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LIMIT THEOREMS FOR A SUPERCRITICAL POISSON RANDOM INDEXED BRANCHING PROCESS

ZHENLONG GAO * ** AND
YANHUA ZHANG,* *Qufu Normal University*

Abstract

Let $\{Z_n, n = 0, 1, 2, \ldots\}$ be a supercritical branching process, $\{N_t, t \geq 0\}$ be a Poisson process independent of $\{Z_n, n = 0, 1, 2, \ldots\}$, then $\{Z_{N_t}, t \geq 0\}$ is a supercritical Poisson random indexed branching process. We show a law of large numbers, central limit theorem, and large and moderate deviation principles for $\log Z_{N_t}$.

Keywords: Central limit theorem; large deviation principle; moderate deviation principle; random indexed branching process; stock prices

2010 Mathematics Subject Classification: Primary 60J80
Secondary 60F10

1. Introduction

The model of random indexed branching process is one of the important extensions of the Galton–Watson branching process, which was introduced by Epps [4] in order to study the evolution of stock prices. The statistical investigation on various estimates and some parameters of the process were performed by Dion and Epps [3]. Recently, this randomly indexed branching process has been brought to attention in both the theoretical and applied sense. On theoretical side, Mitov *et al.* [8], [11] considered a critical branching process subordinated by a general renewal process. The authors investigated the probability of nonextinction, the asymptotic behavior of the moments, and also limiting distributions under normalization. Results on the subcritical case were presented in [10]. In a more applied direction, Mitov and Mitov [7] derived an equation for the fair price of a European call option based on modeling the underlying stock price by this process with a Poisson subordinator and with a geometric offspring distribution. Subsequently, an equation for the fair price of an up-and-out call option, a particular form of a barrier option, was derived in [9]. For more details, we refer the reader to the doctoral theses of Williams [12] and Wu [13]. In [13], based on the idea of Athreya [1], the author derived the large deviation for Z_{N_t+1}/Z_{N_t}. Large deviation results for sums indexed by the generations of a Galton–Watson process were presented in [5]. In this paper, for a supercritical random indexed branching process with a Poisson subordinator, we shall mainly show the asymptotic properties of $\log Z_{N_t}$.

Let us give a description of the model. Let $\{Z_n, n = 0, 1, 2, \ldots\}$ and $\{N_t, t \geq 0\}$ be two independent stochastic processes on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following characteristics.

(i) That $\{Z_n\}$ is a Galton–Watson branching process with an offspring distribution $\{p_i, i = 0, 1, 2, \ldots\}$. Throughout this paper, we assume that our branching process starts from

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* Postal address: School of Statistics, Qufu Normal University, Qufu 273165, P.R.China
** Email address: gaozhenlong0325@163.com

one ancestor, i.e. $Z_0 = 1$ almost surely (a.s.) and that it belongs to the Böttcher case with a finite mean, i.e. $p_0 + p_1 = 0$ and $1 < m = \sum_{n=2}^{\infty} np_n < \infty$.

(ii) That $\{N_t, t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$.

Definition 1.1. The continuous-time process $\{Z_{N_t}, t \geq 0\}$ is called a Poisson randomly indexed branching process.

Define $W_n = Z_n/m^n$ for $n = 0, 1, 2, \dots$, then $\{W_n\}$ is a nonnegative martingale with limit W. One can prove that $\{W_{N_t}, t \geq 0\}$ is also a nonnegative martingale and has the same limit W; see [13].

First, we present a large deviation principle. Let

$$\Lambda_t(\theta) = \log \mathbb{E}\biggl(\exp\biggl(\frac{\theta \log Z_{N_t}}{t}\biggr)\biggr),$$

$$\Lambda(\theta) = \lim_{t\to+\infty} \frac{\Lambda_t(t\theta)}{t} = \lim_{t\to+\infty} \frac{\log \mathbb{E}(Z_{N_t}^{\theta})}{t}, \qquad \Lambda^*(x) = \sup_{\theta\in R}\{\theta x - \Lambda(\theta)\}.$$

According to the Gärtner–Ellis theorem (see [2]) and the results in [6], we have our first main result.

Theorem 1.1. (Large deviation.) *Assume that* $\mathbb{E}(Z_1^a) < \infty$ *for all* $a \geq 1$. *Then, for any measurable subset* B *of* $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in B^o} \Lambda^*(x) &\leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\biggl(\frac{\log Z_{N_t}}{t} \in B\biggr) \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\biggl(\frac{\log Z_{N_t}}{t} \in B\biggr) \\ &\leq -\inf_{x\in \bar{B}} \Lambda^*(x), \end{aligned}$$

where B^o *denotes the interior of* B, $\bar{B}$ *its closure, and*

$$\Lambda^*(x) = \begin{cases} \dfrac{x}{\log m} \log\biggl(\dfrac{x}{\lambda \log m}\biggr) - \dfrac{x}{\log m} + \lambda, & x \geq 0, \\ +\infty, & x < 0. \end{cases}$$

Now we consider moderate deviations. Let $\{a_t, t \geq 0\}$ be a family of positive numbers satisfying

$$\frac{a_t}{t} \to 0 \quad \text{and} \quad \frac{a_t}{\sqrt{t}} \to \infty \quad \text{as } t \to \infty.$$

As in the case of the large deviation principle, based on the Gärtner–Ellis theorem and the moderate deviation principle for Poisson process, we have the following theorem.

Theorem 1.2. (Moderate deviation.) *For any measurable subset* B *of* $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in B^o} \frac{x^2}{2\lambda} &\leq \liminf_{t\to\infty} \frac{t}{a_t^2} \log \mathbb{P}\biggl(\frac{\log Z_{N_t}/\log m - \lambda t}{a_t} \in B\biggr) \\ &\leq \limsup_{t\to\infty} \frac{t}{a_t^2} \log \mathbb{P}\biggl(\frac{\log Z_{N_t}/\log m - \lambda t}{a_t} \in B\biggr) \\ &\leq -\inf_{x\in \bar{B}} \frac{x^2}{2\lambda}. \end{aligned}$$

Similar to the law of large numbers and central limit theorem for Poisson process, we have our final result.

Theorem 1.3. (Law of large numbers and central limit theorem.) *Let* $\Phi(x)$ *be the standard normal distribution function. For any* $x \in R$,

$$\lim_{t\to\infty}\frac{\log Z_{N_t}}{t} = \lambda \log m \quad a.s., \qquad \lim_{t\to\infty}\mathbb{P}\bigg(\frac{\log Z_{N_t}/\log m - \lambda t}{\sqrt{\lambda t}} \le x\bigg) = \Phi(x).$$

2. Large deviation principle

The large deviation principle can be derived from the following two lemmas and the Gärtner–Ellis theorem.

Lemma 2.1. *Assume that* $\mathbb{E}(Z_1^a) < \infty$ *for all* $a \ge 1$*. For any* $\theta \in R$*, there exists a constant* $C(\theta) > 0$ *such that*

$$\lim_{n\to\infty}\frac{\mathbb{E}(Z_n^\theta)}{m^{n\theta}} = C(\theta).$$

Proof. The proof is a consequence of [6, Theorem 1.3]. □

Lemma 2.2. *Assume that* $\mathbb{E}(Z_1^a) < \infty$ *for all* $a \ge 1$*. For any* $\theta \in R$*,* $\Lambda(\theta) = \lambda(m^\theta - 1)$ *and*

$$\Lambda^*(x) = \begin{cases} \dfrac{x}{\log m}\log\bigg(\dfrac{x}{\lambda \log m}\bigg) - \dfrac{x}{\log m} + \lambda, & x \ge 0, \\ +\infty, & x < 0, \end{cases}$$

where $0 \log 0 := 0$*. Furthermore,* $\Lambda^*(\lambda \log m) = 0$*,* $\Lambda^*(x)$ *is strictly increasing for* $x > \lambda \log m$ *and* $\Lambda^*(x)$ *is strictly decreasing for* $x < \lambda \log m$*.*

Proof. For any $\theta \in R$, using the law of the total probability, and the independence of Z_n and N_t, we obtain the θ-order moment of Z_{N_t} as

$$\begin{aligned}
\mathbb{E}(Z_{N_t}^\theta) &= \sum_{n=0}^{\infty}\mathbb{E}(Z_{N_t}^\theta \mid N_t = n)\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty}\mathbb{E}(Z_n^\theta)\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} C(\theta)m^{n\theta}\mathbb{P}(N_t = n) + \sum_{n=0}^{\infty}[\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}]\mathbb{P}(N_t = n) \\
&= C(\theta)\sum_{n=0}^{\infty} m^{n\theta}\frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t} + \sum_{n=0}^{\infty}[\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}]\frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t} \\
&= C(\theta)\exp(\lambda t(m^\theta - 1)) + I, \qquad (2.1)
\end{aligned}$$

where

$$I = \sum_{n=0}^{\infty}[\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}]\frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}.$$

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According to Lemma 2.1, for any $\varepsilon > 0$, there exists a constant $N = N(\theta)$ such that for any $n > N(\theta)$, we have

$$\left|\frac{\mathbb{E}(Z_n^\theta)}{m^{n\theta}} - C(\theta)\right| < \varepsilon.$$

Consequently,

$$\begin{aligned}
|I| &\le \sum_{n=0}^{\infty} |\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}| \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \\
&= \sum_{n=0}^{N} |\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}| \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} + \sum_{n=N+1}^{\infty} |\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}| \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \\
&\le M(\theta) + \varepsilon \sum_{n=0}^{\infty} m^{n\theta} \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \\
&= M(\theta) + \varepsilon \exp(\lambda t(m^\theta - 1)),
\end{aligned} \tag{2.2}$$

where

$$M(\theta) = \sum_{n=0}^{N} |\mathbb{E}(Z_n^\theta) - C(\theta)m^{n\theta}| \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}.$$

According to (2.1) and (2.2), we have

$$(C(\theta) - \varepsilon)\exp(\lambda t(m^\theta - 1)) - M(\theta) \le \mathbb{E}(Z_{N_t}^\theta) \le (C(\theta) - \varepsilon)\exp(\lambda t(m^\theta - 1)) - M(\theta).$$

Given $\varepsilon < C(\theta)$, we have

$$\Lambda(\theta) = \lim_{t \to +\infty} \frac{\log \mathbb{E}(Z_{N_t}^\theta)}{t} = \lambda(m^\theta - 1).$$

Consequently, the Fenchel–Legendre transform of Λ is

$$\Lambda^*(x) = \begin{cases} \dfrac{x}{\log m} \log\left(\dfrac{x}{\lambda \log m}\right) - \dfrac{x}{\log m} + \lambda, & x \ge 0, \\ +\infty, & x < 0. \end{cases} \tag{2.3}$$

The proof of Lemma 2.2 follows from (2.3). □

The following result is a general result on large deviations; see [6].

Lemma 2.3. *Let I be a continuous function on R satisfying*

(i) *that $I(b) = \inf_{x \in R} I(x) = 0$ for some $b \in R$;*

(ii) *that I is strictly increasing on $[b, \infty)$ and strictly decreasing on $(-\infty, b]$.*

Let μ_t, $t \ge 0$ be a family of probability distribution on R and let $\{a_t\}$ be a family of positive numbers satisfying $a_t \to \infty$. Then, the following statements are equivalent.

(i) *For $x < b$,*

$$\lim_{t \to \infty} \frac{\log \mu_t((-\infty, x])}{a_t} = -I(x).$$

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For $x > b$,

$$\lim_{t\to\infty} \frac{\log \mu_t([x,\infty))}{a_t} = -I(x).$$

(ii) *It holds that $\{\mu_t, t \geq 0\}$ satisfies a large deviation principle. For any measurable subset B of R,*

$$-\inf_{x\in B^o} I(x) \leq \liminf_{t\to\infty} \frac{\log \mu_n(B)}{a_t} \leq \limsup_{t\to\infty} \frac{\log \mu_n(B)}{a_t} \leq -\inf_{x\in \bar{B}} I(x).$$

From Theorem 1.1 and Lemma 2.3, we obtain immediately the following corollary.

Corollary 2.1. *Assume that $\mathbb{E}(Z_1^a) < \infty$ for all $a \geq 1$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\bigg(\frac{\log Z_{N_t}}{t} \leq x\bigg) = -\Lambda^*(x) \quad \textit{for } x < \lambda \log m,$$
$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\bigg(\frac{\log Z_{N_t}}{t} \geq x\bigg) = -\Lambda^*(x) \quad \textit{for } x > \lambda \log m.$$

3. Moderate deviation principle

For any $t > 0, \theta \in R$, define

$$\theta_t = \frac{a_t \theta}{t \log m}.$$

The large deviation principle can be derived from the following two lemmas and the Gärtner–Ellis theorem.

Lemma 3.1. *For any $\theta \in R$, there exists a constant $C > 0$ such that*

$$C \leq \liminf_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \leq \limsup_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \leq 1.$$

Proof. The result is obvious for $\theta = 0$.

For $\theta \neq 0$, using the law of the total probability and the independence of Z_n and N_t, we have

$$\mathbb{E}(Z_{N_t}^{\theta_t}) = \mathbb{E}(W_{N_t}^{\theta_t} m^{N_t\theta_t}) = \sum_{n=0}^{\infty} \mathbb{E}(W_n^{\theta_t}) m^{n\theta_t} \mathbb{P}(N_t = n). \tag{3.1}$$

For $\theta < 0$, we have $\theta_t < 0$. By Jensen's inequality, $\mathbb{E}(W_n^{\theta_t}) \geq (\mathbb{E}(W_n))^{\theta_t} = 1$. By (3.1), we have

$$\liminf_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \geq 1.$$

On the other hand, according to [6, Theorem 2.1], there exist two positive constants a and C_a such that $\mathbb{E}(W^{-a}) \leq C_a$. Noting that $-\theta_t/a \in (0,1)$ for large enough t and that by [6, Lemma 2.1], $\mathbb{E}(W_n^{-s}) \leq \mathbb{E}(W^{-s})$ for $s > 0$, again by Jensen's inequality, we have

$$\mathbb{E}(W_n^{\theta_t}) = \mathbb{E}(W_n^{-a})^{-\theta_t/a} \leq (\mathbb{E}(W_n^{-a}))^{-\theta_t/a} \leq (\mathbb{E}(W^{-a}))^{-\theta_t/a} \leq C_a^{-\theta_t/a}$$

According to (3.1), this leads to

$$\limsup_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \leq 1.$$

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For $\theta > 0$, we have $\theta_t \in (0, 1)$ for large enough t , so by Jensen's inequality, $\mathbb{E}(W_n^{\theta_t}) \leq (\mathbb{E}(W_n))^{\theta_t} = 1$. By (3.1), we have

$$\limsup_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \leq 1.$$

On the other hand, from the proof of [6, Lemma 2.3], we know that $\mathbb{E}(W^b) \leq C_b$ for some $b > 1$ and $C_b > 0$. By Hölder's inequality,

$$1 = \mathbb{E}(W_n) = \mathbb{E}(W_n^{\theta_t/p} W_n^{1-\theta_t/p}) \leq (\mathbb{E}(W_n^{\theta_t})^{1/p} (\mathbb{E}(W_n^{(1-\theta_t/p)q}))^{1/q} \tag{3.2}$$

for $p, q > 1,\ p^{-1} + q^{-1} = 1$. Take

$$p = p_t := \frac{b-\theta_t}{b-1}, \qquad q = q_t := \frac{b-\theta_t}{1-\theta_t},$$

then $p, q > 1,\ p^{-1} + q^{-1} = 1$, $(1-\theta_t/p)q = b$, and $p/q = (1-\theta_t)/(b-1)$. According to [6, Lemma 2.1] and (3.2), we obtain

$$\mathbb{E}(W_n^{\theta_t}) \geq (\mathbb{E}(W_n^b))^{-(1-\theta_t)/(b-1)} \geq (\mathbb{E}(W^b))^{-(1-\theta_t)/(b-1)} \geq C_b^{-(1-\theta_t)/(b-1)}.$$

By (3.1), this leads to

$$\liminf_{t\to\infty} \frac{\mathbb{E}(Z_{N_t}^{\theta_t})}{\mathbb{E}(m^{N_t\theta_t})} \geq C_b^{-1/(b-1)} =: C \in (0, 1]. \qquad \square$$

Next, for any $t > 0, \theta \in R$, define

$$\bar{\Lambda}_t(\theta) = \log \mathbb{E}\biggl(\exp\biggl(\frac{\theta(\log Z_{N_t}/\log m - \lambda t)}{a_t}\biggr)\biggr),$$

$$\bar{\Lambda}(\theta) = \lim_{t\to+\infty} a_t^{-2} t \bar{\Lambda}_t\biggl(\frac{a_t^2}{t}\theta\biggr), \qquad \bar{\Lambda}^*(x) = \sup_{\theta\in R}\{\theta x - \bar{\Lambda}(\theta)\},$$

$$\tilde{\Lambda}_t(\theta) = \log \mathbb{E}\biggl(\exp\biggl(\frac{\theta(N_t - \lambda t)}{a_t}\biggr)\biggr),$$

$$\tilde{\Lambda}(\theta) = \lim_{t\to+\infty} a_t^{-2} t \tilde{\Lambda}_t\biggl(\frac{a_t^2}{t}\theta\biggr), \qquad \tilde{\Lambda}^*(x) = \sup_{\theta\in R}\{\theta x - \tilde{\Lambda}(\theta)\}.$$

Lemma 3.2. *For any $\theta, x \in R$,*

$$\bar{\Lambda}(\theta) = \tilde{\Lambda}(\theta) = \frac{1}{2}\lambda\theta^2, \qquad \bar{\Lambda}^*(x) = \tilde{\Lambda}^*(x) = \frac{x^2}{2\lambda}.$$

Proof. According to the moderate deviation principle for a Poisson process, we only need to prove that

$$\bar{\Lambda}(\theta) = \tilde{\Lambda}(\theta) \quad \text{for all } \theta \in R.$$

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For any $\theta \in R$, by Lemma 3.1,

$$\begin{aligned}\bar{\Lambda}(\theta) &= \lim_{t\to+\infty} a_t^{-2} t \bar{\Lambda}_t\left(\frac{a_t^2}{t}\theta\right)\\ &= \lim_{t\to+\infty} a_t^{-2} t \log \mathbb{E}\left(\exp\left(\frac{a_t^2\theta(\log Z_{N_t}/\log m - \lambda t)}{a_t t}\right)\right)\\ &= \lim_{t\to+\infty} a_t^{-2} t[\log \mathbb{E}(Z_{N_t}^{\theta_t}) - \lambda a_t \theta]\\ &= \lim_{t\to+\infty} a_t^{-2} t[\log \mathbb{E}(m^{N_t\theta_t}) - \lambda a_t \theta]\\ &= \lim_{t\to+\infty} a_t^{-2} t \tilde{\Lambda}_t(\frac{a_t^2}{t}\theta)\\ &= \tilde{\Lambda}(\theta).\end{aligned}$$

□

From Theorem 1.3 and Lemma 2.3, we obtain immediately the following corollary.

Corollary 3.1. *For all $x > 0$,*

$$\lim_{t\to\infty} \frac{t}{a_t^2} \log \mathbb{P}\left(\frac{\log Z_{N_t}/\log m - \lambda t}{a_t} \le -x\right) = -\frac{x^2}{2\lambda},$$
$$\lim_{t\to\infty} \frac{t}{a_t^2} \log \mathbb{P}\left(\frac{\log Z_{N_t}/\log m - \lambda t}{a_t} \ge x\right) = -\frac{x^2}{2\lambda}.$$

4. Law of large numbers and central limit theorem

The proof of Theorem 1.3 is based on the law of large numbers and central limit theorem for the Poisson process in the following lemma.

Lemma 4.1. *For a Poisson process $\{N_t, t \ge 0\}$ with parameter $\lambda > 0$, we have*

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \quad a.s. \qquad \lim_{t\to\infty} \mathbb{P}\left(\frac{N_t - \lambda t}{\sqrt{\lambda t}} \le x\right) = \Phi(x).$$

Proof of Theorem 1.3. For any $t > 0$, we have

$$\frac{\log Z_{N_t}}{t} = \frac{\log W_{N_t} + N_t \log m}{t}.$$

Note that our branching process belongs to the Böttcher case, so $W_{N_t} \to W > 0$ a.s. when $t \to \infty$. By Lemma 4.1, we have the law of large numbers.

Next, for any $x \in R$, define

$$A_t(x) = \left\{\frac{N_t - \lambda t}{\sqrt{\lambda t}} \le x\right\} \qquad B_t(x) = \left\{\frac{\log W_{N_t}}{\sqrt{\lambda t}\log m} \le x\right\},$$
$$C_t(x) = \left\{\frac{\log Z_{N_t}/\log m - \lambda t}{\sqrt{\lambda t}} \le x\right\}.$$

Note that as $W_{N_t} = Z_{N_t}/m^{N_t}$, we have for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(C_t(x)) &= \mathbb{P}(C_t(x) \cap B_t(-\varepsilon)) + \mathbb{P}(C_t(x) \cap B_t^c(-\varepsilon))\\ &\le \mathbb{P}(B_t(\varepsilon)) + \mathbb{P}(A_t(x+\varepsilon)). \qquad (4.1)\end{aligned}$$

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Since $W_{N_t} \to W > 0$ a.s. when $t \to \infty$, we have

$$\lim_{t\to\infty} \mathbb{P}(B_t(-\varepsilon)) = 0 \tag{4.2}$$

Let $t \to \infty$ and then $\varepsilon \to 0$ in (4.1), by Lemma 3.1 and (4.2), we have

$$\limsup_{t\to\infty} \mathbb{P}(C_t(x)) \le \Phi(x).$$

Similarly, one can show that $\liminf_{t\to\infty} \mathbb{P}(C_t(x)) \ge \Phi(x)$. □

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