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# Long-time behaviour of solutions to a one-dimensional strongly nonlinear model for phase transitions with micro-movements

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This paper studies the long-time behaviour of solutions to a one-dimensional strongly nonlinear partial differential equation system arising from phase transitions with microscopic movements. Our system features a strongly nonlinear internal energy balance equation. Uniform bounds of the global solutions and the compactness of the orbit are obtained for the first time using a lemma established recently by Jiang. The existence of global attractors and convergence of global solutions to a single steady state as time goes to infinity are also proved.

## 1. Introduction and main results

We investigate the long-time behaviour of global solutions to a strongly nonlinear partial differential equation (PDE) system arising from a reversible phase transition model based on Frémond's theory (see [7, 8, 15]). The most notable feature of this model is that it takes into account the microscopic movements of particles by including their effect on the macroscopic behaviour of the body.

Considering a two-phase material located in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\Gamma$, the state of phase change is described by the absolute temperature $\theta$ and the order parameter $\varphi$ standing for the local proportion of one of the two phases. Following Frémond and neglecting the macroscopic velocities (the material behaves like a rigid body at the macroscopic level), the energy balance equation results in (see [6, 31])

$$\partial_t e + \operatorname{div} \boldsymbol{q} = r + B\partial_t\varphi + \boldsymbol{H} \cdot \nabla\partial_t\varphi, \tag{1.1}$$

where $e$ denotes the internal energy, $\boldsymbol{q}$ is the heat flux vector, $r$ corresponds to an external heat source, and $B$ and $\boldsymbol{H}$ are a density of energy function and an energy flux vector, respectively, resulting from the microscopic interior forces to be specified later.

The evolution of the phase variable $\varphi$ is derived from the principle of virtual power (see [15]) which reads

$$-\operatorname{div} \boldsymbol{H} + B = A, \tag{1.2}$$

where $A$ collects the amount of external forces.





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In order to state the constitutive laws, we introduce the expressions of the free energy $\varPsi$ and a pseudo-potential of dissipation $\varPhi$. We choose the free energy which describes the state of the system as follows:

$$\varPsi(\theta, \varphi, \nabla\varphi) = -c_s\theta\ln\theta - \frac{L}{\theta_c}(\theta - \theta_c)\lambda(\varphi) + G(\varphi)L + \tfrac{1}{2}\nu|\nabla\varphi|^2, \tag{1.3}$$

where $L > 0$ stands for the latent heat at the critical transition temperature $\theta_c > 0$, where $c_s > 0$ represents the specific heat, where the parameter $\nu > 0$ is the factor of the interfacial energy term, and where $\lambda(\cdot)$ and $G(\cdot)$ are smooth functions.

The microscopic contribution to the macroscopic behaviour during the thermal evolution process is expressed in terms of the so-called pseudo-potential of dissipation $\varPhi$, which is defined as

$$\varPhi(\partial_t\varphi, \nabla\partial_t\varphi) = \tfrac{1}{2}\mu(\partial_t\varphi)^2 + \tfrac{1}{2}h|\nabla\partial_t\varphi|^2, \tag{1.4}$$

where $\mu > 0$ and $h > 0$ are two coefficients related to the evolution of the interface. Note that, as the phenomenon is reversible, no constraints are required on the sign of $\partial_t\varphi$.

In order to comply with the second principle of thermodynamics, we have the following constitutive relations:

$$B = \frac{\partial\varPsi}{\partial\varphi} + \frac{\partial\varPhi}{\partial(\partial_t\varphi)}, \qquad \boldsymbol{H} = \frac{\partial\varPsi}{\partial(\nabla\varphi)} + \frac{\partial\varPhi}{\partial(\nabla\partial_t\varphi)}, \tag{1.5}$$

while the internal energy $e$ is related to the free energy $\varPsi$ and to the entropy $\eta = -\partial\varPsi/\partial\theta$ by the classical relation

$$e = \varPsi + \eta\theta = \varPsi - \theta\frac{\partial\varPsi}{\partial\theta}. \tag{1.6}$$

Finally, we assume that the heat flux $\boldsymbol{q}$ is determined by the standard Fourier heat flux law

$$\boldsymbol{q} = -k_0\nabla\theta, \quad k_0 > 0. \tag{1.7}$$

Thus, by substituting in (1.1) and (1.2) relations (1.3)–(1.7), the resulting system of PDEs can be written as follows:

$$c_s\theta_t + \frac{L}{\theta_c}\theta\lambda'(\varphi)\varphi_t - k_0\Delta\theta = \mu\varphi_t^2 + h|\nabla\varphi_t|^2 + r, \tag{1.8 a}$$

$$\mu\varphi_t - h\Delta\varphi_t - \nu\Delta\varphi + LG'(\varphi) - \frac{L}{\theta_c}(\theta - \theta_c)\lambda'(\varphi) = A. \tag{1.8 b}$$

We stress that our model is compatible with the second principle of thermodynamics in terms of the Clausius–Duhem inequality [15]

$$\eta_t \geqslant -\nabla\cdot\left(\frac{\boldsymbol{q}}{\theta}\right) + \frac{r}{\theta}, \tag{1.9}$$

since substituting into the Clausius–Duhem inequality leads to the reduced inequality

$$0 \leqslant \frac{\mu\varphi_t^2}{\theta} + \frac{h|\nabla\varphi_t|^2}{\theta} + \frac{k_0|\nabla\theta|^2}{\theta^2}, \tag{1.10}$$

which holds true and proves the thermodynamical consistency of the model.

Without loss of generality, we assume that the external heat source $r = 0$ and all the other physical parameters $c_s$, $L$, $\theta_c$, $k_0$, $\mu$, $h$, $\nu$ and $A$ are normed to 1. In addition, we take the typical forms of $\lambda$ as a linear function and $G$ as a double-well potential (see [6, 9]), i.e.

$$\lambda(\varphi) = \varphi, \qquad G(\varphi) = \tfrac{1}{8}(1 - \varphi^2)^2. \tag{1.11}$$

Since we are focusing on the long-time behaviour of global solutions in the one-dimensional case, in the remainder of this paper we choose $\Omega = (0, 1)$ with $\Gamma = \{0, 1\}$. Thus, the PDE system studied in this paper is

$$\left.\begin{aligned} \theta_t + \theta\varphi_t - \theta_{xx} &= \varphi_t^2 + \varphi_{xt}^2, \\ \varphi_t - \varphi_{xxt} - \varphi_{xx} + \tfrac{1}{2}(\varphi^3 - \varphi) - \theta &= 0. \end{aligned}\right\} \tag{1.12}$$

We supplement system (1.12) with homogeneous Neumann boundary conditions

$$\theta_x|_{x=0,1} = 0, \tag{1.13}$$

$$\varphi_x|_{x=0,1} = \varphi_{xt}|_{x=0,1} = 0, \tag{1.14}$$

and the initial data

$$\theta(0) = \theta_0, \qquad \varphi(0) = \varphi_0. \tag{1.15}$$

We remark that, in view of the relations (1.5) and (1.7), the homogeneous Neumann conditions (1.13) and (1.14) account for no heat flux and no energy flux on the boundary (see [4, 6]). As a result, the total energy of the system is conserved during the evolution process. Indeed, if we multiply the second equation in (1.12) by $\varphi_t$, add the result up to the first equation in (1.12), then integrate the resultant over $\Omega$ by parts, we obtain the following conservation relation:

$$\int_0^1 (\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{8}\varphi^4 - \tfrac{1}{4}\varphi^2 + \theta)\,\mathrm{d}x = \int_0^1 (\tfrac{1}{2}\varphi_{0x}^2 + \tfrac{1}{8}\varphi_0^4 - \tfrac{1}{4}\varphi_0^2 + \theta_0)\,\mathrm{d}x := m. \tag{1.16}$$

In view of (1.16), the corresponding stationary problem to (1.12)–(1.15) is

$$\left.\begin{aligned} u_{xx} &= 0, \\ -\psi_{xx} + \tfrac{1}{2}(\psi^3 - \psi) - u &= 0, \\ u_x|_{x=0,1} = \psi_x|_{x=0,1} &= 0, \\ \int_0^1 (\tfrac{1}{2}\psi_x^2 + \tfrac{1}{8}\psi^4 - \tfrac{1}{4}\psi^2 + u)\,\mathrm{d}x &= m. \end{aligned}\right\} \tag{1.17}$$

We study the infinite-dimensional dynamical system associated with (1.12)–(1.15) and the convergence of any global solution towards a single steady state solution that fulfils (1.17). To formulate our results, we first introduce some notation on the functional settings. Let $L^p(\Omega)$, $W^{k,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}$ be the usual Lebesgue and Sobolev spaces, respectively, and, as usual, $H^k(\Omega) = W^{k,2}(\Omega)$ and

$$H_N^2 = \{f(x) \mid f \in H^2 \text{ and } f_x|_{x=0,1} = 0\}.$$

Let $A$ be the unbounded linear operator defined by $A = I - \Delta$, whose domain is $D(A) = H_N^2$. It is well known (see, for example, [37]) that one can define spaces

$D(A^s)$ for $s \in \mathbb{R}$, with inner product $\langle\cdot,\cdot\rangle_s = (A^{s/2}\cdot, A^{s/2}\cdot)$ and corresponding norm $|\cdot|_s = \sqrt{\langle\cdot,\cdot\rangle_s}$. In particular, $D(A^{1/2}) = H^1, D(A^0) = L^2$. $\|\cdot\|_B$ denotes the norm in the space $B$; we also use the abbreviation $\|\cdot\| := \|\cdot\|_{L^2}$. We denote by $C^k(I,B)$, $k \in \mathbb{N}_0$, the space of $k$ times continuously differentiable functions from $I \subset \mathbb{R}$ into a Banach space $B$, and, likewise, by $L^p(I,B)$, $1 \leqslant p \leqslant \infty$, the corresponding Lebesgue spaces. Throughout this paper, we will frequently use the Sobolev embedding theorem that $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$.

Now we are in a position to state the main result of our paper.

THEOREM 1.1. *Suppose $\theta_0 \in H^1$ and $\varphi_0 \in H^2_N$ are two given functions and suppose that $\theta_0 > 0$ in $[0,1]$. Then the following results hold.*

(i) *Problem (1.12)–(1.15) admits a unique global solution $(\theta,\varphi)$ satisfying*

$$\varphi \in C([0,+\infty), H^2_N), \varphi_t \in C([0,+\infty), H^2_N) \cap L^2((0,+\infty), H^2_N), \tag{1.18}$$

$$\varphi_{tt} \in L^2((0,+\infty), H^2_N), \tag{1.19}$$

$$\theta \in C([0,+\infty), H^1(\Omega)), \theta_t \in L^2((0,+\infty), L^2(\Omega)), \tag{1.20}$$

$$\theta > 0, \quad \forall (x,t) \in [0,1] \times [0,+\infty). \tag{1.21}$$

(ii) *As $t \to +\infty$, it holds that*

$$\|\theta_x\| \to 0, \qquad \|\theta - \bar\theta\|_{L^\infty} \to 0, \tag{1.22}$$

$$\|\varphi_t\|_{H^2} \to 0, \tag{1.23}$$

*where $\bar\theta = \int_0^1 \theta(x,t)\,\mathrm{d}x$.*

(iii) *The orbit is compact in $H^1 \times H^2$.*

(iv) *Let*

$$H := \{(\theta,\varphi) \in H^1(\Omega) \times H^2_N(\Omega) \colon \theta(x,t) > 0,\ x \in [0,1]\} \tag{1.24}$$

*and, for every $\beta_1, \beta_2 > 0$, $\beta_3 < 0$ such that $0 < \beta_1 < \mathrm{e}^{\beta_3 - C_{\beta_2}}$ ($C_{\beta_2}$ is given in lemma 5.3), we define the metric space*

$$H_{\beta_1,\beta_2,\beta_3} := \Big\{(\theta,\varphi) \in H,\ \theta \geqslant \beta_1 > 0,\ \int_0^1 (\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{8}\varphi^4 - \tfrac{1}{4}\varphi^2 + \theta)\,\mathrm{d}x \leqslant \beta_2, \int_0^1 (\ln\theta + \varphi)\,\mathrm{d}x \geqslant \beta_3\Big\}. \tag{1.25}$$

*Then the orbit starting from $H_{\beta_1,\beta_2,\beta_3}$ will reenter itself after a finite time and stay there forever. Moreover, it possesses in $H_{\beta_1,\beta_2,\beta_3}$ a global attractor $A_{\beta_1,\beta_2,\beta_3}$, which is compact.*

(v) *The $\omega$-limit set*

$$\omega(\theta_0,\varphi_0) := \{(\tilde\theta(x), \tilde\varphi(x)) \mid \exists t_n \text{ such that } (\theta(x,\cdot),\varphi(x,\cdot)) \to (\tilde\theta(x), \tilde\varphi(x)) \text{ in } H^1 \times H^2 \text{ as } t_n \to +\infty\} \tag{1.26}$$

*is a singleton. In other words, the global solution $(\theta(t), \varphi(t))$ will converge to a single stationary solution to problem (1.17) in $H^1 \times H^2$ as time goes to infinity.*

REMARK 1.2. Note that our choice of Neumann boundary condition (1.14) has an evident physical justification. From a mathematical point of view, we stress that different choices of boundary conditions for $\varphi$ and $\varphi_t$ can be accounted for in our analysis. Different boundary conditions for $\theta$, such as constant temperature boundary condition, can be dealt with in an analogous way as well (see [32]).

REMARK 1.3. The linear choice of $\lambda(\cdot)$ is made for the sake of simplicity and the nonlinear cases can be treated similarly (see [18]).

REMARK 1.4. When nonlinear heat flux $\boldsymbol{q} = -k_0\theta\nabla\theta$ is considered (see, for example, [1, 14]), our method can be also applied in the long-time behaviour analysis with some modification. We save this study for later work.

Before giving a detailed proof of our main results, let us now briefly recall some related results in the literature.

In recent years, nonlinear coupled PDE systems arising from phase transitions based on Frémond's theory have been extensively studied in many research papers (see [3, 5, 6, 10, 12, 23, 24, 31] and the references cited therein). It was first proposed in [7, 8] that the interior forces depend on microscopic movements during the phase change. Since then, various mathematical models have been derived in the literature for the first-order phase transitions with microscopic movements. In [1], Berti *et al*. developed a phase transition model describing temperature-induced solid–liquid transitions in materials whose thermal conductivity increases linearly with temperature. More precisely, they derived a phase transition model analogous to ours under the assumptions that $\boldsymbol{q} = -k_0\theta\nabla\theta$ and $\lambda(\varphi) = -3\varphi^2 + 8\varphi^3 - 6\varphi^2$, $G(\varphi) = 6\varphi^2(1-\varphi)^2$ in expression of the free energy (1.3). Moreover, the pseudo-potential of dissipation $\Phi$ was assumed to be independent of $\nabla\partial_t\varphi$, i.e. $h = 0$. A well-posedness result for a *simplified* version of the resulting PDE system in a regular bounded domain $\Omega \subset \mathbb{R}^3$ was established for Neumann–Dirichlet boundary conditions. The simplification by neglecting the highly nonlinear terms $\mu\varphi_t^2$ appearing in the original model was interpreted as the so-called small perturbation assumption (see [16]). Phase transition models taking the microscopic movements into account are also proposed for thermoviscoelastic materials [3, 28, 29] and ferromagnetic materials [2, 26]. It is worth noting that, in [26], a thermodynamical model based on micro-force balance of the ferro/paramagnetic transition was proposed. Our model (1.8) can be formally viewed as a typical form of their model in the one-dimensional case.

As we can see from the literature, when the small perturbation assumption is not taken into account and the standard Fourier heat flux law is considered, the existence and uniqueness of the global solutions to the full models is obtained only in the one-dimensional case (see, for example, [20–22, 29]). The global well-posedness of classical solutions in the three-dimensional case is still an open problem due to the high nonlinearities of the original models. Feireisl *et al*. [14] considered a model that is analogous to ours with the presence of the highly nonlinear term in a bounded domain $\Omega \subset \mathbb{R}^3$. The existence and uniqueness of classical solutions was proved provided that the heat conduction is governed by a particular non-Fourier heat flux law. However, concerning the standard Fourier case, they only obtained existence of solutions in a suitable weak sense defined by an entropy inequality (see [13]).

In [6] the Cauchy–Neumann problem of system (1.12) in a bounded domain $\Omega \subset \mathbb{R}^3$ was studied from a different point of view. Indeed, system (1.12) was considered as the limit problem of approximation PDE systems consisted of (1.8 $a$) coupled with the following equation for $\varphi$:

$$\rho_0\varphi_{tt} + \mu\varphi_t - h\Delta\varphi_t - \nu\Delta\varphi + \tfrac{1}{2}(\varphi^3 - \varphi) - \frac{L}{\theta_c}(\theta - \theta_c) = 0, \tag{1.27}$$

when $\rho_0 > 0$ goes to zero. The additional term $\rho_0\varphi_{tt}$ comes from the consideration of the power of microscopic acceleration forces. They established the local well-posedness of the approximation problems by a fixed-point procedure. It is worth noting that the proof of the local existence result was demonstrated in a different way from that of [9] in order to avoid the lifespan's dependence on $\rho_0$. Then, passing to the limit based on the *a priori* estimates, they proved the local existence result for the limit problem.

We now focus on asymptotic analysis as time goes to infinity of global solutions to the original full models of phase transitions with micro-movements. To the best of our knowledge, this was an open problem for years until [18] studied the long-time behaviour of global solutions for a one-dimensional case. They studied the strongly nonlinear PDE system derived in [1] under the standard Fourier heat flux law, which also coincides with the system proposed in [11] with constant mass density. By establishing a lemma of analysis (see lemma 3.1), they obtained the uniform bounds of the global solutions. Moreover, they proved the compactness of the orbit and the existence of global attractors. In a continuation work [19], Jiang and Zhang considered the corresponding stationary states to the problems of [18]. Using a modified plane-analysis method, they proved that the stationary problem admits at most countable infinite solutions. Hence, the global solution to the evolution system will converge to a single steady state as times goes to infinity.

We include some new features, which are listed below.

(i) The asymptotic behaviour of solutions to the original phase transition models has been an open problem for some time due to the high nonlinearities of the original model creating difficulties in obtaining the uniform *a priori* estimates independent of time of the solutions. Thanks to the recently established lemma 3.1 (see [18]) and some delicate *a priori* estimates, we are able to solve this problem. Moreover, we would like to mention that this lemma may have important applications in the analysis of long-time behaviour for various nonlinear coupled systems, including an energy equation for the absolute temperature $\theta$, such as nonlinear thermoviscoelastic systems (see, for example, [14, 17, 33, 36]).

(ii) From a mathematical point of view, the simplification (or the so-called small perturbation assumptions) of the original model not only removes the main obstacle from obtaining the uniform *a priori* estimates of the solutions independent of time, but also fails to keep an energy conservation (1.16) that has important physical significance. Due to this conservation property, the long-time behaviour must be considered in a different way from that of the simplified problem. We cannot expect an absorbing ball for initial data varying *in the whole space*. Instead, we should consider the dynamics in a restricted set that is invariant for the orbit (see [32, 33, 36, 38]).

(iii) The corresponding stationary problem is an ordinary differential equation of second order subject to Neumann boundary conditions with a non-local term that is the integral of the unknown function and its derivative. This is significantly different from the stationary problems of the Cahn–Hilliard equations, the phase field equations, the thin film equations, etc., where the non-local term only depends on the unknown function itself. We use a modified plane analysis method (see [19, 25, 34]) to prove that the stationary problem admits at most an infinitely countable number of solutions.

The remainder of the paper is organized as follows. In Section 2 we establish global well-posedness results. Exploiting lemma 3.1, we obtain the uniform estimates of the solutions in Section 3. Then we prove the compactness of the orbit by a decomposition of the trajectory in Section 4. The existence of maximal attractors is given in Section 5, where we use lemma 3.1 again. In Section 6, we prove that the stationary problem admits at most an infinitely countable number of solutions and the global solution of the evolution problem will converge to an equilibrium as time goes to infinity. For the reader's convenience we give a proof of lemma 3.1 in the appendix.

## 2. Global existence and uniqueness

In this section we establish the global existence and uniqueness of solutions to our problem. First, we state the local well-posedness result.

Theorem 2.1 (local existence and uniqueness). *For any initial data $(\theta_0, \varphi_0) \in H^1 \times H^2_N$, there exists $\delta > 0$ depending on $\Omega$, $\|\theta_0\|_{H^1}$ and $\|\varphi_0\|_{H^2}$ such that the problem admits a unique solution $(\theta, \varphi)$ in $\Omega \times [0, \delta]$ satisfying*

$$\theta \in C([0,\delta], H^1(\Omega)), \quad \theta_t \in L^2((0,\delta), L^2(\Omega)), \tag{2.1}$$

$$\varphi \in C([0,\delta], H^2_N), \qquad \varphi_t \in C([0,\delta], H^2_N) \cap L^2((0,\delta), H^2_N), \tag{2.2}$$

$$\varphi_{tt} \in L^2((0,\delta), H^2_N). \tag{2.3}$$

*Sketch of the proof.* The proof of theorem 2.1 is quite standard using a fixed-point procedure. Let $M_0 = \|\varphi_0\|^2_{H^2} + \|\theta_0\|^2_{H^1}$. We introduce the set

$$X_\delta(M) := \left\{ (\chi, v) \,\middle|\, \begin{array}{l} \chi \in C([0,\delta], H^2_N(\Omega)) \cap C^1([0,\delta], H^2_N(\Omega)), \\ \chi_{tt} \in L^2((0,\delta), H^2_N(\Omega)), \\ v \in C([0,\delta], H^1(\Omega)), \quad v_t \in L^2((0,\delta), L^2(\Omega)), \\ \chi|_{t=0} = \varphi_0, \quad v|_{t=0} = \theta_0, \\ \max_{0\leqslant t\leqslant\delta}(\|\chi\|^2_{H^2} + \|\chi_t\|^2_{H^2}) + \int_0^\delta \|\chi_{tt}\|^2_{H^2}\,\mathrm{d}\tau \leqslant M, \\ \max_{0\leqslant t\leqslant\delta}\|v\|^2_{H^1} + \int_0^\delta \|v_t\|^2\,\mathrm{d}\tau \leqslant 2M_0. \end{array} \right\} \tag{2.4}$$

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For $(\chi, v) \in X_\delta(M)$, we consider the following two auxiliary linear problems:

$$\left.\begin{aligned} \theta_t + A\theta = f_1(v,\chi) &:= v - v\chi_t + \chi_t^2 + \chi_{xt}^2, \\ \theta(0) &= \theta_0, \end{aligned}\right\} \tag{2.5}$$

and

$$\left.\begin{aligned} A\varphi_t + A\varphi = f_2(v,\chi) &:= -\tfrac{1}{2}(\chi^3 - 3\chi) + v, \\ \varphi(0) &= \varphi_0. \end{aligned}\right\} \tag{2.6}$$

It is easy to verify that $f_1(v,\chi), f_2(v,\chi) \in C([0,\delta], H^1) \cap H^1((0,\delta), L^2)$, hence, by the theory of linear parabolic equations, (2.5) has a unique solution

$$\theta \in C([0,\delta], H^1) \cap H^1((0,\delta), L^2).$$

On the other hand, (2.6) can be solved by

$$\varphi(t) = \mathrm{e}^{-t}\varphi_0 - \int_0^t \mathrm{e}^{s-t}[A^{-1}f_2(v,\chi)](s)\,\mathrm{d}s. \tag{2.7}$$

Next, we argue in a similar way to [18] to prove that, by choosing proper $M$ and $\delta$, the mapping $\mathcal{G}\colon (v,\chi) \mapsto (\theta,\varphi)$ maps $X_\delta$ into itself and, moreover, $\mathcal{G}$ is a contraction. Then the local existence and uniqueness result follows. We omit the details here. □

Now we aim to show that the unique local solution can be extended to $[0,T]$ for any $T > 0$. To this end, we will prove the boundedness of $\|\theta\|_{H^1}$ and $\|\varphi\|_{H^2}$. In what follows, we denote by $C_T$ a universal constant that may depend on the initial data, $\Omega$ and $T$, and by $C$ we denote a universal constant that may depend on the initial data and $\Omega$, but not on $T$. Since $\theta$ represents the absolute temperature, we expect $\theta$ to be positive since $\theta_0 > 0$ in $[0,1]$. This is given by the following lemma (see [27]).

LEMMA 2.2. *For any given initial data $(\theta_0, \varphi_0) \in H^1 \times H_N^2$ satisfying $\theta_0 > 0$ in $[0,1]$, let $(\varphi,\theta)$ be the local solution according to theorem 2.1. Then it holds that*

$$\theta > 0, \quad \forall (x,t) \in \bar{\Omega} \times [0,\delta]. \tag{2.8}$$

*Proof.* We argue using the comparison principle as in [20]. First, we rewrite the first equation of (1.12) as

$$\theta_t - \theta_{xx} = -\theta\varphi_t + \varphi_t^2 + \varphi_{xt}^2 := a\theta + b. \tag{2.9}$$

Now, due to theorem 2.1, we have

$$a \in L^1((0,\delta), L^\infty(\Omega)) \quad \text{and} \quad b \geqslant 0 \quad \text{in } \bar{\Omega} \times [0,\delta], \tag{2.10}$$

so that

$$\theta_t - \theta_{xx} \geqslant -\|a\|_{L^\infty(\Omega)}\theta. \tag{2.11}$$

Setting $\underline{\theta}_0 := \min_{x\in[0,1]} \theta_0(x)$ and noting that

$$\Theta\colon t \mapsto \underline{\theta}_0 \exp\left(-\int_0^t \|a(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s\right) \tag{2.12}$$

satisfies $\Theta(0) \leqslant \theta_0$ and $\Theta' + \|a\|_{L^\infty(\Omega)}\Theta = 0$, the comparison principle entails that

$$\theta(\cdot, t) \geqslant \Theta(t) \quad \text{in } \bar{\Omega} \times [0, \delta]. \tag{2.13}$$

Thus, the positivity of $\theta$ follows. □

LEMMA 2.3. *For any $t > 0$, it holds that*

$$\|\varphi\|_{H^1} \leqslant C, \tag{2.14}$$

$$\int_0^1 \theta(t)\,\mathrm{d}x \leqslant C. \tag{2.15}$$

*Proof.* Multiplying the second equation of (1.12) by $\varphi_t$ and adding the result to the first equation, then integrating over $\Omega$, we arrive at the following conservation relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\theta + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{8}\varphi^4 - \tfrac{1}{4}\varphi^2)\,\mathrm{d}x = 0. \tag{2.16}$$

Using Young's inequality, we see that

$$\tfrac{1}{8}\varphi^4 - \tfrac{1}{4}\varphi^2 \geqslant \tfrac{1}{16}\varphi^4 - C. \tag{2.17}$$

Thus, by virtue of the boundary conditions and of the Poincaré inequality, (2.14) and (2.15) easily follow. □

LEMMA 2.4. *For any $t > 0$, the following estimate holds:*

$$\int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{\varphi_t^2}{\theta} + \frac{\varphi_{xt}^2}{\theta}\right)\mathrm{d}x\,\mathrm{d}\tau \leqslant C. \tag{2.18}$$

*Proof.* Multiplication of the first equation in (1.12) by $\theta^{-1}$ and integrating with respect to $x$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\ln\theta + \varphi)(t)\,\mathrm{d}x - \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{\varphi_t^2}{\theta} + \frac{\varphi_{xt}^2}{\theta}\right)(t)\,\mathrm{d}x = 0. \tag{2.19}$$

Since $\ln\theta \leqslant \theta - 1$ for all $\theta > 0$, and noting lemma 2.3, we obtain (2.18). □

LEMMA 2.5. *For any $t \in [0, T]$, it holds that*

$$\int_0^t \|\theta\|^2\,\mathrm{d}\tau \leqslant C_T \tag{2.20}$$

*and*

$$\int_0^t \|\varphi_t\|_{H^1}^2\,\mathrm{d}\tau \leqslant C_T. \tag{2.21}$$

*Proof.* Let $u = \sqrt{\theta}$. Then

$$u_x^2 = \frac{1}{4}\frac{\theta_x^2}{\theta}. \tag{2.22}$$

It follows from (2.18) that

$$\int_0^t \int_0^1 \frac{u_x^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C. \tag{2.23}$$

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Using Young's inequality together with (2.15), we infer that

$$\int_0^t \left(\int_0^1 |u_x|\,\mathrm{d}x\right)^2 \mathrm{d}\tau \leqslant \int_0^t \left(\int_0^1 \frac{u_x^2}{\theta}\,\mathrm{d}x\right)\left(\int_0^1 \theta\,\mathrm{d}x\right)\mathrm{d}\tau \leqslant C. \tag{2.24}$$

Noting that

$$\|u\|^2 = \|\theta\|_{L^1} \tag{2.25}$$

and the Sobolev embedding inequality

$$\|u\|_{L^\infty} \leqslant C\|u\|_{W^{1,1}},$$

we conclude that

$$\int_0^t \|u\|_{L^\infty}^2\,\mathrm{d}\tau \leqslant C_T, \tag{2.26}$$

and, hence,

$$\int_0^t \|\theta\|_{L^\infty}\,\mathrm{d}\tau \leqslant C_T. \tag{2.27}$$

Then it follows that

$$\int_0^t \int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_0^t \left(\|\theta\|_{L^\infty}\int_0^1 \theta\,\mathrm{d}x\right)\mathrm{d}\tau \leqslant C_T. \tag{2.28}$$

On the other hand, multiplying the second equation in (1.12) by $\varphi_t$, integrating over $\Omega$ and applying Young's inequality, we obtain that

$$\|\varphi_t\|^2 + \|\varphi_{xt}\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tfrac{1}{8}\varphi^4 - \tfrac{1}{4}\varphi^2)\,\mathrm{d}x \leqslant \tfrac{1}{2}\|\varphi_t\|^2 + C\|\theta\|^2. \tag{2.29}$$

Integrating the above inequality with respect to $t$ for $t \in [0,T]$ and noting (2.14), we obtain

$$\int_0^t \|\varphi_t\|_{H^1}^2\,\mathrm{d}\tau \leqslant C_T. \tag{2.30}$$

□

LEMMA 2.6. *For any $t \in [0,T]$, it holds that*

$$\|\varphi_t\|_{H^2} + \|\varphi\|_{H^2} + \|\theta\|_{H^1} \leqslant C_T \tag{2.31}$$

*and*

$$\int_0^t (\|\theta_t\|^2 + \|\varphi_{xxt}\|^2 + \|\varphi_{xt}\|^2)\,\mathrm{d}\tau \leqslant C_T. \tag{2.32}$$

*Proof.* Differentiate the second equation in (1.12) with respect to $t$ to obtain that

$$\varphi_{tt} - \varphi_{xxtt} - \varphi_{xxt} + (\tfrac{3}{2}\varphi^2\varphi_t - \tfrac{1}{2}\varphi_t) - \theta_t = 0. \tag{2.33}$$

Multiplying (2.33) by $\varphi_t$, integrating over $\Omega$ and applying Young's inequality, then using the fact that $\|\varphi\|_{L^\infty(\Omega)} \leqslant \|\varphi\|_{H^1} \leqslant C$, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2) + \|\varphi_{xt}\|^2 &= -\int_0^1 (\tfrac{3}{2}\varphi^2\varphi_t - \tfrac{1}{2}\varphi_t - \theta_t)\varphi_t\,\mathrm{d}x \\ &\leqslant \tfrac{1}{4}\|\theta_t\|^2 + C\|\varphi_t\|^2. \end{aligned}\tag{2.34}$$

Similarly, multiplying (2.33) by $-\varphi_{xxt}$ and integrating over $\Omega$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2) + \|\varphi_{xxt}\|^2 = \int_0^1 (\tfrac{3}{2}\varphi^2\varphi_t - \tfrac{1}{2}\varphi_t - \theta_t)\varphi_{xxt}\,\mathrm{d}x$$
$$\leqslant \tfrac{1}{2}\|\varphi_{xxt}\|^2 + \|\theta_t\|^2 + C\|\varphi_t\|^2. \tag{2.35}$$

On the other hand, multiplying the first equation in (1.12) by $\theta_t$, integrating over $\Omega$ and applying Young's inequality, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x\|^2 + \|\theta_t\|^2 = \int_0^1 (\varphi_t^2 + \varphi_{xt}^2 - \theta\varphi_t)\theta_t\,\mathrm{d}x$$
$$\leqslant \tfrac{1}{2}\|\theta_t\|^2 + C\int_0^1 \theta^2\varphi_t^2\,\mathrm{d}x + C\int_0^1 (\varphi_t^4 + \varphi_{xt}^4)\,\mathrm{d}x. \tag{2.36}$$

By virtue of the Sobolev embedding theorem and the Poincaré inequality, the last two terms on the right-hand side of (2.36) yield

$$\int_0^1 \theta^2\varphi_t^2\,\mathrm{d}x \leqslant \|\theta\|_{L^\infty}^2\|\varphi_t\|^2 \leqslant C\biggl(\|\theta_x\| + \int_0^1 \theta\,\mathrm{d}x\biggr)^2\|\varphi_t\|^2 \leqslant C(\|\theta_x\|^2 + 1)\|\varphi_t\|^2, \tag{2.37}$$

and

$$\int_0^1 (\varphi_t^4 + \varphi_{xt}^4)\,\mathrm{d}x \leqslant \|\varphi_t\|_{L^\infty}^2\|\varphi_t\|^2 + \|\varphi_{xt}\|_{L^\infty}^2\|\varphi_{xt}\|^2$$
$$\leqslant C(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2)(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2). \tag{2.38}$$

Now, multiplying (2.35) by $\frac{1}{8}$ and adding the result up with (2.34) and (2.36), combining with (2.37) and (2.38), we finally obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}\|\varphi_t\|^2 + \tfrac{9}{16}\|\varphi_{xt}\|^2 + \tfrac{1}{16}\|\varphi_{xxt}\|^2 + \tfrac{1}{2}\|\theta_x\|^2) + \|\varphi_{xt}\|^2 + \tfrac{1}{16}\|\varphi_{xxt}\|^2 + \tfrac{1}{8}\|\theta_t\|^2$$
$$\leqslant C(\|\theta_x\|^2 + 1)\|\varphi_t\|^2 + C(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2 + 1)(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2). \tag{2.39}$$

Let

$$y(t) = \tfrac{1}{2}\|\varphi_t\|^2 + \tfrac{9}{16}\|\varphi_{xt}\|^2 + \tfrac{1}{16}\|\varphi_{xxt}\|^2 + \tfrac{1}{2}\|\theta_x\|^2.$$

Then (2.39) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leqslant C(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2)y(t) + C(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2). \tag{2.40}$$

By the Gronwall inequality and lemma 2.5, we conclude that $y(t) \leqslant C_T$. Assertion (2.31) follows from the elliptic regularity theory and (2.32) comes from the integral of (2.39) with respect to $t$. This completes the proof. □

We are now in a position to state and prove theorem 2.7 concerning the global existence and uniqueness result.

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THEOREM 2.7 (global existence and uniqueness). *Under the assumptions of theorem 1.1, for any $T > 0$, there exists a unique global solution $(\theta, \varphi)$ such that*

$$\varphi \in C([0,T], H_N^2), \qquad \varphi_t \in C([0,T], H^2) \cap L^2((0,T), H^2), \tag{2.41}$$

$$\varphi_{tt} \in L^2((0,T), H^2), \tag{2.42}$$

$$\theta \in C([0,T], H^1(\Omega)), \qquad \theta_t \in L^2((0,T), L^2(\Omega)), \tag{2.43}$$

$$\theta > 0, \quad \forall (x,t) \in [0,1] \times [0,T]. \tag{2.44}$$

*Proof.* Based on lemmas 2.3–2.6, we can see that the unique solution obtained in theorem 2.1 can be extended to $[0,T]$ for any $T > 0$.

It remains to prove (2.42). We note that

$$\|\varphi_{tt}\|_{H^2}^2 \leqslant C\|\varphi_{tt} - \varphi_{xxtt}\|^2. \tag{2.45}$$

Then, by (2.33) and lemmas 2.5 and 2.6, we obtain (2.42). This completes the proof. □

## 3. Uniform *a priori* estimates

In this section we obtain the uniform *a priori* estimates in order to study the long-time behaviour of the global solutions. To begin with, we state the following key lemma of analysis, which was first established in [18] and which plays a crucial role in the proof of uniform *a priori* estimates.

LEMMA 3.1. *Let $\Omega = (0,1)$. Suppose that $\vartheta(t,x) \in C([0,+\infty), H^1(\Omega))$ is a positive function. Moreover, suppose that the following relations hold:*

$$\int_0^1 \vartheta(t,x)\,\mathrm{d}x \leqslant K_1 \tag{3.1}$$

*and*

$$\int_0^\infty \int_0^1 \frac{\vartheta_x^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant K_2, \tag{3.2}$$

*where $K_1$ and $K_2$ are two positive constants that are independent of $t$.*

*Then we have that*

$$\int_0^\infty \|\vartheta - \bar{\vartheta}\|^2\,\mathrm{d}\tau \leqslant C(K_1, K_2), \tag{3.3}$$

*where*

$$\bar{\vartheta}(t) = \int_0^1 \vartheta(t,x)\,\mathrm{d}x$$

*stands for the mean integral of $\vartheta$ and $C(K_1, K_2)$ is a positive constant depending only on $K_1$, $K_2$ and $\Omega$.*

For the reader's convenience we give the proof of this lemma in the appendix. Now, with the help of this key lemma, we are able to obtain the uniform estimates of the global solutions, which are given by the following theorem.

THEOREM 3.2. *Under the assumptions of theorem 2.7, we have the following uniform estimates with respect to time for the unique global solution $(\theta, \varphi)$ to problem (1.12):*

$$\|\theta\|_{H^1}^2 + \int_0^t \|\theta_t\|^2 \, \mathrm{d}\tau \leqslant C, \tag{3.4}$$

$$\|\varphi\|_{H^2}^2 + \|\varphi_t\|_{H^2}^2 + \int_0^t (\|\varphi_t\|_{H^2}^2 + \|\varphi_{tt}\|_{H^2}^2) \, \mathrm{d}\tau \leqslant C. \tag{3.5}$$

*Moreover, when $t$ goes to infinity, it holds that*

$$\|\theta - \bar{\theta}\|_{L^\infty} + \|\theta_x\| + \|\varphi_t\|_{H^2} \to 0. \tag{3.6}$$

*Proof.* First of all, we see from lemma 2.3, lemma 2.4 and theorem 2.7 that $\theta(t, x)$ fulfils the conditions of lemma 3.1. Thus, it follows that

$$\int_0^\infty \|\theta - \bar{\theta}\|^2 \, \mathrm{d}\tau \leqslant C. \tag{3.7}$$

Applying the Sobolev embedding theorem and Young's inequality, we note that

$$\begin{aligned} \int_0^1 (\varphi_t^2 + \varphi_{xt}^2) \, \mathrm{d}x &\leqslant \|\theta\|_{L^\infty} \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta} \, \mathrm{d}x \\ &\leqslant (\|\theta - \bar{\theta}\|_{L^\infty} + \bar{\theta}) \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta} \, \mathrm{d}x \\ &\leqslant C(\|\theta_x\| + \|\theta - \bar{\theta}\| + 1) \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta} \, \mathrm{d}x \\ &\leqslant C(\|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2 + 1) \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta} \, \mathrm{d}x. \end{aligned} \tag{3.8}$$

Integrating the first equation of (1.12) over $\Omega$ yields

$$\bar{\theta}_t + \int_0^1 \theta \varphi_t \, \mathrm{d}x = \int_0^1 (\varphi_t^2 + \varphi_{xt}^2) \, \mathrm{d}x. \tag{3.9}$$

Multiplying the substraction of (3.9) from the first equation of (1.12) by $\theta - \bar{\theta}$ and integrating over $\Omega$, we are led to

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2 &= -\int_0^1 (\theta - \bar{\theta}) \theta \varphi_t \, \mathrm{d}x + \int_0^1 (\theta - \bar{\theta})(\varphi_t^2 + \varphi_{xt}^2) \, \mathrm{d}x \\ &= -\int_0^1 (\theta - \bar{\theta})^2 \varphi_t \, \mathrm{d}x - \bar{\theta} \int_0^1 (\theta - \bar{\theta}) \varphi_t \, \mathrm{d}x \\ &\quad + \bar{\theta}(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2) + \int_0^1 \theta (\varphi_t^2 + \varphi_{xt}^2) \, \mathrm{d}x. \end{aligned} \tag{3.10}$$

Exploiting Young's inequality and by the Sobolev embedding theorem, we infer that

$$\begin{aligned}\int_0^1 (\theta-\bar\theta)^2\varphi_t\,\mathrm{d}x &\leqslant \|\varphi_t\|_{L^\infty}\|\theta-\bar\theta\|^2\\ &\leqslant C(\|\varphi_t\|_{L^\infty}^2+1)\|\theta-\bar\theta\|^2\\ &\leqslant C(\|\varphi_{xt}\|^2+\|\varphi_t\|^2+1)\|\theta-\bar\theta\|^2. \end{aligned}\tag{3.11}$$

Noting that

$$\bar\theta=\int_0^1\theta\,\mathrm{d}x\leqslant C$$

and applying Young's inequality, we obtain that

$$-\bar\theta\int_0^1(\theta-\bar\theta)\varphi_t\,\mathrm{d}x+\bar\theta(\|\varphi_t\|^2+\|\varphi_{xt}\|^2)\leqslant C(\|\theta-\bar\theta\|^2+\|\varphi_t\|^2+\|\varphi_{xt}\|^2). \tag{3.12}$$

It remains to estimate the last term on the right-hand side of (3.10). Applying Young's inequality again, we have

$$\int_0^1\theta(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x\leqslant\int_0^1\theta^2(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x+C(\|\varphi_t\|^2+\|\varphi_{xt}\|^2), \tag{3.13}$$

where

$$\begin{aligned}\int_0^1\theta^2(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x &\leqslant 2\int_0^1(\theta-\bar\theta)^2(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x+2\bar\theta^2(\|\varphi_t\|^2+\|\varphi_{xt}\|^2)\\ &\leqslant C(\|\varphi_t\|_{L^\infty}^2+\|\varphi_{xt}\|_{L^\infty}^2)\|\theta-\bar\theta\|^2+C(\|\varphi_t\|^2+\|\varphi_{xt}\|^2)\\ &\leqslant C(\|\varphi_{xxt}\|^2+\|\varphi_{xt}\|^2+\|\varphi_t\|^2)\|\theta-\bar\theta\|^2+C(\|\varphi_t\|^2+\|\varphi_{xt}\|^2). \end{aligned}\tag{3.14}$$

Collecting (3.10)–(3.14), we arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta-\bar\theta\|^2+\|\theta_x\|^2 &\leqslant C(\|\varphi_{xxt}\|^2+\|\varphi_{xt}\|^2+\|\varphi_t\|^2)\|\theta-\bar\theta\|^2\\ &\quad+C(\|\varphi_t\|^2+\|\varphi_{xt}\|^2+\|\theta-\bar\theta\|^2). \end{aligned}\tag{3.15}$$

Next, we multiply the first equation of (1.12) by $\theta_t$ and integrate with respect to $x$ over $\Omega$. Then applying Young's inequality yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x\|^2-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\theta(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x+\|\theta_t\|^2\\ &\quad=-2\int_0^1\theta(\varphi_t\varphi_{tt}+\varphi_{xt}\varphi_{xtt})\,\mathrm{d}x-\int_0^1\theta\theta_t\varphi_t\,\mathrm{d}x\\ &\quad\leqslant\epsilon(\|\varphi_{tt}\|^2+\|\varphi_{xtt}\|^2)+\tfrac{1}{2}\|\theta_t\|^2+C_\epsilon\int_0^1\theta^2(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x, \end{aligned}\tag{3.16}$$

where $\epsilon$ is a positive constant to be specified later.

On the other hand, testing (2.33) by $\varphi_t$ and integrating over $\Omega$, then using Young's inequality, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_t\|^2+\|\varphi_{xt}\|^2)+\|\varphi_{xt}\|^2=\int_0^1(\theta_t-\tfrac{3}{2}\varphi^2\varphi_t+\tfrac{1}{2}\varphi_t)\varphi_t\,\mathrm{d}x \leqslant \epsilon\|\theta_t\|^2+C_\epsilon\|\varphi_t\|^2. \tag{3.17}$$

Multiplying (2.33) by $\varphi_{tt}$, integrating by parts and using Young's inequality, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_{xt}\|^2+\|\varphi_{tt}\|^2+\|\varphi_{xtt}\|^2=\int_0^1(\theta_t-\tfrac{3}{2}\varphi^2\varphi_t+\tfrac{1}{2}\varphi_t)\varphi_{tt}\,\mathrm{d}x \leqslant \tfrac{1}{2}\|\varphi_{tt}\|^2+C_1\|\theta_t\|^2+C\|\varphi_t\|^2. \tag{3.18}$$

Similarly, multiplying (2.33) by $-\varphi_{xxt}$ and integrating over $\Omega$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_{xt}\|^2+\|\varphi_{xxt}\|^2)+\|\varphi_{xxt}\|^2=-\int_0^1(\theta_t-\tfrac{3}{2}\varphi^2\varphi_t+\tfrac{1}{2}\varphi_t)\varphi_{xxt} \leqslant \tfrac{1}{2}\|\varphi_{xxt}\|^2+C_1\|\theta_t\|^2+C\|\varphi_t\|^2. \tag{3.19}$$

Now, multiplying (3.17) by $\kappa$, (3.18) by $\alpha$, and (3.19) by $\alpha$, then adding the resultants up with (3.15) and (3.16), and noting (3.14), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(&\|\theta-\bar{\theta}\|^2+\|\theta_x\|^2+\kappa\|\varphi_t\|^2+(\kappa+2\alpha)\|\varphi_{xt}\|^2+\alpha\|\varphi_{xxt}\|^2\\ &-2\int_0^1\theta(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x\biggr)\\ &+\|\theta_x\|^2+(\tfrac{1}{2}-\epsilon\kappa-2C_1\alpha)\|\theta_t\|^2+\tfrac{1}{2}\alpha\|\varphi_{tt}\|^2\\ &+\kappa\|\varphi_{xt}\|^2+\alpha\|\varphi_{xtt}\|^2+\tfrac{1}{2}\alpha\|\varphi_{xxt}\|^2\\ &\leqslant\epsilon(\|\varphi_{tt}\|^2+\|\varphi_{xtt}\|^2)+C_\epsilon(\|\varphi_{xxt}\|^2+\|\varphi_{xt}\|^2+\|\varphi_t\|^2)\|\theta-\bar{\theta}\|^2\\ &\quad+C(\epsilon,\kappa,\alpha)(\|\varphi_t\|^2+\|\varphi_{xt}\|^2+\|\theta-\bar{\theta}\|^2).\end{aligned} \tag{3.20}$$

Let

$$\Gamma(t)=\|\theta-\bar{\theta}\|^2+\|\theta_x\|^2+\kappa\|\varphi_t\|^2+(\kappa+2\alpha)\|\varphi_{xt}\|^2+\alpha\|\varphi_{xxt}\|^2-2\int_0^1\theta(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x. \tag{3.21}$$

By the one-dimensional Agmon inequality and Young's inequality, we deduce that

$$\begin{aligned}2\int_0^1\theta(\varphi_t^2+\varphi_{xt}^2)\,\mathrm{d}x&\leqslant 2(\|\varphi_t\|_{L^\infty}^2+\|\varphi_{xt}\|_{L^\infty}^2)\bar{\theta}\\ &\leqslant C(\|\varphi_t\|_{L^\infty}^2+\|\varphi_{xt}\|_{L^\infty}^2)\\ &\leqslant\delta_0\|\varphi_{xxt}\|^2+C_{\delta_0}(\|\varphi_t\|^2+\|\varphi_{xt}\|^2),\end{aligned} \tag{3.22}$$

with $\delta_0$ being a positive constant to be chosen later.

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Now, we choose appropriate $\epsilon$, $\kappa$ and $\delta_0$ such that

$$2C_1\alpha \leqslant \tfrac{1}{8}, \qquad \epsilon\kappa \leqslant \tfrac{1}{8}, \qquad \delta_0 \leqslant \tfrac{1}{2}\alpha, \qquad \epsilon \leqslant \tfrac{1}{4}\alpha, \qquad \kappa \geqslant 2C_{\delta_0}. \tag{3.23}$$

For example, we pick $\alpha = 1/(16C_1)$, $\delta_0 = \frac{1}{2}\alpha$, $\kappa = 2C_{\delta_0}$ and $\epsilon = \min(1/(8\kappa), \frac{1}{4}\alpha)$, which fulfils (3.23). Therefore, we obtain that

$$\Gamma(t) \geqslant \tfrac{1}{2}\alpha\|\varphi_{xxt}\|^2 + C_{\delta_0}(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2) + \|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2. \tag{3.24}$$

Noting (3.8), we finally derive from (3.20) and (3.24) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) &\leqslant C(\|\varphi_{xxt}\|^2 + \|\varphi_{xt}\|^2 + \|\varphi_t\|^2)\|\theta - \bar{\theta}\|^2 + C(\|\varphi_t\|^2 + \|\varphi_{xt}\|^2 + \|\theta - \bar{\theta}\|^2)\\ &\leqslant C(\|\varphi_{xxt}\|^2 + \|\varphi_{xt}\|^2 + \|\varphi_t\|^2)\|\theta - \bar{\theta}\|^2\\ &\quad + C(\|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2)\int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta}\,\mathrm{d}x\\ &\quad + C\biggl(\|\theta - \bar{\theta}\|^2 + \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta}\,\mathrm{d}x\biggr)\end{aligned} \tag{3.25}$$

Hence, from (2.18), (3.7) and (3.25), we conclude by the Gronwall inequality that

$$\Gamma(t) \leqslant C. \tag{3.26}$$

Then it follows from (3.24) that

$$\|\varphi_{xxt}\|^2 + \|\varphi_{xt}\|^2 + \|\varphi_t\|^2 + \|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2 \leqslant C. \tag{3.27}$$

By the Poincaré inequality and the second equation of (1.12), we infer that

$$\|\theta\|_{H^1} + \|\varphi\|_{H^2} \leqslant C. \tag{3.28}$$

Now, let us return to (3.20). Noting that $\Gamma(t) \leqslant C$, we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) &+ \|\theta_x\|^2 + \tfrac{1}{4}\|\theta_t\|^2 + \tfrac{1}{4}\alpha\|\varphi_{tt}\|^2 + \kappa\|\varphi_{xt}\|^2\\ &+ \tfrac{1}{2}\alpha\|\varphi_{xtt}\|^2 + \tfrac{1}{2}\alpha\|\varphi_{xxt}\|^2 \leqslant C(\|\theta - \bar{\theta}\|^2 + \|\varphi_t\|^2 + \|\varphi_{xt}\|^2).\end{aligned} \tag{3.29}$$

Recalling (3.8) and $\Gamma(t) \leqslant C$, we have

$$\int_0^t (\|\varphi_t\|^2 + \|\varphi_{xt}\|^2)\,\mathrm{d}\tau \leqslant C. \tag{3.30}$$

Then, integrating (3.29) from 0 to $t$ for $t > 0$, and by equation (2.33), we obtain that

$$\int_0^t (\|\theta_x\|^2 + \|\theta_t\|^2 + \|\varphi_t\|_{H^2}^2 + \|\varphi_{tt}\|_{H^2}^2)\,\mathrm{d}\tau \leqslant C. \tag{3.31}$$

Hence, it follows from (3.22) and (3.31) that

$$\int_0^t \Gamma(\tau)\,\mathrm{d}\tau \leqslant \int_0^t (\|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|\varphi_t\|_{H^2}^2 + \|\varphi_{tt}\|_{H^2}^2)\,\mathrm{d}\tau \leqslant C. \tag{3.32}$$

Concerning the asymptotic property (3.6), we shall make use of the following lemma by Shen and Zheng [34].

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LEMMA 3.3. *Suppose that $y(t)$ and $h(t)$ are non-negative functions, $y'(t)$ is locally integrable on $(0,+\infty)$, and $y(t)$ and $h(t)$ satisfy*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant A_1 y^2 + A_2 + h(t), \quad \forall t \geqslant 0, \tag{3.33}$$

$$\int_0^T y(\tau)\,\mathrm{d}\tau \leqslant A_3, \quad \int_0^T h(\tau)\,\mathrm{d}\tau \leqslant A_4, \quad \forall T > 0, \tag{3.34}$$

*with $A_1$, $A_2$, $A_3$ and $A_4$ being positive constants independent of $t$ and $T$. Then, for any $r > 0$,*

$$y(t+r) \leqslant \left(\frac{A_3}{r} + A_2 r + A_4\right)\mathrm{e}^{A_1 A_3}, \quad \forall t \geqslant 0. \tag{3.35}$$

*Moreover,*

$$\lim_{t\to+\infty} y(t) = 0. \tag{3.36}$$

Now, letting $y(t) = \Gamma(t)$ and applying lemma 3.3 on (3.29), we obtain that

$$\Gamma(t) \to 0 \quad \text{as } t \to +\infty. \tag{3.37}$$

Recalling (3.24) and the Sobolev embedding inequality

$$\|\theta - \bar{\theta}\|_{L^\infty} \leqslant C\|\theta_x\|, \tag{3.38}$$

we finally prove that, as $t$ goes to infinity,

$$\|\theta - \bar{\theta}\|_{L^\infty} + \|\theta_x\| + \|\varphi_t\|_{H^2} \to 0. \tag{3.39}$$

This completes the proof. □

## 4. Compactness of the orbit

The results in Section 3 imply that the unique global solution to problem (1.12) defines a strongly continuous nonlinear semigroup $S(t)$ acting on $H^1 \times H^2_N$ such that $(\theta(t), \varphi(t)) = S(t)(\theta_0, \varphi_0)$.

In this section we will prove the compactness of the orbit of $(\theta(t), \varphi(t))$ for $t > 0$. In what follows, we shall exploit some formal *a priori* estimates that can be justified rigorously using the approximate procedure and the standard density argument.

LEMMA 4.1. *Under the assumptions of theorem 2.1, the following uniform estimate holds.*

$$\|\theta(t)\|_{H^2} \leqslant C, \quad \forall t \geqslant 1, \tag{4.1}$$

*where $C$ is a constant depending on the initial data, but not on $t$.*

*Proof.* First, using the first equation of (1.12) and theorem 3.2, it holds that

$$\int_0^\infty \|\theta_{xx}\|^2\,\mathrm{d}\tau \leqslant C. \tag{4.2}$$

Now, we multiply the first equation of (1.12) by $-\theta_{xxt}$ and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{xx}\|^2 + \|\theta_{xt}\|^2 = \int_0^1 (-\theta_x\varphi_t - \theta\varphi_{xt} + 2\varphi_t\varphi_{xt} + 2\varphi_{xt}\varphi_{xxt})\theta_{xt}\,\mathrm{d}x. \tag{4.3}$$

Recalling theorem 3.2 and applying Young's inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{xx}\|^2 + \|\theta_{xt}\|^2 &\leqslant C(\|\varphi_t\|_{L^\infty}^2\|\theta_x\|^2 + \|\theta\|_{L^\infty}^2\|\varphi_{xt}\|^2 + \|\varphi_{xt}\|_{L^\infty}^2\|\varphi_t\|^2 + \|\varphi_{xt}\|_{L^\infty}^2\|\varphi_{xxt}\|^2) \\ &\leqslant C(\|\theta_x\|^2 + \|\varphi_t\|_{H^2}^2). \end{aligned} \tag{4.4}$$

Then, exploiting the uniform Gronwall inequality (see [37]), we finally deduce that

$$\|\theta_{xx}(t)\| \leqslant C, \quad \forall t \geqslant 1. \tag{4.5}$$

Thus, (4.1) follows easily. □

Next, we prove the pre-compactness of $\varphi(t)$. Unlike the equation for $\theta$, we have to decompose the solution $\varphi$ into a uniformly stable part and a compact part. We decompose

$$\varphi = \varphi^d + \varphi^c, \tag{4.6}$$

where $\varphi^d$ and $\varphi^c$ satisfy

$$\left.\begin{aligned} \varphi_t^d - \varphi_{xxt}^d - \varphi_{xx}^d + \varphi^d = 0, \\ \varphi_x^d|_{x=0,1} = 0, \\ \varphi^d(0) = \varphi_0, \end{aligned}\right\} \tag{4.7}$$

and

$$\left.\begin{aligned} \varphi_t^c - \varphi_{xxt}^c - \varphi_{xx}^c + \varphi^c + \tfrac{1}{2}(\varphi^3 - 3\varphi) - \theta = 0, \\ \varphi_x^c|_{x=0,1} = 0, \\ \varphi^c(0) = 0. \end{aligned}\right\} \tag{4.8}$$

We have the following properties.

Lemma 4.2. *For any given $\varphi_0 \in H_N^2$, the unique global solution $\varphi^d$ to problem (4.7) has the following estimate:*

$$\|\varphi^d(t)\|_{H^2} \leqslant C\mathrm{e}^{-t}\|\varphi_0\|_{H^2}. \tag{4.9}$$

*Proof.* The existence and uniqueness of $\varphi^d$ to (4.7) can be easily proved as in Section 2. Next, multiplying the equation in (4.7) by $-\varphi_{xx}^d + \varphi^d$ and integrating over $\Omega$, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_{xx}^d\|^2 + 2\|\varphi_x^d\|^2 + \|\varphi^d\|^2) + \|\varphi_{xx}^d\|^2 + 2\|\varphi_x^d\|^2 + \|\varphi^d\|^2 = 0. \tag{4.10}$$

Hence, we obtain that

$$\begin{aligned}\|\varphi^d(t)\|_{H^2}^2 &\leqslant \|\varphi_{xx}^d\|^2 + 2\|\varphi_x^d\|^2 + \|\varphi^d\|^2 \\ &= (\|\varphi_{0xx}\|^2 + 2\|\varphi_{0x}\|^2 + \|\varphi_0\|^2)\mathrm{e}^{-2t} \\ &\leqslant C\mathrm{e}^{-2t}\|\varphi_0\|_{H^2}^2. \end{aligned} \tag{4.11}$$

□

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LEMMA 4.3. *For any $t \geqslant 0$, it holds that*

$$\|\varphi^c(t)\|_{H^3} \leqslant C. \tag{4.12}$$

*Proof.* First, by theorem 3.2 and lemma 4.2, we have that

$$\|\varphi^c(t)\|_{H^2} \leqslant C \quad \text{for } t \geqslant 0. \tag{4.13}$$

Multiplying the equation of (4.8) by $\varphi^c_{xxxx}$ and integrating over $\Omega$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi^c_{xx}\|^2 + \|\varphi^c_{xxx}\|^2) + \|\varphi^c_{xx}\|^2 + \|\varphi^c_{xxx}\|^2 = -\int_0^1 \varphi^c_{xx}(3\varphi\varphi_x^2 + \tfrac{3}{2}\varphi^2\varphi_{xx} - 3\varphi_{xx} - \theta_{xx})\,\mathrm{d}x. \tag{4.14}$$

Recalling theorem 3.2 and applying Young's inequality and the Sobolev embedding theorem, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi^c_{xx}\|^2 + \|\varphi^c_{xxx}\|^2) + \|\varphi^c_{xx}\|^2 + \|\varphi^c_{xxx}\|^2 \leqslant \tfrac{1}{2}\|\varphi^c_{xx}\|^2 + C(\|\varphi_{xx}\|^2 + \|\varphi_x\|^2 + \|\theta_{xx}\|^2) + C. \tag{4.15}$$

Then, by the Gronwall inequality, we infer that

$$\|\varphi^c_{xx}\|^2 + \|\varphi^c_{xxx}\|^2 \leqslant C. \tag{4.16}$$

Combining (4.13) and (4.16), we complete the proof. □

Since the solution $(\theta, \varphi)$ defines a $C_0$-semigroup and problem (1.12)–(1.15) has a Lyapunov function. Thus, it is a gradient system. Then, it follows from the well-known results in dynamic systems (see [37, 39]) that the $\omega$-limit set $\omega(\theta_0, \varphi_0)$ is a compact and connected set that consists of equilibria of problem (1.12)–(1.15).

## 5. Existence of global attractors

In order to prove the existence of a global attractor, we shall apply theorem I.1.1 from [37], which was rephrased in [35] as follows.

THEOREM 5.1. *Suppose that we have the following.*

(i) *The mapping $S(t)$, $t \geqslant 0$, defined by the solution to problem (1.12) is a non-linear continuous semigroup from $H$ into itself.*

(ii) *The operator $S(t)$ is uniformly compact for $t$ large, i.e. for every bounded set $B$ contained in $H_{\beta_1,\beta_2,\beta_3}$, there exists $t_0$ which may depend on $B$ such that $\bigcup_{t\geqslant t_0} S(t)B$ is relatively compact in $H$.*

(iii) *The orbit starting from any bounded set of $H_{\beta_1,\beta_2,\beta_3}$ will reenter in $H_{\beta_1,\beta_2,\beta_3}$ after a finite time, which depends only on this bounded set, and stay there forever. There exists a bounded set $B_{\beta_1,\beta_2,\beta_3}$ in $H_{\beta_1,\beta_2,\beta_3}$ such that $B_{\beta_1,\beta_2,\beta_3}$ is absorbing in $H_{\beta_1,\beta_2,\beta_3}$.*

*Then the $\omega$-limit set of $B_{\beta_1,\beta_2,\beta_3}$, $A_{\beta_1,\beta_2,\beta_3}$ is a global attractor that is compact and attracts the bounded sets of $H_{\beta_1,\beta_2,\beta_3}$.*

Based on the results obtained in Section 3 and Section 4, it remains to verify condition (iii).

Hereafter, we always assume that the initial data $(\theta_0,\varphi_0)\in B\subset H_{\beta_1,\beta_2,\beta_3}$, with $\|(\theta_0,\varphi_0)\|_H\leqslant R$, where $B$ is an arbitrary bounded set in $H_{\beta_1,\beta_2,\beta_3}$ and $R>0$ is a constant depending on $B$. We will prove that, for any $(\theta_0,\varphi_0)\in B$, there exists $t(B)>0$ such that, when $t\geqslant t(B)$, the corresponding solution $(\theta,\varphi)\in H_{\beta_1,\beta_2,\beta_3}$ and there exists a bounded set $B_{\beta_1,\beta_2,\beta_3}$ in $H_{\beta_1,\beta_2,\beta_3}$ such that $B_{\beta_1,\beta_2,\beta_3}$ is absorbing in $H_{\beta_1,\beta_2,\beta_3}$.

In what follows, we will denote by $C$ and $C_i$ constants that may depend on $\beta_1$, $\beta_2$, $\beta_3$ and $\Omega$, but not on the initial data.

Lemma 5.2. *For any $t>0$, it holds that*

$$\|\varphi\|_{H^1}\leqslant C, \tag{5.1}$$

$$\theta>0,\quad \forall(x,t)\in[0,1]\times[0,\infty), \tag{5.2}$$

$$\|\theta\|_{L^1}\leqslant C. \tag{5.3}$$

*Proof.* Integrating (2.16) with respect to $t$ yields

$$\int_0^1(\tfrac{1}{2}\varphi_x^2+\tfrac{1}{8}\varphi^4-\tfrac{1}{4}\varphi^2+\theta)\,\mathrm{d}x=\int_0^1(\tfrac{1}{2}\varphi_{0x}^2+\tfrac{1}{8}\varphi_0^4-\tfrac{1}{4}\varphi_0^2+\theta_0)\,\mathrm{d}x\leqslant\beta_2. \tag{5.4}$$

Then we can prove (5.1)–(5.3) as in lemma 2.3. □

Lemma 5.3. *For any $t>0$, it holds that*

$$\int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta^2}+\frac{\varphi_t^2}{\theta}+\frac{\varphi_{xt}^2}{\theta}\right)\mathrm{d}x\,\mathrm{d}\tau\leqslant C, \tag{5.5}$$

$$\mathrm{e}^{\beta_3-C_{\beta_2}}\leqslant\int_0^1\theta\,\mathrm{d}x\leqslant C, \tag{5.6}$$

*where $C_{\beta_2}=16\beta_2+\frac{9}{2}$.*

*Proof.* Multiplication of the first equation in (1.12) by $\theta^{-1}$ and integrating with respect to $x$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(\ln\theta+\varphi)(t)\,\mathrm{d}x-\int_0^1\left(\frac{\theta_x^2}{\theta^2}+\frac{\varphi_t^2}{\theta}+\frac{\varphi_{xt}^2}{\theta}\right)(t)\,\mathrm{d}x=0. \tag{5.7}$$

Since $\ln\theta\leqslant\theta$ for all $\theta>0$ and $\|\varphi\|_{H^1}\leqslant C$, we obtain that

$$\int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta^2}+\frac{\varphi_t^2}{\theta}+\frac{\varphi_{xt}^2}{\theta}\right)\mathrm{d}x\,\mathrm{d}\tau\leqslant\int_0^1(\ln\theta+\varphi)\,\mathrm{d}x-\beta_3\leqslant C. \tag{5.8}$$

Moreover, it follows from (5.7) that

$$\int_0^1(\ln\theta+\varphi)\,\mathrm{d}x\geqslant\int_0^1(\ln\theta_0+\varphi_0)\,\mathrm{d}x\geqslant\beta_3. \tag{5.9}$$

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On the other hand, noting (5.4) and exploiting Young's inequality, we may deduce that

$$\int_0^1 \varphi \, dx \leqslant 16\beta_2 + \tfrac{9}{2} := C_{\beta_2}. \tag{5.10}$$

Hence,

$$\int_0^1 \ln\theta \, dx \geqslant \beta_3 - \int_0^1 \varphi \, dx \geqslant \beta_3 - C_{\beta_2}. \tag{5.11}$$

Applying Jensen's inequality, we infer that

$$\ln \int_0^1 \theta \, dx \geqslant \int_0^1 \ln\theta \, dx \geqslant \beta_3 - C_{\beta_2}. \tag{5.12}$$

Then (5.6) follows from (5.3) and (5.12). □

Lemma 5.4. *For any $(\varphi_0, \theta_0) \in B$, there exists $t_0 = t_0(B) > 0$ depending only on $B$ such that, for all $t \geqslant t_0$, $x \in [0,1]$,*

$$\theta(x,t) \geqslant \beta_1 > 0. \tag{5.13}$$

*Proof.* We use a contradiction argument. Suppose that the assertion does not hold. Then there exists an initial datum $(\theta_0, \varphi_0) \in B$ and a sequence $t_n \to \infty$, $x_n \in [0,1]$, such that the corresponding solution $(\theta, \varphi)$ satisfies

$$\theta(x_n, t_n) < \beta_1. \tag{5.14}$$

On the other hand, it follows from theorem 3.2 that, as $n \to \infty$,

$$\theta(x, t_n) - \bar{\theta}(t_n) \to 0, \quad \forall x \in [0,1]. \tag{5.15}$$

By (5.6), we have

$$\int_0^1 \theta(x, t_n) \, dx = \bar{\theta}(t_n) \geqslant e^{\beta_3 - C_{\beta_2}} > \beta_1 > 0. \tag{5.16}$$

Then, we derive from (5.15) and (5.16) that

$$\liminf_{n\to\infty} \theta(x, t_n) \geqslant e^{\beta_3 - C_{\beta_2}} > \beta_1 > 0, \tag{5.17}$$

which contradicts (5.14). The proof is complete. □

Now, we can see from lemmas 5.2–5.4 that the orbits starting from any bounded set $B \in H_{\beta_1,\beta_2,\beta_3}$ reenter $H_{\beta_1,\beta_2,\beta_3}$ when $t \geqslant t_0(B)$ and stay there forever.

Let

$$B_{\beta_1,\beta_2,\beta_3} = \{(\theta,\varphi) \in H_{\beta_1,\beta_2,\beta_3},\ \|\theta\|_{H^1} \leqslant C_2,\ \|\varphi\|_{H^2} \leqslant C_3\}, \tag{5.18}$$

where $C_2$ and $C_3$ are positive constants depending only on $\beta_i$, $i = 1,2,3$, and will be specified later.

Lemma 5.5. *$B_{\beta_1,\beta_2,\beta_3}$ is an absorbing set in $H_{\beta_1,\beta_2,\beta_3}$, i.e. for any bounded set $B \in H_{\beta_1,\beta_2,\beta_3}$, there exists some time $t_1 = t_1(B)$ such that, when $t \geqslant t_1(B)$, $S(t)B \subset B_{\beta_1,\beta_2,\beta_3}$.*

*Proof.* To begin with, applying lemma 3.1 again, we infer from lemmas 5.2 and 5.3 that

$$\int_0^\infty \|\theta - \bar{\theta}\|^2 \, \mathrm{d}\tau \leqslant C. \tag{5.19}$$

Arguing as in the proof of theorem 3.2 and noticing (3.8) and (3.24), it is not difficult to derive that there exists a generic constant $C_4 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) + C_4\Gamma(t) \leqslant \Gamma(t)\biggl(\|\theta - \bar{\theta}\|^2 + \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta}\,\mathrm{d}x\biggr) \\ + C\biggl(\|\theta - \bar{\theta}\|^2 + \int_0^1 \frac{\varphi_t^2 + \varphi_{xt}^2}{\theta}\,\mathrm{d}x\biggr).\end{aligned} \tag{5.20}$$

Owing to lemma 5.3 and (5.19), we obtain that

$$\Gamma(t) \leqslant C\Gamma(0)\mathrm{e}^{-C_4 t} + C_5. \tag{5.21}$$

This implies that there exists $t_1 = t_1(B)$ such that, when $t \geqslant t_1$,

$$\Gamma(t) \leqslant 2C_5. \tag{5.22}$$

Recalling (3.24), we deduce that, when $t \geqslant t_1$,

$$\|\varphi_t\|_{H^2}^2 + \|\theta - \bar{\theta}\|^2 + \|\theta_x\|^2 \leqslant C_7. \tag{5.23}$$

Then, by virtue of the Poincaré inequality and equation (1.12), we finally deduce that, when $t \geqslant t_1$,

$$\|\theta\|_{H^1} \leqslant C_2, \qquad \|\varphi\|_{H^2} \leqslant C_3. \tag{5.24}$$

Thus, $B_{\beta_1,\beta_2,\beta_3}$ is an absorbing set and the proof is now complete. □

## 6. Stationary problem

In this section we investigate the multiplicity of solutions to the following elliptic problem:

$$\left.\begin{aligned} -\nu\psi_{xx} + \tfrac{1}{2}L(\psi^3 - \psi) - \frac{L}{\theta_c}u = 0, \\ -k_0 u_{xx} = 0, \\ \psi_x|_{x=0,1} = 0,\ u_x|_{x=0,1} = 0, \\ \int_0^1 (\tfrac{1}{2}\nu\psi_x^2 + L(\tfrac{1}{8}\psi^4 - \tfrac{1}{4}\psi^2) + c_s u)\,\mathrm{d}x = m \\ \triangleq \int_0^1 (\tfrac{1}{2}\nu\varphi_{0x}^2 + L(\tfrac{1}{8}\varphi_0^4 - \tfrac{1}{4}\varphi_0^2) + c_s\theta_0)\,\mathrm{d}x, \end{aligned}\right\} \tag{6.1}$$

which is the corresponding stationary problem of system (1.8) under the assumption (1.11) and $A = L$, $r = 0$ with initial-boundary conditions (1.13) and (1.15). We aim to count the number of solutions to problem (6.1) using the plane analysis method (see [19, 25, 34]). Since most of the detailed discussions can be found in [19, 25, 34], we just sketch the proof as follows.

First, by virtue of the homogeneous Neumann boundary conditions, we have

$$u \equiv u^0, \tag{6.2}$$

where $u^0$ is a constant that satisfies

$$u^0 = \frac{1}{c_s}\biggl[m - \int_0^1 (\tfrac{1}{2}\nu\psi_x^2 + L(\tfrac{1}{8}\psi^4 - \tfrac{1}{4}\psi^2))\,\mathrm{d}x\biggr]. \tag{6.3}$$

It follows that the unknown function $\psi$ indeed satisfies a nonlinear elliptic equation with a non-local term

$$\left.\begin{aligned} -\nu\psi_{xx} + \tfrac{1}{2}L(\psi^3 - \psi) - \frac{Lu^0}{\theta_c} &= 0,\\ \psi_x|_{x=0,1} &= 0. \end{aligned}\right\} \tag{6.4}$$

We note that, in (6.3), the integral involves the derivative of $\psi$. In order to cancel this term, we multiply the equation in (6.4) by $\psi$ and integrate on $(0,1)$ to obtain that

$$\nu\|\psi_x\|^2 + \tfrac{1}{2}L\int_0^1 (\psi^4 - \psi^2)\,\mathrm{d}x - \frac{Lu^0}{\theta_c}\int_0^1 \psi\,\mathrm{d}x = 0. \tag{6.5}$$

Then a substitution of (6.5) into (6.3) leads to

$$c_s u^0 - \tfrac{1}{8}L\int_0^1 \psi^4\,\mathrm{d}x + \frac{Lu^0}{2\theta_c}\int_0^1 \psi\,\mathrm{d}x = m. \tag{6.6}$$

For the sake of simplicity, we introduce a new parameter

$$\sigma := \frac{2u^0}{\theta_c}. \tag{6.7}$$

Then the stationary problem (6.1) can be rewritten as

$$\frac{2\nu}{L}\psi_{xx} = f(\psi;\sigma) \triangleq \psi^3 - \psi - \sigma, \tag{6.8 a}$$

$$\psi_x|_{x=0,1} = 0, \tag{6.8 b}$$

$$\tfrac{1}{2}(c_s\theta_c\sigma) + L\int_0^1 (\tfrac{1}{4}\sigma\psi - \tfrac{1}{8}\psi^4)\,\mathrm{d}x = m. \tag{6.8 c}$$

Now multiplying (6.8 *a*) by $\psi_x$, we obtain the following identity:

$$\frac{\nu}{L}\psi_x^2 = F(\psi;\sigma,b) := J(\psi;\sigma) - b, \tag{6.9}$$

where

$$J(\psi;\sigma) = \tfrac{1}{4}\psi^4 - \tfrac{1}{2}\psi^2 - \sigma\psi \tag{6.10}$$

and $b$ is some constant of integration.

Without loss of generality, we look for strictly increasing solutions (see [25]). In view of (6.9), it means that the pair of parameters $(\sigma, b)$ should be such that $F$ has two zeros, $\psi_1 < \psi_2$, so that

$$F(\psi_1) = F(\psi_2) = 0 \quad \text{and} \quad F(\psi) > 0 \quad \text{for } \psi_1 < \psi < \psi_2. \tag{6.11}$$

Such a pair $(\sigma, b)$ will be called admissible and the union of all admissible pairs will be called the admissible region and will be denoted by $\Sigma$. We note that the admissible range of $\sigma$ is the open interval $(-\frac{2}{9}\sqrt{3}, \frac{2}{9}\sqrt{3})$ (see [34]) and, for $\sigma \in (-\frac{2}{9}\sqrt{3}, \frac{2}{9}\sqrt{3})$, $f(\psi;\sigma)$ has three roots $w_0(\sigma) < w_1(\sigma) < w_2(\sigma)$, and $F$ has two further zeros $\psi_0$ and $\psi_3$ such that

$$\psi_0 \leqslant w_0 \leqslant \psi_1 \leqslant w_1 \leqslant \psi_2 \leqslant w_2 \leqslant \psi_3. \tag{6.12}$$

Now, as in [30] (see also [19]), we introduce the time map $I_0$, which is defined as

$$I_0(\sigma, b) = \sqrt{\frac{\nu}{L}} \int_{\psi_1}^{\psi_2} \frac{\mathrm{d}s}{\sqrt{F(s;\sigma,b)}}. \tag{6.13}$$

We also define

$$I_1(\sigma, b) = \sqrt{\frac{\nu}{L}} \int_{\psi_1}^{\psi_2} (\tfrac{1}{2}c_s\theta_c\sigma + \tfrac{1}{4}L\sigma s - \tfrac{1}{8}Ls^4)\frac{\mathrm{d}s}{\sqrt{F(s;\sigma,b)}}, \tag{6.14}$$

and

$$R(\sigma, b) = \frac{I_1(\sigma, b)}{I_0(\sigma, b)}. \tag{6.15}$$

Then it follows from the periodicity of non-trivial solutions to (6.8) (see [25, 34]) that our problem is reduced to finding $(\sigma, b) \in \Sigma$ such that

$$I_0(\sigma, b) = 1/n \quad \text{for some } n \in \mathbb{N}, \tag{6.16}$$

and

$$R(\sigma, b) = m. \tag{6.17}$$

For the sake of argument, we introduce the level sets

$$\mathcal{C}_\lambda = \{(\sigma, b) \in \Sigma \colon I_0(\sigma, b) = \lambda\} \tag{6.18}$$

and

$$\mathcal{D}_\mu = \{(\sigma, b) \in \Sigma \colon R(\sigma, b) = \mu\}. \tag{6.19}$$

Now we examine the sketches of $\Sigma$, $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$. We refer the interested reader to [25, 34] for detailed discussions on $\Sigma$ and $\mathcal{C}_\lambda$. It follows by inspection that $\Sigma$ is symmetric in $\sigma$ and the boundary of $\Sigma$ consists of three arcs $\Gamma_0$, $\Gamma_1$ and $\Gamma_2$ defined by

$$\psi_0 = w_0 = \psi_1 \text{ on } \Gamma_0, \qquad \psi_1 = w_1 = \psi_2 \text{ on } \Gamma_1, \qquad \psi_2 = w_2 = \psi_3 \text{ on } \Gamma_2, \tag{6.20}$$

and $\Sigma$ can be expressed as

$$\Sigma = (\sigma, b) \mid \sigma \in (-\tfrac{2}{9}\sqrt{3}, \tfrac{2}{9}\sqrt{3}),\ b_0(\sigma) \leqslant b < b_1(\sigma) \text{ for } \sigma \in [0, \tfrac{2}{9}\sqrt{3}) \text{ and } b_2(\sigma) \leqslant b < b_1(\sigma) \text{ for } \sigma \in (-\tfrac{2}{9}\sqrt{3}, 0] \tag{6.21}$$

where

$$b_0(\sigma) = \inf\{b\colon (\sigma, b) \in \Sigma \cap \{\sigma \geqslant 0\}\}, \tag{6.22}$$

$$b_1(\sigma) = \sup\{b\colon (\sigma, b) \in \Sigma\}, \tag{6.23}$$

$$b_2(\sigma) = \inf\{b\colon (\sigma, b) \in \Sigma \cap \{\sigma \leqslant 0\}\}. \tag{6.24}$$

By [25, lemma 8.1], for $\lambda > \pi\sqrt{2\nu/L}$ the level set $\mathcal{C}_\lambda$ is a connected curve which is symmetric in $\sigma$ and joins the points

$$\biggl(\pm\frac{1}{3\sqrt{3}}\biggl(1-\frac{2\pi^2}{\lambda^2 L}\biggr)^{1/2}\biggl(2+\frac{2\pi^2}{\lambda^2 L}\biggr), \frac{1}{12}\biggl(1-\frac{4\pi^4}{\lambda^4 L^2}\biggr)\biggr), \quad \lambda > \pi\sqrt{\frac{2\nu}{L}}. \tag{6.25}$$

On the other hand, we can prove in a similar way as in [19] that $R(\sigma, b)$ is analytic on $(\sigma, b)$ in $\operatorname{Int}\Sigma$ and the set $\mathcal{D}_\mu$ consists of at most an infinitely countable number of curves that are either closed or otherwise have end points on the boundary of $\Sigma$. In addition, for fixed $\sigma \in (-\frac{2}{9}\sqrt{3}, \frac{2}{9}\sqrt{3})$, as $b \uparrow b_1(\sigma)$,

$$R(\sigma, b) \to \tfrac{1}{2}c_s\theta_c\sigma + \tfrac{1}{4}L\sigma w_1(\sigma) - \tfrac{1}{8}Lw_1^4(\sigma). \tag{6.26}$$

Noting that $w_1(\cdot)$ is an odd function, we infer that the level set $\mathcal{C}_\lambda$ does not coincide with any componential curve of $\mathcal{D}_\mu$ because the end points of them do not coincide.

Since $\partial I_0/\partial b < 0$ (see [25, lemma 5.1]), we get $b = b_{1/n}(\sigma)$ from $I_0(\sigma, b) = 1/n$, for $n \in \mathbb{N}$. Then substituting it into $R(\sigma, b) = m$ leads to $R(\sigma, b_{1/n}(\sigma)) = m$. Therefore, from the analyticity of $R(\sigma, b_{1/n}(\sigma))$ on $\sigma$, we know that the set $\mathcal{C}_{1/n} \cap \mathcal{D}_m$ has no accumulation point in $\operatorname{Int}\Sigma$. In other words, the only possibility of accumulation points should be the end points of the curves in $\mathcal{C}_{1/n}$, $n \in \mathbb{N}$. We may now conclude this part with the following proposition. Further discussions on the multiplicity of the problem (6.8) may be carried out in an analogous way as in [25, 34].

Proposition 6.1. *Problem (6.1) admits at most an infinitely countable number of solutions.*

Since we have proven that the $\omega$-limit set is a non-empty connected set and consists of stationary solutions, it follows from proposition 6.1 that the $\omega$-limit set is a singleton. More precisely, we have the following corollary.

Corollary 6.2. *For any $(\theta_0, \varphi_0) \in H^1 \times H_N^2$ satisfying $\theta_0 > 0$ in $[0, 1]$, the unique global solution $(\theta, \varphi)$ of problems (1.12)–(1.15) will converge to an equilibrium in $H^1 \times H^2$ as $t \to +\infty$.*

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## Appendix A. Proof of lemma 3.1

By the Sobolev embedding theorem, we have

$$\vartheta(t, x) \in C([0, +\infty), C(\bar{\Omega})).$$

Now we introduce the function $y(t) := \|u(t)\|_{L^\infty(\Omega)} - 2(\bar{\vartheta}(t))^{1/2}$, where

$$u(t,x) = \sqrt{\vartheta(t,x)}. \tag{A 1}$$

Obviously, $y(t)$ is a continuous function on $[0,+\infty)$. Hence, we can divide the interval $[0,+\infty)$ into two sets as the curve of $y(t)$ crosses the $t$-axis. We denote them by $S$ and $U$ which consist of closed intervals (the end points of these intervals are transverse intersection points), where

$$y(t) \geqslant 0, \quad \forall t \in S,$$

and

$$y(t) \leqslant 0, \quad \forall t \in U.$$

Now we argue as follows.

(i) For $t \in U$, since $y(t) \leqslant 0$ and noting (3.1), we have that

$$u(t) \leqslant 2\biggl(\int_0^1 \vartheta(t)\,\mathrm{d}x\biggr)^{1/2} \leqslant 2K_1^{1/2}, \tag{A 2}$$

hence, it holds that

$$\|\vartheta\|_{L^\infty} \leqslant 4K_1, \quad \forall t \in U. \tag{A 3}$$

Then, from (3.2), we have

$$\begin{aligned} \int_U \int_0^1 \vartheta_x^2 \,\mathrm{d}x\,\mathrm{d}\tau &= \int_U \int_0^1 \frac{\vartheta_x^2}{\vartheta^2}\vartheta^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \int_U \biggl(\|\vartheta\|_{L^\infty}^2 \int_0^1 \frac{\vartheta_x^2}{\vartheta^2}\,\mathrm{d}x\biggr)\mathrm{d}\tau \\ &\leqslant \sup_U \|\vartheta\|_{L^\infty}^2 \int_0^\infty \int_0^1 \frac{\vartheta_x^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant 16K_1^2 K_2. \end{aligned} \tag{A 4}$$

By the Poincaré inequality, we infer that

$$\int_U \|\vartheta - \bar{\vartheta}\|_{L^2}^2 \,\mathrm{d}\tau \leqslant C \int_U \int_0^1 \vartheta_x^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqslant C(K_1, K_2). \tag{A 5}$$

(ii) For fixed $t \in S$, by the continuity of $u(t,x)$ with respect to the $x$ variable, we can deduce that, at time $t$, there exists $x_0(t) \in [0,1]$ such that

$$u(x_0(t), t) = \biggl(\int_0^1 \vartheta(t)\,\mathrm{d}x\biggr)^{1/2}.$$

This can be shown by a contradiction argument. Indeed, for the time being, if, for any point $x \in [0,1]$, it holds that

$$u(x,t) > \biggl(\int_0^1 \vartheta(t)\,\mathrm{d}x\biggr)^{1/2},$$

then, using Young's inequality, we infer that

$$\left(\int_0^1 \vartheta(t)\,\mathrm{d}x\right)^{1/2} = \left(\int_0^1 u^2\,\mathrm{d}x\right)^{1/2} \geqslant \int_0^1 u\,\mathrm{d}x > \left(\int_0^1 \vartheta(t)\,\mathrm{d}x\right)^{1/2}, \tag{A 6}$$

which leads to a contradiction.

Hence, for $t \in S$, we have

$$u(x,t) - \bar{\vartheta}^{1/2}(t) = \int_{x_0(t)}^{x} u_x\,\mathrm{d}x. \tag{A 7}$$

Therefore, it follows from the Young inequality that

$$\begin{aligned}\int_S \|u - \bar{\vartheta}^{1/2}\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau &\leqslant \int_S \left(\int_0^1 |u_x|\,\mathrm{d}x\right)^2 \mathrm{d}\tau \\ &\leqslant \int_0^\infty \left(\int_0^1 \frac{u_x^2}{\vartheta}\,\mathrm{d}x\right)\left(\int_0^1 \vartheta\,\mathrm{d}x\right)\mathrm{d}\tau \\ &= \int_0^\infty \left(\int_0^1 \frac{\vartheta_x^2}{4\vartheta^2}\,\mathrm{d}x\right)\left(\int_0^1 \vartheta\,\mathrm{d}x\right)\mathrm{d}\tau \\ &\leqslant \tfrac{1}{4}K_1K_2. \end{aligned}\tag{A 8}$$

Moreover, for the time being, since $y(t) \geqslant 0$, we have

$$\bar{\vartheta}^{1/2} \leqslant \|u(t)\|_{L^\infty} - \bar{\vartheta}(t)^{1/2}, \tag{A 9}$$

hence,

$$\bar{\vartheta}^{1/2} \leqslant \|u - \bar{\vartheta}^{1/2}\|_{L^\infty}. \tag{A 10}$$

Noting

$$\vartheta - \bar{\vartheta} = u^2 - \bar{\vartheta} = |u - \bar{\vartheta}^{1/2}|^2 + 2\bar{\vartheta}^{1/2}(u - \bar{\vartheta}^{1/2}), \tag{A 11}$$

we infer from (A 10) that

$$\|\vartheta - \bar{\vartheta}\|_{L^\infty} \leqslant 3\|u - \bar{\vartheta}^{1/2}\|^2_{L^\infty}. \tag{A 12}$$

Then it follows from (A 8) that

$$\int_S \|\vartheta - \bar{\vartheta}\|_{L^\infty}\,\mathrm{d}\tau \leqslant 3\int_S \|u - \bar{\vartheta}^{1/2}\|^2_{L^\infty}\,\mathrm{d}\tau \leqslant \tfrac{3}{4}K_1K_2. \tag{A 13}$$

Now, it is obvious to see that

$$\begin{aligned}\int_S \|\vartheta - \bar{\vartheta}\|^2\,\mathrm{d}\tau &= \int_S\int_0^1 |\vartheta - \bar{\vartheta}|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \int_S \left(\|\vartheta - \bar{\vartheta}\|_{L^\infty}\int_0^1 |\vartheta - \bar{\vartheta}|\,\mathrm{d}x\right)\mathrm{d}\tau \\ &\leqslant \int_S \left(\|\vartheta - \bar{\vartheta}\|_{L^\infty}\int_0^1 (\vartheta + \bar{\vartheta})\,\mathrm{d}x\right)\mathrm{d}\tau \\ &\leqslant 2\sup_S \|\vartheta\|_{L^1}\int_S \|\vartheta - \bar{\vartheta}\|_{L^\infty}\,\mathrm{d}\tau \\ &\leqslant \tfrac{3}{2}K_1^2K_2. \end{aligned}\tag{A 14}$$

Finally, combining (A 5) and (A 14) we conclude that

$$\int_0^\infty \|\vartheta - \bar{\vartheta}\|^2 \, \mathrm{d}\tau = \int_S \|\vartheta - \bar{\vartheta}\|^2 \, \mathrm{d}\tau + \int_U \|\vartheta - \bar{\vartheta}\|^2 \, \mathrm{d}\tau \leqslant C(K_1, K_2). \tag{A 15}$$

This completes the proof.

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