Math. Struct. in Comp. Science (1999), *vol.* 9, *pp.* 103–108. Printed in the United Kingdom

Order in open intervals of computable reals

JAN VON PLATO

University of Helsinki, Unioninkatu 40 B, 00014 University of Helsinki, Finland
E-mail: vonplato@helsinki.fi

Received 10 September 1997; revised 10 February 1998

In constructive theories, an apartness relation is often taken as basic and its negation used as equality. An apartness relation should be continuous in its arguments, as in the case of computable reals. A similar approach can be taken to order relations. We shall here study the partial order on open intervals of computable reals. Since order on reals is undecidable, there is no simple uniformly applicable lattice meet operation that would always produce non-negative intervals as values. We show how to solve this problem by a suitable definition of apartness for intervals. We also prove the *strong extensionality* of the lattice operations, where by strong extensionality of an operation f on elements a, b we mean that apartness of values implies apartness in some of the arguments: $f(a,b) \neq f(c,d) \supset a \neq c \vee b \neq d$.

Most approaches to computable reals start from a concrete definition. We shall instead represent them by an abstract axiomatically introduced order structure.

1. Constructive linear order with minimum and maximum

A set L with a relation $a < b$ is a *constructive linear order* if it satisfies

LO1. $\sim(a < b \;\&\; b < a)$,
LO2. $a < b \supset a < c \vee c < b$.

The *strict linear order* $a < b$ allows us to define *apartness, equality* and *weak linear order* by

$$a \neq b \quad \equiv \quad a < b \vee b < a, \tag{1.1}$$

$$a = b \quad \equiv \quad \sim a \neq b, \tag{1.2}$$

$$a \leqslant b \quad \equiv \quad \sim b < a. \tag{1.3}$$

These have the usual constructively valid properties.

A constructive linear order with *minimum and maximum* has the two-place operations $min(a,b)$, $max(a,b)$ with the additional axioms

LO3. $min(a,b) \leqslant a, \quad min(a,b) \leqslant b$,
LO4. $a \leqslant max(a,b), \quad b \leqslant max(a,b)$.

LO5. $min(a,b) < c \supset a < c \vee b < c$,
LO6. $c < max(a,b) \supset c < a \vee c < b$.

Constructively, we do not have the property $min(a,b) = a \vee min(a,b) = b$, nor the dual

$max(a, b) = a \vee max(a, b) = b$, but the following result shows that axioms LO5 and LO6 are still constructively acceptable.

Theorem 1.1.

(i) $\sim(min(a, b) < a \;\&\; min(a, b) < b)$ is equivalent to LO5,

(ii) $\sim(a < max(a, b) \;\&\; b < max(a, b))$ is equivalent to LO6.

Proof. (i) In order to derive LO5, assume $min(a, b) < c$. Then, by LO2, $min(a, b) < a \vee a < c$.

Case 1. If $min(a, b) < a$, then $b \leqslant min(a, b)$ by $\sim(min(a, b) < a \;\&\; min(a, b) < b)$. Therefore $b < c$, so $a < c \vee b < c$.

Case 2. $a < c$ gives $a < c \vee b < c$.

To show the reverse implication, assume LO5 and $min(a, b) < a \;\&\; min(a, b) < b$. By LO5, $a < a \vee b < a$ and $a < b \vee b < b$, which is impossible.

(ii) The proof for *max* is similar. □

Contraposition of axioms LO5 and LO6 gives uniqueness principles for *min* and *max* analogous to those for meet and join in a lattice. (See also von Plato (1996) for more details.)

In Negri and Soravia (1998), Axioms LO1–LO6 are explicitly verified for *formal reals*, a version of computable reals based on constructive pointfree topology. They also show the equivalence of formal reals to several other, more common approaches to computable reals.

2. Open intervals

Open intervals will be elements of the product $L \times L$ of a set L with constructive linear order with minimum and maximum. We shall say that an interval (a, b) is *positive* if $a < b$, *negative* if $b < a$, *nonpositive* if $b \leqslant a$, *nonnegative* if $a \leqslant b$ and *degenerate* if $a = b$.

Definition 2.1. $(a, b) \nleq (c, d) \equiv a < b \;\&\; (a < c \vee d < b)$.

We say that interval (a, b) *exceeds* interval (c, d) if $(a, b) \nleq (c, d)$.

Lemma 2.2.

(i) $\sim(a, b) \nleq (a, b)$,

(ii) $(a, b) \nleq (c, d) \supset (a, b) \nleq (e, f) \vee (e, f) \nleq (c, d)$.

Proof. (i) By $\sim(a < a \vee b < b)$.

(ii) Let $(a, b) \nleq (c, d)$, or $a < b \;\&\; (a < c \vee d < b)$. We have to prove

$$a < b \;\&\; (a < e \vee f < b) \vee e < f \;\&\; (e < c \vee d < f). \qquad (*)$$

Case 1. $a < b \;\&\; a < c$. Then $a < e \vee e < b$ by axiom LO2.

Case 1.1. $a < e$. Then $(*)$ follows.

Case 1.2. $e < b$. Then $e < f \vee f < b$ by LO2.

Case 1.2.1. $e < f$. Then by $a < c$, $a < e \vee e < c$.

Case 1.2.1.1 $a < e$. Then $(*)$ follows.

Case 1.2.1.2 $e < c$. Then $(*)$ follows.

Case 1.2.2. $f < b$. Then $(*)$ follows.

Case 2. $a < b \;\&\; d < b$. The proof is similar. □

https://doi.org/10.1017/S0960129598002564 Published online by Cambridge University Press

Note that 2.2(ii) has the same form as Axiom LO2. We call it *splitting of excess*, or just *splitting* for short. The following are immediate consequences of Definition 2.1.

Corollary 2.3.

(i) $(a,b) \nleqslant (c,c) \supset a < b$,

(ii) $\sim (c,c) \nleqslant (a,b)$.

Using the same notation for relations in L and $L \times L$, we define apartness, equality, and weak and strict partial order relations in $L \times L$ by the following:

$$(a,b) \neq (c,d) \quad \equiv \quad (a,b) \nleqslant (c,d) \vee (c,d) \nleqslant (a,b) \tag{2.1}$$

$$(a,b) = (c,d) \quad \equiv \quad \sim (a,b) \neq (c,d) \tag{2.2}$$

$$(a,b) \leqslant (c,d) \quad \equiv \quad \sim (a,b) \nleqslant (c,d) \tag{2.3}$$

$$(a,b) < (c,d) \quad \equiv \quad (a,b) \leqslant (c,d) \ \& \ (a,b) \neq (c,d). \tag{2.4}$$

These relations have the usual constructive properties. If $a < b$, it follows that

$$(a,b) \leqslant (c,d) \quad \supset\subset \quad c \leqslant a \ \& \ b \leqslant d \tag{2.5}$$

$$(a,b) < (c,d) \quad \supset\subset \quad (c \leqslant a \ \& \ b < d) \vee (c < a \ \& \ b \leqslant d). \tag{2.6}$$

Definition (2.4) is equivalent to

$$(a,b) < (c,d) \quad \supset\subset \quad (c,d) \nleqslant (a,b) \ \& \ \sim (a,b) \nleqslant (c,d). \tag{2.7}$$

There is by Corollary 2.3(ii) a *bottom element* (c,c) of the partial ordering of intervals:

$$(c,c) \leqslant (a,b). \tag{2.8}$$

Its uniqueness is guaranteed by our definition of interval equality. Further, it follows from Corollary 2.3 that all nonpositive, degenerate and negative intervals are equal, so we have the following corollary.

Corollary 2.4.

(i) $b \leqslant a \supset (a,b) = (c,c)$,

(ii) $(a,a) = (c,c)$,

(iii) $b < a \supset (a,b) = (c,c)$.

3. The lattice of open intervals

So far we have established an excess relation for intervals and shown that this relation is irreflexive and obeys the splitting property, Lemma 2.2. The usual reflexive and transitive partial order was obtained as negation of excess, in (2.3). We shall now proceed to giving a special lattice structure to the open intervals of the constructive continuum.

We first define the meet and join operations for intervals:

$$(a,b) \wedge (c,d) \quad \equiv \quad (max(a,c), min(b,d)) \tag{3.1}$$

$$(a,b) \vee (c,d) \quad \equiv \quad (min(a,c), max(b,d)). \tag{3.2}$$

This definition gives meets that can be negative intervals, which is not a problem

https://doi.org/10.1017/S0960129598002564 Published online by Cambridge University Press

if the partial order is decidable. But we avoid the dependency of meet on a proof of nonnegativity through Definition 2.1: say, if in (3.1) we have $min(b,d) < max(a,c)$, the meet is by Corollary 2.4 equal to the bottom element (c,c). Our Definition 2.1 leads to equating all negative, nonpositive and degenerate intervals, thus permitting a uniformly applicable lattice meet operation for intervals even if the order relation is undecidable. It also follows that any positive interval covers the bottom element, $a < b \supset (c,c) < (a,b)$. More, generally, if we have a notion of positivity for opens of a topological space, a suitable definition of apartness and equality of opens can give a result to this effect.

The following lemma collects the lattice properties resulting from our definition of intervals and of lattice meet and join.

Lemma 3.1.

(i) $(a,b)\wedge(c,d) \nleqslant (e,f) \supset (a,b) \nleqslant (e,f)$,
(ii) $(a,b)\wedge(c,d) \nleqslant (e,f) \supset (c,d) \nleqslant (e,f)$,
(iii) $(a,b) \nleqslant (c,d)\vee(e,f) \supset (a,b) \nleqslant (c,d)$,
(iv) $(a,b) \nleqslant (c,d)\vee(e,f) \supset (a,b) \nleqslant (e,f)$,
(v) $(a,b) \nleqslant (c,d)\wedge(e,f) \supset (a,b) \nleqslant (c,d) \vee (a,b) \nleqslant (e,f)$,
(vi) $(a,b)\vee(c,d) \nleqslant (e,f) \supset (a,b) \nleqslant (e,f) \vee (c,d) \nleqslant (e,f)$.

Proof. (i) If $(a,b)\wedge(c,d) \nleqslant (e,f)$, Definition (3.1) gives $(max(a,c), min(b,d)) \nleqslant (a,b)$ so by the definition of interval excess (Definition 2.1),

$$max(a,c) < min(b,d) \ \& \ (max(a,c) < e \vee f < min(b,d)).$$

We have to prove that $a < b \ \& \ (a < e \vee f < d)$. By $max(a,c) < min(b,d)$, we get $a < b$. For the rest, there are two cases: if $max(a,c) < e$, then $a < e$, and if $f < min(b,d)$, then $f < d$. Laws (ii)–(iv) are proved similarly. For (v), assume $(a,b) \nleqslant (c,d)\wedge(e,f)$, so

$$a < b \ \& \ (a < max(c,e) \vee min(d,f) < b).$$

If $a < max(c,e)$, we have $a < c \vee a < e$ by LO6. If $a < c$, then $(a,b) \nleqslant (c,d)$, and if $a < e$, then $(a,b) \nleqslant (e,f)$. If $min(d,f) < b$, the conclusion follows similarly by LO5. (vi) is proved analogously. □

The usual lattice laws are, up to a simple modification, contrapositions of the above ones. If $(a,b)\wedge(c,d) \nleqslant (a,b)$, substitution of (a,b) for (e,f) in Lemma 3.1(i) gives an impossibility so that $(a,b)\wedge(c,d) \leqslant (a,b)$ follows, and a similar remark applies to (ii)–(iv). For (v)–(vi), we take contrapositions to arrive at the following corollary.

Corollary 3.2.

(i) $(a,b)\wedge(c,d) \leqslant (a,b)$,
(ii) $(a,b)\wedge(c,d) \leqslant (c,d)$,
(iii) $(a,b) \leqslant (a,b)\vee(c,d)$,
(iv) $(c,d) \leqslant (a,b)\vee(c,d)$,
(v) $(a,b) \leqslant (c,d) \ \& \ (a,b) \leqslant (e,f) \supset (a,b) \leqslant (c,d)\wedge(e,f)$,
(vi) $(a,b) \leqslant (e,f) \ \& \ (c,d) \leqslant (e,f) \supset (a,b)\vee(c,d) \leqslant (e,f)$.

In the other direction, of the laws given by Lemma 3.1, the first four can be derived from the corresponding ones in the corollary, using splitting of excess, but the last two are constructively stronger than the usual uniqueness principles for lattice meet and join. A characteristic consequence of 3.1(i)–(vi), one that cannot be proved from 3.2(i)–(vi), is the following result about the meet and join operations.

Theorem 3.3.

(i) $(a_1, b_1)\wedge(c_1, d_1) \nleq (a_2, b_2)\wedge(c_2, d_2) \supset (a_1, b_1) \nleq (a_2, b_2) \vee (c_1, d_1) \nleq (c_2, d_2)$,
(ii) $(a_1, b_1)\vee(c_1, d_1) \nleq (a_2, b_2)\vee(c_2, d_2) \supset (a_1, b_1) \nleq (a_2, b_2) \vee (c_1, d_1) \nleq (c_2, d_2)$.

Proof. By Lemma 3.1(v), antecedent of (i) gives

$$(a_1, b_1)\wedge(c_1, d_1) \nleq (a_2, b_2) \vee (a_1, b_1)\wedge(c_1, d_1) \nleq (c_2, d_2).$$

In the first case, we have by splitting,

$$(a_1, b_1)\wedge(c_1, d_1) \nleq (a_1, b_1) \vee (a_1, b_1) \nleq (a_2, b_2),$$

but the former is impossible by Corollary 3.2(i), so $(a_1, b_1) \nleq (a_2, b_2)$ follows. In the second case, we similarly get $(c_1, d_1) \nleq (c_2, d_2)$. Law (ii) is proved analogously. □

It is easy to see that we can replace the excess relations in the above result by apartnesses. This leads to what can be called, following the general terminology of Troelstra and van Dalen (1988, p. 386), *strong extensionality* of the meet and join for intervals. We can go further in the arguments of meet and join: it follows almost by definition that in general,

$$(a, b) \neq (c, d) \supset a \neq c \vee b \neq d. \tag{3.3}$$

We then get the promised result as the following corollary.

Corollary 3.4.

(i) $(a_1, b_1)\wedge(c_1, d_1) \neq (a_2, b_2)\wedge(c_2, d_2) \supset a_1 \neq a_2 \vee b_1 \neq b_2 \vee c_1 \neq c_2 \vee d_1 \neq d_2$,
(ii) $(a_1, b_1)\vee(c_1, d_1) \neq (a_2, b_2)\vee(c_2, d_2) \supset a_1 \neq a_2 \vee b_1 \neq b_2 \vee c_1 \neq c_2 \vee d_1 \neq d_2$.

Strong extensionality has as a consequence, through contraposition, the substitution principle of equals in the meet and join operations.

4. Relation to earlier literature

In earlier literature, Scott (1968) comes closest to the structure we have given to order in the intuitionistic continuum. We find there the two axioms of constructive linear order, and apartness, equality and weak order as defined from strict linear order, but no axiomatization of minimum and maximum. The two axioms are also found in Bridges (1989), and various metamathematical independence proofs are given in Bridges (1991). Brouwer (1927, 1950) studies properties of the relation we write as $\sim\sim a < b$, which he calls 'the non–contradictory of the measurable natural order on the continuum'. Heyting (1956, p. 110) studies what we call constructive linear order, but treats equality as primitive. Axiom LO2 is explicit, and LO1 follows from Heyting's (somewhat redundant) set of axioms. (Say, if $a < b$ & $b < a$, transitivity gives $a < a$, and then $\sim a = a$ by Heyting's Axiom (1), which is impossible.) In Kleene and Vesley (1965, p. 143), we find as properties of the strict order relation on the continuum Axiom LO2, then $a < b \supset \sim b < a$ and (redundantly) $\sim a < a$.

https://doi.org/10.1017/S0960129598002564 Published online by Cambridge University Press

As for the lattice structure of open intervals of the intuitionistic continuum, it needs axiomatically introduced operations of minimum and maximum, and a solution to the uniformity problem of the lattice meet operation. The only consideration in this direction we have found is in Heyting (1956, p. 26), where he proves properties of computable reals. There we find as theorems some results corresponding to the first two axioms of *min* and *max*, namely LO3 and LO4.

Since order in the continuum is not decidable, it behaves rather like a partial order. This is indicated already in Brouwer's early paper Brouwer (1927). However, constructivization of partial order cannot be based on a strict partial order relation and an apartness relation. For example, such constructively valid properties as strong extensionality of lattice operations, do not follow from axiomatizations with strict partial order and apartness. The notion of excess and its characteristic properties of irreflexivity and splitting, as in Lemma 2.2, show a simple way out here.

References

Bridges, D. (1989) The constructive theory of preference relations on a locally compact space. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen* **A92** 141–165.

Bridges, D. (1991) The constructive inequivalence of various notions of preference ordering. *Mathematical Social Sciences* **21** 169–176.

Brouwer, L. (1927) Virtuelle Ordnung und unerweiterbare Ordnung. As reprinted in Brouwer's *Collected Works* **1** 406–408.

Brouwer, L. (1950) Remarques sur la notion d'ordre. As reprinted in Brouwer's *Collected Works* **1** 499–500.

Heyting, A. (1956) *Intuitionism, an Introduction*, North-Holland.

Kleene, S. and R. Vesley (1965) *The Foundations of Intuitionistic Mathematics*, North-Holland.

Negri, S. and D. Soravia (1998) The continuum as a formal space. *Archive for Mathematical Logic* (to appear).

von Plato, J. (1996) Organization and development of a constructive axiomatization. In: Berardi, S. and Coppo, M. (eds.) Types for Proofs and Programs. *Springer-Verlag Lecture Notes in Computer Science* **1158** 288–296.

Scott, D. (1968) Extending the topological interpretation to intuitionistic analysis. *Compositio Mathematica* **20** 194–210.

Troelstra, A. and van Dalen, D. (1988) *Constructivism in Mathematics* **2**, North-Holland.