Euro. Jnl of Applied Mathematics (2003), *vol.* 14, *pp.* 387–406.
DOI: 10.1017/S0956792503005151 Printed in the United Kingdom

Weak and strong formulations for the time-harmonic eddy-current problem in general multi-connected domains

A. ALONSO RODRÍGUEZ*, P. FERNANDES† and A. VALLI‡

*Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italy
†Istituto per la Matematica Applicata del C.N.R., via De Marini 6, Torre di Francia, 16149 Genova, Italy
‡Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy

(*Received 8 October 2001; revised 23 September 2002*)

The eddy-current problem for the time-harmonic Maxwell equations in domains and with conductors of general topology is considered. The existence of a unique magnetic field is proved for a suitable weak formulation. An equivalent strong formulation is then derived, where the conditions related to the specific geometry of the domain are made explicit. In particular, a new condition that must be satisfied by the magnetic field on the interface between a multiply-connected conductor and the non-conducting region is determined. Finally, the strong formulation of the problem for the electric field in the non-conducting region is derived, and the existence of a unique solution is proved. In conclusion, this leads to the determination of the complete set of equations describing the eddy-current problem in terms of the magnetic and the electric fields. Whether some commonly-used formulations satisfy the additional condition on the interface is also checked.

1 Introduction

Let us consider a bounded connected open set $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$. The unit outward normal vector on $\partial\Omega$ will be denoted by $\mathbf{n}$. We assume that $\overline{\Omega}$ is split into two parts, $\overline{\Omega} = \overline{\Omega_C} \cup \overline{\Omega_I}$, where Ω_C, a non-homogeneous non-isotropic conductor, and Ω_I, a perfect insulator, are open disjoint subsets, such that $\overline{\Omega_C} \subset \Omega$. For the sake of simplicity, we also suppose that Ω_I is connected[1].

We denote by $\Gamma := \partial\Omega_I \cap \partial\Omega_C$ the interface between the two subdomains; note that, in the present situation, $\partial\Omega_C = \Gamma$ and $\partial\Omega_I = \partial\Omega \cup \Gamma$. Moreover, let Γ_j, $j = 1, \ldots, p_\Gamma$, be the connected components of Γ, and $(\partial\Omega)_r$, $r = 0, 1, \ldots, p_{\partial\Omega}$, be the connected components of $\partial\Omega$; in particular, we denote by $(\partial\Omega)_0$ the exterior component. Finally, we indicate by $n^*_{\partial\Omega}$ the number of cycles on $\partial\Omega$ non-homotopic to zero in Ω_I, and by n^*_Γ the number of cycles on Γ non-homotopic to zero in Ω_I.

[1] The general case can be treated in a similar way, focusing on each connected component of Ω_I, but some technical modifications are needed when the boundary of a connected component of Ω_I has empty intersection with $\partial\Omega$.

As is well known, the complete Maxwell system of electromagnetism reads:

$$\begin{cases} \dfrac{\partial \mathscr{D}}{\partial t} + \mathscr{J} = \operatorname{curl} \mathscr{H} \\ \dfrac{\partial \mathscr{B}}{\partial t} + \operatorname{curl} \mathscr{E} = 0 \\ \operatorname{div} \mathscr{D} = \rho \\ \operatorname{div} \mathscr{B} = 0, \end{cases}$$

where $\mathscr{E}$ and $\mathscr{H}$ are the electric and magnetic field, $\mathscr{D}$ and $\mathscr{B}$ the electric and magnetic induction, respectively, $\mathscr{J}$ is the total electric current density, and ρ is the charge density.

We assume the constitutive relations $\mathscr{D} = \boldsymbol{\varepsilon}\mathscr{E}$, $\mathscr{B} = \boldsymbol{\mu}\mathscr{H}$, where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the dielectric and magnetic permeability tensors, respectively, as well as the generalised Ohm's law $\mathscr{J} = \boldsymbol{\sigma}\mathscr{E} + \mathscr{J}_e$, where $\boldsymbol{\sigma}$ is the electric conductivity and $\mathscr{J}_e$ is the given electric current density, driving the problem. Notice that the generalised Ohm's law holds for both conductors and insulators (see, for instance, Bossavit [6, § 1.1.3; 7, § 1.2]).

The time-harmonic Maxwell equations are derived from the complete system assuming that the electric field $\mathscr{E}$, the magnetic field $\mathscr{H}$ and the given electric current density $\mathscr{J}_e$ are of the form

$$\begin{aligned} \mathscr{E}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathscr{H}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] \\ \mathscr{J}_e(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)], \end{aligned}$$

where $\omega \neq 0$ is a given angular frequency.

In this paper, we study the time-harmonic *eddy-current problem*, in which the displacement current term $\frac{\partial \mathscr{D}}{\partial t}$ is neglected. In particular, we consider the *magnetic* boundary value problem, in which the tangential component $\mathbf{H} \times \mathbf{n}$ of the magnetic field is assumed to vanish on $\partial\Omega$. Under these assumptions, from the complete Maxwell system one easily finds the following equations:

$$\begin{cases} \operatorname{curl} \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{H} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

However, we shall see in § 5 that this problem has to be closed by additional equations, involving the electric field in the insulator Ω_I.

The magnetic permeability $\boldsymbol{\mu}$ is assumed to be a symmetric tensor, uniformly positive definite in Ω, with entries in $L^\infty(\Omega)$. The same assumption holds for the dielectric coefficient $\boldsymbol{\varepsilon}$ in Ω_I, which is not present in (1.1), but will appear in the final problem. Since Ω_I is a perfect insulator, we require that $\boldsymbol{\sigma}_{|\Omega_I} \equiv \mathbf{0}$; moreover, as Ω_C is a non-homogeneous non-isotropic conductor, $\boldsymbol{\sigma}_{|\Omega_C}$ is assumed to be a symmetric tensor, uniformly positive definite in Ω_C, with entries in $L^\infty(\Omega_C)$. The driving current density $\mathbf{J}_e$ is not assumed to vanish in Ω_C, so that not only eddy-currents induced by external windings can be modelled, but also the skin effect in current-driven conductors and hybrid situations like that reported in Bossavit [6, § 5.21, Figs. 5.3, 5.4].

The same problem $(1.1)_1$, $(1.1)_2$, together with the *electric* boundary condition $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, has been considered in Alonso & Valli [2] (see also Bossavit [8]). However, the resolution technique in Alonso & Valli [2] is based on the elimination of $\mathbf{H}$; here, we solve the problem by a different approach, based on the elimination of the electric field $\mathbf{E}$. A similar procedure, in the case $\Omega_I = \mathbb{R}^3 \setminus \overline{\Omega_C}$, Ω_I simply-connected, is presented in Bossavit [6, Chapter 5].

Our aim is to give an existence and uniqueness result for the magnetic and the electric fields $\mathbf{H}$ and $\mathbf{E}$ without assuming topological restrictions either on the domain Ω or on the conductor Ω_C.

The strong form of the eddy-current system is given in § 5: we note that, to our knowledge, this complete set of equations has not yet been presented in the literature. The proof that it is well-posed depends upon its reformulation as a couple of equivalent problems. The first one, that will be called the magnetic eddy-current problem, is presented in § 4. It only concerns the magnetic field $\mathbf{H}$ and can be solved independently, giving also the electric field $\mathbf{E}_C$ in Ω_C. The second one gives the electric field $\mathbf{E}_I$ in Ω_I, and is solvable once $\mathbf{H}$ and $\mathbf{E}_C$ have been determined. To our knowledge, also the strong form of the magnetic eddy-current problem has not yet appeared in the literature for topologically general Ω_C and Ω_I. In particular, this general strong formulation could be interesting for a numerical approximation algorithm based on finite differences or collocation methods. Moreover, the standard way of developing widely-used vector potential formulations is based on the introduction of potentials in strong formulations of this kind.

We want to focus attention especially on the possibility of 'singular' cycles on Γ, a situation which, to our knowledge, has never been completely analysed before. They lead to an additional interface condition between $\mathbf{H}_{|\Omega_I}$ and $\mathbf{H}_{|\Omega_C}$, without which no electric field exists that satisfies both the Ampère's law $(1.1)_1$ in Ω_C and Faraday's law $(1.1)_2$ in the whole of Ω. Hence, any correct formulation of the time-harmonic eddy-current problem in term of the magnetic field must contain (implicitly or explicitly) this additional interface condition. We want to underline that this is not the case, for instance, in the formulation considered by Reissel [21], or else in the one reported in Kanayama & Kikuchi [17] ($(\mathbf{H}_C, \mathbf{J}_C)$-$\mathbf{H}_I$ formulation). On the other hand, the well-known vector potential formulations $\mathbf{A}^*_C$-$\mathbf{A}_I$, $(\mathbf{A}_C, V_C)$-$\mathbf{A}_I$ and $(\mathbf{T}_C, \Phi_C)$-$\mathbf{A}_I$ reported in Bíró [5] furnish a magnetic field $\mathbf{H}$ that implicitly satisfies the additional interface condition. Let us finally point out that no 'exotic' configuration is required in order for singular cycles to exist. In fact, they occur for any multiply-connected conductor, a rather common situation in engineering practice.

In § 3 we show that the weak problem, obtained from (1.1) by a well-known procedure (e.g. see Bossavit [6, Chapter 5]), has a unique solution, easily related to the magnetic field $\mathbf{H}$. Moreover, we prove in § 4 that the solution $\mathbf{H}$ is also the unique solution to a strong problem, which, therefore, turns out to be equivalent to the weak problem. Concerning with the unique solvability of the strong problem for the electric field $\mathbf{E}$ in Ω_I, which we study in § 5, we have only to quote the results given in Alonso & Valli [1], where a suitable weak formulation is also presented. As a consequence, the complete eddy-current problem also turns out to be well-posed. § 6 is devoted to the question of whether some frequently-used formulations furnish a solution $\mathbf{H}$ that satisfies the additional interface condition due to the presence of the singular cycles.

2 Notation and preliminaries

As usual, we indicate by $L^2(\Omega)$ the space of real or complex measurable functions which are square-integrable in Ω, and we set $H^1(\Omega) := \{\varphi \in L^2(\Omega) \,|\, \nabla\varphi \in (L^2(\Omega))^3\}$.

The space $H(\mathrm{curl};\Omega)$ (respectively, $H(\mathrm{div};\boldsymbol{\mu};\Omega)$) indicates the set of the real or complex vector functions $\boldsymbol{\phi} \in (L^2(\Omega))^3$ such that $\mathrm{curl}\,\boldsymbol{\phi} \in (L^2(\Omega))^3$ (respectively, $\mathrm{div}(\boldsymbol{\mu}\boldsymbol{\phi}) \in L^2(\Omega)$). By $H_0(\mathrm{curl};\Omega)$, we indicate the subspace of $H(\mathrm{curl};\Omega)$ comprising those functions $\boldsymbol{\phi}$ satisfying $(\boldsymbol{\phi}\times\mathbf{n})_{|\partial\Omega} = \mathbf{0}$. Similarly, by $H_0(\mathrm{div};\boldsymbol{\mu};\Omega)$ we indicate the subspace of $H(\mathrm{div};\boldsymbol{\mu};\Omega)$ comprising those functions $\boldsymbol{\phi}$ satisfying $(\boldsymbol{\mu}\boldsymbol{\phi}\cdot\mathbf{n})_{|\partial\Omega} = 0$.

It is also useful to recall that a function $\boldsymbol{\phi}$ belongs to $H(\mathrm{curl};\Omega)$ if and only if its restrictions $\boldsymbol{\phi}_{|\Omega_I}$ and $\boldsymbol{\phi}_{|\Omega_C}$ belong to $H(\mathrm{curl};\Omega_I)$ and $H(\mathrm{curl};\Omega_C)$, respectively, and $(\boldsymbol{\phi}_{|\Omega_I}\times\mathbf{n}_I)_{|\Gamma} = (\boldsymbol{\phi}_{|\Omega_C}\times\mathbf{n}_I)_{|\Gamma}$, where we denote by $\mathbf{n}_I$ the unit outward normal vector to Ω_I on Γ. Similarly, a function $\boldsymbol{\phi}$ belongs to $H(\mathrm{div};\boldsymbol{\mu};\Omega)$ if and only if its restrictions $\boldsymbol{\phi}_{|\Omega_I}$ and $\boldsymbol{\phi}_{|\Omega_C}$ belong to $H(\mathrm{div};\boldsymbol{\mu}_{|\Omega_I};\Omega_I)$ and $H(\mathrm{div};\boldsymbol{\mu}_{|\Omega_C};\Omega_C)$, respectively, and $(\boldsymbol{\mu}_{|\Omega_I}\boldsymbol{\phi}_{|\Omega_I}\cdot\mathbf{n}_I)_{|\Gamma} = (\boldsymbol{\mu}_{|\Omega_C}\boldsymbol{\phi}_{|\Omega_C}\cdot\mathbf{n}_I)_{|\Gamma}$.

The following Greens formulae are well known for continuously differentiable functions and for a domain Ω with a boundary $\partial\Omega$ of class $C^{1,1}$:

$$\int_\Omega (\mathbf{v}\cdot\nabla\varphi + \varphi\,\mathrm{div}\,\mathbf{v}) = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,\varphi_{|\partial\Omega} \tag{2.1}$$

$$\int_\Omega (\mathbf{u}\cdot\mathrm{curl}\,\mathbf{w} - \mathrm{curl}\,\mathbf{u}\cdot\mathbf{w}) = \int_{\partial\Omega} (\mathbf{u}\times\mathbf{n})\cdot(\mathbf{n}\times\mathbf{w}\times\mathbf{n}). \tag{2.2}$$

They still hold, in a suitable weak sense, for $\mathbf{v}\in H(\mathrm{div};\Omega)$, $\varphi\in H^1(\Omega)$, $\mathbf{u}$, $\mathbf{w}\in H(\mathrm{curl};\Omega)$[2], and also when Ω is a polyhedral domain with a Lipschitz boundary[3] (see Cessenat [13], Buffa & Ciarlet [11, 12] and Buffa [10]).

Moreover, when $\boldsymbol{\lambda}$ is a vector field on $\partial\Omega$ satisfying $\boldsymbol{\lambda}\cdot\mathbf{n} = 0$, we have

$$\int_{\partial\Omega} (\mathrm{div}_\tau\boldsymbol{\lambda})\,\varphi_{|\partial\Omega} = -\int_{\partial\Omega} \boldsymbol{\lambda}\cdot(\mathbf{n}\times\nabla\varphi\times\mathbf{n}), \tag{2.3}$$

for all $\varphi\in H^1(\Omega)$.

Finally, we recall that for all $\mathbf{v}\in H(\mathrm{curl};\Omega)$ the following relation holds:

$$\mathrm{div}_\tau(\mathbf{v}\times\mathbf{n}) = \mathrm{curl}\,\mathbf{v}\cdot\mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.4}$$

All these formulae will be frequently needed in the sequel: therefore, from now on we assume that:

either the boundary $\partial\Omega$ is of class $C^{1,1}$, or else Ω is a polyhedral domain with a Lipschitz boundary. (H1)

[2] Since any vector field can be written on $\partial\Omega$ as $\mathbf{w} = (\mathbf{w}\cdot\mathbf{n})\,\mathbf{n} + \mathbf{n}\times\mathbf{w}\times\mathbf{n}$, the right-hand side in (2.2) could be written as $\int_{\partial\Omega}(\mathbf{u}\times\mathbf{n})\cdot\mathbf{w}$. However, we have preferred to keep the former expression, which emphasizes that it is the tangential values of $\mathbf{u}$ and $\mathbf{w}$ that are the natural boundary conditions for the space $H(\mathrm{curl};\Omega)$.

[3] Note that a polyhedral domain can have a non-Lipschitz boundary: an example is furnished by two bricks in a pile, where the upper one is rotated by an angle equal to $\pi/2$.

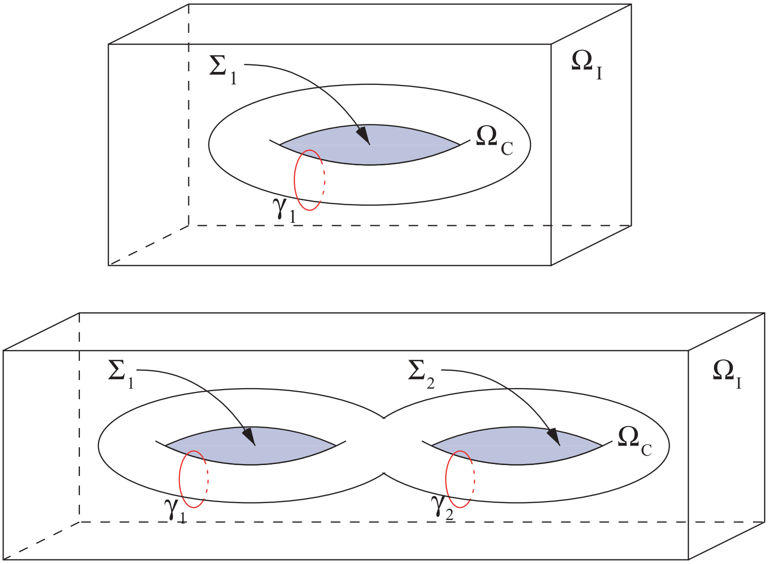

FIGURE 1. The 'cutting' surfaces Σ_m, $m = 1, 2$, when the conductor Ω_C is a torus (top) or a double-torus (bottom).

We also introduce two linear spaces of harmonic vector fields. Since we want to construct an explicit basis for each, it is useful to make some other geometrical assumptions on Ω_I. We assume that (e.g. see Foias & Temam [16], Picard [20], Amrouche *et al.* [4] and Fernandes & Gilardi [15]):

there exist n_Γ 'cuts' Σ_m, which are the interior of two-dimensional, mutually disjoint, compact and connected Lipschitz manifolds $\overline{\Sigma_m}$ with boundary $\partial\Sigma_m$, such that $\Sigma_m \subset \Omega_I$ and $\partial\Sigma_m \subset \Gamma$, and such that in the open set $\widehat{\Omega_I} := \Omega_I \setminus \cup_m \Sigma_m$, assumed to be connected, every curl-free vector field with vanishing tangential component on $\partial\Omega$ has a global potential. (H2)

These cuts are named 'essential' cuts in Fernandes & Gilardi [15]. Note that, in general, we have $n_\Gamma \leqslant n_\Gamma^*$; for example, if Ω and Ω_C are two coaxial tori, we have $n_\Gamma^* = 2$ and $n_\Gamma = 0$.

In a similar way, we assume that

there exist $n_{\partial\Omega}$ 'cuts' Σ_i^*, which are the interior of two-dimensional, mutually disjoint, compact and connected Lipschitz manifolds $\overline{\Sigma_i^*}$ with boundary $\partial\Sigma_i^*$, such that $\Sigma_i^* \subset \Omega_I$ and $\partial\Sigma_i^* \subset \partial\Omega$, and such that in the open set $\widetilde{\Omega_I} := \Omega_I \setminus \cup_i \Sigma_i^*$, assumed to be connected, every curl-free vector field with vanishing tangential component on Γ has a global potential. (H3)

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

Let us introduce now the following space of harmonic fields:

$$\begin{aligned}\mathscr{H}_{\mu_I}(\partial\Omega;\Gamma) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl}\mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu}_I\mathbf{v}_I) = 0,\\ \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega, \boldsymbol{\mu}_I\mathbf{v}_I \cdot \mathbf{n}_I = 0 \text{ on } \Gamma\},\end{aligned} \tag{2.5}$$

where we have set $\boldsymbol{\mu}_I := \boldsymbol{\mu}_{|\Omega_I}$. It is well-known that this space has finite dimension. The dimension of $\mathscr{H}_{\mu_I}(\partial\Omega;\Gamma)$ is equal to $n_\Gamma + p_{\partial\Omega}$ (see Fernandes & Gilardi [15, Proposition 5.6]; see also Alonso & Valli [1], Kress [18] and Picard [19, 20]). A basis for this space is given by ∇z_r, $r = 1,\dots,p_{\partial\Omega}$, and $\boldsymbol{\rho}_l$, $l = 1,\dots,n_\Gamma$, where

$$\begin{cases}\operatorname{div}(\boldsymbol{\mu}_I\nabla z_r) = 0 & \text{in } \Omega_I\\ \boldsymbol{\mu}_I\nabla z_r \cdot \mathbf{n}_I = 0 & \text{on } \Gamma\\ z_r = 0 & \text{on } \partial\Omega \setminus (\partial\Omega)_r\\ z_r = 1 & \text{on } (\partial\Omega)_r,\end{cases} \tag{2.6}$$

and $\boldsymbol{\rho}_l \in \mathscr{H}_{\mu_I}(\partial\Omega;\Gamma)$ satisfy

$$\int_{\gamma_m} \boldsymbol{\rho}_l \cdot d\boldsymbol{\gamma} = \delta_{lm} \tag{2.7}$$

for each ‘basis’ cycle γ_m on Γ. We recall that, for an irrotational vector field belonging to $(L^2(\Omega_I))^3$, the definition of the line integrals in (2.7) has a meaning (e.g. see Dautray & Lions [14]).

In a similar way, one can define another space of harmonic fields

$$\begin{aligned}\mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl}\mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\varepsilon}_I\mathbf{v}_I) = 0,\\ \mathbf{v}_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma, \boldsymbol{\varepsilon}_I\mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

The dimension of $\mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega)$ is $n_{\partial\Omega} + p_\Gamma - 1$. A basis for this space is given by ∇w_j, $j = 1,\dots,p_\Gamma - 1$, and $\boldsymbol{\pi}_k$, $k = 1,\dots,n_{\partial\Omega}$, where

$$\begin{cases}\operatorname{div}(\boldsymbol{\varepsilon}_I\nabla w_j) = 0 & \text{in } \Omega_I\\ \boldsymbol{\varepsilon}_I\nabla w_j \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\\ w_j = 0 & \text{on } \Gamma \setminus \Gamma_j\\ w_j = 1 & \text{on } \Gamma_j,\end{cases}$$

and $\boldsymbol{\pi}_k \in \mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega)$ satisfy

$$\int_{\alpha_i} \boldsymbol{\pi}_k \cdot d\boldsymbol{\alpha} = \delta_{ki}$$

for each ‘basis’ cycle α_i on $\partial\Omega$.

The construction of the basis functions $\boldsymbol{\rho}_l \in \mathscr{H}_{\mu_I}(\partial\Omega;\Gamma)$ and $\pi_k \in \mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega)$ can be done in a more explicit way by resorting to the solution of a suitable elliptic problem in $\widehat{\Omega_I}$ and $\widetilde{\Omega_I}$, respectively. For example, it is well known that in $\widehat{\Omega_I}$ the basis function $\boldsymbol{\rho}_l$ is

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

the gradient of p_l, the solution of

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}\nabla p_l) = 0 & \text{in } \widehat{\Omega_I} \\ \boldsymbol{\mu}_I \nabla p_l \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \setminus \cup_m \partial\Sigma_m \\ p_l = 0 & \text{on } \partial\Omega \\ [\boldsymbol{\mu}_I \nabla p_l \cdot \mathbf{n}_\Sigma]_{\Sigma_m} = 0 & \text{for each } m = 1, \ldots, n_\Gamma \\ [p_l]_{\Sigma_m} = \delta_{lm} & \text{for each } m = 1, \ldots, n_\Gamma, \end{cases}$$

where $[\cdot]_{\Sigma_m}$ denotes the jump across the surface Σ_m (e.g. see Foias & Temam [16], where this construction is used for the space of tangential harmonic fields).

3 The weak formulation of the magnetic eddy-current problem

From now on we assume either that $\partial\Omega \in C^{1,1}$ and $\Gamma \in C^{1,1}$, or that Ω, Ω_I and Ω_C are polyhedral domains with a Lipschitz boundary, and that assumptions $(H2)$ and $(H3)$, introduced in the preceding section, are satisfied.

First, from $(1.1)_1$ and $(1.1)_3$, we have that in Ω_I (i.e. where $\boldsymbol{\sigma} = \mathbf{0}$), the current density $\mathbf{J}_e$ must be the curl of a vector field with vanishing tangential component on $\partial\Omega$ (indeed, the magnetic field $\mathbf{H}$). Therefore, well-known necessary conditions for $\mathbf{J}_e$ are

$$\operatorname{div}(\mathbf{J}_{e|\Omega_I}) = 0 \quad \text{in } \Omega_I, \ \mathbf{J}_{e|\Omega_I} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.1}$$

However, this is not enough: in fact, as shown in Alonso & Valli [1], any vector field being the curl of a tangential vector field has to satisfy two additional necessary conditions, which are related to the topology of Ω_I. Precisely, the conditions are

$$\int_{\Gamma_j} \mathbf{J}_{e|\Omega_I} \cdot \mathbf{n}_I = 0 \quad \forall\, j = 1, \ldots, p_\Gamma - 1, \int_{\Omega_I} \mathbf{J}_{e|\Omega_I} \cdot \boldsymbol{\pi}_k = 0 \quad \forall\, k = 1, \ldots, n_{\partial\Omega} \tag{3.2}$$

(see § 2 for notation).

Hence, in the following we assume that the current density $\mathbf{J}_e \in (L^2(\Omega))^3$ satisfies (3.1) and (3.2). As a consequence, Theorem 4.2 in Alonso & Valli [1] shows that there exists a vector field $\mathbf{H}_{e,I} \in H(\text{curl}; \Omega_I)$ satisfying

$$\begin{cases} \operatorname{curl} \mathbf{H}_{e,I} = \mathbf{J}_{e|\Omega_I} & \text{in } \Omega_I \\ \mathbf{H}_{e,I} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

and we can also construct a vector field $\mathbf{H}_{e,C} \in H(\text{curl}; \Omega_C)$ such that

$$\mathbf{H}_{e,C} \times \mathbf{n}_C + \mathbf{H}_{e,I} \times \mathbf{n}_I = \mathbf{0} \quad \text{on } \Gamma. \tag{3.4}$$

We note that the existence result reported in Alonso & Valli [1] is true not only for a domain Ω_I with a $C^{1,1}$ boundary, but also for a polyhedral domain with a Lipschitz boundary. In fact, it is only based on the compactness of the immersion of X_I in $(L^2(\Omega_I))^3$, where

$$X_I := \{\mathbf{v}_I \in H(\text{curl}; \Omega_I) \cap H(\text{div}; \Omega_I) \,|\, \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathbf{v}_I \cdot \mathbf{n}_I = 0 \text{ on } \partial\Omega\}.$$

This compactness result can be found in Fernandes & Gilardi [15, Proposition 7.3], or else follows from the regularity results in Alonso & Valli [3, Theorems 4.3 and 4.4].

By following the same approach presented in Bossavit [6, Chapter 5], from (1.1) we obtain the weak formulation we are looking for. Let us start introducing the Hilbert space of *complex-valued* vector functions:

$$V := \{\mathbf{v} \in H_0(\mathrm{curl};\Omega) \mid \mathrm{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I\}. \tag{3.5}$$

Let us write $\mathbf{E}_I := \mathbf{E}_{|\Omega_I}$, $\mathbf{E}_C := \mathbf{E}_{|\Omega_C}$, and similarly for $\mathbf{H}$, $\mathbf{J}_e$, $\boldsymbol{\mu}$, $\boldsymbol{\varepsilon}$ and the test functions $\mathbf{v} \in V$. Then, consider $(1.1)_2$, multiply by $\overline{\mathbf{v}}$, where $\mathbf{v} \in V$, and integrate in Ω. We find

$$\int_\Omega \mathrm{curl}\,\mathbf{E} \cdot \overline{\mathbf{v}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0.$$

On the other hand, using the Green formula (2.2) and recalling that $\mathbf{v} \in V$, we have

$$\int_\Omega \mathrm{curl}\,\mathbf{E} \cdot \overline{\mathbf{v}} = \int_{\Omega_C} \mathbf{E}_C \cdot \mathrm{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_I} \mathbf{E}_I \cdot \mathrm{curl}\,\overline{\mathbf{v}_I} = \int_{\Omega_C} \mathbf{E}_C \cdot \mathrm{curl}\,\overline{\mathbf{v}_C}.$$

Using $(1.1)_1$ in Ω_C to express $\mathbf{E}_C$, it follows at once that the solution $\mathbf{H}$ to (1.1) satisfies

$$\int_{\Omega_C} (\boldsymbol{\sigma}^{-1}\,\mathrm{curl}\,\mathbf{H}_C \cdot \mathrm{curl}\,\overline{\mathbf{v}_C} + i\omega\boldsymbol{\mu}_C\mathbf{H}_C \cdot \overline{\mathbf{v}_C}) + \int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \overline{\mathbf{v}_I} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathrm{curl}\,\overline{\mathbf{v}_C}.$$

Let us set

$$\mathbf{Z} := \begin{cases} \mathbf{H}_I - \mathbf{H}_{e,I} & \text{in } \Omega_I \\ \mathbf{H}_C - \mathbf{H}_{e,C} & \text{in } \Omega_C, \end{cases} \tag{3.6}$$

and denote as usual $\mathbf{Z}_I := \mathbf{Z}_{|\Omega_I}$, $\mathbf{Z}_C := \mathbf{Z}_{|\Omega_C}$. From $(1.1)_1$ in Ω_I and $\mathbf{H} \in H_0(\mathrm{curl};\Omega)$, it follows that $\mathbf{Z} \in V$. Let us define in $H(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega)$ the bilinear form $\mathscr{A}(\cdot,\cdot)$ as

$$\begin{aligned} \mathscr{A}(\mathbf{w},\mathbf{v}) := & \int_{\Omega_C} (\boldsymbol{\sigma}^{-1}\,\mathrm{curl}\,\mathbf{w}_C \cdot \mathrm{curl}\,\overline{\mathbf{v}_C} + i\omega\boldsymbol{\mu}_C\mathbf{w}_C \cdot \overline{\mathbf{v}_C}) \\ & + \int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{w}_I \cdot \overline{\mathbf{v}_I}. \end{aligned} \tag{3.7}$$

Therefore, we have that $\mathbf{Z} \in V$ satisfies

$$\mathscr{A}(\mathbf{Z},\mathbf{v}) = -\mathscr{A}(\mathbf{H}^*,\mathbf{v}) + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathrm{curl}\,\overline{\mathbf{v}_C} \quad \forall\, \mathbf{v} \in V, \tag{3.8}$$

having defined $\mathbf{H}^* \in H_0(\mathrm{curl};\Omega)$ as

$$\mathbf{H}^* := \begin{cases} \mathbf{H}_{e,I} & \text{in } \Omega_I \\ \mathbf{H}_{e,C} & \text{in } \Omega_C \end{cases}. \tag{3.9}$$

The weak formulation we are interested in is therefore given by (3.8), and we can prove at once the following existence and uniqueness result:

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

Theorem 3.1 *The weak problem* (3.8) *has a unique solution.*

Proof Let us recall that the (complex) Hilbert space V is endowed with the natural norm

$$||\mathbf{v}||_V^2 := \int_\Omega |\mathbf{v}|^2 + \int_{\Omega_C} |\operatorname{curl} \mathbf{v}_C|^2.$$

Then one has only to note that the bilinear form $\mathscr{A}(\cdot,\cdot)$ is continuous and coercive in $V \times V$. □

We note that a variational formulation which is essentially the same as (3.8) has been obtained in Bossavit & Vérité [9] (see also Bossavit [6, Chapter 5]; in these cases, the insulator Ω_I is an unbounded domain, precisely $\Omega_I = \mathbf{R}^3 \setminus \overline{\Omega_C}$).

4 The strong formulation of the magnetic eddy-current problem

We want to show now that the solution $\mathbf{Z}$ of (3.8) is indeed a solution of a suitable strong problem. From now on, let us substitute $\mathbf{H} := \mathbf{Z} + \mathbf{H}^*$ back in the weak formulation (3.8).

Choose in (3.8) the test function $\mathbf{v} \in V$ such that $\mathbf{v}_C \in (C_0^\infty(\Omega_C))^3$ and $\mathbf{v}_I = \mathbf{0}$. We have

$$\int_{\Omega_C} (\boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{H}_C \cdot \operatorname{curl} \overline{\mathbf{v}_C} + i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \overline{\mathbf{v}_C}) = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}_C},$$

which by integration by parts gives

$$\operatorname{curl}(\boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{H}_C) + i\omega \boldsymbol{\mu}_C \mathbf{H}_C = \operatorname{curl}(\boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C}) \qquad \text{in } \Omega_C. \tag{4.1}$$

In particular, $\operatorname{div}(\boldsymbol{\mu}_C \mathbf{H}_C) = 0$ in Ω_C.

Take in (3.8) the test function $\mathbf{v} = \nabla\phi$, where $\phi_I \in C_0^\infty(\Omega_I)$ and $\phi_C = 0$. Clearly $\nabla\phi \in V$, and then

$$\int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \nabla\phi_I = 0.$$

Integrating by parts, we find

$$\operatorname{div}(\boldsymbol{\mu}_I \mathbf{H}_I) = 0 \qquad \text{in } \Omega_I. \tag{4.2}$$

Take now an arbitrary complex function η on Γ, and denote by $\psi_{\eta,I} \in H^1(\Omega_I)$ a function such that $\psi_{\eta,I} = \eta$ on Γ and $\psi_{\eta,I} = 0$ on $\partial\Omega$. Denote also by $\mathbf{v}_{\eta,C} \in H(\operatorname{curl}; \Omega_C)$ a function satisfying $\mathbf{v}_{\eta,C} \times \mathbf{n}_C + \nabla\psi_{\eta,I} \times \mathbf{n}_I = \mathbf{0}$ on Γ. The function

$$\mathbf{v}_\eta := \begin{cases} \nabla\psi_{\eta,I} & \text{in } \Omega_I \\ \mathbf{v}_{\eta,C} & \text{in } \Omega_C \end{cases}$$

belongs to V. We want to use it as a test function in (3.8). First, from (2.1) and (4.2) we find

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \nabla\overline{\psi_{\eta,I}} = i\omega \int_\Gamma \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_I \, \overline{\eta}.$$

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

On the other hand, from (2.2) and (4.1) we have

$$\begin{aligned}
\int_{\Omega_C} & [\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C}) \cdot \operatorname{curl}\overline{\mathbf{v}_{\eta,C}} + i\omega\boldsymbol{\mu}_C\mathbf{H}_C \cdot \overline{\mathbf{v}_{\eta,C}}] \\
&= \int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_C \times \overline{\mathbf{v}_{\eta,C}} \times \mathbf{n}_C) \\
&= \int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \nabla\overline{\psi_{\eta,I}} \times \mathbf{n}_I).
\end{aligned}$$

From (2.3), (2.1) and (4.1) we can conclude that

$$\begin{aligned}
\int_\Gamma & [\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \nabla\overline{\psi_{\eta,I}} \times \mathbf{n}_I) \\
&= -\int_\Gamma \operatorname{div}_\tau([\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C)\,\overline{\eta} \\
&= -\int_\Gamma \operatorname{curl}[\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \cdot \mathbf{n}_C\,\overline{\eta} \\
&= i\omega \int_\Gamma \boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C\,\overline{\eta}.
\end{aligned}$$

Summing up, taking $\mathbf{v}_\eta$ as a test function in (3.8), we have obtained

$$\int_\Gamma (\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}_I + \boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C)\,\overline{\eta} = 0,$$

and hence, due to the arbitrarity of η, the interface equation

$$\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}_I + \boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C = 0 \quad \text{on } \Gamma. \tag{4.3}$$

For each function z_r, $r = 1, \dots, p_{\partial\Omega}$, defined in (2.6), let us denote by $\mathbf{v}_{r,C}$ a function belonging to $H(\operatorname{curl};\Omega_C)$ and satisfying $\mathbf{v}_{r,C} \times \mathbf{n}_C + \nabla z_r \times \mathbf{n}_I = \mathbf{0}$ on Γ. Then, the function

$$\mathbf{v}_r := \begin{cases} \nabla z_r & \text{in } \Omega_I \\ \mathbf{v}_{r,C} & \text{in } \Omega_C \end{cases}$$

belongs to V. By proceeding as before and using also (4.3), we easily find

$$\int_{(\partial\Omega)_r} \boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n} = 0 \quad \forall\, r = 1, \dots, p_{\partial\Omega}.$$

Finally, denoting by $\mathbf{v}_{l,C} \in H(\operatorname{curl};\Omega_C)$ a function satisfying $\mathbf{v}_{l,C} \times \mathbf{n}_C + \boldsymbol{\rho}_l \times \mathbf{n}_I = \mathbf{0}$ on Γ, where the function $\boldsymbol{\rho}_l$, $l = 1, \dots, n_\Gamma$, is defined in (2.7), the function

$$\mathbf{v}_l := \begin{cases} \boldsymbol{\rho}_l & \text{in } \Omega_I \\ \mathbf{v}_{l,C} & \text{in } \Omega_C \end{cases}$$

belongs to V. Taking it as a test function in (3.8), from (2.2) and (4.1) one obtains at once

$$\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \boldsymbol{\rho}_l + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) = 0 \quad \forall\, l = 1, \dots, n_\Gamma.$$

Moreover, since $\mathbf{Z} \in V$, we also have $\operatorname{curl} \mathbf{Z}_I = \mathbf{0}$ in Ω_I, $\mathbf{Z}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and $\mathbf{Z}_C \times \mathbf{n}_C + \mathbf{Z}_I \times \mathbf{n}_I = \mathbf{0}$ on Γ.

Summing up, using also (3.3) and (3.4), the magnetic field $\mathbf{H} = \mathbf{Z} + \mathbf{H}^*$ satisfies the strong problem

$$\operatorname{curl}(\boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{H}_C) + i\omega\boldsymbol{\mu}_C \mathbf{H}_C = \operatorname{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) \quad \text{in } \Omega_C \tag{4.4}$$

$$\operatorname{curl} \mathbf{H}_I = \mathbf{J}_{e,I} \quad \text{in } \Omega_I \tag{4.5}$$

$$\operatorname{div}(\boldsymbol{\mu}_I \mathbf{H}_I) = 0 \quad \text{in } \Omega_I \tag{4.6}$$

$$\int_{(\partial\Omega)_r} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0 \quad \forall\, r = 1, \dots, p_{\partial\Omega} \tag{4.7}$$

$$\int_{\Omega_I} i\omega\boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_l + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl} \mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) = 0 \qquad \forall\, l = 1, \dots, n_\Gamma \tag{4.8}$$

$$\mathbf{H}_I \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \tag{4.9}$$

$$\boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_I + \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C = 0 \quad \text{on } \Gamma \tag{4.10}$$

$$\mathbf{H}_I \times \mathbf{n}_I + \mathbf{H}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on } \Gamma. \tag{4.11}$$

Equations (4.7) and (4.8) take into account the topology of Ω_I. The physical interpretation of (4.7) is simply that there is no 'magnetic charge' hidden in the 'holes' of Ω (namely, in the regions surrounded by $(\partial\Omega)_r$, $r = 1, \dots, p_{\partial\Omega}$).

On the other hand, (4.8) is much more interesting, since it can be seen as an additional interface condition between $\mathbf{H}_C$ and $\mathbf{H}_I$, and, to the best of the authors' knowledge, has never been written before. In fact, the independent interface conditions implicitly contained in the Maxwell system (1.1) are condition (4.11) and the continuity of the tangential component of $\mathbf{E}$:

$$\mathbf{E}_I \times \mathbf{n}_I + \mathbf{E}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on } \Gamma. \tag{4.12}$$

It should be noted that the matching condition (4.10) is weaker than (4.12) (it could be obtained from (4.12) through (2.4) and $(1.1)_2$), and therefore, for an interface Γ of general geometry, must be strenghtened in a suitable way.

We claim that problem (4.4)–(4.11) has a unique solution having existence been proved above. In the next Section we will prove that a solution to (4.4)–(4.11), together with a suitable electric field $\mathbf{E}$, gives a solution to (1.1), hence a solution to the weak problem (3.8). Since this last problem has a unique solution, also the solution to (4.4)–(4.11) is unique, and the strong problem (4.4)–(4.11) is equivalent to the weak problem (3.8).

Moreover, we emphasize that, if we drop condition (4.8) from problem (4.4)–(4.11), the remaining problem is not well-posed, as uniqueness does not hold as we can see from the following: let us assume for simplicity that $p_{\partial\Omega} = 0$ (so that (4.7) disappears) and that $n_\Gamma = 1$ (namely, in this case Ω_C is a torus, and there is only one basis cycle γ_1 on Γ). Consider the Hilbert space

$$V_0 := \left\{ \mathbf{v} \in H_0(\operatorname{curl}; \Omega) \mid \operatorname{curl} \mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I\ ,\ \int_{\gamma_1} \mathbf{v}_I \cdot d\gamma = 0 \right\},$$

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

and recall the definition of the bilinear form $\mathscr{A}(\cdot,\cdot)$ in (3.7). By the Lax–Milgram lemma, for each complex number $h_{I,1}$ one can find a unique solution of the problem

$$\mathbf{W} \in V_0 \ : \ \mathscr{A}(\mathbf{W}, \mathbf{v}_0) = -h_{I,1}\mathscr{A}(\boldsymbol{\varrho}^*, \mathbf{v}_0) \quad \forall\ \mathbf{v}_0 \in V_0,$$

having defined $\boldsymbol{\varrho}^* \in H_0(\mathrm{curl};\Omega)$ as

$$\boldsymbol{\varrho}^* := \begin{cases} \boldsymbol{\varrho}_1 & \text{in } \Omega_I \\ \boldsymbol{\varrho}_C & \text{in } \Omega_C \end{cases},$$

where $\boldsymbol{\varrho}_C \in H(\mathrm{curl};\Omega_C)$ satisfies $\boldsymbol{\varrho}_1 \times \mathbf{n}_I + \boldsymbol{\varrho}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ. On the other hand, setting $\mathbf{H} := \mathbf{W} + h_{I,1}\boldsymbol{\varrho}^*$, by proceeding as before it is easily proved that $\mathbf{H}$ is a solution to (4.4), (4.5), (4.6), (4.9), (4.10) and (4.11) for $\mathbf{J}_e = \mathbf{0}$, and this is true for each choice of the complex number $h_{I,1}$. Since $\int_{\gamma_1} \mathbf{H}_I \cdot d\boldsymbol{\gamma} = h_{I,1}$, uniqueness does not hold for (4.4), (4.5), (4.6), (4.9), (4.10) and (4.11).

It is apparent that by dropping (4.8) we have lost just the information determining the 'circulation' of $\mathbf{H}_I$ along the basis cycle γ_1. Hence, condition (4.8) should determine that circulation in some way. However, we shall see later on that the most natural physical interpretation of (4.8) is given rather in terms of the flux of $\boldsymbol{\mu}_I\mathbf{H}_I$ across the 'cut' Σ_1 cutting the basis cycle γ_1.

5 The strong formulation of the eddy-current problem

The problem (4.4)–(4.11) has been analysed. We have now to determine the electric field $\mathbf{E}$ in Ω.

First, from $(1.1)_1$, in Ω_C we can write

$$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathrm{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}. \tag{5.1}$$

Therefore, (4.4) can be rewritten in the usual form

$$\begin{aligned} &\mathrm{curl}\,\mathbf{E}_C + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{0} && \text{in } \Omega_C \\ &\mathrm{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = \mathbf{J}_{e,C} && \text{in } \Omega_C. \end{aligned}$$

Let us consider now the problem in Ω_I. As the electric conductivity coefficient $\boldsymbol{\sigma}$ vanishes in the insulator Ω_I, uniqueness for the electric field $\mathbf{E}$ clearly does not hold for problem (1.1), as, for instance, we can add to $\mathbf{E}$ the gradient of any function ψ having compact support in Ω_I. Therefore we are led to modify (1.1) by adding some other equations.

In Alonso & Valli [2], by using a perturbation argument, it is proposed to add

$$\mathrm{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 \qquad \text{in } \Omega_I \tag{5.2}$$

and

$$\begin{cases} \displaystyle\int_{(\partial\Omega)_r} \boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n} = 0 & \forall\ r = 0,1,\ldots,p_{\partial\Omega} \\ \displaystyle\int_{\Gamma_j} \boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n}_I = 0 & \forall\ j = 1,\ldots,p_\Gamma - 1. \end{cases} \tag{5.3}$$

However, in the present situation the same perturbation argument shows that, in Ω_I, the perturbed electric induction $i\omega\boldsymbol{\varepsilon}_I\mathbf{E}_I$ has to be the curl of the vector field $\mathbf{H}_I - \mathbf{H}_{e,I}$, whose tangential component *vanishes* on $\partial\Omega$. Therefore, we also have to impose

$$\boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{5.4}$$

and $(5.3)_1$ can be dropped. Moreover, one has also to add a last relation (see Alonso & Valli [1])

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \boldsymbol{\pi}_k = 0 \quad \forall\, k = 1, \ldots, n_{\partial\Omega}. \tag{5.5}$$

The physical interpretation of these additional equations is the following: equations (5.2) and (5.4) say, respectively, that the electric charge density vanishes everywhere in the insulator Ω_I, and that the electric charge surface density similarly vanishes everywhere on the boundary $\partial\Omega$ (the latter entails $(5.3)_1$, which means that no electric charge is hidden in the 'holes' of Ω and that the total electric charge in Ω is vanishing); equation $(5.3)_2$ expresses that the total electric charge in each conductor surrounded by Γ_j, $j = 1, \ldots, p_\Gamma - 1$, separately vanishes (as a consequence of these conditions, the same is true also for the conductor surrounded by Γ_{p_Γ}); equation (5.5) makes zero the circulations of the harmonic component of $\mathbf{E}_I$ along all the basis cycles on $\partial\Omega$, and thus, by the Faraday's law, means that no additional magnetic flux is linked by any 'loop' of Ω.

The complete set of equations in Ω_I is therefore given by

$$\begin{cases} \operatorname{curl} \mathbf{E}_I = -i\omega \boldsymbol{\mu}_I \mathbf{H}_I & \text{in } \Omega_I \\ \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_{\Gamma_j} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n}_I = 0 & \forall\, j = 1, \ldots, p_\Gamma - 1 \\ \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \boldsymbol{\pi}_k = 0 & \forall\, k = 1, \ldots, n_{\partial\Omega} \\ \mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma. \end{cases} \tag{5.6}$$

Equations (4.7), (4.9), $(5.6)_3$ and $(5.6)_5$ altogether mean that no field source external to Ω is introduced through either the boundary conditions or the domain topology (which is sometimes possible by adding suitable right-hand sides). Moreover, $(5.6)_2$ and $(5.6)_4$ say that the electric charge plays no active role in Ω. Hence, the obtained fields will be due to $\mathbf{J}_e$ only.

We are now in a position to prove the following result.

Theorem 5.1 *If the magnetic field* $\mathbf{H}$ *is the solution to* (4.4)–(4.11)*, and* $\mathbf{E}_C$ *is given by* (5.1)*, then problem* (5.6) *has a unique solution.*

Proof We have already noted in § 3 that the result presented in Alonso & Valli [1, Theorem 4.2] is true not only for a domain Ω_I with a $C^{1,1}$ boundary, but also for a polyhedral domain with a Lipschitz boundary. Moreover, one can easily modify the proof there to take into consideration the presence of the matrix $\boldsymbol{\varepsilon}_I$.

The existence and uniqueness of the solution of (5.6) therefore follows from the verification of the compatibility conditions

$$\begin{aligned} &\operatorname{div}(-i\omega \boldsymbol{\mu}_I \mathbf{H}_I) = 0 \quad \text{in } \Omega_I \\ &\int_{(\partial\Omega)_r} -i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0 \quad \forall\, r = 1, \ldots, p_{\partial\Omega} \\ &\operatorname{div}_\tau(\mathbf{E}_C \times \mathbf{n}_C) - i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_I = 0 \quad \text{on } \Gamma \\ &\int_{\Omega_I} -i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_l = \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) \quad \forall\, l = 1, \ldots, n_\Gamma. \end{aligned} \tag{5.7}$$

Condition $(5.7)_1$ is the same as (4.6), and condition $(5.7)_2$ is given by (4.7). Moreover, from (5.1), (4.4) and (4.10), it follows

$$\begin{aligned}\mathrm{div}_\tau(\mathbf{E}_C \times \mathbf{n}_C) &= \mathrm{div}_\tau([\boldsymbol{\sigma}^{-1}(\mathrm{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C) \\ &= \mathrm{curl}[\boldsymbol{\sigma}^{-1}(\mathrm{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \cdot \mathbf{n}_C \\ &= -i\omega\boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C = i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}_I \quad \text{on } \Gamma.\end{aligned}$$

Finally, from (4.8) and (5.1) for each $l = 1, \ldots, n_\Gamma$ we have

$$\begin{aligned}\int_{\Omega_I} -i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \boldsymbol{\rho}_l &= \int_\Gamma [\boldsymbol{\sigma}^{-1}(\mathrm{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) \\ &= \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I).\end{aligned}$$

The proof of the theorem is thus complete. □

Remark We emphasize that conditions (4.6), (4.7), (4.8) and (4.10) are necessary compatibility conditions for the solution of problem (5.6), namely, for determining the electric field **E** satisfying Faraday's law in the whole of the domain Ω. In other words, a formulation of the eddy-current problem in term of the magnetic field **H** is not correct if any of these conditions is missing. In particular, we want to focus on the 'interface' condition (4.8), related to the topology of Ω_I. □

We are now in a position to conclude the discussion left open at the end of §4 by giving the following physical interpretation of (4.8). Faraday law should link the flux of $\boldsymbol{\mu}_I\mathbf{H}_I$ across the cut Σ_1 to the circulation of $\mathbf{E}_C$ along $\partial\Sigma_1 \subset \Gamma$, but $\mathbf{E}_C$ is not regular enough to give a meaning to its line integral. As a matter of fact, if $\mathbf{E}_C$ was more regular, $(5.7)_4$ could be rewritten, after some computations, as the Faraday's law applied to Σ_1. Hence, (4.8) ensures that **H** and $\mathbf{E}_C$ obtained from (4.4)–(4.11) and (5.1) satisfy Faraday's law applied to Σ_1 in the weak sense given by $(5.7)_4$.

It is easily seen that the complete set of equations (4.4)–(4.11), (5.1) and (5.6) is somehow redundant. First, (4.6) and (4.7) are a consequence of $(5.6)_1$ (the latter by means of an approximation argument and of the Stokes theorem for regular fields on closed surfaces). Also, from $(5.6)_6$ we have

$$\mathrm{div}_\tau(\mathbf{E}_I \times \mathbf{n}_I + \mathbf{E}_C \times \mathbf{n}_C) = 0 \quad \text{on } \Gamma,$$

that from (2.4) is equivalent to

$$\mathrm{curl}\,\mathbf{E}_I \cdot \mathbf{n}_I + \mathrm{curl}\,\mathbf{E}_C \cdot \mathbf{n}_C = 0 \quad \text{on } \Gamma;$$

hence from $(5.6)_1$, we obtain (4.10). Finally, from $(5.6)_1$ and $(5.6)_6$ we obtain

$$\begin{aligned}\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \boldsymbol{\rho}_l = -\int_{\Omega_I} \mathrm{curl}\,\mathbf{E}_I \cdot \boldsymbol{\rho}_l &= \int_{\partial\Omega_I} (\mathbf{E}_I \times \mathbf{n}_I) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) \\ &= -\int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I),\end{aligned}$$

hence from (5.1) we find (4.8).

Therefore, recalling that (5.1) and (4.4) are equivalent to a first-order system, we can rewrite the global problem in the non-redundant form

$$\operatorname{curl} \mathbf{H}_C - \boldsymbol{\sigma} \mathbf{E}_C = \mathbf{J}_{e,C} \quad \text{in } \Omega_C \tag{5.8}$$

$$\operatorname{curl} \mathbf{E}_C + i\omega \boldsymbol{\mu}_C \mathbf{H}_C = \mathbf{0} \quad \text{in } \Omega_C \tag{5.9}$$

$$\operatorname{curl} \mathbf{H}_I = \mathbf{J}_{e,I} \quad \text{in } \Omega_I \tag{5.10}$$

$$\operatorname{curl} \mathbf{E}_I + i\omega \boldsymbol{\mu}_I \mathbf{H}_I = \mathbf{0} \quad \text{in } \Omega_I \tag{5.11}$$

$$\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0 \quad \text{in } \Omega_I \tag{5.12}$$

$$\mathbf{H}_I \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \tag{5.13}$$

$$\boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{5.14}$$

$$\int_{\Gamma_j} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n}_I = 0 \quad \forall\, j = 1, \ldots, p_\Gamma - 1 \tag{5.15}$$

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \boldsymbol{\pi}_k = 0 \quad \forall\, k = 1, \ldots, n_{\partial\Omega} \tag{5.16}$$

$$\mathbf{E}_I \times \mathbf{n}_I + \mathbf{E}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on } \Gamma \tag{5.17}$$

$$\mathbf{H}_I \times \mathbf{n}_I + \mathbf{H}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on } \Gamma. \tag{5.18}$$

Notice that all the equations of problem (5.6) are present, while those we have dropped are essentially the compatibility conditions of problem (5.6).

We conclude this section with the following theorem.

Theorem 5.2 *Problem* (5.8)–(5.18) *has a unique solution. Moreover,* $(\mathbf{E}, \mathbf{H})$ *is the solution to* (5.8)–(5.18) *if and only if* $\mathbf{H}$ *is the solution to* (4.4)–(4.11)*,* $\mathbf{E}_C$ *is obtained by* (5.1) *and* $\mathbf{E}_I$ *is the solution to* (5.6).

Proof The procedure is quite similar to that employed at the end of § 4 for showing well-posedness of problem (4.4)–(4.11). In fact, we have already seen that a solution to (5.8)–(5.18) is given by $\mathbf{H} = \mathbf{Z} + \mathbf{H}^*$ (where $\mathbf{Z}$ is the solution to (3.8) and $\mathbf{H}^*$ is defined in (3.9)), by $\mathbf{E}_C$ defined in (5.1) and by the solution $\mathbf{E}_I$ to (5.6). On the other hand, since, as we have proved in § 3, any solution to (1.1) gives, through (3.6), the solution to (3.8), the magnetic field $\mathbf{H}$ is uniquely determined. Consequently, using (5.8) also $\mathbf{E}_C$ is unique. Finally, uniqueness of $\mathbf{E}_I$ follows from that of problem (5.6).

Noting that $\mathbf{H}$ is the unique solution to (4.4)–(4.11), the second statement follows. □

Remark Analogous results to those presented in § 3–5 can be obtained for the eddy-current problem

$$\begin{cases} \operatorname{curl} \mathbf{H} - \boldsymbol{\sigma} \mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega \boldsymbol{\mu} \mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

Concerning the weak problem for the magnetic field $\mathbf{H}$, it becomes

$$\mathscr{A}(\mathbf{H}, \mathbf{v}^*) = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}_C^*} \quad \forall\, \mathbf{v}^* \in V^*, \tag{5.20}$$

where

$$V^* := \{\mathbf{v}^* \in H(\operatorname{curl}; \Omega) \mid \operatorname{curl} \mathbf{v}_I^* = \mathbf{0} \text{ in } \Omega_I\}. \tag{5.21}$$

(Clearly, one has also to modify the necessary assumptions on $\mathbf{J}_{e,I}$, to ensure that $\operatorname{div}\mathbf{J}_{e,I} = 0$ in Ω_I and $\int_{\Gamma_j}\mathbf{J}_{e,I}\cdot\mathbf{n}_I = 0$ for all $j = 1,\ldots,p_\Gamma - 1$, $\int_{(\partial\Omega)_r}\mathbf{J}_{e,I}\cdot\mathbf{n} = 0$ for all $r = 0,\ldots,p_{\partial\Omega}$. In this way, the existence of a vector field $\mathbf{H}_{e,I}$ such that $\operatorname{curl}\mathbf{H}_{e,I} = \mathbf{J}_{e,I}$ is assured.)

Concerning the strong formulation, one obtains as before (4.4), (4.5), (4.6), (4.10) and (4.11). However, (4.9) is no longer satisfied, and is replaced by $\boldsymbol{\mu}_I\mathbf{H}_I\cdot\mathbf{n} = 0$ on $\partial\Omega$. This last condition is obtained by taking in (5.20) the test function

$$\mathbf{v}_\eta^* := \begin{cases} \nabla\psi_{\eta,I}^* & \text{in } \Omega_I \\ \mathbf{0} & \text{in } \Omega_C, \end{cases}$$

where $\psi_{\eta,I}^* \in H^1(\Omega_I)$, $\psi_{\eta,I}^* = 0$ on Γ and $\psi_{\eta,I}^* = \eta$ on $\partial\Omega$, with η an arbitrary complex function defined on $\partial\Omega$.

Finally, instead of $\mathscr{H}_{\mu_I}(\partial\Omega;\Gamma)$, the relevant space of harmonic fields is in this case

$$\widehat{\mathscr{H}}_{\mu_I} := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl}\mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu}_I\mathbf{v}_I) = 0, \boldsymbol{\mu}_I\mathbf{v}_I\cdot\mathbf{n}_I = 0 \text{ on } \Gamma\cup\partial\Omega\}. \qquad (5.22)$$

This space has dimension equal to $n_{\partial\Omega_I}$, the total number of 'cuts' $\hat{\Sigma}_s$ contained in Ω_I such that in $\Omega_I \setminus \cup_s\hat{\Sigma}_s$ every curl-free vector field has a global potential. By taking as test function the one whose restriction to Ω_I is equal to the basis function $\widehat{\boldsymbol{\rho}}_s$ of $\widehat{\mathscr{H}}_{\mu_I}$, $s = 1,\ldots,n_{\partial\Omega_I}$, it is easily seen that conditions (4.7) and (4.8) have to be replaced by

$$\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I\cdot\widehat{\boldsymbol{\rho}}_s + \int_\Gamma[\boldsymbol{\sigma}^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})]\times\mathbf{n}_C\cdot(\mathbf{n}_I\times\widehat{\boldsymbol{\rho}}_s\times\mathbf{n}_I) = 0. \qquad (5.23)$$

The strong form of the problem concerning the electric field $\mathbf{E}_I$ is now given by $(5.6)_1$, $(5.6)_2$ and $(5.6)_6$, while clearly the boundary condition $(5.6)_3$ has to be substituted by $\mathbf{E}_I\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Moreover, the remaining conditions related to the geometry of Ω_I are now given by $(5.6)_4$ and by $\int_{(\partial\Omega)_r}\boldsymbol{\varepsilon}_I\mathbf{E}_I\cdot\mathbf{n} = 0$ for all $r = 0,\ldots,p_{\partial\Omega}$.

It can be noted that the necessary and sufficient solvability conditions for this problem (see Alonso & Valli [1, Theorem 4.1]) are indeed satisfied, as a consequence of the properties of the magnetic field $\mathbf{H}$. □

Remark As in the 'method of images', when the problem domain and the material properties are symmetric with respect to a plane and the driving current density is either symmetric or skew-symmetric with respect to the same plane, the problem can be reduced to an equivalent one posed in only a half of the problem domain, with suitable boundary conditions on that plane. The fields in the whole of the problem domain are then readily obtained from the solution of the equivalent problem since they are symmetric or skew-symmetric with respect to the symmetry plane. If a finite number of these symmetry planes are present, they can be exploited in sequence to get an equivalent problem posed in only a part of the original problem domain, which we call 'symmetry cell'. In numerical applications, to reduce the computational cost, the equivalent problem posed in the smallest symmetry cell is almost always considered. The problems thus obtained, unfortunately, do not fit our assumptions because either the conducting region may touch the boundary or the boundary conditions may be of mixed type. It can be shown, however, that equivalent problems obtained by correct exploitation of symmetries are well-posed if the original one is. It should be noticed that the relevant topology is the one of the original problem and that conditions due to it (i.e. $(3.2)_1$, $(3.2)_2$, (4.7), (4.8), $(5.6)_4$ and $(5.6)_5$ may survive in some form in the equivalent problem, even if its topology in the symmetry cell looks trivial. The terms involved in the conditions due to the topology are essentially (i.e. neglecting regularity considerations) integrals of quantities that may be either symmetric or skew-symmetric with respect to a symmetry plane. If the latter case happens for all the terms in a condition, this condition is automatically satisfied and disappears from the equivalent problem, but, if the former case happens for any term, this term (and, then, the condition) survives with a halved integration domain. □

6 Practical implications: do usual formulations satisfy condition (4.8)?

In this section we want to investigate whether or not some frequently-used formulations for eddy-current problems furnish a magnetic field **H** that satisfies (4.8).

(i) The $\mathbf{A}_C^*$-$\mathbf{A}_I$ formulation
This formulation, reported in Bíró [5], is based on the unknowns $\mathbf{A}_C^*$ and $\mathbf{A}_I$ such that

$$i\omega\mathbf{A}_C^* = -\mathbf{E}_C, \quad \operatorname{curl}\mathbf{A}_I = \boldsymbol{\mu}_I\mathbf{H}_I,$$

with the interface conditions on Γ

$$\mathbf{A}_C^* \times \mathbf{n}_C + \mathbf{A}_I \times \mathbf{n}_I = \mathbf{0}, \ \left(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C^*\right) \times \mathbf{n}_C + \left(\boldsymbol{\mu}_I^{-1}\operatorname{curl}\mathbf{A}_I\right) \times \mathbf{n}_I = \mathbf{0}.$$

We have

$$\begin{aligned}\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \boldsymbol{\rho}_l &= \int_{\Omega_I} i\omega \operatorname{curl}\mathbf{A}_I \cdot \boldsymbol{\rho}_l = i\omega\int_\Gamma (\mathbf{n}_I \times \mathbf{A}_I)\cdot(\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I)\\ &= i\omega\int_\Gamma (\mathbf{A}_C^* \times \mathbf{n}_C)\cdot(\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I)\\ &= -\int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C)\cdot(\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I).\end{aligned} \tag{6.1}$$

Therefore, condition (4.8) is satisfied.

(ii) The $(\mathbf{A}_C, V_C)$-$\mathbf{A}_I$ formulation
This formulation, reported in Bíró [5], is based on the unknowns $(\mathbf{A}_C, V_C)$ and $\mathbf{A}_I$ such that

$$i\omega(\mathbf{A}_C + \nabla V_C) = -\mathbf{E}_C, \quad \operatorname{curl}\mathbf{A}_I = \boldsymbol{\mu}_I\mathbf{H}_I,$$

with the interface conditions on Γ

$$\mathbf{A}_C \times \mathbf{n}_C + \mathbf{A}_I \times \mathbf{n}_I = \mathbf{0}, \ \left(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\right) \times \mathbf{n}_C + \left(\boldsymbol{\mu}_I^{-1}\operatorname{curl}\mathbf{A}_I\right) \times \mathbf{n}_I = \mathbf{0}.$$

With respect to the preceding case, on the right-hand side of (6.1) the only additional term is

$$-i\omega\int_\Gamma (\nabla V_C \times \mathbf{n}_C)\cdot(\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I);$$

however, (6.1) and (4.8) are still satisfied, as the term above indeed vanishes. In fact, we have by (2.3) and (2.4)

$$\begin{aligned}\int_\Gamma (\nabla V_C \times \mathbf{n}_C) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) &= \int_\Gamma (\boldsymbol{\rho}_l \times \mathbf{n}_I) \cdot (\mathbf{n}_C \times \nabla V_C \times \mathbf{n}_C) \\ &= -\int_\Gamma \operatorname{div}_\tau(\boldsymbol{\rho}_l \times \mathbf{n}_I)\, V_{C|\Gamma} \\ &= -\int_\Gamma \operatorname{curl} \boldsymbol{\rho}_l \cdot \mathbf{n}_I\, V_{C|\Gamma} = 0,\end{aligned}$$

as $\operatorname{curl} \boldsymbol{\rho}_l = \mathbf{0}$.

(iii) The $(\mathbf{T}_C, \Phi_C)$-$\mathbf{A}_I$ formulation

This formulation, reported in Bíró [5], is based on the unknowns $(\mathbf{T}_C, \Phi_C)$ and $\mathbf{A}_I$ such that

$$\mathbf{T}_C - \nabla \Phi_C = \mathbf{H}_C - \mathbf{H}_{e,C}, \quad \operatorname{curl} \mathbf{A}_I = \boldsymbol{\mu}_I \mathbf{H}_I,$$

with the interface conditions on Γ

$$\begin{aligned}&[\boldsymbol{\sigma}^{-1}(\operatorname{curl} \mathbf{T}_C + \operatorname{curl} \mathbf{H}_{e,C} - \mathbf{J}_{e,C})] \times \mathbf{n}_C - i\omega \mathbf{A}_I \times \mathbf{n}_I = \mathbf{0} \\ &(\mathbf{T}_C - \nabla \Phi_C + \mathbf{H}_{e,C}) \times \mathbf{n}_C + \left(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{A}_I\right) \times \mathbf{n}_I = \mathbf{0}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_l = \int_{\Omega_I} i\omega \operatorname{curl} \mathbf{A}_I \cdot \boldsymbol{\rho}_l &= i\omega \int_\Gamma (\mathbf{n}_I \times \mathbf{A}_I) \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) \\ &= -\int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl} \mathbf{T}_C + \operatorname{curl} \mathbf{H}_{e,C} - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) \\ &= -\int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl} \mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I),\end{aligned}$$

that is (4.8).

(iv) The $(\mathbf{H}_C, \mathbf{J}_C)$-$\mathbf{H}_I$ formulation

Kanayama & Kikuchi [17] reported (and used in numerical computations) a formulation in which the unknowns are the magnetic field and the eddy-current $\mathbf{J}_C = \boldsymbol{\sigma} \mathbf{E}_C + \mathbf{J}_{e,C}$, and the interface conditions on Γ are given by

$$\mathbf{H}_C \times \mathbf{n}_C + \mathbf{H}_I \times \mathbf{n}_I = \mathbf{0}, \ \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C + \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_I = 0. \tag{6.2}$$

However, it is not indicated that, in general geometry, the additional 'interface' condition (4.8) has to be imposed, and in this case, it does not follow from the given formulation. Therefore, this formulation cannot be employed for a general domain Ω, without explicitly adding (4.8), that, with respect to $\mathbf{J}_C$ and $\mathbf{H}_I$, reads

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_l + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\mathbf{J}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot (\mathbf{n}_I \times \boldsymbol{\rho}_l \times \mathbf{n}_I) = 0$$

for $l = 1, \ldots, n_\Gamma$.

(v) The formulation considered by Reissel

In Reissel [21], the formulation given by (5.8)–(5.11) and (5.13) is considered, with the interface conditions (6.2). The uniqueness of the magnetic field is obtained by imposing the additional conditions

$$\int_{\gamma_l} \mathbf{H_I} \cdot d\gamma = h_{I,l}, \quad \forall\, l = 1, \ldots, n_\Gamma. \tag{6.3}$$

(The domain Ω_I is assumed to be an exterior domain, hence $\partial\Omega$ is empty.)

It can be shown that (4.8) is satisfied only for a specific value of the data $h_{I,l}$, that in particular depends upon the magnetic field $\mathbf{H}_C$. In other words, solving the problem with the interface conditions (6.2) and the topological conditions (6.3) does not assure that the unique solution thus obtained satisfies the 'interface' condition (4.8).

In fact, from a physical viewpoint $h_{I,l}$ is the total current crossing any surface having γ_l as a boundary (by the Ampère's law). Hence, it is a quantity to be determined by solving the problem rather than a datum that can be arbitrarily given.

From a practical viewpoint, our analysis leads to the following general conclusions. As condition (4.8) is implicitly included in the weak formulation (3.8) in terms of the magnetic field **H**, numerical methods based on it may correctly solve problems involving multiply-connected conductors. However, methods based on the strong formulation in term of **H** fail with multiply-connected conductors unless (4.8) is explicitly included. Notice, however, that the sparsity of the matrix of the algebraic system obtained after discretization is to some extent spoiled by the inclusion of the global constraint (4.8).

Methods where the magnetic field is expressed in term of a vector potential, may cope with multiply-connected conductors even if they have been developed starting from the strong formulation (4.4)–(4.11) and (5.1), without (4.8). In fact, introducing vector potentials may make (4.8) automatically satisfied. In all the considered cases, in particular, the key point is satisfying (4.8) is the introduction of the magnetic vector potential in the non-conducting region.

7 Conclusions

We have analysed the eddy-current problem for the time-harmonic Maxwell equations in domains with general topology. We have shown that the usual weak formulation in term of the magnetic field is equivalent to a suitable strong formulation, putting in evidence the equations that have to be added to the classical model in consequence of the non-trivial topology of the non-conducting region. The existence and uniqueness of a solution has been proved for this strong problem, as well as for the problem concerning the electric field in the non-conducting region. Some remarks related to an apparently new condition that must be satisfied by the magnetic field on the interface between the conducting and non-conducting regions have also been presented.

References

[1] Alonso, A. & Valli, A. (1996) Some remarks on the characterization of the space of tangential traces of $H(\mathrm{rot};\Omega)$ and the construction of an extension operator. *Manuscr. Math.* **89**, 159–178.

https://doi.org/10.1017/S0956792503005151 Published online by Cambridge University Press

[2] Alonso, A. & Valli, A. (1997) A domain decomposition approach for heterogeneous time-harmonic Maxwell equations. *Comput. Meth. Appl. Mech. Eng.* **143**, 97–112.
[3] Alonso, A. & Valli, A. (1999) An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations. *Math. Comput.* **68**, 607–631.
[4] Amrouche, C., Bernardi, C., Dauge, M. & Girault, V. (1998) Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.* **21**, 823–864.
[5] Bíró, O. (1999) Edge element formulations of eddy current problems. *Comput. Meth. Appl. Mech. Eng.* **169**, 391–405.
[6] Bossavit, A. (1993) *Électromagnétisme, en Vue de la Modélisation.* Springer-Verlag.
[7] Bossavit, A. (1998) *Computational Electromagnetism.* Academic Press.
[8] Bossavit, A. (1999) "Hybrid" electric–magnetic methods in eddy-current problems. *Comput. Meth. Appl. Mech. Eng.* **178**, 383–391.
[9] Bossavit, A. & Vérité, J.-C. (1982) A mixed FEM–BIEM method to solve eddy-current problems. *IEEE Trans. MAG* **18**, 431–435.
[10] Buffa, A. (2001) Hodge decomposition on the boundary of non-smooth domains: the multi-connected case. *Math. Models Meth. Appl. Sci.* **11**, 1491–1504.
[11] Buffa, A. & Ciarlet, P. Jr. (2001) On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra. *Math. Methods Appl. Sci.* **24**, 9–30.
[12] Buffa, A. & Ciarlet, P. Jr. (2001) On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Methods Appl. Sci.* **24**, 31–48.
[13] Cessenat, M. (1996) *Mathematical Methods in Electromagnetism: Linear Theory and Applications.* World Scientific.
[14] Dautray, R. & Lions, J.-L. (1992) *Mathematical Analysis and Numerical Methods for Science and Technology. Volume 5: Evolution Problems I.* Springer–Verlag.
[15] Fernandes, P. & Gilardi, G. (1997) Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, *Math. Models Meth. Appl. Sci.* **7**, 957–991.
[16] Foias, C. & Temam, R. (1978) Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcations. *Ann. Scuola Norm. Sup. Pisa* **5**(IV), 29–63.
[17] Kanayama, H. & Kikuchi, F. (1999) 3-D eddy current computation using Nedelec elements. *Information*, **2**, 37–45.
[18] Kress, R. (1971) Ein kombiniertes Dirichlet-Neumannsches Randwertproblem bei harmonischen Vektorfeldern. *Arch. Rational Mech. Anal.* **42**, 40–49.
[19] Picard, R. (1979) Zur Theorie der harmonischen Differentialformen. *Manuscr. Math.* **27**, 31–45.
[20] Picard, R. (1982) On the boundary value problems of electro- and magnetostatics. *Proc. Roy. Soc. Edinburgh A*, **92**, 165–174.
[21] Reissel, M. (1993) On a transmission boundary-value problem for the time-harmonic Maxwell equations without displacement currents. *SIAM J. Math. Anal.* **24**, 1440–1457.