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MODAL OPERATORS ON RINGS OF CONTINUOUS FUNCTIONS

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Abstract. It is a classic result in modal logic, often referred to as Jónsson-Tarski duality, that the category of modal algebras is dually equivalent to the category of descriptive frames. The latter are Kripke frames equipped with a Stone topology such that the binary relation is continuous. This duality generalizes the celebrated Stone duality for boolean algebras. Our goal is to generalize descriptive frames so that the topology is an arbitrary compact Hausdorff topology. For this, instead of working with the boolean algebra of clopen subsets of a Stone space, we work with the ring of continuous real-valued functions on a compact Hausdorff space. The main novelty is to define a modal operator on such a ring utilizing a continuous relation on a compact Hausdorff space.

Our starting point is the well-known Gelfand duality between the category KHaus of compact Hausdorff spaces and the category ***ubaℓ*** of uniformly complete bounded archimedean ℓ-algebras. We endow a bounded archimedean ℓ-algebra with a modal operator, which results in the category ***mbaℓ*** of modal bounded archimedean ℓ-algebras. Our main result establishes a dual adjunction between ***mbaℓ*** and the category KHF of what we call compact Hausdorff frames; that is, Kripke frames equipped with a compact Hausdorff topology such that the binary relation is continuous. This dual adjunction restricts to a dual equivalence between KHF and the reflective subcategory ***mubaℓ*** of ***mbaℓ*** consisting of uniformly complete objects of ***mbaℓ***. This generalizes both Gelfand duality and Jónsson-Tarski duality.

§1. Introduction. In modal logic there is a well-established duality theory between categories of Kripke frames and the corresponding categories of boolean algebras with operators, which forms the backbone of modern studies of modal logic. One of the most fundamental such dualities establishes that the category of modal algebras is dually equivalent to the category of descriptive frames. This duality is often called Jónsson-Tarski duality because it originates in the work of Jónsson and Tarski [24] (see also Kripke [29]). In its current form it was developed by Halmos [19], Esakia [14], and Goldblatt [18]. For a modern account we refer to [35] or the textbooks [9, 11, 27].

This duality generalizes the celebrated Stone duality between the categories of boolean algebras and Stone spaces (zero-dimensional compact Hausdorff spaces). Descriptive frames are Stone spaces equipped with a continuous relation. It is well known that a binary relation R on a Stone space X is continuous iff the corresponding map from X to the Vietoris space $\mathcal{V}X$, given by sending each $x \in X$ to its R-image, is a well-defined continuous map (see [14, Section 1] or [30, Section 3]). Since the Vietoris space $\mathcal{V}X$ of a compact Hausdorff space X is compact Hausdorff, the consideration above allows us to generalize the notion of a descriptive frame to what

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we call a *compact Hausdorff frame*; that is, a compact Hausdorff space equipped with a continuous relation. The category KHF of compact Hausdorff frames was studied in [4], under the name of modal compact Hausdorff spaces, where Isbell [21] and de Vries [12] dualities for the category KHaus of compact Hausdorff spaces were generalized to KHF.

One of the best known (and oldest) dualities for KHaus goes back to the 1930s–1940s and is known under various names. The basic idea is to work with the ring of continuous functions on a compact Hausdorff space, but we arrive at different algebras depending on whether we work with complex-valued or real-valued functions. Gelfand and Naimark [16] worked with continuous complex-valued functions and established that KHaus is dually equivalent to the category of commutative unital C^*-algebras. Independently, Stone [36] worked with continuous real-valued functions and established that KHaus is dually equivalent to the category of uniformly complete bounded archimedean ℓ-algebras. These two categories of algebras are equivalent, which can be seen directly without passing to KHaus. Indeed, the self-adjoint elements of a commutative unital C^*-algebra form a uniformly complete bounded archimedean ℓ-algebra, and each such algebra A gives rise to a commutative unital C^*-algebra by taking the complexification $A \otimes_{\mathbb{R}} \mathbb{C}$ (see [6, Section 7] for details).

Yet another duality for KHaus can be obtained by dropping multiplication from the signature of ℓ-algebras, thus giving rise to the notion of a vector lattice, also called a Riesz space [31]. It follows from the work of Kakutani [25, 26], Krein and Krein [28], and Yosida [37] that KHaus is dually equivalent to the category of uniformly complete bounded archimedean vector lattices.

These dualities for KHaus are often collectively referred to as Gelfand duality (see, e.g., [23, Section IV.4]), the terminology that we will follow in this paper. The version of Gelfand duality we will work with is obtained by associating to each compact Hausdorff space X the ring $C(X)$ of continuous real-valued functions on X. For some time now there has been a desire to generalize Gelfand duality to a duality for KHF, but it remained elusive for at least two reasons. On the conceptual side, there was no agreement on what should be the definition of modal operators on the ring $C(X)$. On the technical side, it was unclear how to axiomatize attempted definitions of modal operators.

The goal of this paper is to resolve these issues. After recalling Gelfand duality, we define a modal operator on the ring $C(X)$ for each compact Hausdorff frame (X, R), and study its basic properties. This motivates the definition of a modal operator on an arbitrary bounded archimedean ℓ-algebra, which is the main definition of the paper, giving rise to the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras. We show that there is a contravariant functor $C : \mathsf{KHF} \to \boldsymbol{mba\ell}$.

Next we define a contravariant functor $Y : \boldsymbol{mba\ell} \to \mathsf{KHF}$ in the opposite direction. Proving that $Y : \boldsymbol{mba\ell} \to \mathsf{KHF}$ is well defined is technically the most challenging part of the paper. Our main result establishes that the contravariant functors C and Y yield a dual adjunction between $\boldsymbol{mba\ell}$ and KHF, which restricts to a dual equivalence between KHF and the reflective subcategory $\boldsymbol{muba\ell}$ of $\boldsymbol{mba\ell}$ consisting of uniformly complete objects of $\boldsymbol{mba\ell}$.

Our result generalizes both Gelfand duality and Jónsson-Tarski duality. We also take first steps in developing correspondence theory for $\boldsymbol{mba\ell}$ by characterizing the

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classes of algebras in ***mbaℓ*** such that the corresponding relations on the dual side are serial, reflexive, transitive, or symmetric. We conclude the paper outlining several possible future directions of this line of research.

In the future, it would be of interest to develop the logical formalisms that can be modeled by modal bounded archimedean ℓ-algebras. This can be done along the same lines as in [13] where a modal extension of Abelian logic [10, 33] is developed. Another approach can be found in [15] with applications to probabilistic logic and Markov processes. This would give rise to novel modal logics based on compact Hausdorff spaces (rather than Stone spaces), and could serve as an alternative to the approaches developed in [4, 5].

§2. Gelfand duality. In this section we give a brief outline of Gelfand duality. We start by recalling several basic definitions (see [8, Chapter XIII and onwards] or [6]). All rings that we will consider in this paper are commutative and unital (have multiplicative identity 1).

DEFINITION 2.1.

1. A ring A with a partial order $\leq$ is an *ℓ-ring* (that is, a *lattice-ordered ring*) if $(A, \leq)$ is a lattice, $a \leq b$ implies $a + c \leq b + c$ for each c, and $0 \leq a, b$ implies $0 \leq ab$.
2. An ℓ-ring A is *bounded* if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).
3. An ℓ-ring A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.
4. An ℓ-ring A is an *ℓ-algebra* if it is an $\mathbb{R}$-algebra and for each $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$ we have $0 \leq \lambda \cdot a$.
5. Let ***baℓ*** be the category of bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms.

Let $A \in$ ***baℓ***. For $a \in A$, define the *absolute value* of a by

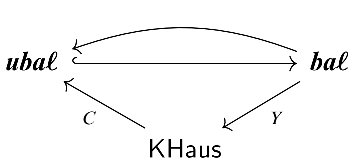
$$|a| = a \vee (-a)$$

and the *norm* of a by

$$||a|| = \inf\{\lambda \in \mathbb{R} \mid |a| \leq \lambda\},$$

where we view $\mathbb{R}$ as an ℓ-subalgebra of A by identifying $\lambda \in \mathbb{R}$ with $\lambda \cdot 1 \in A$. Then A is *uniformly complete* if the norm is complete. Let ***ubaℓ*** be the full subcategory of ***baℓ*** consisting of uniformly complete ℓ-algebras.

THEOREM 2.2 (Gelfand duality [16, 36]) *There is a dual adjunction between* ***baℓ*** *and* KHaus *which restricts to a dual equivalence between* KHaus *and* ***ubaℓ***.



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We describe briefly the functors $C : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ and $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ yielding the dual adjunction. The functor C is defined in an obvious way. It associates with each compact Hausdorff space X the ring $C(X)$ of (necessarily bounded) continuous real-valued functions on X, and with each continuous map $\varphi : X \to Y$ the unital ℓ-algebra homomorphism $C(\varphi) : C(Y) \to C(X)$ given by $C(\varphi)(f) = f \circ \varphi$ for each $f \in C(Y)$. It is straightforward to check that $C : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ is a well-defined contravariant functor.

To define the functor $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$, we recall that an ideal I of $A \in \boldsymbol{ba\ell}$ is an *ℓ-ideal* if $|a| \leq |b|$ and $b \in I$ imply $a \in I$. It is well known that ℓ-ideals are exactly the kernels of ℓ-algebra homomorphisms. An ℓ-ideal I of A is *proper* if $I \neq A$, and a *maximal ℓ-ideal* is a proper ℓ-ideal that is maximal with respect to inclusion. A standard Zorn's lemma argument yields that each proper ℓ-ideal is contained in a maximal ℓ-ideal.

Let Y_A be the space of maximal ℓ-ideals of A, whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{x \in Y_A \mid I \subseteq x\},$$

where I is an ℓ-ideal of A. The space Y_A is often referred to as the *Yosida space* of A in honor of Yosida's fundamental work, which in particular implies that $Y_A \in \mathsf{KHaus}$ (see [37]).

The functor $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is defined by associating with each $A \in \boldsymbol{ba\ell}$ the Yosida space Y_A and with each morphism $\alpha : A \to B$ in $\boldsymbol{ba\ell}$ the continuous map $Y(\alpha) = \alpha^{-1} : Y_B \to Y_A$. It is well known that Y is a well-defined contravariant functor, and that the functors Y and C yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus.

For $X \in \mathsf{KHaus}$ and $x \in X$ let $M_x := \{f \in C(X) \mid f(x) = 0\}$ be the ℓ-ideal of $C(X)$ consisting of functions that vanish at $x \in X$. Then M_x is a maximal ℓ-ideal of $C(X)$, and since X is compact, every maximal ℓ-ideal of $C(X)$ is of this form (see, e.g., [17, Theorem 7.2]). Thus, $\varepsilon_X : X \to Y_{C(X)}$ given by

$$\varepsilon_X(x) = M_x$$

is a homeomorphism.

Let $A \in \boldsymbol{ba\ell}$. In order to define $\zeta_A : A \to C(Y_A)$ we require Hölder's well-known theorem that a totally ordered archimedean ℓ-group is isomorphic to a subgroup of $\mathbb{R}$. If $x \in Y_A$, then the quotient A/x is a totally ordered archimedean ℓ-group, so A/x is isomorphic to a subgroup of $\mathbb{R}$ by Hölder's theorem. Since A/x is also an $\mathbb{R}$-algebra, we conclude that $A/x \cong \mathbb{R}$. Therefore, for each $a \in A$ there is a unique $\lambda \in \mathbb{R}$ with $a + x = \lambda + x$ (as usual, $a + x \in A/x$ denotes the coset of a with respect to x). Thus, we may define $\zeta_A : A \to C(Y_A)$ by

$$\zeta_A(a)(x) = \lambda$$

for each $x \in Y_A$. Since $\bigcap Y_A = 0$ (see [22, Theorem II.2.11]), ζ_A is a monomorphism in $\boldsymbol{ba\ell}$. In addition, ζ_A separates points of Y_A, meaning that if $x, y \in Y_A$ are distinct, then there is $a \in A$ with $\zeta_A(a)(x) \neq \zeta_A(a)(y)$. Consequently, the Stone-Weierstrass theorem yields:

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Proposition 2.3. *The uniform completion of $A \in \boldsymbol{ba\ell}$ is $\zeta_A : A \to C(Y_A)$. Therefore, if A is uniformly complete, then ζ_A is an isomorphism.*

We abuse the terminology and call $C(Y_A)$ the uniform completion of A. Recalling that a subcategory D of a category C is a *reflective subcategory* if the inclusion functor D $\hookrightarrow$ C has a left adjoint, the above result yields:

Corollary 2.4. *$\boldsymbol{uba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, and the reflector assigns to each $A \in \boldsymbol{ba\ell}$ its uniform completion $C(Y_A) \in \boldsymbol{uba\ell}$.*

As a result, we obtain that the dual adjunction between $\boldsymbol{ba\ell}$ and KHaus restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus, yielding Gelfand duality.

Remark 2.5. It follows from Gelfand duality that each $A \in \boldsymbol{ba\ell}$ is isomorphic to a subalgebra of $C(Y_A)$. Thus, each $A \in \boldsymbol{ba\ell}$ is a function ring or f-ring for short (see, e.g., [8, Section XVII.5]). We will use this in Section 6.

§3. Modal operators on $C(X)$. In this section we define modal operators on rings of continuous real-valued functions on compact Hausdorff frames and study their basic properties. This motivates the definition of a modal operator on $A \in \boldsymbol{ba\ell}$, giving rise to the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras. We end the section by describing a contravariant functor from KHF to $\boldsymbol{mba\ell}$.

We recall that a *Kripke frame* is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a binary relation on X. As usual, for $x \in X$ we write

$$R[x] = \{y \in X \mid xRy\} \quad \text{and} \quad R^{-1}[x] = \{y \in X \mid yRx\},$$

and for $U \subseteq X$ we write

$$R[U] = \bigcup\{R[u] \mid u \in U\} \quad \text{and} \quad R^{-1}[U] = \bigcup\{R^{-1}[u] \mid u \in U\}.$$

Definition 3.1. [4] A binary relation R on a compact Hausdorff space X is *continuous* if:

1. $R[x]$ is closed for each $x \in X$.
2. $F \subseteq X$ closed implies $R^{-1}[F]$ is closed.
3. $U \subseteq X$ open implies $R^{-1}[U]$ is open.

If R is a continuous relation on X, we call (X, R) a *compact Hausdorff frame*.

Notation 3.2. For a binary relation R on a set X let

$$D = \{x \in X \mid R[x] \neq \varnothing\} = R^{-1}[X],$$
$$E = X \setminus D = \{x \in X \mid R[x] = \varnothing\}.$$

The next lemma is straightforward and we omit the proof.

Lemma 3.3. *If (X, R) is a compact Hausdorff frame, then D and E are clopen subsets of X.*

Definition 3.4. For a compact Hausdorff frame (X, R), define $\Box_R$ on $C(X)$ by

$$(\Box_R f)(x) = \begin{cases} \inf f R[x] & \text{if } x \in D, \\ 1 & \text{if } x \in E. \end{cases}$$

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REMARK 3.5. We define $\Diamond_R$ by

$$(\Diamond_R f)(x) = \begin{cases} \sup f R[x] & \text{if } x \in D, \\ 0 & \text{if } x \in E. \end{cases}$$

We have

$$\Diamond_R f = 1 - \Box_R(1 - f) \quad \text{and} \quad \Box_R f = 1 - \Diamond_R(1 - f).$$

For, if $x \in D$, then

$$\begin{aligned} 1 - \Box_R(1 - f)(x) &= 1 - \inf\{1 - f(y) \mid xRy\} = 1 - (1 - \sup\{f(y) \mid xRy\}) \\ &= \sup\{f(y) \mid xRy\} = \Diamond_R f(x). \end{aligned}$$

If $x \in E$, then

$$(1 - \Box_R(1 - f))(x) = 1 - 1 = 0 = (\Diamond_R f)(x).$$

Thus, $\Diamond_R f = 1 - \Box_R(1 - f)$, as desired. A similar argument will show that $\Box_R f = 1 - \Diamond_R(1 - f)$. Therefore, each of $\Box_R$ and $\Diamond_R$ can be determined from the other.

REMARK 3.6. Let (X, R) be a compact Hausdorff frame, $f \in C(X)$, and $x \in X$ with $R[x] \neq \varnothing$. Then $f R[x]$ is a nonempty compact subset of $\mathbb{R}$, and so it has least and greatest elements. Thus, we have

$$(\Box_R f)(x) = \min f R[x] \quad \text{and} \quad (\Diamond_R f)(x) = \max f R[x].$$

A version of the next result goes back to Michael [34]. Let $X \in \mathsf{KHaus}$ and $f \in C(X)$. Denoting by $\mathcal{V}^* X$ the space of nonempty closed subsets of X with the Vietoris topology, it follows from [34, Proposition 4.7] that the map $\mathcal{V}^* X \to \mathbb{R}$ that sends F to $\inf f(F)$ is continuous.

LEMMA 3.7. *Suppose that (X, R) is a compact Hausdorff frame. If $f \in C(X)$, then $\Box_R f \in C(X)$.*

PROOF. To see that $\Box_R f$ is continuous, it is sufficient to show that for each $\lambda \in \mathbb{R}$, both $(\Box_R f)^{-1}(\lambda, \infty)$ and $(\Box_R f)^{-1}(-\infty, \lambda)$ are open in X. We first show that $(\Box_R f)^{-1}(\lambda, \infty)$ is open. Let $x \in X$ and first suppose that $x \in D$. Then $f R[x]$ is a nonempty compact subset of $\mathbb{R}$, so it has a least element. Therefore,

$$\begin{aligned} x \in (\Box_R f)^{-1}(\lambda, \infty) &\text{ iff } (\Box_R f)(x) > \lambda \\ &\text{ iff } \min f R[x] > \lambda \\ &\text{ iff } R[x] \subseteq f^{-1}(\lambda, \infty) \\ &\text{ iff } x \in X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]. \end{aligned}$$

Next suppose that $x \in E$. Then $(\Box_R f)(x) = 1$. Thus, $E \subseteq (\Box_R f)^{-1}(\lambda, \infty)$ if $\lambda < 1$, and $E \cap (\Box_R f)^{-1}(\lambda, \infty) = \varnothing$ otherwise. Consequently,

$$(\Box_R f)^{-1}(\lambda, \infty) = \left[D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)])\right] \cup E$$

if $\lambda < 1$, and

$$(\Box_R f)^{-1}(\lambda, \infty) = D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)])$$

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if $1 \leq \lambda$. Since $f \in C(X)$ and R is continuous, $X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]$ is open. Thus, $(\Box_R f)^{-1}(\lambda, \infty)$ is open by Lemma 3.3.

We next show that $(\Box_R f)^{-1}(-\infty, \lambda)$ is open. If $x \in D$, then

$$\begin{aligned} x \in (\Box_R f)^{-1}(-\infty, \lambda) &\text{ iff } (\Box_R f)(x) < \lambda \\ &\text{ iff } \min f R[x] < \lambda \\ &\text{ iff } R[x] \cap f^{-1}(-\infty, \lambda) \neq \varnothing \\ &\text{ iff } x \in R^{-1}[f^{-1}(-\infty, \lambda)]. \end{aligned}$$

If $\lambda \leq 1$, then $E \cap (\Box_R f)^{-1}(-\infty, \lambda) = \varnothing$. On the other hand, if $1 < \lambda$, then $E \subseteq (\Box_R f)^{-1}(-\infty, \lambda)$. Therefore,

$$(\Box_R f)^{-1}(-\infty, \lambda) = D \cap R^{-1}[f^{-1}(-\infty, \lambda)]$$

if $\lambda \leq 1$, and

$$(\Box_R f)^{-1}(-\infty, \lambda) = \left[D \cap (R^{-1}[f^{-1}(-\infty, \lambda)])\right] \cup E$$

if $\lambda > 1$. Since $f \in C(X)$ and R is continuous, $R^{-1}[f^{-1}(-\infty, \lambda)]$ is open. Consequently, $(\Box_R f)^{-1}(-\infty, \lambda)$ is open by Lemma 3.3. This completes the proof that if $f \in C(X)$, then $\Box_R f \in C(X)$. ⊣

In the next lemma we describe the properties of $\Box_R$. For this we recall (see, e.g., [6, Remark 2.2]) that if $A \in$ ***baℓ*** and $a \in A$, then the *positive* and *negative* parts of a are defined as

$$a^+ = a \vee 0 \quad \text{and} \quad a^- = -(a \wedge 0) = (-a) \vee 0.$$

Then $a^+, a^- \geq 0$, $a^+ \wedge a^- = 0$, $a = a^+ - a^-$, and $|a| = a^+ + a^-$.

LEMMA 3.8. *Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.*

1. $\Box_R(f \wedge g) = \Box_R f \wedge \Box_R g$. *In particular, $\Box_R$ is order preserving.*
2. $\Box_R \lambda = \lambda + (1 - \lambda)(\Box_R 0)$. *In particular, $\Box_R 1 = 1$.*
3. $\Box_R(f^+) = (\Box_R f)^+$.
4. $\Box_R(f + \lambda) = \Box_R f + \Box_R \lambda - \Box_R 0$.
5. *If $0 \leq \lambda$, then $\Box_R(\lambda f) = (\Box_R \lambda)(\Box_R f)$.*

PROOF. (1) For $x \in D$, we have

$$\begin{aligned} \Box_R(f \wedge g)(x) &= \inf\{(f \wedge g)(y) \mid y \in R[x]\} = \inf\{\min\{f(y), g(y)\} \mid y \in R[x]\} \\ &= \min\{\inf\{f(y) \mid y \in R[x]\}, \inf\{g(y) \mid y \in R[x]\}\} \\ &= \min\{(\Box_R f)(x), (\Box_R g)(x)\} \\ &= (\Box_R f \wedge \Box_R g)(x). \end{aligned}$$

If $x \in E$, then

$$\Box_R(f \wedge g)(x) = 1 = (\Box_R f \wedge \Box_R g)(x).$$

Thus, $\Box_R(f \wedge g) = \Box_R f \wedge \Box_R g$.

(2) For $x \in D$, if $\mu \in \mathbb{R}$, we have $(\Box_R \mu)(x) = \inf\{\mu \mid y \in R[x]\} = \mu$. From this we see that $(\Box_R \lambda)(x) = \lambda = (\lambda + (1 - \lambda)(\Box_R 0))(x)$. If $x \in E$, then $(\Box_R \lambda)(x) =$

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$1 = (\lambda + (1 - \lambda)(\Box_R 0))(x)$. Thus, $\Box_R \lambda = \lambda + (1 - \lambda)(\Box_R 0)$. Setting $\lambda = 1$ yields $\Box_R 1 = 1$.

(3) For $x \in D$, we have

$$\begin{aligned}(\Box_R(f^+))(x) &= \Box_R(f \vee 0)(x) = \inf\{\max\{f(y), 0\} \mid y \in R[x]\}\\ &= \max\{\inf\{f(y) \mid y \in R[x]\}, 0\} = \max\{\Box_R f(x), 0\}\\ &= (\Box_R f \vee 0)(x) = (\Box_R f)^+(x).\end{aligned}$$

If $x \in E$, then $(\Box_R(f^+))(x) = 1 = (\Box_R f)^+(x)$. Thus, $\Box_R(f^+) = (\Box_R f)^+$.

(4) For $x \in D$, we have

$$\begin{aligned}\Box_R(f + \lambda)(x) &= \inf\{f(y) + \lambda \mid y \in R[x]\}\\ &= \inf\{f(y) \mid y \in R[x]\} + \lambda\\ &= \Box_R f(x) + \lambda.\end{aligned}$$

On the other hand,

$$(\Box_R f + \Box_R \lambda - \Box_R 0)(x) = (\Box_R f)(x) + (\Box_R \lambda)(x) - (\Box_R 0)(x) = (\Box_R f)(x) + \lambda.$$

Therefore, $\Box_R(f + \lambda)(x) = (\Box_R f + \Box_R \lambda - \Box_R 0)(x)$. If $x \in E$, then

$$\Box_R(f + \lambda)(x) = 1 = (\Box_R f + \Box_R \lambda - \Box_R 0)(x).$$

Thus, $\Box_R(f + \lambda) = \Box_R f + \Box_R \lambda - \Box_R 0$.

(5) Let $0 \leq \lambda$. For $x \in D$, we have

$$\begin{aligned}(\Box_R \lambda f)(x) &= \inf\{\lambda f(y) \mid y \in R[x]\} = \lambda \inf\{f(y) \mid y \in R[x]\}\\ &= \lambda(\Box_R f)(x) = (\Box_R \lambda)(x)(\Box_R f)(x) = (\Box_R \lambda \Box_R f)(x).\end{aligned}$$

If $x \in E$, then

$$(\Box_R \lambda f)(x) = 1 = (\Box_R \lambda)(\Box_R f)(x).$$

Thus, $\Box_R(\lambda f) = (\Box_R \lambda)(\Box_R f)$. ⊣

REMARK 3.9. Lemma 3.8 can be stated dually in terms of $\Diamond_R$ as follows. Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.

1. $\Diamond_R(f \vee g) = \Diamond_R f \vee \Diamond_R g$. In particular, $\Diamond_R$ is order preserving.
2. $\Diamond_R \lambda = \lambda(\Diamond_R 1)$. In particular, $\Diamond_R 0 = 0$.
3. $\Diamond_R(f \wedge 1) = (\Diamond_R f) \wedge 1$.
4. $\Diamond_R(f + \lambda) = \Diamond_R f + \Diamond_R \lambda$.
5. If $0 \leq \lambda$, then $\Diamond_R(\lambda f) = \Diamond_R \lambda \Diamond_R f$.

The identities (1), (3), and (5) are direct translations of the corresponding identities for $\Box_R$. However, the identities (2) and (4) are simpler. We next show why $\Diamond_R$ affords such simplifications.

For (2), since $\Diamond_R 1 = 1 - \Box_R 0$, by Lemma 3.8(2),

$$\Diamond_R \lambda = 1 - \Box_R(1 - \lambda) = 1 - (1 - \lambda + \lambda \Box_R 0) = \lambda(1 - \Box_R 0) = \lambda \Diamond_R 1.$$

For (4), by using (4) and (2) of Lemma 3.8, we have

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$$\begin{aligned}\Diamond_R(f+\lambda) &= 1-\Box_R(1-(f+\lambda)) = 1-\Box_R((1-f)-\lambda)\\ &= 1-(\Box_R(1-f)+\Box_R(-\lambda)-\Box_R 0) = \Diamond_R f-\Box_R(-\lambda)+\Box_R 0\\ &= \Diamond_R f-(-\lambda+(1+\lambda)\Box_R 0)+\Box_R 0 = \Diamond_R f+\lambda(1-\Box_R 0)\\ &= \Diamond_R f+\Diamond_R \lambda.\end{aligned}$$

In Remark 4.2 we will explain why we prefer to work with $\Box_R$.

Lemmas 3.7 and 3.8 motivate the main definition of this paper.

DEFINITION 3.10.

1. Let $A \in \boldsymbol{ba\ell}$. We say that a unary function $\Box : A \to A$ is a *modal operator* on A provided $\Box$ satisfies the following axioms for each $a, b \in A$ and $\lambda \in \mathbb{R}$:
   (M1) $\Box(a \wedge b) = \Box a \wedge \Box b$.
   (M2) $\Box\lambda = \lambda + (1-\lambda)\Box 0$.
   (M3) $\Box(a^+) = (\Box a)^+$.
   (M4) $\Box(a+\lambda) = \Box a + \Box\lambda - \Box 0$.
   (M5) $\Box(\lambda a) = (\Box\lambda)(\Box a)$ provided $\lambda \geq 0$.
2. If $\Box$ is a modal operator on $A \in \boldsymbol{ba\ell}$, then we call the pair $\mathfrak{A} = (A, \Box)$ a *modal bounded archimedean ℓ-algebra*.
3. Let $\boldsymbol{mba\ell}$ be the category of modal bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms preserving $\Box$.

REMARK 3.11. We can define $\Diamond : A \to A$ dual to $\Box$ by $\Diamond a = 1 - \Box(1-a)$ for each $a \in A$. Then $(A, \Diamond)$ satisfies the axioms for $\Diamond$ dual to the ones for $\Box$ in Definition 3.10(1) (see Remark 3.9). While algebras in $\boldsymbol{mba\ell}$ can be axiomatized either in the signature of $\Box$ or $\Diamond$, we prefer to work with $\Box$ for the reasons given in Remark 4.2.

REMARK 3.12. If $\Box 0 = 0$, then (M2), (M4), and (M5) simplify to the following:
(M2′) $\Box\lambda = \lambda$.
(M4′) $\Box(a+\lambda) = \Box a + \lambda$.
(M5′) $\Box(\lambda a) = \lambda\Box a$ provided $\lambda \geq 0$.

Moreover, assuming $\Box 0 = 0$ and (M4′), we obtain (M2′) by setting $a = 0$ in (M4′). Furthermore, $\Diamond a = -\Box(-a)$. In Section 7 we will see that $\Box 0 = 0$ holds iff the binary relation $R_\Box$ given in Definition 4.1 is serial.

LEMMA 3.13. *Let* $(A, \Box) \in \boldsymbol{mba\ell}$, $a, b \in A$, *and* $\lambda \in \mathbb{R}$.

1. $a \leq b$ *implies* $\Box a \leq \Box b$.
2. $\Box 1 = 1$.
3. $a \geq 0$ *implies* $\Box a \geq 0$.
4. $(\Box 0)(\Box a) = \Box 0$. *In particular,* $\Box 0$ *is an idempotent.*
5. $\Box(a+\lambda) = \Box a + \lambda(1-\Box 0)$.
6. $\Diamond a = -\Box(-a)(1-\Box 0)$.
7. $(\Diamond a)(\Box 0) = 0$.

PROOF. (1) If $a \leq b$, then $a \wedge b = a$. Therefore, by (M1), we have $\Box a = \Box(a \wedge b) = \Box a \wedge \Box b$. Thus, $\Box a \leq \Box b$.
(2) This follows by substituting $\lambda = 1$ in (M2).
(3) From (M3) and $a \geq 0$ we have $\Box a = \Box(a^+) = (\Box a)^+ \geq 0$.
(4) By (M5), $\Box 0 = \Box(0a) = (\Box 0)(\Box a)$. Setting $a = 0$ gives $(\Box 0)^2 = \Box 0$.

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(5) By (M4), $\Box(a+\lambda) = \Box a + \Box\lambda - \Box 0$. By (M2),

$$\Box\lambda = \lambda + (1-\lambda)(\Box 0) = \lambda(1-\Box 0) + \Box 0.$$

Therefore, $\Box\lambda - \Box 0 = \lambda(1-\Box 0)$, and so (5) follows.

(6) By (M4), (2), and (4) we have

$$\begin{aligned}\Diamond a &= 1 - \Box(1-a) = 1 - (\Box(-a) + \Box 1 - \Box 0)\\ &= -\Box(-a) + \Box 0 = -\Box(-a) + \Box(-a)\Box 0\\ &= -\Box(-a)(1-\Box 0).\end{aligned}$$

(7) Since $\Box 0$ is an idempotent by (4), we have $(1-\Box 0)\Box 0 = 0$. Multiplying both sides of (6) by $\Box 0$ yields $\Diamond a \Box 0 = 0$. ⊣

As follows from Lemmas 3.7 and 3.8, if (X, R) is a compact Hausdorff frame, then $(C(X), \Box_R) \in \boldsymbol{mba\ell}$. We now extend this correspondence to a contravariant functor. For this we recall the definition of a bounded morphism.

Definition 3.14.

1. A *bounded morphism* (or *p-morphism*) between Kripke frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ is a map $f : X \to Y$ satisfying $f(R[x]) = S[f(x)]$ for each $x \in X$ (equivalently, $f^{-1}(S^{-1}[y]) = R^{-1}[f^{-1}(y)]$ for each $y \in Y$).
2. Let KHF be the category of compact Hausdorff frames and continuous bounded morphisms.

Lemma 3.15. *If $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ are compact Hausdorff frames and $\varphi : X \to Y$ is a continuous bounded morphism, then $C(\varphi)$ is a morphism in* $\boldsymbol{mba\ell}$.

Proof. That $C(\varphi)$ is a $\boldsymbol{ba\ell}$-morphism follows from Gelfand duality. Therefore, it is sufficient to prove that $C(\varphi)$ preserves $\Box$; that is, $C(\varphi)(\Box_S f) = \Box_R C(\varphi)(f)$ for each $f \in C(Y)$. Since φ is a bounded morphism, $\varphi(R[x]) = S[\varphi(x)]$ for each $x \in X$. Let $x \in X$ and $f \in C(Y)$. If $R[x] \neq \varnothing$, then $S[\varphi(x)] \neq \varnothing$, so

$$\begin{aligned}C(\varphi)(\Box_S f)(x) &= (\Box_S f \circ \varphi)(x) = (\Box_S f)(\varphi(x)) = \inf(f(S[\varphi(x)]))\\ &= \inf(f(\varphi(R[x]))) = \inf((f\circ\varphi)(R[x])) = \Box_R(f\circ\varphi)(x)\\ &= \Box_R(C(\varphi)(f))(x).\end{aligned}$$

If $R[x] = \varnothing$, then $S[\varphi(x)] = \varnothing$, so

$$C(\varphi)(\Box_S f)(x) = (\Box_S f)(\varphi(x)) = 1 = (\Box_R C(\varphi)(f))(x).$$

Thus, $C(\varphi)(\Box_S f) = \Box_R C(\varphi)(f)$. ⊣

Theorem 3.16. *There is a contravariant functor C : KHF $\to$ $\boldsymbol{mba\ell}$ which sends $\mathfrak{F} = (X, R)$ to $C(\mathfrak{F}) = (C(X), \Box_R)$ and a morphism φ in KHF to $C(\varphi)$.*

Proof. If $\mathfrak{F} \in$ KHF, then $C(\mathfrak{F}) \in \boldsymbol{mba\ell}$ by Lemmas 3.7 and 3.8. If φ is a morphism in KHF, then $C(\varphi)$ is a morphism in $\boldsymbol{mba\ell}$ by Lemma 3.15. It follows from Gelfand duality that $C(\psi\circ\varphi) = C(\varphi)\circ C(\psi)$ and that C preserves identity morphisms. Thus, C is a contravariant functor. ⊣

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§4. Continuous relations on the Yosida space. In this section we define a contravariant functor $Y : \boldsymbol{mba\ell} \to \mathsf{KHF}$ in the other direction, which is technically the most involved part of the paper.

Let $A \in \boldsymbol{ba\ell}$. For $S \subseteq A$ let

$$S^+ = \{a \in S \mid a \geq 0\}.$$

We point out that if I is an ℓ-ideal of A, then $I^+ = \{a^+ \mid a \in I\}$.

Let $(A, \Box) \in \boldsymbol{mba\ell}$. For $S \subseteq A$ we use the following standard notation from modal logic

$$\Box S := \{\Box a \mid a \in S\} \text{ and } \Box^{-1} S := \{a \mid \Box a \in S\}.$$

DEFINITION 4.1. Let $(A, \Box) \in \boldsymbol{mba\ell}$ and let Y_A be the Yosida space of A. Define $R_\Box$ on Y_A by

$$x R_\Box y \quad \text{iff} \quad \Box y^+ \subseteq x, \quad \text{iff} \quad y^+ \subseteq \Box^{-1} x.$$

REMARK 4.2. Comparing the definition above of $R_\Box$ to the definition of $R_\Box$ on the space of ultrafilters of a modal algebra in Jónsson-Tarski duality, we see that the inclusion is reversed because here we work with ideals rather than filters.

We have that

$$x R_\Box y \text{ iff } (\forall a \geq 0)(a + y = 0 + y \Rightarrow \Box a + x = 0 + x).$$

If we work with $\Diamond$ instead of $\Box$, since $\Diamond a = 1 - \Box(1 - a)$, the definition becomes

$$x R_\Box y \text{ iff } (\forall b \leq 1)(b + y = 1 + y \Rightarrow \Diamond b + x = 1 + x).$$

Thus, $x R_\Box y$ iff $\{1 - \Diamond b \mid 1 - b \in y, b \leq 1\} \subseteq x$. This more complicated definition is one reason why we prefer to work with $\Box$ rather than $\Diamond$. Another is that, as is standard in working with ordered algebras, using $\Box$ allows us to work with the positive cone rather than the set of elements below 1.

Let $A \in \boldsymbol{ba\ell}$. We recall that the *zero set* of $a \in A$ is defined as

$$Z_\ell(a) = \{x \in Y_A \mid a \in x\}.$$

If $S \subseteq A$, then we set

$$Z_\ell(S) = \bigcap \{Z_\ell(a) \mid a \in S\} = \{x \in Y_A \mid S \subseteq x\}.$$

It is easy to see that if I is the ℓ-ideal generated by S, then $Z_\ell(S) = Z_\ell(I)$. We define the *cozero set of S* as

$$\text{coz}_\ell(S) = Y_A \setminus Z_\ell(S) = \{x \in Y_A \mid S \not\subseteq x\}.$$

Since the zero sets are exactly the closed sets, the cozero sets are exactly the open sets of Y_A. The family $\{\text{coz}_\ell(a) \mid a \in A\}$ then constitutes a basis for the topology on Y_A.

REMARK 4.3. Let $A \in \boldsymbol{ba\ell}$, Y_A be the Yosida space of A, $x \in Y_A$, and $a \in A$.

1. x is a prime ideal of A because $A/x \cong \mathbb{R}$ (see, e.g., [20, Corollary 2.7]).
2. Either $a^+ \in x$ or $a^- \in x$. This follows from (1) and $a^+ a^- = 0$.
3. $a^+ \in x$ and $a^- \notin x$ iff $a + x < 0 + x$ (see [7, Remark 2.11]).

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4. $a^+ \in x$ iff $a + x \leq 0 + x$. For, if $a^+ \in x$, then

$$a + x = (a^+ - a^-) + x = -a^- + x \leq 0 + x$$

since $a^- \geq 0$. Conversely, if $a + x \leq 0 + x$, then either $a + x < 0 + x$, in which case $a^+ \in x$ by (3), or $a + x = 0 + x$, in which case $a \in x$, so $a^+ \in x$.
5. $a^- \in x$ and $a^+ \notin x$ iff $a + x > 0 + x$ (see [7, Remark 2.11]).
6. $a^- \in x$ iff $a + x \geq 0 + x$. The proof is similar to that of (4) but uses (5) instead of (3).

Let $(A, \Box) \in \boldsymbol{mba\ell}$ and $R_\Box$ be defined on Y_A as in Definition 4.1. Recalling Notation 3.2, we denote $R_\Box^{-1}[Y_A]$ by D_A and $Y_A \setminus D_A$ by E_A.

Lemma 4.4. *Let $(A, \Box) \in \boldsymbol{mba\ell}$, $a \in A$, $\lambda \in \mathbb{R}$, and $x \in Y_A$.*

1. *If $x \in D_A$, then $\Box 0 \in x$.*
2. *If $\Box 0 \in x$, then $\Box(a + \lambda) + x = (\Box a + \lambda) + x$.*
3. *If $\Box 0 \in x$, then $\Box((a - \lambda)^+) \in x$ iff $(\Box a - \lambda)^+ \in x$.*
4. *If $\Box 0 \in x$, then $\Diamond a + x = -\Box(-a) + x$.*
5. *If $\Box 0 \notin x$, then $1 - \Box a \in x$.*
6. *If $\Diamond a \notin x$, then $\Box 0 \in x$.*

Proof. (1) If $x \in D_A$, then there is y with $x R_\Box y$. Therefore, since $0 \in y^+$, we have $\Box 0 \in x$.

(2) By (M4) and (M2), $\Box(a + \lambda) = \Box a + \lambda - \lambda \Box 0$. Therefore, if $\Box 0 \in x$, then $\Box(a + \lambda) + x = (\Box a + \lambda) + x$.

(3) This follows from (M3), Remark 4.3(4), and (2).

(4) Apply Lemma 3.13(6).

(5) By Lemma 3.13(4), $\Box 0 = (\Box 0)(\Box a)$, so $(\Box 0)(1 - \Box a) = 0 \in x$. Since $\Box 0 \notin x$ and x is a prime ideal, $1 - \Box a \in x$.

(6) By Lemma 3.13(7), $(\Diamond a)(\Box 0) = 0 \in x$. Since x is a prime ideal and $\Diamond a \notin x$, we have $\Box 0 \in x$. ⊣

Proposition 4.5. *$R_\Box[x]$ is closed for every $x \in Y_A$.*

Proof. We prove that $Y_A \setminus R_\Box[x]$ is open for every $x \in Y_A$. Let $y \notin R_\Box[x]$, so $y^+ \nsubseteq \Box^{-1} x$. Therefore, there is $a \geq 0$ such that $a \in y$ and $\Box a \notin x$. By Lemma 3.13(3), $\Box a \geq 0$, so there exists $0 \leq \lambda \in \mathbb{R}$ such that $(\Box a - \lambda) + x > 0 + x$ but $(a - \lambda) + y < 0 + y$. By Remark 4.3(3,5), $(a - \lambda)^- \notin y$ and $(\Box a - \lambda)^+ \notin x$. Thus, $y \in \mathrm{coz}_\ell((a - \lambda)^-)$, and it remains to show that $\mathrm{coz}_\ell((a - \lambda)^-) \cap R_\Box[x] = \varnothing$. Suppose not. Then there is z such that $x R_\Box z$ and $z \in \mathrm{coz}_\ell((a - \lambda)^-)$. Since $(a - \lambda)^- \notin z$, we have $(a - \lambda)^+ \in z$ (see Remark 4.3(2)). But $x R_\Box z$ means $z^+ \subseteq \Box^{-1} x$, so $\Box 0, \Box((a - \lambda)^+) \in x$. Thus, by Lemma 4.4(3), $(\Box a - \lambda)^+ \in x$, hence $(\Box a - \lambda) + x \leq 0 + x$. This is a contradiction. Consequently, we have that $\mathrm{coz}_\ell((a - \lambda)^-) \cap R_\Box[x] = \varnothing$, completing the proof. ⊣

Remark 4.6. Since $R_\Box[x]$ is closed for each $x \in Y_A$, it is of the form $Z_\ell(S)$ for some $S \subseteq A$. In fact, $R_\Box[x] = Z_\ell(S)$ where $S = \{a \in A \mid a \geq 0 \text{ and } \Diamond a \in x\}$.

For a topological space X and a continuous real-valued function f on X, we recall that the *zero set* of f is

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

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and the *cozero set* of f is

$$\operatorname{coz}(f) = X \setminus Z(f) = \{x \in X \mid f(x) \neq 0\}.$$

The following lemma is a consequence of [17, Problem 1D, p. 21].

LEMMA 4.7. *Let $A \in$ **baℓ** and $a, s \in A$. If $Z_\ell(a) \subseteq \operatorname{int} Z_\ell(s)$, then there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a)f$ in $C(Y_A)$.*

PROOF. Observe that for each $t \in A$ we have $Z_\ell(t) = Z(\zeta_A(t))$. Therefore, $Z_\ell(a) \subseteq \operatorname{int} Z_\ell(s)$ implies $Z(\zeta_A(a)) \subseteq \operatorname{int} Z(\zeta_A(s))$. Now apply [17, Problem 1D, p. 21]. ⊣

LEMMA 4.8. *Let $(A, \Box) \in$ **mbaℓ**, $x \in Y_A$, $S = (A \setminus \Box^{-1}x)^+$, and $a \in (\Box^{-1}x)^+$.*

1. $\bigcap\{\operatorname{coz}_\ell(s) \mid s \in S\} = \bigcap\{\overline{\operatorname{coz}_\ell(s)} \mid s \in S\}$.
2. $\overline{\operatorname{coz}_\ell(s)} \cap Z_\ell(a) \neq \varnothing$ *for every* $s \in S$.
3. *The family* $\{\overline{\operatorname{coz}_\ell(s)} \cap Z_\ell(a) \mid s \in S\}$ *has the finite intersection property.*

PROOF. (1) The inclusion $\subseteq$ is clear. To prove the reverse inclusion, it is sufficient to prove that for each $s \in S$ there is $t \in S$ such that $\overline{\operatorname{coz}_\ell(t)} \subseteq \operatorname{coz}_\ell(s)$. Since $s \in S$, we have $\Box s \notin x$, so there is $\varepsilon \in \mathbb{R}$ with $\Box s + x > \varepsilon + x > 0 + x$. Set $t = (s - \varepsilon)^+$. Then $t \geq 0$ and

$$\Box t = \Box(s - \varepsilon)^+ = (\Box(s - \varepsilon))^+$$

by (M3). If $\Box t \in x$, then $\Box(s - \varepsilon) + x \leq 0 + x$, so $\Box s - \varepsilon(1 - \Box 0) + x \leq 0 + x$ by Lemma 3.13(5). We have $\Box 0 \in x$ by Lemma 4.4(5) as $\Box a \in x$. Therefore, $(\Box s - \varepsilon) + x \leq 0 + x$, and hence $\Box s + x \leq \varepsilon + x$. The obtained contradiction shows $\Box t \notin x$, so $t \in S$. Let $z \in Z_\ell(s)$. Then $z \in \zeta_A(s)^{-1}(-\varepsilon, \varepsilon)$, an open set. But $\zeta_A(s)^{-1}(-\varepsilon, \varepsilon) \subseteq Z_\ell(t)$ by the definition of t and Remark 4.3(3), so $Z_\ell(s) \subseteq \operatorname{int} Z_\ell(t)$. Thus, $\overline{\operatorname{coz}_\ell(t)} \subseteq \operatorname{coz}_\ell(s)$.

(2) Note that $\overline{\operatorname{coz}_\ell(s)} \cap Z_\ell(a) \neq \varnothing$ means that $Z_\ell(a) \not\subseteq \operatorname{int} Z_\ell(s)$. We argue by contradiction. Suppose $Z_\ell(a) \subseteq \operatorname{int} Z_\ell(s)$. Then by Lemma 4.7, there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a)f$ in $C(Y_A)$. Since $C(Y_A)$ is the uniform completion of A (see Proposition 2.3), there is a sequence $\{b_n\} \subseteq A$ such that $f = \lim \zeta_A(b_n)$. It is well known that multiplication is continuous with respect to the norm, so we have $\lim \zeta_A(ab_n) = \zeta_A(a)f = \zeta_A(s)$. Since $s \in S$, there is $\varepsilon > 0$ such that $\Box s + x > \varepsilon + x$, so $(\Box s - \varepsilon) + x > 0 + x$. There is N such that $||s - ab_N|| < \varepsilon$. Therefore, $s < ab_N + \varepsilon$. Take $0 < \lambda \in \mathbb{R}$ such that $b_N \leq \lambda$. Then $s < \lambda a + \varepsilon$. Since $0 \leq \Box 0 \leq \Box a \in x$, we have $\Box 0 \in x$. Thus by Lemmas 3.13(1) and 4.4(2), and (M5),

$$\Box s + x \leq \Box(\lambda a + \varepsilon) + x = (\Box(\lambda a) + \varepsilon) + x = (\Box\lambda\Box a + \varepsilon) + x.$$

But $\Box a \in x$, so $\Box s + x \leq \varepsilon + x$, contradicting $\varepsilon + x < \Box s + x$.

(3) We first show that the intersection of any two members of the family contains another member of the family. Let $s, t \in S$. Then $\Box s, \Box t \notin x$. Since x is a maximal ℓ-ideal, $A/x \cong \mathbb{R}$ is totally ordered, so

$$(\Box s \wedge \Box t) + x = \min\{\Box s + x, \Box t + x\} \neq 0 + x,$$

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and hence $\Box s \wedge \Box t \notin x$. By (M1), this shows $\Box(s \wedge t) \notin x$, which gives $s \wedge t \in S$. Since $\mathrm{coz}_\ell(s \wedge t) = \mathrm{coz}_\ell(s) \cap \mathrm{coz}_\ell(t)$, we have:

$$\begin{aligned}(\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a)) \cap (\overline{\mathrm{coz}_\ell(t)} \cap Z_\ell(a)) &= \overline{\mathrm{coz}_\ell(s)} \cap \overline{\mathrm{coz}_\ell(t)} \cap Z_\ell(a) \\ &\supseteq \overline{\mathrm{coz}_\ell(s) \cap \mathrm{coz}_\ell(t)} \cap Z_\ell(a) \\ &= \overline{\mathrm{coz}_\ell(s \wedge t)} \cap Z_\ell(a).\end{aligned}$$

Because $s \wedge t \in S$, we have that $\overline{\mathrm{coz}_\ell(s \wedge t)} \cap Z_\ell(a)$ is in the family. An easy induction argument then completes the proof because every element of the family is nonempty by (2). ⊣

PROPOSITION 4.9. *Let $(A, \Box) \in \boldsymbol{mba\ell}$ and $x \in Y_A$. Then*

$$(\Box^{-1} x)^+ = \bigcup \{y^+ \mid y \in R_\Box[x]\}.$$

PROOF. The right-to-left inclusion follows from the definition of $R_\Box$. For the left-to-right inclusion, let $a \in (\Box^{-1} x)^+$. By Lemma 4.8(1),

$$\bigcap \{\mathrm{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\} = \bigcap \{\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a) \mid s \in S\}.$$

By Lemma 4.8(3) and compactness of Y_A, this intersection is nonempty. Therefore, there is $y \in \bigcap\{\mathrm{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\}$. This means that $a \in y$ and $y \cap S = \varnothing$, so $y^+ \subseteq \Box^{-1} x$. Thus, a is contained in some $y \in R_\Box[x]$, completing the proof. ⊣

LEMMA 4.10. *Let $(A, \Box) \in \boldsymbol{mba\ell}$.*

1. $R_\Box^{-1}[Z_\ell(a)] = Z_\ell(\Box a)$ *for every* $0 \leq a \in A$.
2. $D_A = Z_\ell(\Box 0)$.

PROOF. (1) Let $x \in R_\Box^{-1}[Z_\ell(a)]$. Then there is $y \in Y_A$ such that $x R_\Box y$ and $a \in y$. Therefore, $a \in y^+ \subseteq \Box^{-1} x$. Thus, $\Box a \in x$, and so $x \in Z_\ell(\Box a)$.

For the other inclusion, let $x \in Z_\ell(\Box a)$. Since $\Box a \in x$ and $\Box a \geq 0$, we have $a \in (\Box^{-1} x)^+$. By Proposition 4.9, there is $y \in Y_A$ such that $x R_\Box y$ and $a \in y$. Thus, $x \in R_\Box^{-1}[Z_\ell(a)]$.

(2) This follows from (1) by setting $a = 0$ and using $Y_A = Z_\ell(0)$. ⊣

We will use Lemma 4.10 to prove that $R_\Box^{-1}[F]$ is closed for each closed subset F of Y_A. For this we require Esakia's lemma, which is an important tool in modal logic (see, e.g., [11, Section 10.3]). The original statement is for descriptive frames, but it has a straightforward generalization to the setting of compact Hausdorff frames (see [4, Lemma 2.17]). We call a relation R on a compact Hausdorff space X *point-closed* if $R[x]$ is closed for each $x \in X$.

LEMMA 4.11 (Esakia's lemma) *If R is a point-closed relation on a compact Hausdorff space X, then for each (nonempty) down-directed family $\{F_i \mid i \in I\}$ of closed subsets of X we have*

$$R^{-1}\left[\bigcap \{F_i \mid i \in I\}\right] = \bigcap \{R^{-1}[F_i] \mid i \in I\}.$$

REMARK 4.12. Let $(A, \Box) \in \boldsymbol{mba\ell}$ and S be a set of nonnegative elements of A closed under addition. Since $Z_\ell(a + b) \subseteq Z_\ell(a) \cap Z_\ell(b)$ for each $a, b \in S$, we have

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that $\{Z_\ell(a) \mid a \in S\}$ is a down-directed family of closed subsets of Y_A. Then, by Esakia's lemma and Lemma 4.10, we have:

$$R_\Box^{-1}[Z_\ell(S)] = R_\Box^{-1}\left[\bigcap\{Z_\ell(a) \mid a \in S\}\right] = \bigcap\{R_\Box^{-1}[Z_\ell(a)] \mid a \in S\}$$
$$= \bigcap\{Z_\ell(\Box a) \mid a \in S\} = Z_\ell(\Box S).$$

In particular, for an ℓ-ideal I, since $Z_\ell(I) = Z_\ell(I^+)$, we have

$$R_\Box^{-1} Z_\ell(I) = R_\Box^{-1} Z_\ell(I^+) = \bigcap\{Z_\ell(\Box a) \mid a \in I^+\}.$$

PROPOSITION 4.13. *$R_\Box^{-1}[F]$ is closed for every closed subset F of Y_A.*

PROOF. Since F is a closed subset of Y_A, there is an ℓ-ideal I such that $F = Z_\ell(I)$. By Remark 4.12,

$$R_\Box^{-1} Z_\ell(I) = \bigcap\{Z_\ell(\Box a) \mid a \in I^+\},$$

which is closed because it is an intersection of closed subsets of Y_A. ⊣

LEMMA 4.14. *If $\Diamond a \in x$ and $x R_\Box y$, then $a^+ \in y$.*

PROOF. Suppose that $x R_\Box y$ and $a^+ \notin y$. By Remark 4.3(4) $a + y > 0 + y$, so there is $0 < \lambda \in \mathbb{R}$ such that $a + y = \lambda + y$. Therefore, $\lambda - a \in y$, so $(\lambda - a)^+ \in y$. Since $y^+ \subseteq \Box^{-1} x$, we have $\Box((\lambda - a)^+) \in x$. By Lemma 4.4(1) $\Box 0 \in x$, so $(\lambda + \Box(-a))^+ \in x$ by Lemma 4.4(3). Thus, $(\lambda + \Box(-a)) + x \leq 0 + x$, so $\lambda + x \leq -\Box(-a) + x$, and hence $\lambda + x \leq \Diamond a + x$ by Lemma 4.4(4). Since $\lambda + x > 0 + x$, this shows $\Diamond a \notin x$. ⊣

LEMMA 4.15. *$R_\Box^{-1}[\mathrm{coz}_\ell(a)] = \mathrm{coz}_\ell(\Diamond a)$ for every $0 \leq a \in A$.*

PROOF. For the left-to-right inclusion, suppose $x \notin \mathrm{coz}_\ell(\Diamond a)$. Then $\Diamond a \in x$. Consider $y \in R_\Box[x]$. By Lemma 4.14, $a = a^+ \in y$, so $y \notin \mathrm{coz}_\ell(a)$. Therefore, $x \notin R_\Box^{-1}[\mathrm{coz}_\ell(a)]$.

For the right-to-left inclusion, let $x \in \mathrm{coz}_\ell(\Diamond a)$. Then $\Diamond a \notin x$, so $\Box 0 \in x$ by Lemma 4.4(6). Therefore, by Lemma 4.4(4), $0 + x \neq \Diamond a + x = -\Box(-a) + x$, and hence $\Box(-a) \notin x$. Since $-a \leq 0$, we have $\Box(-a) + x \leq \Box 0 + x = 0 + x$. Thus, there is $\lambda \in \mathbb{R}$ with $\lambda < 0$ and $\Box(-a) + x = \lambda + x$, so $\Box(-a) - \lambda \in x$. By Lemma 4.4(3), we have

$$\Box((-a - \lambda)^+) \in x \text{ iff } (\Box(-a) - \lambda)^+ \in x.$$

Consequently, by Proposition 4.9,

$$(-a - \lambda)^+ \in (\Box^{-1} x)^+ = \bigcup\{y^+ \mid y \in R_\Box[x]\}.$$

Hence, there is $y \in R_\Box[x]$ such that $(-a - \lambda)^+ \in y$. This means that $(-a - \lambda) + y \leq 0 + y$, so $a + y \geq -\lambda + y > 0 + y$. Therefore, $a \notin y$, and so $y \in \mathrm{coz}_\ell(a)$. Thus, $x \in R_\Box^{-1}[\mathrm{coz}_\ell(a)]$. ⊣

PROPOSITION 4.16. *$R_\Box^{-1}[U]$ is open for every open subset U of Y_A.*

PROOF. Open subsets of Y_A are of the form $\mathrm{coz}_\ell(I) = \bigcup\{\mathrm{coz}_\ell(a) \mid a \in I\}$ for some ℓ-ideal I. Since $\mathrm{coz}_\ell(I) = \bigcup\{\mathrm{coz}_\ell(a) \mid a \in I,\ a \geq 0\}$ and $R_\Box^{-1}$ commutes with

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arbitrary unions, by Lemma 4.15, we have

$$\begin{aligned} R_\Box^{-1} \operatorname{coz}_\ell(I) &= R_\Box^{-1} \bigcup \{\operatorname{coz}_\ell(a) \mid a \in I,\ a \geq 0\} \\ &= \bigcup \{R_\Box^{-1} \operatorname{coz}_\ell(a) \mid a \in I,\ a \geq 0\} \\ &= \bigcup \{\operatorname{coz}_\ell(\Diamond a) \mid a \in I,\ a \geq 0\}, \end{aligned}$$

which is open because it is a union of open subsets of Y_A. ⊣

Putting Propositions 4.5, 4.13, and 4.16 together yields:

THEOREM 4.17. *If* $(A, \Box) \in \boldsymbol{mba\ell}$, *then* $(Y_A, R_\Box) \in \mathsf{KHF}$.

We finish the section by showing how to extend the object correspondence of Theorem 4.17 to a contravariant functor $Y : \boldsymbol{mba\ell} \to \mathsf{KHF}$.

LEMMA 4.18. *Let* $(A, \Box), (B, \Box) \in \boldsymbol{mba\ell}$ *and* $\alpha : A \to B$ *be a morphism in* $\boldsymbol{mba\ell}$. *Then* $Y(\alpha) : (Y_B, R_\Box) \to (Y_A, R_\Box)$ *is a bounded morphism.*

PROOF. For each $y \in Y_A$, we have that y^+ and $\alpha(y^+)$ are sets of nonnegative elements closed under addition, so Remark 4.12 applies. Therefore, since $Z_\ell(y^+) = \{y\}$,

$$Y(\alpha)^{-1}(R_\Box^{-1}[y]) = Y(\alpha)^{-1}(R_\Box^{-1}[Z_\ell(y^+)]) = Y(\alpha)^{-1}(Z_\ell(\Box y^+))$$

and

$$Z_\ell(\Box\alpha(y^+)) = R_\Box^{-1}[Z_\ell(\alpha(y^+))].$$

The definition of $Y(\alpha)$ shows that

$$Y(\alpha)^{-1}(Z_\ell(\Box y^+)) = Z_\ell(\alpha(\Box y^+)) \quad \text{and} \quad Y(\alpha)^{-1}(Z_\ell(y^+)) = Z_\ell(\alpha(y^+)).$$

This yields

$$Y(\alpha)^{-1}(R_\Box^{-1}[y]) = Y(\alpha)^{-1}(Z_\ell(\Box y^+)) = Z_\ell(\alpha(\Box y^+))$$

and

$$R_\Box^{-1}[Y(\alpha)^{-1}(y)] = R_\Box^{-1}[Y(\alpha)^{-1}(Z_\ell(y^+))] = R_\Box^{-1}[Z_\ell(\alpha(y^+))] = Z_\ell(\Box\alpha(y^+)).$$

Consequently, since α commutes with $\Box$, we have

$$Y(\alpha)^{-1}(R_\Box^{-1}[y]) = R_\Box^{-1}[Y(\alpha)^{-1}(y)],$$

which proves that $Y(\alpha)$ is a bounded morphism. ⊣

Putting Theorem 4.17 and Lemma 4.18 together and remembering that $Y : \boldsymbol{ba\ell} \to$ KHaus is a contravariant functor yields:

THEOREM 4.19. $Y : \boldsymbol{mba\ell} \to \mathsf{KHF}$ *is a contravariant functor.*

§5. Duality. In this section we prove our main results. We show that Y and C yield a dual adjunction between $\boldsymbol{mba\ell}$ and KHF which restricts to a dual equivalence between the category of uniformly complete members of $\boldsymbol{mba\ell}$ and KHF.

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Definition 5.1. Let ***mubal*** be the full subcategory of ***mbal*** consisting of uniformly complete objects of ***mbal***.

Proposition 5.2. ***mubal*** *is a reflective subcategory of* ***mbal***.

Proof. By Corollary 2.4, ***ubal*** is a reflective subcategory of ***bal***, where CY : ***bal*** $\to$ ***ubal*** is the reflector. We first show that ζ_A is an ***mbal***-morphism for each $(A, \Box) \in$ ***mbal***. Let $x \in Y_A$. Recall that

$$(\Box_{R_\Box}\zeta_A(a))(x) = \begin{cases} \inf\{\zeta_A(a)(y) \mid xR_\Box y\} & \text{if } x \in D_A, \\ 1 & \text{if } x \in E_A. \end{cases}$$

If $x \in E_A$, then $\Box 0 \notin x$ by Lemma 4.10(2). Therefore, $\Box a - 1 \in x$ by Lemma 4.4(5), and hence $\zeta_A(\Box a)(x) = 1 = (\Box_{R_\Box}\zeta_A(a))(x)$. Now let $x \in D_A$. Then $(\Box_{R_\Box}\zeta_A(a))(x) = \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. We first show that $\zeta_A(\Box a)(x) \leq \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. Suppose that $xR_\Box y$, so $y^+ \subseteq \Box^{-1}x$. Let $\lambda = \zeta_A(a)(y)$. Then $a - \lambda \in y$, so $(a-\lambda)^+ \in y^+ \subseteq \Box^{-1}x$, and hence $(\Box a - \lambda)^+ \in x$ by Lemma 4.4(1,3). Therefore,

$$0 = \zeta_A((\Box a - \lambda)^+)(x) = \max\{\zeta_A(\Box a)(x) - \lambda, 0\},$$

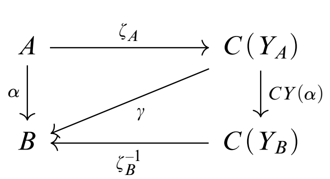
so $\zeta_A(\Box a)(x) - \lambda \leq 0$, and hence $\zeta_A(\Box a)(x) \leq \lambda = \zeta_A(a)(y)$. Thus, $\zeta_A(\Box a)(x) \leq \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$.

We next show that $\zeta_A(\Box a)(x) \geq \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. Let $\mu = \zeta_A(\Box a)(x)$. Then $(\Box a - \mu)^+ \in x$, and hence $\Box((a-\mu)^+) \in x$. Therefore, by Proposition 4.9,

$$(a-\mu)^+ \in (\Box^{-1}x)^+ = \bigcup\{y^+ \mid xR_\Box y\}.$$

So there is $y \in R_\Box[x]$ such that $(a-\mu)^+ \in y$. Thus, by Remark 4.3(4), $\zeta_A(a)(y) \leq \mu = \zeta_A(\Box a)(x)$. Consequently, $\inf\{\zeta_A(a)(y) \mid y \in R_\Box[x]\} \leq \zeta_A(\Box a)(x)$, and hence $\zeta_A(\Box a) = \Box_{R_\Box}\zeta(a)$. This yields that ζ_A is an ***mbal***-morphism.

Next, let $\alpha : A \to B$ be an ***mbal***-morphism with $B \in$ ***mubal***. Since α is a ***bal***-morphism, there is a unique ***bal***-morphism $\gamma : C(Y_A) \to B$ such that $\gamma \circ \zeta_A = \alpha$. Since ζ is a natural transformation and ζ_B is an isomorphism, it follows that $\gamma = \zeta_B^{-1} \circ CY(\alpha)$.



As we saw in the paragraph above, ζ_B is an ***mbal***-morphism. Also, $CY(\alpha) : C(Y_A) \to C(Y_B)$ is an ***mbal***-morphism by Lemmas 4.18 and 3.15. Therefore, γ is an ***mbal***-morphism, concluding the proof. ⊣

Theorem 5.3. *The functors* Y : ***mbal*** $\to$ KHF *and* C : KHF $\to$ ***mbal*** *yield a dual adjunction of the categories, which restricts to a dual equivalence between* ***mubal*** *and* KHF.

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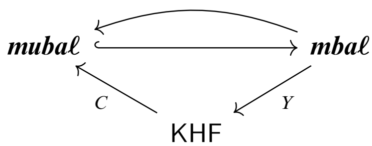

PROOF. By Gelfand duality, the functors $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ and $C : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus that restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus. The natural transformations are given by $\zeta : 1_{\boldsymbol{ba\ell}} \to CY$ and $\varepsilon : 1_{\mathsf{KHaus}} \to YC$ where we recall from Section 2 that $\varepsilon_X : X \to X_{C(X)}$ is defined by

$$\varepsilon_X(x) = M_x = \{f \in C(X) \mid f(x) = 0\}.$$

Therefore, it is sufficient to show that ζ_A is a morphism in $\boldsymbol{mba\ell}$ for each $(A, \Box) \in \boldsymbol{mba\ell}$ and that ε_X is a bounded morphism for each $(X, R) \in \mathsf{KHF}$. We showed in the proof of Proposition 5.2 that ζ_A is a morphism in $\boldsymbol{mba\ell}$, and hence it remains to show that xRy iff $\varepsilon_X(x) R_{\Box_R} \varepsilon_X(y)$ for each $(X, R) \in \mathsf{KHF}$.

To see this recall that $\varepsilon_X(x) R_{\Box_R} \varepsilon_X(y)$ means that $M_y^+ \subseteq \Box_R^{-1} M_x$. First suppose that xRy and $f \in M_y^+$. Then $f(y) = 0$ and $f \geq 0$. By the definition of $\Box_R$ we have $(\Box_R f)(x) = \inf\{f(z) \mid xRz\} = 0$. Therefore, $\Box_R f \in M_x$, and so $f \in \Box_R^{-1} M_x$. This gives $M_y^+ \subseteq \Box_R^{-1} M_x$. Next suppose that $x \not R y$, so $y \notin R[x]$. If $R[x] = \varnothing$, then $(\Box_R 0)(x) = 1$, so $0 \in M_y^+$ but $\Box_R 0 \notin M_x$, yielding $M_y^+ \not\subseteq \Box_R^{-1} M_x$. On the other hand, if $R[x] \neq \varnothing$, since $R[x]$ is closed, by Urysohn's Lemma there is $f \geq 0$ such that $f(y) = 0$ and $f(R[x]) = \{1\}$. Thus, $f \in M_y^+$ and $\Box_R f \notin M_x$. Consequently, $M_y^+ \not\subseteq \Box_R^{-1} M_x$. ⊣

§6. Connection to modal algebras and Jónsson-Tarski duality. In this section we connect Theorem 5.3 to Jónsson-Tarski duality. Recall that a *modal algebra* is a pair $\mathfrak{A} = (A, \Box)$ where A is a boolean algebra and $\Box$ is a unary function on A preserving finite meets (including 1). As usual, the dual function $\Diamond$ is defined by $\Diamond a = \neg\Box\neg a$, and is axiomatized as a unary function preserving finite joins (including 0). Let MA be the category of modal algebras and modal homomorphisms (boolean homomorphisms preserving $\Box$).

We recall from the Introduction that a *descriptive frame* is a pair $\mathfrak{F} = (X, R)$ where X is a Stone space and R is a continuous relation on X, and that DF is the category of descriptive frames and continuous bounded morphisms. As we already pointed out, Stone duality generalizes to the following duality:

THEOREM 6.1 (Jónsson–Tarski duality) MA *is dually equivalent to* DF.

The functors $(-)^* : \mathsf{DF} \to \mathsf{MA}$ and $(-)_* : \mathsf{MA} \to \mathsf{DF}$ are defined as follows. For a descriptive Kripke frame $\mathfrak{F} = (X, R)$ let $\mathfrak{F}^* = (\mathsf{Clop}(X), \Box_R)$ where $\mathsf{Clop}(X)$ is the boolean algebra of clopen subsets of X and $\Box_R U = X \setminus R^{-1}[X \setminus U]$ (alternatively, $\Diamond_R U = R^{-1}[U]$). For a bounded morphism f let $f^* = f^{-1}$. Then $(-)^* : \mathsf{DF} \to \mathsf{MA}$ is a well-defined contravariant functor.

For a modal algebra $\mathfrak{A} = (A, \Box)$ let $\mathfrak{A}_* = (Y_A, R_\Box)$ where Y_A is the set of ultrafilters of A with the Stone topology and

$$xR_\Box y \quad \text{iff} \quad (\forall a \in A)(\Box a \in x \Rightarrow a \in y) \quad \text{iff} \quad \Box^{-1} x \subseteq y$$

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(alternatively, $xR_{\Box}y$ iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$ iff $y \subseteq \Diamond^{-1}x$). For a modal algebra homomorphism h let $h_* = h^{-1}$. Then $(-)_* : \mathsf{MA} \to \mathsf{DF}$ is a well-defined contravariant functor, and the functors $(-)_*$ and $(-)^*$ yield a dual equivalence of MA and DF.

To define a functor from $\boldsymbol{mba\ell}$ to MA we recall that for each commutative ring A with 1, the idempotents of A form a boolean algebra $\mathrm{Id}(A)$, where the boolean operations on $\mathrm{Id}(A)$ are defined as follows:

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad \neg e = 1 - e.$$

We point out that if $A \in \boldsymbol{ba\ell}$, then the lattice operations on A restrict to those on $\mathrm{Id}(A)$.

REMARK 6.2. Since each $A \in \boldsymbol{ba\ell}$ is an f-ring (see Remark 2.5), we will freely use the following two identities of f-rings (see [8, Section XIII.3] and [8, Corollary XVII.5.1]):

$$(a \wedge b) + c = (a + c) \wedge (b + c) \quad \text{and} \quad (a \wedge b)d = (ad) \wedge (bd) \text{ for } d \geq 0.$$

LEMMA 6.3. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $\Box$ sends idempotents to idempotents.*

PROOF. First observe that $e \in A$ is an idempotent iff $1 \wedge 2e = e$. To see this, if e is an idempotent, by Remark 6.2,

$$(1 \wedge 2e) - e = (1 - e) \wedge e = \neg e \wedge e = 0.$$

Therefore, $1 \wedge 2e = e$. Conversely, suppose that $1 \wedge 2e = e$. Then $(1 - e) \wedge e = 0$ by the same calculation. Since each $A \in \boldsymbol{ba\ell}$ is an f-ring, from $(1 - e) \wedge e = 0$ it follows that $(1 - e)e = 0$ (see, e.g., [8, Lemma XVII.5.1]). Thus, $e^2 = e$, and hence e is an idempotent.

For each $a \in A$, by (M5), (M2), and Lemma 3.13(4) we have

$$\Box(2a) = \Box 2 \Box a = (2 - \Box 0)\Box a = (2 - 2\Box 0 + \Box 0)\Box a = 2\Box a(1 - \Box 0) + \Box 0.$$

By Lemma 3.13(3), $\Box 0 \geq 0$, so Lemma 3.13(4) and Remark 6.2 imply

$$(1 \wedge 2\Box a)\Box 0 = \Box 0 \wedge 2\Box a \Box 0 = \Box 0 \wedge 2\Box 0 = \Box 0.$$

Now suppose e is an idempotent, so $e = 1 \wedge 2e$. Since $\Box 0 \leq \Box 1 = 1$, we have that $1 - \Box 0 \geq 0$. Thus, by Remark 6.2 and the two identities just proved,

$$\begin{aligned}
\Box e &= \Box(1 \wedge 2e) = 1 \wedge \Box(2e) \\
&= ((1 - \Box 0) + \Box 0) \wedge \Box(2e) \\
&= ((1 - \Box 0) + \Box 0) \wedge (2\Box e(1 - \Box 0) + \Box 0) \\
&= ((1 - \Box 0) \wedge 2\Box e(1 - \Box 0)) + \Box 0 \\
&= (1 \wedge 2\Box e)(1 - \Box 0) + \Box 0 \\
&= (1 \wedge 2\Box e)(1 - \Box 0) + (1 \wedge 2\Box e)\Box 0 \\
&= 1 \wedge 2\Box e.
\end{aligned}$$

Therefore, $\Box e$ is an idempotent. ⊣

LEMMA 6.4. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $(\mathrm{Id}(A), \Box) \in$ MA.*

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PROOF. Since $A \in \boldsymbol{ba\ell}$, we have that $\mathrm{Id}(A)$ is a boolean algebra. That $\Box$ is well defined on $\mathrm{Id}(A)$ follows from Lemma 6.3. That $\Box$ preserves finite meets in $\mathrm{Id}(A)$ follows from (M1) and Lemma 3.13(2). Thus, $(\mathrm{Id}(A), \Box) \in \mathsf{MA}$. ⊣

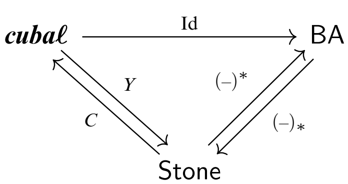
Define $\mathrm{Id} : \boldsymbol{mba\ell} \to \mathsf{MA}$ by sending $(A, \Box) \in \boldsymbol{mba\ell}$ to $(\mathrm{Id}(A), \Box) \in \mathsf{MA}$ and a morphism $A \to B$ in $\boldsymbol{mba\ell}$ to its restriction $\mathrm{Id}(A) \to \mathrm{Id}(B)$. The next lemma is an easy consequence of Lemma 6.4.

LEMMA 6.5. $\mathrm{Id} : \boldsymbol{mba\ell} \to \mathsf{MA}$ *is a well-defined covariant functor.*

We recall (see [32] and the references therein) that a commutative ring A is *clean* if each element is the sum of an idempotent and a unit.

DEFINITION 6.6. Let $\boldsymbol{cuba\ell}$ be the full subcategory of $\boldsymbol{uba\ell}$ consisting of those $A \in \boldsymbol{uba\ell}$ where A is clean.

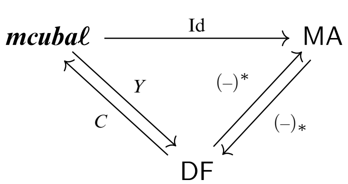
REMARK 6.7. By Stone duality, the category BA of boolean algebras and boolean homomorphisms is dually equivalent to the category Stone of Stone spaces and continuous maps. Thus, by [6, Proposition 5.20], the following diagram commutes (up to natural isomorphism), and the functor Id yields an equivalence of $\boldsymbol{cuba\ell}$ and BA.



DEFINITION 6.8. Let $\boldsymbol{mcuba\ell}$ be the full subcategory of $\boldsymbol{muba\ell}$ consisting of those $(A, \Box) \in \boldsymbol{muba\ell}$ where A is clean.

As a corollary of Theorems 5.3, 6.1 and Remark 6.7, we obtain:

THEOREM 6.9. *The diagram below commutes* (*up to natural isomorphism*) *and the functor* Id *yields an equivalence of* $\boldsymbol{mcuba\ell}$ *and* MA.



§7. Some correspondence results. In this section we take the first steps towards the correspondence theory for $\boldsymbol{mba\ell}$ by characterizing algebraically what it takes for the relation $R_\Box$ to satisfy additional first-order properties, such as seriality, reflexivity, transitivity, and symmetry.

We recall that a relation R on X is *serial* if $R[x] \neq \varnothing$ for each $x \in X$.

LEMMA 7.1. *Let* $(X, R) \in \mathsf{KHF}$.

1. *If R is serial, then* $\Box_R 0 = 0$ *in* $C(X)$.
2. *If R is reflexive, then* $\Box_R f \leq f$ *for each* $f \in C(X)$.

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3. *If R is transitive, then $\Box_R f \leq \Box_R(\Box_R f(1 - \Box_R 0) + f\Box_R 0)$ for each $f \in C(X)$.*
4. *If R is symmetric, then $(\Diamond_R \Box_R f)(1 - \Box_R 0) \leq f(1 - \Box_R 0)$ for each $f \in C(X)$.*

PROOF. (1) Suppose that R is serial. Then $R[x] \neq \varnothing$, so $(\Box_R 0)(x) = 0$ for each $x \in Y$. Thus, $\Box_R 0 = 0$.

(2) Suppose that R is reflexive and $f \in C(Y)$. For each $x \in Y$, we have $x \in R[x]$. Thus, $(\Box_R f)(x) = \inf f R[x] \leq f(x)$.

(3) Suppose that R is transitive. Let $f \in C(Y)$ and $x \in Y$. If $R[x] = \varnothing$, then by the definition of $\Box_R$,

$$(\Box_R f)(x) = 1 = \Box_R(\Box_R f(1 - \Box_R 0) + f\Box_R 0)(x).$$

Suppose that $R[x] \neq \varnothing$. Then $(\Box_R f)(x) = \inf f R[x]$ and

$$\begin{aligned}\Box_R(\Box_R f(1 - \Box_R 0) + f\Box_R 0)(x) \\ = \inf\{(\Box_R f)(y)(1 - \Box_R 0)(y) + f(y)(\Box_R 0)(y) \mid xRy\}.\end{aligned}$$

We have

$$(\Box_R f)(y)(1 - \Box_R 0)(y) + f(y)(\Box_R 0)(y) = \begin{cases} f(y) & \text{if } R[y] = \varnothing, \\ (\Box_R f)(y) & \text{if } R[y] \neq \varnothing. \end{cases}$$

It is therefore sufficient to prove that, for each $y \in R[x]$, if $R[y] = \varnothing$ then $(\Box_R f)(x) \leq f(y)$ and if $R[y] \neq \varnothing$ then $(\Box_R f)(x) \leq (\Box_R f)(y)$. Suppose $R[y] = \varnothing$. Since $R[x] \neq \varnothing$, we have

$$(\Box_R f)(x) = \inf\{f(z) \mid z \in R[x]\} \leq f(y).$$

If $R[y] \neq \varnothing$, then by transitivity of R we have $R[y] \subseteq R[x]$, so

$$(\Box_R f)(x) = \inf\{f(z) \mid z \in R[x]\} \leq \inf\{f(w) \mid w \in R[y]\} = (\Box_R f)(y).$$

Thus, $\Box_R f \leq \Box_R(\Box_R f(1 - \Box_R 0) + f\Box_R 0)$.

(4) Suppose that R is symmetric. Let $f \in C(Y)$ and $x \in Y$. If $R[x] = \varnothing$, then $(1 - \Box_R 0)(x) = 0$ so

$$(\Diamond_R \Box_R f)(x)(1 - \Box_R 0)(x) = 0 = f(x)(1 - \Box_R 0)(x).$$

If $R[x] \neq \varnothing$, then $(1 - \Box_R 0)(x) = 1$, so it is sufficient to prove that $(\Diamond_R \Box_R f)(x) \leq f(x)$. For any $y \in R[x]$ we have $x \in R[y]$ by symmetry. Therefore,

$$(\Box_R f)(y) = \inf\{f(z) \mid z \in R[y]\} \leq f(x).$$

Thus, recalling Remark 3.5, we have

$$(\Diamond_R \Box_R f)(x) = \sup\{(\Box_R f)(y) \mid y \in R[x]\} \leq f(x). \qquad \dashv$$

Since each $(A, \Box) \in \boldsymbol{mba\ell}$ is isomorphic to a subalgebra of $(C(Y), \Box_R)$ and equations satisfied by an algebra are also satisfied by a subalgebra, as an immediate consequence of Lemma 7.1 we obtain:

PROPOSITION 7.2. *Let $(A, \Box) \in \boldsymbol{mba\ell}$ and $(Y_A, R_\Box)$ be its dual.*

1. *If $R_\Box$ is serial, then $\Box 0 = 0$ in A.*
2. *If $R_\Box$ is reflexive, then $\Box a \leq a$ for each $a \in A$.*

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3. *If $R_\Box$ is transitive, then $\Box a \le \Box(\Box a(1 - \Box 0) + a\Box 0)$ for each $a \in A$.*
4. *If $R_\Box$ is symmetric, then $\Diamond\Box a(1 - \Box 0) \le a(1 - \Box 0)$ for each $a \in A$.*

PROPOSITION 7.3. *Let $(A, \Box) \in$ **mbaℓ** and $(Y_A, R_\Box)$ be its dual.*

1. *If $\Box 0 = 0$, then $R_\Box$ is serial.*
2. *If $\Box a \le a$ for each $a \in A$, then $R_\Box$ is reflexive.*
3. *If $\Box a \le \Box(\Box a(1 - \Box 0) + a\Box 0)$ for each $a \in A$, then $R_\Box$ is transitive.*
4. *If $\Diamond\Box a(1 - \Box 0) \le a(1 - \Box 0)$ for each $a \in A$, then $R_\Box$ is symmetric.*

PROOF. (1) Suppose that $\Box 0 = 0$ in A. Since $Y_A = Z_\ell(0)$, by Lemma 4.10(2), we have $D_A = Z_\ell(\Box 0) = Z_\ell(0) = Y_A$. Thus, $R_\Box$ is serial.

(2) Suppose $\Box a \le a$ for each $a \in A$. Let $x \in Y_A$ and $a \in x^+$. We then have $0 \le \Box a \le a \in x$. Thus, $x^+ \subseteq \Box^{-1}x$, and so $xR_\Box x$.

(3) Suppose $\Box a \le \Box(\Box a(1 - \Box 0) + a\Box 0)$ for each $a \in A$. Let $x, y, z \in Y_A$ with $xR_\Box y$ and $yR_\Box z$. Then $y^+ \subseteq \Box^{-1}x$ and $z^+ \subseteq \Box^{-1}y$. Let $a \in z^+$. Then $\Box a \in y^+$. As $0 \in z^+$, we have $\Box 0 \in y^+$. Thus, since y is an ideal, $\Box a(1 - \Box 0) + a\Box 0 \in y$. Because $\Box a(1 - \Box 0) + a\Box 0 \ge 0$, we have $\Box(\Box a(1 - \Box 0) + a\Box 0) \in x$. By hypothesis, $0 \le \Box a \le \Box(\Box a(1 - \Box 0) + a\Box 0) \in x$. Thus, $\Box a \in x$. This shows that $z^+ \subseteq \Box^{-1}x$, and hence $xR_\Box z$.

(4) Suppose $\Diamond\Box a(1 - \Box 0) \le a(1 - \Box 0)$ for each $a \in A$. Let $x, y \in Y_A$ with $xR_\Box y$. Then $y^+ \subseteq \Box^{-1}x$, so $0 \in y^+$ implies $\Box 0 \in x$. Thus,

$$\Diamond\Box a + x = \Diamond\Box a(1 - \Box 0) + x \le a(1 - \Box 0) + x = a + x.$$

To see that $yR_\Box x$, let $a \in x^+$. If $\Box a \notin y$, then $0 + y < \Box a + y$ because $\Box a \ge 0$. So there is $0 < \lambda \in \mathbb{R}$ such that $\lambda - \Box a \in y$. Thus, $(\lambda - \Box a)^+ \in y^+$. Since $xR_\Box y$, by (2) and (4) of Lemma 4.4, we have

$$\begin{aligned}(\lambda - \Diamond\Box a)^+ + x &= (\lambda + \Box(-\Box a))^+ + x = (\Box(\lambda - \Box a))^+ + x \\ &= \Box(\lambda - \Box a)^+ + x = 0 + x.\end{aligned}$$

Because $\Diamond\Box a + x \le a + x$ we have $(\lambda - a) + x \le (\lambda - \Diamond\Box a) + x$. Therefore,

$$0 + x \le (\lambda - a)^+ + x \le (\lambda - \Diamond\Box a)^+ + x = 0 + x.$$

This implies $(\lambda - a)^+ \in x$. Thus, by Remark 4.3(4), $0 + x < \lambda + x \le a + x$, which contradicts $a \in x^+$. Therefore, $\Box a \in y$, which yields $x^+ \subseteq \Box^{-1}y$. Thus, $yR_\Box x$. ⊣

Putting Propositions 7.2 and 7.3 together yields:

THEOREM 7.4. *Let $(A, \Box) \in$ **mbaℓ** and $(Y_A, R_\Box)$ be its dual.*

1. *$R_\Box$ is serial iff $\Box 0 = 0$ in A.*
2. *$R_\Box$ is reflexive iff $\Box a \le a$ for each $a \in A$.*
3. *$R_\Box$ is transitive iff $\Box a \le \Box(\Box a(1 - \Box 0) + a\Box 0)$ for each $a \in A$.*
4. *$R_\Box$ is symmetric iff $\Diamond\Box a(1 - \Box 0) \le a(1 - \Box 0)$ for each $a \in A$.*

REMARK 7.5. If we work with $\Diamond$ instead of $\Box$, then Theorem 7.4 can be stated as follows.

1. $R_\Box$ is serial iff $\Diamond 1 = 1$.
2. $R_\Box$ is reflexive iff $a \le \Diamond a$ for each $a \in A$.

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3. $R_\Box$ is transitive iff $\Diamond(\Diamond a + a(1 - \Diamond 1)) \leq \Diamond a$ for each $a \in A$.
4. $R_\Box$ is symmetric iff $\Diamond\Box a \leq a\Diamond 1$ for each $a \in A$.

REMARK 7.6. Let $(A, \Box) \in \boldsymbol{mbal}$. If $\Box 0 = 0$, then the transitivity and symmetry axioms simplify to $\Box a \leq \Box\Box a$ and $\Diamond\Box a \leq a$, which are standard transitivity and symmetry axioms in modal logic.

DEFINITION 7.7.

1. Let $\boldsymbol{mbal}^{\mathsf{D}}$ be the full subcategory of $\boldsymbol{mbal}$ consisting of objects $(A, \Box) \in \boldsymbol{mbal}$ satisfying $\Box 0 = 0$.
2. Let $\boldsymbol{mbal}^{\mathsf{T}}$ be the full subcategory of $\boldsymbol{mbal}$ consisting of objects $(A, \Box) \in \boldsymbol{mbal}$ satisfying $\Box a \leq a$.
3. Let $\boldsymbol{mbal}^{\mathsf{K4}}$ be the full subcategory of $\boldsymbol{mbal}$ consisting of objects $(A, \Box) \in \boldsymbol{mbal}$ satisfying $\Box a \leq \Box(\Box a(1 - \Box 0) + a\Box 0)$.
4. Let $\boldsymbol{mbal}^{\mathsf{B}}$ be the full subcategory of $\boldsymbol{mbal}$ consisting of objects $(A, \Box) \in \boldsymbol{mbal}$ satisfying $\Diamond\Box a(1 - \Box 0) \leq a(1 - \Box 0)$.
5. Let $\boldsymbol{mbal}^{\mathsf{S4}} = \boldsymbol{mbal}^{\mathsf{T}} \cap \boldsymbol{mbal}^{\mathsf{K4}}$.
6. Let $\boldsymbol{mbal}^{\mathsf{S5}} = \boldsymbol{mbal}^{\mathsf{S4}} \cap \boldsymbol{mbal}^{\mathsf{B}}$.

REMARK 7.8. Since the reflexivity axiom implies the seriality axiom, we obtain that $(A, \Box) \in \boldsymbol{mbal}^{\mathsf{S4}}$ iff $(A, \Box) \in \boldsymbol{mbal}^{\mathsf{T}}$ and $\Box a \leq \Box\Box a$ for each $a \in A$. Similarly, $(A, \Box) \in \boldsymbol{mbal}^{\mathsf{S5}}$ iff $(A, \Box) \in \boldsymbol{mbal}^{\mathsf{S4}}$ and $\Diamond\Box a \leq a$ for each $a \in A$.

REMARK 7.9. The notation of Definition 7.7 is motivated by the standard notation in modal logic:

1. D denotes the least normal modal logic containing the axiom $\Diamond\top$.
2. T denotes the least normal modal logic containing the axiom $\Box p \to p$.
3. K4 denotes the least normal modal logic containing the axiom $\Box p \to \Box\Box p$.
4. B denotes the least normal modal logic containing the axiom $\Diamond\Box p \to p$.
5. S4 denotes the join of T and K4.
6. S5 denotes the join of S4 and B.

The inclusions between the subclasses of algebras in $\boldsymbol{mbal}$ given in Definition 7.7 are the same as for the corresponding classes of modal algebras; see Figure 1. Similarly to Definition 7.7, for $\mathsf{X} \in \{\mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{B}, \mathsf{S4}, \mathsf{S5}\}$ we define the following categories:

- The categories $\boldsymbol{mubal}^{\mathsf{X}}$ are defined similarly to $\boldsymbol{mbal}^{\mathsf{X}}$ but with $\boldsymbol{mbal}$ replaced by $\boldsymbol{mubal}$.
- The categories $\boldsymbol{mcubal}^{\mathsf{X}}$ are defined similarly to $\boldsymbol{mbal}^{\mathsf{X}}$ but with $\boldsymbol{mbal}$ replaced by $\boldsymbol{mcubal}$.
- The categories MA^{X} are defined similarly to $\boldsymbol{mbal}^{\mathsf{X}}$ but with $\boldsymbol{mbal}$ replaced by MA.
- The categories $\mathsf{KHF}^{\mathsf{X}}$ are defined by adding the corresponding properties of the relation R to the definition of KHF.
- The categories DF^{X} are defined as $\mathsf{KHF}^{\mathsf{X}}$ by restricting KHF to DF.

Theorems 5.3, 6.9, and 7.4, and the corresponding versions of Theorem 6.1 yield the following result.

THEOREM 7.10. *Suppose that* $\mathsf{X} \in \{\mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{B}, \mathsf{S4}, \mathsf{S5}\}$.

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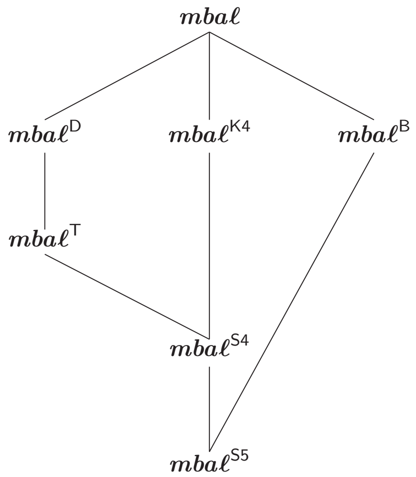

FIGURE 1. Inclusion relationships between some subcategories of ***mbaℓ***.

1. *The category* ***mubaℓ***$^{\mathsf{X}}$ *is dually equivalent to* $\mathsf{KHF}^{\mathsf{X}}$.
2. *The categories* ***mcubaℓ***$^{\mathsf{X}}$ *and* MA^{X} *are dually equivalent to* DF^{X}, *and hence are equivalent*.

§8. Concluding remarks. We finish the paper with several remarks, which indicate a number of possible directions for future research.

REMARK 8.1.

1. As we pointed out in the Introduction, there are other dualities for KHaus. For example, in pointfree topology we have Isbell duality [21] (see also [1] or [23, Section III.1]) and de Vries duality [12] (see also [2]). The two are closely related, see [3]. Isbell and de Vries dualities were generalized to the setting of KHF in [4]. We plan to compare the results of [4] to the ones obtained in this paper.
2. As we pointed out in the Introduction, another relevant duality was established by Kakutani [25, 26], the Krein brothers [28], and Yosida [37], who worked in the signature of vector lattices. Gelfand duality has a natural counterpart in this setting. Let ***bav*** be the category of bounded archimedean vector lattices and let ***ubav*** be its reflective subcategory consisting of uniformly complete objects. Then there is a dual adjunction between ***bav*** and KHaus, which restricts to a dual equivalence between ***ubav*** and KHaus. This duality is sometimes referred to as Yosida duality (or Kakutani–Krein–Yosida duality). In our axiomatization of ***mbaℓ*** (see Definition 3.10), the only axiom involving multiplication is (M5). In the serial case (M5) simplifies to (M5′) of Remark 3.12, which only involves scalar multiplication. In the non-serial case, (M5) can be replaced by the following two axioms
   - $\Box(\lambda a) = \lambda \Box a + (1 - \lambda)\Box 0$ provided $\lambda \geq 0$,
   - $\Box 0 \wedge (1 - \Box a)^+ = 0$,

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which again only involve vector lattice operations. This yields the category ***mbav*** of modal bounded archimedean vector lattices and its reflective subcategory ***mubav*** consisting of uniformly complete objects. The results of Section 5 then generalize to the setting of ***mbav*** and ***mubav***, and provide a generalization of Yosida duality.

3. Our definition of a modal operator on a bounded archimedean ℓ-algebra can be further adjusted to the settings of ℓ-rings, ℓ-groups, and MV-algebras. In this regard, it would be interesting to develop logical systems corresponding to these algebras. As we pointed out in the introduction, this can be done along the same lines as in [13] (see also [15]).
4. It would be natural to develop the correspondence theory for $\boldsymbol{mba\ell}$ by generalizing the results of Section 7, with the final goal towards a Sahlqvist type correspondence (see, e.g., [9, Chapter 3]).

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