t \in \mathrm{dom}_D(x)$ such that $[\![t = t']\!] \in D\}$. Clearly, $A \subseteq_{\mathbf{V}} \kappa$. Construct an element $x_A \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}_D(x_A) = \bigcup f^{-1}(A)$ and $\mathrm{ran}(A) = \{\mathbf{1}\}$. Hence, by the construction we get $[\![x = x_A]\!] \in D$.

Let us take $A, B \subseteq_{\mathbf{V}}$ such that $A \neq B$. It will be proved that $[\![x_A = x_B]\!] = \mathbf{0}$. Without loss of generality, let there exist $a \in A$ where $a \notin B$. By the construction,

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$\text{dom}(x_A) = \bigcup f^{-1}(A)$ and $\text{dom}(x_B) = \bigcup f^{-1}(B)$, whereas $\text{ran}(x_A) = \{\mathbf{1}\} = \text{ran}(x_B)$. So for any $t \in f^{-1}(a)$, $x_A(t) = \mathbf{1}$, whereas by the assumption, $[\![t \in x_B]\!] = \mathbf{0}$. Hence, we can conclude that $[\![x_A = x_B]\!] = \mathbf{0}$. ⊣

Corollary 39. *If v is a power set of an element u in $\mathbf{V}^{(\mathbb{A})}$, where $|\text{Part}(\text{dom}_D(u))|_{\mathbf{V}} = \kappa$, then $|\text{Part}(\text{dom}_D(v))|_{\mathbf{V}} = 2^\kappa$.*

If $u \in \mathbf{V}^{(\mathbb{A})}$ and two elements $v, w \in \mathbf{V}^{(\mathbb{A})}$ are such that $\mathbf{V}^{(\mathbb{A})} \models \text{Pow}(u, v) \wedge \text{Pow}(u, w)$ then, using Theorem 31 and Corollary 39, we can conclude that $\mathbf{V}^{(\mathbb{A})} \models \exists f \ \text{BijFunc}(f; v, w)$.

Applying Theorem 33 and Corollary 39, we get the desired result: Cantor's theorem in $\mathbf{V}^{(\mathbb{A})}$.

Theorem 40 (Cantor's theorem). *Let $\mathbb{A}$ be any $\mathcal{PS}$-algebra and $u \in \mathbf{V}^{(\mathbb{A})}$ be an arbitrary element. If $v \in \mathbf{V}^{(\mathbb{A})}$ is such that $\mathbf{V}^{(\mathbb{A})} \models \text{Pow}(u, v)$ holds, then $|u|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$ also holds.*

5.5. Generalized continuum hypothesis in $\mathcal{PS}$-algebra-valued models. We shall prove in this section that the necessary and sufficient condition of the validity of the generalized continuum hypothesis (GCH) in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ is its validity in the ground model $\mathbf{V}$. Hence, GCH becomes independent of the set theory corresponding to $\mathbf{V}^{(\mathbb{A})}$. The statement of GCH is as follows.

For any infinite set s there does not exist any set t such that the cardinal number of s is less than the cardinal number of t and the cardinal number of t is less than the cardinal number of the power set of s.

Definition 41. In any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, an element $u \in \mathbf{V}^{(\mathbb{A})}$ is said to be infinite if for any natural-number-like element $\hat{n}$, where $n \in \text{ORD}$ is a natural number, there exists a subset v of u in $\mathbf{V}^{(\mathbb{A})}$ such that $\hat{n}$ is a cardinal number of v, i.e., $\hat{n} \in \mathsf{Cardinal}_v$.

Theorem 42. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and a model $\mathbf{V}$ of ZFC, $\mathbf{V} \models \text{GCH}$ if and only if $\mathbf{V}^{(\mathbb{A})} \models \text{GCH}$.*

Proof. Let $\mathbf{V} \models \text{GCH}$; we shall prove that $\mathbf{V}^{(\mathbb{A})} \models \text{GCH}$. Suppose $u \in \mathbf{V}^{(\mathbb{A})}$ is an infinite element and $\hat{\kappa} \in \mathsf{Cardinal}_u$. By this assumption we get that for any natural number in ORD there exists a subset of κ which has the cardinality of that natural number in $\mathbf{V}$. Hence, κ is an infinite cardinal in $\mathbf{V}$. Let v be a power set of u in $\mathbf{V}^{(\mathbb{A})}$. Then, by using Corollary 39, we get that $\widehat{(2^\kappa)} \in \mathsf{Cardinal}_v$. If possible, let there exist $w \in \mathbf{V}^{(\mathbb{A})}$ such that $|u|_{\mathbf{V}^{(\mathbb{A})}} < |w|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$. Let $\hat{\eta} \in \mathsf{Cardinal}_w$. Hence, by Theorem 23, we get $\kappa < \eta < 2^\kappa$ in $\mathbf{V}$, which contradicts our assumption that $\mathbf{V} \models \text{GCH}$.

Conversely, suppose $\mathbf{V} \not\models \text{GCH}$. Then in $\mathbf{V}$, there are infinite cardinals κ and η such that $\kappa < \eta < 2^\kappa$. Now consider the element $\hat{\kappa} \in \mathbf{V}^{(\mathbb{A})}$. Clearly, by definition, $\hat{\kappa}$ is an infinite cardinal number in $\mathbf{V}^{(\mathbb{A})}$ as well. Let v be a power set of $\hat{\kappa}$ in $\mathbf{V}^{(\mathbb{A})}$. Again by using Corollary 39, it can be concluded that $\widehat{(2^\kappa)} \in \mathsf{Cardinal}_v$. Hence, by our assumption, $|\hat{\kappa}|_{\mathbf{V}^{(\mathbb{A})}} < |\hat{\eta}|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$, i.e., $\mathbf{V}^{(\mathbb{A})} \not\models \text{GCH}$. ⊣

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Let us consider any $\mathcal{PS}$-algebra $\mathbb{A}$ and two models $\mathbf{V}_1$ and $\mathbf{V}_2$ of ZFC such that $\mathbf{V}_1 \models$ GCH but $\mathbf{V}_2 \not\models$ GCH. Then, as an application of Theorem 42, $\mathbf{V}_1^{(\mathbb{A})} \models$ GCH whereas $\mathbf{V}_2^{(\mathbb{A})} \not\models$ GCH. Let T be the theory consisting of the set-theoretic sentences, in the language of ZFC, which are valid in both of $\mathbf{V}_1^{(\mathbb{A})}$ and $\mathbf{V}_2^{(\mathbb{A})}$. By Observation 7, NFF-ZF $\subseteq \mathsf{T}$, as both of $\mathbf{V}_1^{(\mathbb{A})}$ and $\mathbf{V}_2^{(\mathbb{A})}$ validate NFF-ZF. Since the theory T has two models, viz. $\mathbf{V}_1^{(\mathbb{A})}$ and $\mathbf{V}_2^{(\mathbb{A})}$, which disagree with respect to the validity of GCH, GCH is independent from the theory T. Depending on the choice of the $\mathcal{PS}$-algebra $\mathbb{A}$ and the designated set of it, the theory T could be a non-classical set theory. This fact leads us to a study of independence proofs in non-classical set theories [13].

§6. Cardinal arithmetic in $\mathcal{PS}$-algebra-valued models. In this section we define $\kappa + \eta$, $\kappa \,.\, \eta$, and κ^η in $\mathbf{V}^{(\mathbb{A})}$ for any two cardinal numbers $\kappa, \eta \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is any $\mathcal{PS}$-algebra and $\mathbf{V}$ is a model of ZFC. Like the other constructions, these operations are defined in $\mathbf{V}^{(\mathbb{A})}$ to be similar to their notions in classical set theory.

In classical set theory, for any two cardinal numbers κ and η, cardinal addition $\kappa + \eta$ is defined as the cardinal number of the set $(\kappa \times \{0\}) \cup (\eta \times \{1\})$, and cardinal multiplication $\kappa \,.\, \eta$ is defined to be the cardinal number of the set $\kappa \times \eta$. For any two sets x and y in $\mathbf{V}$, the set x^y is defined as the collection of all functions from y into x. This is then used to define cardinal exponentiation in $\mathbf{V}$ as $|x|^{|y|} = |x^y|$.

To define these operations in $\mathbf{V}^{(\mathbb{A})}$, we recall Definition 21 of pair-like elements. Then we get the following proposition.

Proposition 43. *Let $\mathbb{A}$ be any $\mathcal{PS}$-algebra and $u, v \in \mathbf{V}^{(\mathbb{A})}$ be two elements such that u is (a, b)-like and v is (c, d)-like, for some $a, b, c, d \in \mathbf{V}^{(\mathbb{A})}$. Then $[\![u = v]\!] \in D$ if and only if $[\![a = c]\!] \in D$ and $[\![b = d]\!] \in D$.*

Let us now define the *cross product* in $\mathbf{V}^{(\mathbb{A})}$ as follows.

Definition 44. Let $u, v \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, be any two arbitrary elements. An element $w \in \mathbf{V}^{(\mathbb{A})}$ is said to be $(u \times v)$-like if for each $a \in \mathrm{dom}_D(u)$ and $b \in \mathrm{dom}_D(v)$ there exists an (a, b)-like element in $\mathrm{dom}(w)$ and nothing else; whereas $\mathrm{ran}(w) = \{\mathbf{1}\}$.

For any two $(u \times v)$-like elements p and q, $|\mathrm{Part}(\mathrm{dom}_D(p))|_{\mathbf{V}} = |\mathrm{Part}(\mathrm{dom}_D(q))|_{\mathbf{V}}$, using Proposition 43. Hence, by applying Theorems 23 and 31 we get $\mathsf{Cardinal}_p = \mathsf{Cardinal}_q$. We are now able to define cardinal addition, cardinal multiplication, and cardinal exponentiation in $\mathbf{V}^{(\mathbb{A})}$.

6.1. Cardinal addition. We define addition of two cardinal numbers in a $\mathcal{PS}$-algebra-valued model as it is defined in classical set theory.

Definition 45. Let u and v be two cardinal numbers in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. Suppose $p, q \in \mathbf{V}^{(\mathbb{A})}$ are two elements such that $\mathrm{dom}(p) = \{\hat{1}\}$, $\mathrm{dom}(q) = \{\hat{0}\}$, and $\mathrm{ran}(p) = \{\mathbf{1}\} = \mathrm{ran}(q)$. Let s be a $(u \times p)$-like element and t be a $(v \times q)$-like element. If $|\mathrm{Part}(\mathrm{dom}_D(s \cup t))|_{\mathbf{V}} = \xi$, then the addition of u and v, denoted by $u + v$, is the cardinal number $\hat{\xi}$.

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Theorem 46 shows that the definition of cardinal addition in a $\mathcal{PS}$-algebra-valued model is well defined.

THEOREM 46. *Let u and v be two cardinal numbers in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, where u is κ-like and v is η-like, where $\kappa \,.\, \eta = \zeta$ and $\kappa + \eta = \xi$ (say) in $\mathbf{V}$. Then the following hold.*

(i) *For any $(u \times v)$-like element $w \in \mathbf{V}^{(\mathbb{A})}$, $|\mathrm{Part}(\mathrm{dom}_D(w))|_{\mathbf{V}} = \zeta$.*
(ii) *Suppose $p, q \in \mathbf{V}^{(\mathbb{A})}$ are such that $\mathrm{dom}(p) = \{\hat{1}\}$, $\mathrm{dom}(q) = \{\hat{0}\}$, and $\mathrm{ran}(p) = \{\mathbf{1}\} = \mathrm{ran}(q)$. Then for any $(u \times p)$-like element s and $(v \times q)$-like element t, $|\mathrm{Part}(\mathrm{dom}_D(s \cup t))|_{\mathbf{V}} = \xi$.*

PROOF. The proof is a consequence of Proposition 43 by using Theorem 9(i). ⊣

COROLLARY 47. *For any two cardinal numbers u and v in $\mathbf{V}^{(\mathbb{A})}$, where u is κ-like and v is η-like, if $\kappa + \eta = \xi$ in $\mathbf{V}$ then $u + v$ is $\hat{\xi}$ in $\mathbf{V}^{(\mathbb{A})}$.*

Once we have Corollary 47, we can prove the following theorem, which gives us some basic properties of cardinal addition.

THEOREM 48. *For any $\mathcal{PS}$-algebra $\mathbb{A}$ and any cardinal numbers $u, v, w \in \mathbf{V}^{(\mathbb{A})}$, the following are valid in $\mathbf{V}^{(\mathbb{A})}$.*

(i) *$u + \mathsf{zero} = \hat{\kappa}$, where $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$ and zero is any 0-like element.*
(ii) *$(u + v) + w = u + (v + w)$.*
(iii) *$u + v = v + w$.*
(iv) *If $u \leq v$ then $u + w \leq v + w$.*

By the definition, we know that for any two cardinal numbers $u, v \in \mathbf{V}^{(\mathbb{A})}$, one of $u \leq v$ and $v < u$ is valid. We say $\max\{u, v\} = v$ if and only if $u \leq v$ in $\mathbf{V}^{(\mathbb{A})}$.

THEOREM 49. *Let u and v be two cardinal numbers in a $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, where at least one of them is infinite. Then $u + v = \max\{u, v\}$ is valid in $\mathbf{V}^{(\mathbb{A})}$.*

PROOF. Using Observation 30, we know that all cardinal numbers are cardinal-like elements. Hence, the proof follows from the fact that the theorem holds in the ground model $\mathbf{V}$ of ZFC. ⊣

6.2. Cardinal multiplication. As for cardinal addition, we shall define cardinal multiplication in a $\mathcal{PS}$-algebra-valued model as it is defined in classical set theory.

DEFINITION 50. Let u and v be two cardinal numbers in $\mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra. Suppose $w \in \mathbf{V}^{(\mathbb{A})}$ is a $(u \times v)$-like element. If $|\mathrm{Part}(\mathrm{dom}_D(w))|_{\mathbf{V}} = \kappa$ then the cardinal multiplication of u and v in $\mathbf{V}^{(\mathbb{A})}$, denoted by $u \,.\, v$, is defined to be the cardinal number $\hat{\kappa}$.

The definition of cardinal multiplication in $\mathcal{PS}$-algebra-valued models is well defined by Theorem 46(i). Moreover, if u and v are, respectively, κ-like and η-like elements, where $\kappa \,.\, \eta = \xi$ (say) in $\mathbf{V}$, then $u \,.\, v$ is $\hat{\xi}$ in $\mathbf{V}^{(\mathbb{A})}$. Hence, we have the following theorem.

THEOREM 51. *For any cardinal numbers $u, v, w \in \mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ is a $\mathcal{PS}$-algebra, the following are valid in $\mathbf{V}^{(\mathbb{A})}$.*

(i) $u \,.\, \mathsf{zero} = \hat{0}$, *where* zero *is any* 0-*like element*.
(ii) $u \,.\, \mathsf{one} = \hat{\kappa}$, *where* $|\mathrm{Part}(\mathrm{dom}_D(u))|_{\mathbf{V}} = \kappa$ *and* one *is any* 1-*like element*.
(iii) $(u \,.\, v) \,.\, w = u \,.\, (v \,.\, w)$.
(iv) $u \,.\, v = v \,.\, u$.
(v) *If* $u \leq v$ *then* $u \,.\, w \leq v \,.\, w$.
(vi) $u \,.\, (v + w) = u \,.\, v + u \,.\, w$.

Following the proof of Theorem 49, we can also derive the following theorem.

THEOREM 52. *Let* $\mathbb{A}$ *be a* $\mathcal{PS}$-*algebra and* $u, v \in \mathbf{V}^{(\mathbb{A})}$ *be two cardinal numbers such that at least one of them is infinite and the other one is not* 0-*like. Then* $u \,.\, v = \max\{u, v\}$ *is valid in* $\mathbf{V}^{(\mathbb{A})}$.

6.3. Cardinal exponentiation. We shall first discuss the exponentiation of two elements in $\mathbf{V}^{(\mathbb{A})}$. Cardinal exponentiation will follow from it.

DEFINITION 53. Let $\mathbb{A}$ be a $\mathcal{PS}$ algebra and $u, v \in \mathbf{V}^{(\mathbb{A})}$ be two arbitrary elements. Then an element $w \in \mathbf{V}^{(\mathbb{A})}$ is called a v^u-like element if

$$\mathrm{dom}(w) = \{f \in \mathbf{V}^{(\mathbb{A})} \colon \mathbf{V}^{(\mathbb{A})} \models \mathrm{Func}(f) \wedge \mathrm{Dom}(f; u) \wedge \mathrm{Codom}(f; v)\}$$

and $\mathrm{ran}(w) = \{\mathbf{1}\}$.

DEFINITION 54. Let $\mathbb{A}$ be a $\mathcal{PS}$-algebra. Let $u, v \in \mathbf{V}^{(\mathbb{A})}$ be two cardinal numbers, and let $w \in \mathbf{V}^{(\mathbb{A})}$ be a v^u-like element. If $|\mathrm{Part}(\mathrm{dom}_D(w))|_{\mathbf{V}} = \xi$ then the cardinal number v to the power u, denoted by v^u, is defined to be the cardinal number $\hat{\xi}$ in $\mathbf{V}^{(\mathbb{A})}$.

This definition is proved to be well defined by Theorem 56.

LEMMA 55. *Let* $\mathbb{A}$ *be a* $\mathcal{PS}$-*algebra. Let* $u, v \in \mathbf{V}^{(\mathbb{A})}$ *be any two arbitrary elements, and let* $w \in \mathbf{V}^{(\mathbb{A})}$ *be a* v^u-*like element. Then there exists a bijection between* $\mathrm{Part}(\mathrm{dom}_D(w))$ *and* $\mathrm{Part}(\mathrm{dom}_D(v))^{\mathrm{Part}(\mathrm{dom}_D(u))}$ *in* $\mathbf{V}$.

PROOF. For any $f \in \mathrm{dom}_D(w)$ we get $\mathbf{V}^{(\mathbb{A})} \models \mathrm{Func}(f) \wedge \mathrm{Dom}(f; u) \wedge \mathrm{Codom}(f; v)$, by definition. Hence, all of (i)–(v) in the proof of Theorem 23 hold for every $f \in \mathrm{dom}_D(w)$. Applying Proposition 43, the lemma is proved. ⊣

THEOREM 56. *Let* u *and* v *be two cardinal numbers in a* $\mathcal{PS}$-*algebra-valued model* $\mathbf{V}^{(\mathbb{A})}$, *where* u *is* η-*like and* v *is* κ-*like. If* $\kappa^\eta = \xi$ *in* $\mathbf{V}$, *then for any* v^u-*like element* $w \in \mathbf{V}^{(\mathbb{A})}$, $|\mathrm{Part}(\mathrm{dom}_D(w)|_{\mathbf{V}} = \xi$.

PROOF. There exists a bijection between $\mathrm{Part}(\mathrm{dom}_D(w))$ and $\mathrm{Part}(\mathrm{dom}_D(v))^{\mathrm{Part}(\mathrm{dom}_D(u))}$ in $\mathbf{V}$, by Lemma 55. Hence, $|\mathrm{Part}(\mathrm{dom}_D(w))|_{\mathbf{V}} = \kappa^\eta = \xi$. ⊣

Using Theorem 56, we get the following theorem.

THEOREM 57. *If* u, v, w *are three cardinal numbers in* $\mathbf{V}^{(\mathbb{A})}$, *where* $\mathbb{A}$ *is a* $\mathcal{PS}$-*algebra, then the following are valid in* $\mathbf{V}^{(\mathbb{A})}$.

(i) $u^{\mathsf{zero}} = \hat{1}$; $\mathsf{one}^u = \hat{1}$; $\mathsf{zero}^u = \hat{0}$ *if* $\hat{0} < |u|_{\mathbf{V}^{(\mathbb{A})}}$, *where* zero *and* one *are any* 0-*like element and* 1-*like element, respectively*.

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(ii) $(u \,.\, v)^w = u^w \,.\, v^w$.
(iii) $u^{v+w} = u^v \,.\, u^w$.
(iv) $(u^v)^w = u^{v \,\cdot\, w}$.
(v) *If* $u \leq v$ *then* $u^w \leq v^w$.
(vi) *If* $\hat{0} < u \leq v$ *then* $w^u \leq w^v$.

§7. Conclusion.

7.1. Comparison between $\mathcal{PS}$-algebra-valued set theories and IZF. It is discussed that there are $\mathcal{PS}$-algebras $\mathbb{A}$ and designated sets D such that the logic of $(\mathbb{A}, D)$ is classical and the corresponding algebra-valued models are the models of classical set theory. The $\mathcal{PS}$-algebra-valued models of non-classical set theories are quite different from the well-known non-classical set theory IZF when it comes to the foundation of mathematics. We can enlist the following evidences in support of this claim.

(i) An ordinal in IZF is defined to be a transitive set of transitive sets, where a set x is said to be transitive if for any element $y \in x$, $y \subseteq x$ holds. Notice that, unlike classical set theory, the definition of ordinal number in IZF does not demand the linearity of the set with respect to $\in$. An ordinal α is said to be linear if for any pair of ordinals $\beta, \gamma \in \alpha$, $(\beta \in \gamma) \vee (\beta = \gamma) \vee (\gamma \in \beta)$ is satisfied. It can be proved that,

$$\mathsf{IZF} + \text{ `Every ordinal is linear'} \vdash \text{ZF},$$

which can be derived from the fact that $\mathsf{IZF} + $ 'Every ordinal is linear' $\vdash \varphi \vee \neg\varphi$, for any $\varphi \in \mathsf{Prop}$ (cf. [2, p. 164]). On the contrary, Theorem 13 shows that for any $\mathcal{PS}$-algebra $\mathbb{A}$ and any ordinal-like element $u \in \mathbf{V}^{(\mathbb{A})}$, $\mathbf{V}^{(\mathbb{A})} \models \mathsf{Ord}(u)$ and hence, $\mathbf{V}^{(\mathbb{A})} \models \mathrm{LO}(u)$. Intuitively, this leads to the conclusion that in any $\mathcal{PS}$-algebra-valued model, every ordinal is linear, which is not the case in IZF. In connection with that, we also have the following point.

(ii) Consider the first-order formula:

$$\varphi := \forall x \forall y((\mathsf{Nat}(1, x) \wedge y \subseteq x) \leftrightarrow (\mathsf{Nat}(0, y) \vee \mathsf{Nat}(1, y))).$$

The naive interpretation of φ is that 'the subsets of 1 are either 0 or 1'. One can prove that φ is not a theorem of IZF. To check this, let us consider the three-valued Heyting algebra $\mathbb{H} = \langle \{\mathbf{1}, 1/2, \mathbf{0}\}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$. The designated set is considered to be $D = \{\mathbf{1}\}$. We can then prove that $[\![\varphi]\!]_{\mathbb{H}} \notin D$ and hence conclude that $\mathbf{V}^{(\mathbb{H})} \not\models \varphi$. On the other hand, it can be proved that for any $\mathcal{PS}$-algebra $\mathbb{A}$, $\mathbf{V}^{(\mathbb{A})} \models \varphi$. Furthermore, we have proved in Section 5.4 that in any $\mathcal{PS}$-algebra-valued models, the cardinal numbers of all the names representing the power set of a κ-like element, where $\kappa \in \mathrm{ORD}$, are exactly the 2^κ-like elements.

(iii) It can be proved that for any Heyting algebra $\mathbb{H}$, $\mathbf{V}^{(\mathbb{H})} \models \mathrm{AC}$ if and only if $\mathbb{H}$ is a Boolean algebra [2, p. 166]. On the other hand, Theorem 22 shows that any $\mathcal{PS}$-algebra-valued model validates AC if the ground model validate AC.

7.2. Comparison between $\mathcal{PS}$-algebra-valued set theories and an established paraconsistent set theory. In 2012, Weber studied cardinal numbers in a paraconsistent set theory [14]. In his construction, the general comprehension axiom is taken as one of the axioms, viz. Abstraction. As a result, this theory admits Russell's set, the collection of all sets, the collection of all ordinals, etc. into the universe of sets, unlike classical set theory. In [11], we find a study of ordinal numbers in an algebra-valued model of a paraconsistent set theory, viz. the set theory of $\mathbf{V}^{(\mathrm{PS}_3)}$. It was proved that the model $\mathbf{V}^{(\mathrm{PS}_3)}$ of a certain paraconsistent set theory, which was developed in [9, 11], agrees with classical set theory on the invalidity of the general comprehension axiom. As a result, Russell's set, the collection of all sets, and the collection of all ordinals are not sets in this model. Since PS_3 is a particular $\mathcal{PS}$-algebra, we can easily extend these results to all $\mathcal{PS}$-algebras following the same proofs done in [9, 11].

On the other hand, the presence of the general comprehension axiom ensures that the Separation Axiom Schema is valid in Weber's set theory, whereas it can be proved that there are instances of the later which are not valid in some $\mathcal{PS}$-algebra-valued models. Since all the $\mathcal{PS}$-algebras are deductive reasonable implication algebras, as an application of Theorem 1, it is well understood that those instances of the Separation Axiom Schema are not negation-free formulas.

Definition 58. A $\mathcal{PS}$-algebra $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ is said to be a paraconsistent $\mathcal{PS}$-algebra if there exists $a \in \mathbf{A} \setminus \{\mathbf{0}\}$ such that $a^* \neq \mathbf{0}$ as well.

Observe that, independent of the designated set D, the logic of any paraconsistent $\mathcal{PS}$-algebra $\mathbb{A}$ will be a paraconsistent logic. Consider a valuation function v and two propositional formulas φ and ψ such that $v(\varphi) = a$ and $v(\psi) = \mathbf{0}$. Then $v((\varphi \wedge \neg\varphi) \to \psi) = \mathbf{0}$, which implies that $(\varphi \wedge \neg\varphi) \to \psi \notin \mathbf{L}(\mathbb{A}, D)$.

Theorem 59. *Let $\mathbb{A}$ be a paraconsistent $\mathcal{PS}$-algebra. Then there is a formula $\varphi(x)$ having one free variable x, in the language of* ZFC, *for which the corresponding instance of the* Separation Axiom Schema *fails in $\mathbf{V}^{(\mathbb{A})}$, irrespective of the choice of any designated set.*

Proof. Let D be any designated set in $\mathbb{A}$. Since $\mathbb{A}$ is a paraconsistent $\mathcal{PS}$-algebra, there exists an element $a \in \mathbf{A}$ such that $a, a^* \neq \mathbf{0}$. Consider the two elements of $\mathbf{V}^{(\mathbb{A})}$: $u = \{\langle \varnothing, \mathbf{1}\rangle\}$ and $v = \{\langle \varnothing, a\rangle\}$. Then,

$$[\![u = v]\!] = (\mathbf{1} \Rightarrow a) \wedge (a \Rightarrow \mathbf{1}) = \mathbf{1} \in D.$$

Let us now consider the formula $\varphi(x) := \neg\exists y(y \in x)$. Then,

$$\begin{aligned}
[\![\varphi(v)]\!] &= [\![\neg\exists y(y \in v)]\!] \\
&= \Bigg(\bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} \big(v(\varnothing) \ \wedge \ [\![\varnothing = y]\!]_{\mathrm{PS}_3}\big)\Bigg)^* \\
&= \big(a \ \wedge \ \mathbf{1}\big)^* \\
&= a^* \\
&\neq \mathbf{0}.
\end{aligned}$$

Similarly, since $u(\varnothing) = \mathbf{1} = [\![\varnothing = \varnothing]\!]$ we calculate $[\![\varphi(u)]\!] = \mathbf{1}^* = \mathbf{0} \notin D$. We go on to show that the instance of the Separation Axiom Schema corresponding to the formula φ fails. Fix the element w of $\mathbf{V}^{(\mathbb{A})}$, where $w = \{\langle u, \mathbf{1}\rangle, \langle v, \mathbf{1}\rangle\}$. We show that

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'there does not exist any subset of w in $\mathbf{V}^{(\mathbb{A})}$ consisting of those elements which satisfy the formula $\varphi(x)$'. Formally, we shall show that,

$$\bigvee_{y\in\mathbf{V}^{(\mathbb{A})}}\Bigg(\bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}([\![x\in y]\!]\Rightarrow([\![x\in w]\!]\wedge[\![\varphi(x)]\!]))\wedge \bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}(([\![x\in w]\!]\wedge[\![\varphi(x)]\!])\Rightarrow[\![x\in y]\!])\Bigg) = \mathbf{0}.$$

Suppose that, for an arbitrary $y_0\in\mathbf{V}^{(\mathbb{A})}$,

$$\bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}\big(([\![x\in w]\!]\wedge\ [\![\varphi(x)]\!])\Rightarrow[\![x\in y_0]\!]\big)\neq\mathbf{0}.$$

Then, in particular, $\big(([\![v\in w]\!]\wedge[\![\varphi(v)]\!])\Rightarrow[\![v\in y_0]\!]\big)\neq\mathbf{0}$. Since $[\![v\in w]\!]=\mathbf{1}$ and $[\![\varphi(v)]\!]\neq\mathbf{0}$, by our assumption $[\![v\in y_0]\!]\neq\mathbf{0}$. Hence, there exists a $z_0\in\mathrm{dom}(y_0)$ such that $y_0(z_0)\wedge[\![v=z_0]\!]\neq\mathbf{0}$. So we get $[\![u=v]\!]\wedge[\![v=z_0]\!]\neq\mathbf{0}$ and thus $[\![u=z_0]\!]\neq\mathbf{0}$. This implies $[\![u\in y_0]\!]\neq\mathbf{0}$. But then, since $[\![\varphi(u)]\!]=\mathbf{0}$ we have:

$$[\![u\in y_0]\!]\Rightarrow([\![u\in w]\!]\wedge[\![\varphi(u)]\!])=\mathbf{0}.$$

Thus,

$$\bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}([\![x\in y_0]\!]\ \Rightarrow([\![x\in w]\!]\wedge[\![\varphi(x)]\!]))\wedge \bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}(([\![x\in w]\!]\wedge[\![\varphi(x)]\!])\Rightarrow[\![x\in y_0]\!])=\mathbf{0}.$$

Since y_0 is arbitrarily chosen from $\mathbf{V}^{(\mathbb{A})}$, we finally have

$$\bigvee_{y\in\mathbf{V}^{(\mathbb{A})}}\big(\bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}([\![x\in y]\!]\Rightarrow([\![x\in w]\!]\wedge[\![\varphi(x)]\!]))\wedge \bigwedge_{x\in\mathbf{V}^{(\mathbb{A})}}(([\![x\in w]\!]\wedge[\![\varphi(x)]\!])\Rightarrow[\![x\in y]\!])\big) = \mathbf{0}.$$

This completes the proof. ⊣

7.3. Comparison between $\mathcal{PS}$-algebra-valued set theories and the classical set theory. The motivation for building $\mathcal{PS}$-algebra-valued models $\mathbf{V}^{(\mathbb{A})}$ is not only to provide models of a class of different non-classical set theories but also to investigate the existence of non-classical mathematical realms which do not differ from classical mathematics with respect to the basic mathematical demands. This makes $\mathbf{V}^{(\mathbb{A})}$ quite different from the other non-classical set theories that already exist in the literature. However, the formation of ordinals in the Boolean-valued models are quite different from the ordinal-like elements. To explain this formally, we shall explore the ordinal numbers in a given Boolean-valued model. Let $\mathbb{B}$ be a complete Boolean algebra

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having the underlying set $\mathbf{B}$. The *mixture* of a set $\{u_i \in \mathbf{V}^{(\mathbb{B})} : i \in \Lambda\}$ with respect to a set $\{b_i \in \mathbf{B} : i \in \Lambda\}$ is defined by an element $u \in \mathbf{V}^{(\mathbb{B})}$ where $\mathrm{dom}(u) = \bigcup_{i\in\Lambda} \mathrm{dom}(u_i)$ and for $x \in \mathrm{dom}(u)$,

$$u(x) = \bigvee_{i\in\Lambda} (b_i \wedge [\![x \in u_i]\!]).$$

A subset $A \subseteq \mathbf{B}$ is called an *antichain* in $\mathbf{B}$ if for any two elements $a, b \in A$, $a \wedge b = \mathbf{0}$. An antichain A in $\mathbf{B}$ is said to be a *partition of unity in* $\mathbf{B}$ if $\bigvee A = \mathbf{1}$. The elements $\hat{\alpha}$ of $\mathbf{V}^{(\mathbb{B})}$, for all ordinals $\alpha \in \mathbf{V}$, are said to be the *standard ordinals* of $\mathbf{V}^{(\mathbb{B})}$. Then, the ordinals in $\mathbf{V}^{(\mathbb{B})}$ are characterised by the mixtures of standard ordinals with respect to the partitions of unity [2, p. 47]. This shows that, depending on the Boolean algebra $\mathbb{B}$, the class of ordinals in $\mathbf{V}^{(\mathbb{B})}$ does not only contain the standard ordinals. Moreover, there exist ordinals $u \in \mathbf{V}^{(\mathbb{B})}$ such that $\{\mathbf{1}\} \subsetneq \mathrm{ran}(u)$. But, for any ordinal-like element u in $\mathbf{V}^{(\mathbb{B})}$, $\mathrm{ran}(u) = \{\mathbf{1}\}$, since the designated set is always considered to be $\{\mathbf{1}\}$ in a Boolean-valued model of classical set theory. Hence, there are complete Boolean algebras $\mathbb{B}$ so that the class of ordinals and the class of ordinal-like elements in the corresponding Boolean-valued model $\mathbf{V}^{(\mathbb{B})}$ are different. This is not the case if we consider the two-valued Boolean-algebra.

All the $\mathcal{PS}$-algebra-valued models are expressive enough to have the properties of the natural numbers, the cardinal numbers, and cardinal arithmetic, which are similar to those in classical set theory, as shown in Sections 4–6. Depending on that, the advancement of different branches of mathematics in $\mathcal{PS}$-algebra-valued models $\mathbf{V}^{(\mathbb{A})}$ will be continued in future.

In addition, Theorems 22 and 42 give an indication that, as in classical set theory, one can also come up with independent set-theoretic statements in non-classical set theories, through the construction of algebra-valued models. This might lead us to the study of forcing in non-classical set theories in the light of [13].

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