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Decay of correlations in suspension semi-flows of angle-multiplying maps

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Abstract. We consider suspension semi-flows of angle-multiplying maps on the circle for C^r ceiling functions with $r \geq 3$. Under a C^r generic condition on the ceiling function, we show that there exists a Hilbert space (anisotropic Sobolev space) contained in the L^2 space such that the Perron–Frobenius operator for the time-t-map acts naturally on it and that the essential spectral radius of that action is bounded by the square root of the inverse of the minimum expansion rate. This leads to a precise description of decay of correlations. Furthermore, the Perron–Frobenius operator for the time-t-map is quasi-compact for a C^r open and dense set of ceiling functions.

1. *Introduction*

Decay of correlations and related topics for hyperbolic dynamical systems have been studied for more than three decades since the works of Bowen [**6**], Ruelle [**16**] and Sinai [**17**]. For the cases of discrete dynamical systems such as iterations of expanding maps and Anosov diffeomorphisms, exponential decay of correlations for Hölder observables was already known in the early stage of the study [**6, 16, 17**] and nowadays we have a fairly good understanding of the speed of decay and also of the Perron–Frobenius operators spectra [**3, 4, 8, 9**]. On the contrary, for the cases of continuous dynamical systems such as Anosov flows, the corresponding argument is much more subtle and our knowledge is less satisfactory at present. A simple reason for the subtleness in the cases of flows is that the time-t-maps of hyperbolic flows are *not* hyperbolic (but partially hyperbolic) as there is no expansion or contraction in the flow direction. Dolgopyat [**7**] showed exponential decay of correlations for Anosov flows under the assumption that the stable and unstable foliations are both C^1 and are jointly non-integrable. More recently, Liverani [**13**] extended the result to Anosov flows preserving contact structures and, in particular, to all geodesic flows in strictly negative curvature. However, even in such cases, we do not know whether the semi-groups of Perron–Frobenius operators are quasi-compact on some appropriate Banach spaces. Neither do we know whether we observe exponential decay of correlations for mixing or for generic Anosov flows.

The aim of this paper is to study decay of correlations not for Anosov flows but for a class of expanding semi-flows known as suspension semi-flows of angle-multiplying maps on the circle, which we would like to view as a simplified model of the Anosov flow. We consider Perron–Frobenius operators for the time-t-maps of such semi-flows and let them act on the anisotropic Sobolev spaces introduced by Baladi and Tsujii [**4**]. Our main result is that, under a C^r generic condition on the ceiling function, the essential spectral radius of the action is bounded by the square root of the inverse of the minimum expansion rate. This leads to a precise description on decay of correlations, which resembles the results known for hyperbolic discrete dynamical systems [**4, 8**], and extends the earlier result of Pollicott [**15**] on exponential decay. As a byproduct of our methods, we show that the Perron–Frobenius operators are quasi-compact for a C^r open and dense set of ceiling functions.

A prototype of the argument in this paper has actually appeared in Avila *et al* [**1**], where a class of volume-expanding hyperbolic endomorphisms, called *fat solenoidal attractors*, were studied. In this paper, we will apply essentially the same idea to analyze the time-t-maps of the class of expanding semi-flows mentioned above. We emphasize that our intention is to display the idea in a simple setting and present this paper as a study for the cases of hyperbolic flows. (It is possible to apply the idea presented in this paper to Anosov flows with sufficiently smooth stable foliations. However, we suspect that we need to overcome essential difficulties to treat more general Anosov flows.) We therefore confine our argument to a rather restrictive setting, although it is not very difficult to extend it to more general classes of expanding semi-flows and transfer operators (see Remark 1.5).

We fix integers $\ell \geq 2$ and $r \geq 3$. Let $\tau : S^1 \to S^1$ be the angle-multiplying map on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ defined by $\tau(x) = \ell x$. Let $C^r_+(S^1)$ be the space of positive-valued C^r functions on S^1. For each $f \in C^r_+(S^1)$, we consider the subset

$$X_f = \{(x, s) \in S^1 \times \mathbb{R} \mid 0 \leq s < f(x)\}$$

of the cylinder $S^1 \times \mathbb{R}$. The suspension semi-flow $\mathbf{T}_f = \{T^t_f : X_f \to X_f\}_{t \geq 0}$ of τ is the semi-flow on X_f in which each point on X_f moves right upward (or the s-direction) with the unit speed and, at the instant it reaches the upper boundary of X_f, it jumps down to the lower boundary with the x-coordinate transfered by τ (see Figure 1). The precise expression for its time-t-map is

$$T^t_f(x, s) = (\tau^{n(x,s+t;f)}(x), s + t - f^{(n(x,s+t;f))}(x))$$

where $f^{(n)}(x) = \sum_{i=0}^{n-1} f(\tau^i(x))$ and $n(x, t; f) = \max\{n \geq 0 \mid f^{(n)}(x) \leq t\}$.

Let $m = m_f$ be the normalization of the restriction of the standard Lebesgue measure on $S^1 \times \mathbb{R}$ to X_f. This is an ergodic invariant probability measure for $\mathbf{T}_f$. For a point $z = (x, s) \in X_f$ and $t \geq 0$, $E(z, t; f) := \ell^{n(x,s+t;f)}$ is the expansion rate along the orbit of z up to time t. The minimum expansion rate of $\mathbf{T}_f$ is naturally defined by

$$\lambda_{\min}(\mathbf{T}_f) = \lim_{t\to\infty} \left(\min_{z \in X_f} E(z, t; f) \right)^{1/t}.$$

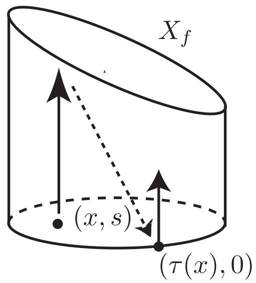

FIGURE 1. The semi-flow $\mathbf{T}_f$.

For functions ψ and φ in $L^2(X_f) = L^2(X_f, m_f)$, we consider the correlation

$$\mathbf{Cor}_t(\psi, \varphi) = \int \psi \cdot \varphi \circ T_f^t \, dm_f - \left(\int \varphi \, dm_f\right)\left(\int \psi \, dm_f\right) \quad \text{for } t \geq 0.$$

Suppose that $\int \psi \, dm_f = 0$ for simplicity. If the semi-flow $\mathbf{T}_f$ is mixing, we have that $\lim_{t\to\infty} \mathbf{Cor}_t(\psi, \varphi) = 0$. The question is the rate of convergence in this limit. Pollicott [**15**] showed that the rate is exponential under a mild condition on f: $|\mathbf{Cor}_t(\psi, \varphi)| <$ const. $\exp(-\epsilon t)$ for some $\epsilon > 0$. (See also Baladi and Vall'ee [**5**] for a generalization.) Under a C^rgeneric condition on f, our results give a more precise description on asymptotic behavior of the correlation as $t \to \infty$: for any real number $\mu > (\lambda_{\min}(\mathbf{T}_f))^{-1/2}$, there exists finitely many complex numbers $\lambda_i \in \mathbb{C}$ (which may not be distinct from each other) with $\mu \leq |\lambda_i| < 1$ and integers $m_i \geq 0$ for $1 \leq i \leq k$, such that

$$\left|\mathbf{Cor}_t(\psi, \varphi) - \sum_{i=1}^{k} H_i(\psi, \varphi) \cdot t^{m_i} \lambda_i^t\right| \leq H_0(\psi, \varphi)\mu^t \quad \text{for } t \geq 0 \tag{1}$$

for any $\varphi \in L^2(X_f)$ and any C^1 function ψ with $\int \psi \, dm_f = 0$ supported on the interior of X_f, where $H_i(\psi, \varphi)$ are coefficients that depend on ψ and φ (and μ). As we will see later, the complex numbers λ_i above are peripheral eigenvalues of the Perron–Frobenius operator for the time-1-map and each integer m_i is bounded by the geometric multiplicity of the eigenvalue λ_i.

In order to state the main results, we introduce some more notation. The differential $(DT_f^t)_z$ of T_f^t at $z \in X_f$ is defined in the usual way if both z and $T^t(z)$ belong to the interior of X_f and, otherwise, is defined by

$$(DT_f^t)_z = \lim_{\epsilon\to+0} (DT_f^t)_{z+(0,\epsilon)}.$$

Fix a real number $\ell^{-1} < \gamma_0 < 1$; it is better to choose γ_0 close to 1, although not necessary. Set

$$\theta_f = \max_{x\in S^1} |f'(x)|/(\gamma_0 \ell - 1)$$

and

$$\mathbf{C}_f = \mathbf{C}(\theta_f) := \{(x, y) \in \mathbb{R}^2 \mid |y| \leq \theta_f |x|\}.$$

By the definition of θ_f, we see that the cone $\mathbf{C}_f$ is strictly invariant for $\mathbf{T}_f$ in the sense that

$$(DT_f^t)_z(\mathbf{C}_f) \subset \mathbf{C}(\gamma_0\theta_f) \subset \mathbf{C}_f \quad \text{for all } z = (x, s) \in X_f \text{ and } t \geq f(x) - s.$$

Note that, for large t, the inverse image $(T_f^t)^{-1}(z)$ of a point $z \in X_f$ consists of many points and thus there are many narrow cones $(DT_f^t)_\zeta(\mathbf{C}_f)$ for $\zeta \in (T_f^t)^{-1}(z)$ in the tangent space at z. As a measure for transversality between such cones, we introduce the quantity

$$\mathbf{m}(f, t) = \max_{z \in X_f} \max_{w \in (T_f^t)^{-1}(z)} \sum_{\zeta : \zeta \not\pitchfork w} \frac{1}{E(\zeta, t; f)}$$

where $\sum_{\zeta:\zeta \not\pitchfork w}$ is the sum over the points $\zeta \in (T_f^t)^{-1}(z)$ such that

$$(DT_f^t)_\zeta(\mathbf{C}_f) \cap (DT_f^t)_w(\mathbf{C}_f) \neq \{0\}.$$

Note that we always have $\mathbf{m}(f, t) \leq 1$ because

$$\sum_{\zeta \in (T_f^t)^{-1}(z)} \frac{1}{E(\zeta, t; f)} = 1 \quad \text{for any } t \geq 0 \text{ and } z \in X_f. \tag{2}$$

Finally, we define the exponent

$$\mathbf{m}(f) = \limsup_{t \to \infty} \mathbf{m}(f, t)^{1/t} \leq 1.$$

The Perron–Frobenius operator $\mathcal{P}_f^t : L^1(X_f) \to L^1(X_f)$ for $t \geq 0$ is defined by

$$\mathcal{P}_f^t(u)(z) = \sum_{w \in (T_f^t)^{-1}(z)} \frac{u(w)}{\det (DT_f^t)_w},$$

so that we have $\mathbf{Cor}_t(\psi, \varphi) = \int \mathcal{P}_f^t \psi \cdot \varphi \, dm_f$ provided that $\int \psi \, dm_f = 0$.

Let $C^1(X_f)$ be the set of functions φ on X_f such that $\mathcal{P}_f^t(\varphi)$ is C^1 on the interior of X_f for any $t \geq 0$. (This condition imposes a restriction on the behavior of the function in the neighborhood of the boundary of X_f, in addition to that it should be C^1 on the interior of X_f.) This contains all the functions that are supported and C^1 on the interior of X_f.

The main results are now stated as follows.

THEOREM 1.1. *There exists a Hilbert space $W_*(X_f)$ such that*

$$C^1(X_f) \subset W_*(X_f) \subset L^2(X_f)$$

and such that the Perron–Frobenius operator $\mathcal{P}_f^t$ for large $t \geq 0$ restricts to the bounded operator $\mathcal{P}_f^t : W_*(X_f) \to W_*(X_f)$ whose essential spectral radius is bounded by $\mathbf{m}(f)^{t/2}$.

THEOREM 1.2. *For each $\rho > 1$, there exists an open and dense subset $\mathcal{R}$ in $C_+^r(S^1)$ such that, for $f \in \mathcal{R}$, the corresponding semi-flow $\mathbf{T}_f = \{T_f^t\}$ is weakly mixing and satisfies $\mathbf{m}(f) \leq \rho \cdot \lambda_{\min}^{-1}(\mathbf{T}_f)$.*

From these theorems, we obtain the following corollary.

COROLLARY 1.3. *For a C^r generic $f \in C^r_+(S^1)$, the semi-flow $\mathbf{T}_f$ is weakly mixing and the essential spectral radius of the Perron–Frobenius operator $\mathcal{P}^t_f$ acting on $W_*(X_f)$ is bounded by $\lambda_{\min}(\mathbf{T}_f)^{-t/2}$ for any sufficiently large t.*

The estimate equation (1) for C^r generic f is an immediate consequence of this corollary.

Proof of equation (1). Take large $t > 0$. By Corollary 1.3, we have the decomposition $W_*(X_f) = E \oplus V$ where E is the sum of the generalized eigenspaces for $\mathcal{P}^t_f$ corresponding to the eigenvalues not smaller than μ^t in absolute value and

$$V = \{\varphi \in W_*(X_f) \mid \mu^{-s} \|\mathcal{P}^s_f \varphi\|_* \to 0 \text{ as } s \to \infty\}$$

where $\|\cdot\|_*$ denotes the norm on $W_*(X_f)$. The finite dimensional subspace E is invariant under $\mathcal{P}^s_f$ for $s \geq 0$ by commutativity of $\mathcal{P}^s_f$ and $\mathcal{P}^t_f$. Thus we have $\mathcal{P}^s_f|_E = \exp(sB)$ for a linear map $B : E \to E$. Now it is easy to derive equation (1), expressing ψ as the sum of an element of V and generalized eigenvectors of B. □

The following interesting observation was pointed out to the author by Avila [**2**]. We say that $f \in C^r_+(S^1)$ is cohomologous to a constant function if $f(x) = \varphi(\tau(x)) - \varphi(x) + c$ for some $\varphi \in C^r(S^1)$ and $c \in \mathbb{R}$.

THEOREM 1.4. *The following conditions for $f \in C^r_+(S^1)$ are equivalent:*
(i) $\mathbf{m}(f) < 1$*;*
(ii) $\mathbf{T}_f$ *is weakly mixing; and*
(iii) f *is not cohomologous to a constant function.*
In particular, either (or all) of these conditions holds for a C^r open and dense set of ceiling functions $f \in C^r_+(S^1)$.

This theorem and Theorem 1.1 (or Dolgopyat's argument [**15**]) imply that, once $\mathbf{T}_f$ is weakly mixing, it is exponentially mixing and there are no intermediate rates in correlation decay (for functions in $C^1(X_f)$ at least). We will give the proof in Appendix A.

Remark 1.5. We can generalize the argument in this paper to a more general class of expanding semi-flows without much difficulty. For instance, the main results remains true with obvious changes in the related definitions when we consider arbitrary C^3 expanding maps on the circle in the place of τ. The proof of Theorem 1.1 can be translated almost literally to such cases using the standard estimates on the distortion of expanding maps, while we need a slight modification to translate the proof of Theorem 1.2.

2. *Proof of Theorem 1.1*

In this section we consider the semi-flow $\mathbf{T}_f = \{T^t_f\}_{t \geq 0}$ for some fixed $f \in C^r_+(S^1)$. For simplicity, we write T^t and $\mathcal{P}^t$ for T^t_f and $\mathcal{P}^t_f$, respectively.

2.1. *Local charts on X_f.*

We set up a system of local charts on X_f. (The system of local charts does *not* give the structure of (branched) manifold on X_f.) To begin with, we consider two small real numbers $\eta > 0$ and $\delta > 0$, and set

$$R = (-\eta, \eta) \times (\delta, 2\delta) \subset Q = (-2\eta, 2\eta) \times (0, 3\delta).$$

For each $a = (x_0, s_0) \in X_f$ such that $[x_0 - 2\eta, x_0 + 2\eta] \times \{s_0\} \subset X_f$, we consider two mappings

$$\kappa_a : Q \to X_f, \quad \kappa_a(x, s) = T^s(x_0 + x, s_0)$$

and

$$\tilde{\kappa}_a : Q \to S^1 \times \mathbb{R}, \quad \tilde{\kappa}_a(x, s) = (x_0 + x, s_0 + s).$$

Note that κ_a and $\tilde{\kappa}_a$ coincide when the image of $\tilde{\kappa}_a$ does not meet the upper boundary of X_f. Let η and δ be so small that κ_a is injective on Q whenever it is defined.

Next we take a finite subset A of X_f so that the mappings κ_a for $a \in A$ are defined and that the images $\tilde{\kappa}_a(R)$, $a \in A$, cover the subset

$$\tilde{X}_f := \{(x, s) \in S^1 \times \mathbb{R} \mid \delta/3 \leq s \leq f(x) + 2\delta/3\}.$$

We may and do suppose that the intersection multiplicity of $\{\tilde{\kappa}_a(R)\}_{a\in A}$ is bounded by an absolute constant (say 100). (This is in fact possible if we let the ratio η/δ be small.) Clearly the images of $\kappa_a(R)$ for $a \in A$ cover X_f.

Let $C^r(R)$ be the set of C^r functions supported on R. We take the family $\{h_a\}_{a\in A}$ of functions in $C^\infty(R)$ as follows. First, take a C^∞ function $\beta_0 : \mathbb{R} \to [0, 1]$ such that $\beta_0(s) = 1$ if $s \leq \delta/3$ and $\beta_0(s) = 0$ if $s \geq 2\delta/3$. We define $\beta : S^1 \times \mathbb{R} \to [0, 1]$ by

$$\beta(x, s) = \begin{cases} \beta_0(s - f(x)) & \text{if } s \geq f(x), \\ 1 & \text{if } \delta < s < f(x), \\ 1 - \beta_0(s) & \text{if } s \leq \delta. \end{cases}$$

This is a C^∞ function supported on $\tilde{X}_f$. From the choice of the finite subset A, we can take a family $\{\tilde{h}_a : S^1 \times \mathbb{R} \to [0, 1]\}_{a\in A}$ of C^∞ functions so that the support of each $\tilde{h}_a$ is contained in $\tilde{\kappa}_a(R)$ and so that we have $\sum_a \tilde{h}_a \equiv \beta$ on $S^1 \times \mathbb{R}$. We then define the C^∞ function $h_a : \mathbb{R}^2 \to [0, 1]$ for $a \in A$ by

$$h_a = \begin{cases} \tilde{h}_a \circ \tilde{\kappa}_a & \text{on } R, \\ 0 & \text{on } \mathbb{R}^2 \setminus R. \end{cases}$$

2.2. *Anisotropic Sobolev spaces.* We recall the anisotropic Sobolev space and the related definitions introduced by Baladi and Tsujii [**4**]. For a cone $\mathbf{C} \subset \mathbb{R}^2$, we define its dual by

$$\mathbf{C}^* = \{v \in \mathbb{R}^2 \mid (v, u) = 0 \text{ for some } u \in \mathbf{C} \setminus \{0\}\}.$$

For two cones $\mathbf{C}, \mathbf{C}' \subset \mathbb{R}^2$, we write $\mathbf{C} \Subset \mathbf{C}'$ if the closure of $\mathbf{C}$ is contained in the interior of $\mathbf{C}'$ *except for* the origin.

A *polarization* Θ is a combination $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ of closed cones $\mathbf{C}_\pm$ in $\mathbb{R}^2$ and C^∞ functions $\varphi_\pm : S^1 \to [0, 1]$ on the unit circle $S^1 \subset \mathbb{R}^2$ that satisfy $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$ and

$$\varphi_+(\xi) = \begin{cases} 1 & \text{if } \xi \in S^1 \cap \mathbf{C}_+, \\ 0 & \text{if } \xi \in S^1 \cap \mathbf{C}_-, \end{cases} \quad \varphi_-(\xi) = 1 - \varphi_+(\xi). \tag{3}$$

For two polarizations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$, we write $\Theta < \Theta'$ if $\mathbb{R}^2 \setminus \mathbf{C}'_+ \Subset \mathbf{C}_-$.

Fix a C^∞ function $\chi : \mathbb{R} \to [0, 1]$ satisfying

$$\chi(s) = \begin{cases} 1 & \text{for } s \le 1, \\ 0 & \text{for } s \ge 2. \end{cases}$$

For a polarization $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$, an integer $n \ge 0$ and $\sigma \in \{+, -\}$, we define the C^∞ function $\psi_{\Theta,n,\sigma} : \mathbb{R}^2 \to [0, 1]$ by

$$\psi_{\Theta,n,\sigma}(\xi) = \begin{cases} \varphi_\sigma(\xi/|\xi|) \cdot (\chi(2^{-n}|\xi|) - \chi(2^{-n+1}|\xi|)) & \text{if } n \ge 1, \\ \chi(|\xi|)/2 & \text{if } n = 0. \end{cases}$$

The family of functions $\psi_{\Theta,n,\sigma}$ for $n \ge 0$ and $\sigma \in \{+, -\}$ is a C^∞ partition of unity on $\mathbb{R}^2$.

For a function $u \in C^r(R)$, we define

$$u_{\Theta,n,\sigma}(x) = \psi_{\Theta,n,\sigma}(D)u(x) := (2\pi)^{-2} \int e^{i(x-y)\xi} \psi_{\Theta,n,\sigma}(\xi) u(y)\, dy\, d\xi$$

where $\psi_{\Theta,n,\sigma}(D)$ is the pseudo-differential operator with symbol $a(x, \xi) = \psi_{\Theta,n,\sigma}(\xi)$. Note that the pseudo-differential operator $\psi_{\Theta,n,\sigma}(D)$ may be viewed as the composition $\mathcal{F}^{-1} \circ \Psi_{\Theta,n,\sigma} \circ \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform and $\Psi_{\Theta,n,\sigma}$ is the multiplication operator by $\psi_{\Theta,n,\sigma}$.

For a polarization $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and a real number p, we define the semi-norms $\|\cdot\|^+_{\Theta,p}$ and $\|\cdot\|^-_{\Theta,p}$ on $C^r(R)$ by

$$\|u\|^\sigma_{\Theta,p} = \left(\sum_{n\ge0} 2^{2pn} \|u_{\Theta,n,\sigma}\|^2_{L^2} \right)^{1/2} = \left(\sum_{n\ge0} 2^{2pn} \|\psi_{\Theta,n,\sigma} \cdot \mathcal{F}u\|^2_{L^2} \right)^{1/2}.$$

We then define the anisotropic Sobolev norm $\|\cdot\|_{\Theta,p,q}$ for real numbers p and q by

$$\|u\|_{\Theta,p,q} = \left((\|u\|^+_{\Theta,p})^2 + (\|u\|^-_{\Theta,q})^2 \right)^{1/2}.$$

Clearly this norm is associated with a scalar product.

We will not actually use the anisotropic Sobolev norms for general p, q and Θ but those for the following specific cases. Fix small $0 < \epsilon < 1/2$ and set

$$\|\cdot\|^+_\Theta := \|\cdot\|^+_{\Theta,1}, \quad \|\cdot\|^-_\Theta := \|\cdot\|^-_{\Theta,0}, \quad \|\cdot\|_\Theta := \|\cdot\|_{\Theta,1,0}$$

and

$$|\cdot|^+_\Theta := \|\cdot\|^+_{\Theta,1-\epsilon}, \quad |\cdot|^-_\Theta := \|\cdot\|^-_{\Theta,-\epsilon}, \quad |\cdot|_\Theta := \|\cdot\|_{\Theta,1-\epsilon,-\epsilon}.$$

In view of Parseval's identity, we have

$$\|\cdot\|_\Theta \ge \|\cdot\|_{L^2}/\sqrt{6} \tag{4}$$

where 6 is (a bound for) the intersection multiplicity of the supports of $\psi_{\Theta,n,\sigma}$. The anisotropic Sobolev spaces $W_*(R; \Theta)$ and $W_\dagger(R; \Theta)$ are the completion of $C^\infty(R)$ with respect to the norms $\|\cdot\|_\Theta$ and $|\cdot|_\Theta$ respectively. By equation (4), we see that the space $W_*(R; \Theta)$ is naturally embedded in $L^2(R)$. Further, we recall the next lemma from Baladi and Tsujii [**4**]. Let $W^s(R)$ be the (usual) Sobolev space of order s, that is,

$$W^s(R) = \{u \in \mathcal{D}'(\mathbb{R}^2) \mid \operatorname{supp}(u) \subset \operatorname{closure}(R),\ (1 + |\xi|^2)^{s/2}\mathcal{F}u(\xi) \in L^2(\mathbb{R}^2)\}.$$

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LEMMA 2.1. *For any polarizations* $\Theta' < \Theta$, *we have:*
(a) $C^1(R) \subset W^1(R) \subset W_*(R;\Theta) \subset L^2(R)$ *and* $W^{1-\epsilon}(R) \subset W_\dagger(R;\Theta) \subset W^{-\epsilon}(R)$*;*
(b) $W_*(R;\Theta) \subset W_*(R;\Theta')$ *and* $W_\dagger(R;\Theta) \subset W_\dagger(R;\Theta')$*; and*
(c) *the inclusion* $W_*(R;\Theta) \subset W_\dagger(R;\Theta)$ *is compact.*

Proof. We can check (a) and (b) easily by using Parseval's identity. For the claim (c), we refer to Baladi and Tsujii [**4**, Proposition 5.1]. □

Regarding the polarization Θ, we will consider three polarizations:

$$\check{\Theta}_0 = \left(\check{\mathbf{C}}_{0,\pm}, \check{\varphi}_{0,\pm}\right) < \Theta_0 = \left(\mathbf{C}_{0,\pm}, \varphi_{0,\pm}\right) < \hat{\Theta}_0 = \left(\hat{\mathbf{C}}_{0,\pm}, \hat{\varphi}_{0,\pm}\right)$$

such that

$$(\mathbf{C}(\gamma_0\theta_f))^* \Subset \hat{\mathbf{C}}_{0,-} \Subset \left(\mathbb{R}^2 \setminus \check{\mathbf{C}}_{0,+}\right) \Subset (\mathbf{C}(\theta_f))^*.$$

The Hilbert space $W_*(X_f)$ in Theorem 1.1 is defined as follows. Consider the projection operator (regarding functions as densities)

$$\Pi : (L^2(R))^A \to L^2(X_f), \qquad \Pi((u_a)_{a\in A}) = \sum_{a\in A} \pi_a(u_a)$$

where $\pi_a : L^2(R) \to L^2(X_f)$ for $a \in A$ is defined by

$$\pi_a(u)(z) = \begin{cases} u(w)/\det D\kappa_a(w) & \text{if } z = \kappa_a(w) \text{ for some } w \in R, \\ 0 & \text{otherwise.} \end{cases}$$

We equip the product spaces $(W_*(R;\Theta_0))^A \subset (L^2(R))^A$ and $(W_\dagger(R;\Theta_0))^A$ with the norms

$$\|\mathbf{u}\| := \left(\sum_{a\in A} \|u_a\|^2_{\Theta_0}\right)^{1/2} \quad \text{and} \quad |\mathbf{u}| := \left(\sum_{a\in A} |u_a|^2_{\Theta_0}\right)^{1/2} \quad \text{where } \mathbf{u} = (u_a)_{a\in A}$$

respectively, so that they are Hilbert spaces. Then we set

$$W_*(X_f) = \Pi((W_*(R;\Theta_0))^A)$$

and equip it with the norm $\|u\| = \inf\{\|\mathbf{u}\| \mid \Pi(\mathbf{u}) = u\}$. This space $W_*(X_f)$ is isomorphic to the orthogonal complement of the kernel of Π in $(W_*(R;\Theta_0))^A$ and, hence, is a Hilbert space.

2.3. *Transfer operators on local charts.* We consider the time-t-map T^t and the corresponding Perron–Frobenius operator $\mathcal{P}^t$ for t large enough. (It is enough to consider t with $t \geq \max_{x\in S^1} f(x) + 3\delta$.) One technical difficulty in treating the time-t-map T^t directly is that it is *not* continuous at the boundary of X_f. To avoid this problem, we consider a system of transfer operators on the local charts $\{\kappa_a\}_{a\in A}$ as a lift of $\mathcal{P}^t$, in which we do not find any trace of the discontinuity of T^t.

For $a, b \in A$, let $Q(a,b,t)$ be the set of points $z \in Q$ such that, for some $(x,s) \in Q$:
(i) $T^t \circ \kappa_a(z) = \kappa_b((x,s))$; and
(ii) $T^{t-\eta} \circ \kappa_a(z) \in \kappa_b(Q)$ for $0 \leq \eta < s$.

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Notice that the second condition (ii) does not follow from (i) when $\tilde{\kappa}_b(Q)$ meets the upper boundary of X_f. In fact, the second condition (ii) is important to avoid discontinuity in the following argument.

For $a, b \in A$ with $Q(a, b, t) \neq \emptyset$, we consider the C^r mapping

$$T^t_{ab} : Q(a, b, t) \to Q, \quad T^t_{ab}(z) = \kappa_b^{-1} \circ T^t \circ \kappa_a(z).$$

This is simply the mapping T^t viewed in the local charts κ_a and κ_b. Notice, however, that we restrict the domain of definition to $Q(a, b, t)$.

For $a, b \in A$ with $Q(a, b, t) \neq \emptyset$, we consider the transfer operator

$$\mathcal{P}^t_{ab} : L^2(R) \to L^2(R), \quad \mathcal{P}^t_{ab} u(z) = \sum_{w \in (T^t_{ab})^{-1}(z)} \frac{h_b(z) u(w)}{\det(DT^t_{ab})_w}$$

where the sum is taken over $w \in Q(a, b, t)$ such that $T^t_{ab}(w) = z$. We define the system of transfer operators on local charts,

$$\mathbf{P}^t : L^2(R)^A \to L^2(R)^A,$$

by

$$\mathbf{P}^t(\mathbf{u}) = \left(\sum_{a \in A} \mathcal{P}^t_{ab}(u_a) \right)_{b \in A} \quad \text{for } \mathbf{u} = (u_a)_{a \in A} \in L^2(R)^A.$$

It is not difficult to check that the following diagram commutes:

$$\begin{array}{ccc} L^1(R)^A & \xrightarrow{\mathbf{P}^t} & L^1(R)^A \\ \downarrow \Pi & & \downarrow \Pi \\ L^1(X_f) & \xrightarrow{\mathcal{P}^t} & L^1(X_f) \end{array} \tag{5}$$

In the following subsections, we will prove the following.

PROPOSITION 2.2. *The operator $\mathbf{P}^t$ restricts to the bounded operator*

$$\mathbf{P}^t : W_*(R; \Theta_0)^A \to W_*(R; \Theta_0)^A.$$

Also $\mathbf{P}^t$ extends to the bounded operator

$$\mathbf{P}^t : W_\dagger(R; \Theta_0)^A \to W_\dagger(R; \Theta_0)^A.$$

Further, we have the Lasota–Yorke type inequality

$$\|\mathbf{P}^t(\mathbf{u})\| \le C_\sharp \cdot \mathbf{m}(f, t)^{1/2} \|\mathbf{u}\| + C|\mathbf{u}| \quad \textit{for } \mathbf{u} \in W_*(R)^A \tag{6}$$

where the constant $C_\sharp$ does not depend on t while the constant C may.

We can deduce Theorem 1.1 from this proposition and from Lemma 2.1(c).

Proof of Theorem 1.1. Since $W_*(R; \Theta_0)^A$ is compactly embedded in $W_\dagger(R; \Theta_0)^A$ from Lemma 2.1(c), the Lasota–Yorke type inequality in Proposition 2.2 implies that the essential spectral radius of the operator $\mathbf{P}^t : W_*(R; \Theta_0)^A \to W_*(R; \Theta_0)^A$ is bounded by $C_\sharp \cdot \mathbf{m}(f, t)^{1/2}$ [**11, 12**]. By the definition of the space $W_*(X_f)$, the commutative diagram (5) restricts to

$$\begin{array}{ccc} W_*(R;\Theta_0)^A & \xrightarrow{\mathbf{P}^t} & W_*(R;\Theta_0)^A \\ \downarrow \Pi & & \downarrow \Pi \\ W_*(X_f) & \xrightarrow{\mathcal{P}^t} & W_*(X_f). \end{array} \tag{7}$$

Recall also that the space $W_*(X_f)$ is identified with the orthogonal complement of the kernel of Π in $W_*(R)^A$. Through this identification, $\mathcal{P}^t$ corresponds to the composition of $\mathbf{P}^t$ with the orthogonal projection along the kernel of Π. Thus the essential spectral radius of $\mathcal{P}^t : W_*(X_f) \to W_*(X_f)$ is bounded by that of $\mathbf{P}^t : W_*(R;\Theta_0)^A \to W_*(R;\Theta_0)^A$ or $C_\sharp \mathbf{m}(f, t)^{1/2}$. Since this holds for any t large enough, the essential spectral radius of $\mathcal{P}^t : W_*(X_f) \to W_*(X_f)$ is bounded by $\mathbf{m}(f)^{1/2}$. The inclusions $C^1(X_f) \subset W_*(X_f) \subset L^2(X_f)$ follow from Lemma 2.1(a). □

2.4. *Two lemmas on the anisotropic Sobolev norms.* In this section, we give two basic lemmas on the anisotropic Sobolev norms. These lemmas and their proofs are slight modifications of those given by Baladi and Tsujii [**4**]. For the convenience of the reader, we give the proofs in Appendices B and C (see also Remark 2.5).

LEMMA 2.3. *Let* $g_i : \mathbb{R}^2 \to [0, 1]$, $1 \le i \le I$, *be a family of* C^r *functions such that* $\sum_{i=1}^I g_i(x) \le 1$ *for* $x \in Q$ *and that* $\mathrm{supp}(g_i) \subset Q$. *Let* Θ *and* Θ' *be polarizations such that* $\Theta' < \Theta$. *Then we have*

$$\left[\sum_{1\le i\le I} \|g_i u\|_{\Theta'}^2\right]^{1/2} \le C_0 \|u\|_\Theta + C|u|_\Theta \quad \text{for } u \in W_*(R;\Theta), \tag{8}$$

where C_0 *is a constant that does not depend on* $\{g_i\}$ *while the constant* C *may. Further, if* $\sum_{i=1}^I g_i(x) \equiv 1$ *for all* $x \in R$, *we also have*

$$\|u\|_{\Theta'} \le \nu \left[\sum_{1\le i\le I} \|g_i u\|_\Theta^2\right]^{1/2} + C\sum_{i=1}^I |g_i u|_\Theta \quad \text{for } u \in W_*(R;\Theta), \tag{9}$$

where ν *is the intersection multiplicity of the supports of the functions* g_i, $1 \le i \le I$.

In the next lemma, we consider the following situation. For a C^{r-1} function $h : \mathbb{R}^2 \to \mathbb{R}$ supported on a closed subset $K \subset R$ and for a C^r diffeomorphism $S : U \to S(U) \subset \mathbb{R}^2$ defined on an open neighborhood U of K, we consider a transfer operator $L : C^{r-1}(R) \to C^{r-1}(R)$ defined by

$$Lu(x) = \begin{cases} h(x) \cdot u \circ S(x) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that, for polarizations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$, we have

$$(DS_\zeta)^{\mathrm{tr}}(\mathbb{R}^2 \setminus \mathbf{C}_+) \Subset \mathbf{C}'_- \quad \text{for all } \zeta \in K,$$

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where $(DS_\zeta)^{\mathrm{tr}}$ denotes the transpose of DS_ζ. Set

$$\gamma(S) = \min_{\zeta\in K} |\det DS_\zeta|$$

and

$$\Lambda(S, \Theta', K) = \sup\left\{ \frac{\|(DS_\zeta)^{\mathrm{tr}}(v)\|}{\|v\|} \;\middle|\; \zeta \in K,\ (DS_\zeta)^{\mathrm{tr}}(v) \notin \mathbf{C}'_- \right\}.$$

LEMMA 2.4. *The operator L extends boundedly to $L : W_*(R; \Theta) \to W_*(R; \Theta')$ and to $L : W_\dagger(R; \Theta) \to W_\dagger(R; \Theta')$. Further, we have for $u \in W_*(R; \Theta)$,*

$$\|Lu\|^-_{\Theta'} \le \gamma(S)^{-1/2}\|h\|_{L^\infty}\|u\|_\Theta \tag{10}$$

and

$$\|Lu\|^+_{\Theta'} \le C_0\gamma(S)^{-1/2}\Lambda(S, \Theta', K)\|h\|_{L^\infty}\|u\|_\Theta + C|u|_\Theta \tag{11}$$

where the constant C_0 does not depend on S, h, Θ nor Θ' while the constant C may. In particular, we have for $u \in W_*(R; \Theta)$,

$$\|Lu\|_{\Theta'} \le C_0\gamma(S)^{-1/2}\max\{1, \Lambda(S, \Theta', K)\}\|h\|_{L^\infty}\|u\|_\Theta + C|u|_\Theta. \tag{12}$$

Remark 2.5. The latter claim (9) of Lemma 2.3 is a special case of Baladi and Tsujii [**4**, Lemma 7.1]. Also, Lemma 2.4 and the former claim (8) in Lemma 2.3 correspond to Baladi and Tsujii [**4**, Propositions 7.2 and 6.1]. However, since we considered only anisotropic Sobolev norms $\|\cdot\|_{\Theta,p,q}$ with $q < 0 < p$ in those propositions [**4**], we need to modify the statements and proofs slightly. This is the reason why we give the proofs of Lemma 2.4 and the former claim (8) of Lemma 2.3 in the Appendices B and C.

2.5. *The Lasota–Yorke type inequality in local charts.* To complete the proof of Proposition 2.2, we only have to show the Lasota–Yorke type inequality (6), as the other claims now follow from Lemma 2.4 where S is the branches of the inverse of T^t_{ab}. For the proof of the Lasota–Yorke type inequality (6), it is enough to show the following lemma for the components $\mathcal{P}^t_{ab}$ of $\mathbf{P}^t$.

LEMMA 2.6. *For $a, b \in A$ and for sufficiently large t, we have*

$$\|\mathcal{P}^t_{ab}(u)\|^2_{\Theta_0} \le C_\sharp \cdot \mathbf{m}(f, t)\|u\|^2_{\Theta_0} + C|u|^2_{\Theta_0} \quad \textit{for } u \in C^r(R)$$

where the constant $C_\sharp$ does not depend on t while the constant C may.

Below, we prove Lemma 2.6. To begin with, we set up some notation. Fix $a, b \in A$ and consider large $t > 0$. Let $\{D(\omega), \omega \in \Omega\}$, be a finite family of small closed disks whose interiors cover the closure of R. We may and do assume that the intersection multiplicity of this cover is bounded by some absolute constant (say 4). Note that we can take such a family of disks with arbitrarily small diameters. Let $D(\omega, i)$, $1 \le i \le I(\omega)$, be the connected components of the preimage $(T^t_{ab})^{-1}(D(\omega))$ that meet the closure of R. Then $\det DT^t_{ab}$ takes a constant value on each component $D(\omega, i)$, which is denoted by $e(\omega, i)$. From equation (2), it holds that

$$\sum_{1\le i\le I(\omega)} e(\omega, i)^{-1} \le 1 \quad \text{for any } \omega \in \Omega. \tag{13}$$

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We write $i \pitchfork_\omega j$ for $1 \le i, j \le I(\omega)$ if

$$(DT^t_{ab})_z(\mathbf{C}_f) \cap (DT^t_{ab})_w(\mathbf{C}_f) = \{0\} \tag{14}$$

for any $z \in D(\omega, i)$ and $w \in D(\omega, j)$. Note that equation (14) holds if and only if

$$(((DT^t_{ab})_z)^{\mathrm{tr}})^{-1}((\mathbf{C}_f)^*) \cap (((DT^t_{ab})_w)^{\mathrm{tr}})^{-1}((\mathbf{C}_f)^*) = \{0\}.$$

Letting the diameters of the disks $D(\omega)$ be small, we may assume

$$\sum_{j \not\pitchfork_\omega i} e(\omega, j)^{-1} \le \mathbf{m}(f, t) \quad \text{for } 1 \le i \le I(\omega) \text{ and } \omega \in \Omega$$

where $\sum_{j \not\pitchfork_\omega i}$ denotes the sum over $1 \le j \le I(\omega)$ such that $j \not\pitchfork_\omega i$. Further, we can take polarizations $\Theta(\omega, i) = (\mathbf{C}_{\omega,i,+}, \mathbf{C}_{\omega,i,-}, \varphi_{\omega,i,+}, \varphi_{\omega,i,-})$ for each $\omega \in \Omega$ and $1 \le i \le I(\omega)$ such that

$$(((DT^t_{ab})_z)^{\mathrm{tr}})^{-1}(\mathbb{R}^2 \setminus \check{\mathbf{C}}_{0,+}) \Subset \mathbf{C}_{\omega,i,-} \Subset \left(\mathbb{R}^2 \setminus \mathbf{C}_{\omega,i,+}\right) \Subset \hat{\mathbf{C}}_{0,-} \quad \text{for any } z \in D(\omega, i)$$

and

$$\overline{(\mathbb{R}^2 \setminus \mathbf{C}_{\omega,i,+})} \cap \overline{(\mathbb{R}^2 \setminus \mathbf{C}_{\omega,j,+})} = \{0\} \quad \text{if } i \pitchfork_\omega j.$$

Take a family of C^∞ functions $g_\omega : \mathbb{R}^2 \to [0, 1]$ for $\omega \in \Omega$ such that the support of each g_ω is contained in $D(\omega)$ and $\sum_{\omega \in \Omega} g_\omega(z) \equiv 1$ for all $z \in R$. We then define the functions $g_{\omega,i} : \mathbb{R}^2 \to [0, 1]$ for $\omega \in \Omega$ and $1 \le i \le I(\omega)$ by

$$g_{\omega,i}(z) = \begin{cases} g_\omega(T^t_{ab}(z)) & \text{if } z \in D(\omega, i), \\ 0 & \text{otherwise.} \end{cases}$$

We now begin the proof of Lemma 2.6. In the following, we will write $C_\sharp$ for constants that do not depend on t and C for constants that may depend on t. (Notice that the values of the constants denoted by $C_\sharp$ and C are different from place to place.) We view the operator $\mathcal{P}^t_{ab}$ under consideration as the composition of the four operations:

(i) breaking a function $u \in C^r(R)$ into $u_{\omega,i} := g_{\omega,i} u$, $\omega \in \Omega$, $1 \le i \le I(\omega)$;
(ii) transforming each $u_{\omega,i}$ to $v_{\omega,i} := \mathcal{P}^t_{ab}(u_{\omega,i})$;
(iii) summing up $v_{\omega,i}$ for $1 \le i \le I(\omega)$ to get $v_\omega := \sum_{1 \le i \le I(\omega)} v_{\omega,i} = g_\omega \mathcal{P}^t_{ab} u$; and
(iv) summing up v_ω for $\omega \in \Omega$ to get $\mathcal{P}^t_{ab} u = \sum_\omega v_\omega$.

For operation (i), the former claim (8) of Lemma 2.3 gives the estimate

$$\sum_{\omega \in \Omega} \sum_{i=1}^{I(\omega)} \|u_{\omega,i}\|^2_{\check{\Theta}_0} \le (C_\sharp \|u\|_{\Theta_0} + C|u|_{\Theta_0})^2 \le C_\sharp \|u\|^2_{\Theta_0} + C|u|^2_{\Theta_0}.$$

For operation (iv), the latter claim (9) of Lemma 2.3 gives the estimate

$$\|\mathcal{P}^t_{ab} u\|^2_{\Theta_0} = \left\| \sum_{\omega \in \Omega} g_\omega \mathcal{P}^t_{ab} u \right\|^2_{\Theta_0} = \left\| \sum_{\omega \in \Omega} v_\omega \right\|^2_{\Theta_0} \le C_\sharp \sum_{\omega \in \Omega} \|v_\omega\|^2_{\hat{\Theta}_0} + C \sum_{\omega \in \Omega} |v_\omega|^2_{\hat{\Theta}_0}.$$

Below, we consider the operations (ii) and (iii). Letting S be a branch of $(T^t_{ab})^{-1}$ and $K = \operatorname{supp}(g_\omega)$ in Lemma 2.4, we obtain the estimates

$$|v_{\omega,i}|_{\Theta(\omega,i)} \le C |u_{\omega,i}|_{\check{\Theta}_0}, \tag{15}$$

$$\|v_{\omega,i}\|_{\hat{\Theta}_0} \le C_\sharp \cdot e(\omega, i)^{-1/2} \|u_{\omega,i}\|_{\check{\Theta}_0} + C|u_{\omega,i}|_{\check{\Theta}_0}, \tag{16}$$

$$\|v_{\omega,i}\|^+_{\hat{\Theta}_0} \le C_\sharp \cdot e(\omega, i)^{-3/2} \|u_{\omega,i}\|_{\check{\Theta}_0} + C|u_{\omega,i}|_{\check{\Theta}_0}. \tag{17}$$

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The next lemma is the core of our argument, in which we essentially make use of the transversality condition (14).

LEMMA 2.7. *If $i \pitchfork_\omega j$, we have*

$$\sum_{n\geq 0} \left|(\psi_{\hat{\Theta}_0,n,-}(D)v_{\omega,i},\, \psi_{\hat{\Theta}_0,n,-}(D)v_{\omega,j})_{L^2}\right| \leq C|v_{\omega,i}|_{\Theta(\omega,i)}|v_{\omega,j}|_{\Theta(\omega,j)}.$$

Proof. Set $w_{i,n} = \psi_{\hat{\Theta}_0,n,-}(D)v_{\omega,i}$, $w'_{i,n} = \psi_{\Theta(\omega,i),n,-}(D)v_{\omega,i}$, $w''_{i,n} = w_{i,n} - w'_{i,n}$. For $n > 0$, from the assumption $i \pitchfork_\omega j$ we have $(w'_{i,n}, w'_{j,n})_{L^2} = 0$, that is,

$$(w_{i,n}, w_{j,n})_{L^2} = (w''_{i,n}, w'_{j,n})_{L^2} + (w'_{i,n}, w''_{j,n})_{L^2} + (w''_{i,n}, w''_{j,n})_{L^2}.$$

Since we have that $\|w'_{i,n}\|_{L^2} \leq 2^{\epsilon n}|v_{\omega,i}|_{\Theta(\omega,i)}$ and that $\|w''_{i,n}\|_{L^2} \leq 2^{-(1-\epsilon)n}|v_{\omega,i}|_{\Theta(\omega,i)}$ by the definition of the norm $|\cdot|_{\Theta(\omega,i)}$, and since $0 < \epsilon < 1/2$, we derive the lemma using the Schwarz inequality. □

It follows from Lemma 2.7 that

$$\left(\|v_\omega\|^-_{\hat{\Theta}_0}\right)^2 \leq \sum_i \sum_{j \not\pitchfork_\omega i} \|v_{\omega,i}\|_{\hat{\Theta}_0}\|v_{\omega,j}\|_{\hat{\Theta}_0} + C\sum_i |v_{\omega,i}|^2_{\Theta(\omega,i)}$$

for some constant $C > 0$ that may depend on $I(\omega)$, where $\sum_{j\not\pitchfork_\omega i}$ denotes the sum over $1 \leq j \leq I(\omega)$ such that $j \not\pitchfork_\omega i$. Applying the inequalities (16) and (15) in the right-hand side, we obtain

$$\begin{aligned}\left(\|v_\omega\|^-_{\hat{\Theta}_0}\right)^2 &\leq C_\sharp \sum_i \sum_{j\not\pitchfork_\omega i} \frac{e(\omega,j)^{-1}\|u_{\omega,i}\|^2_{\check{\Theta}_0} + e(\omega,i)^{-1}\|u_{\omega,j}\|^2_{\check{\Theta}_0}}{2} + C\sum_i |u_{\omega,i}|^2_{\check{\Theta}_0}\\ &\leq C_\sharp \mathbf{m}(f,t)\sum_i \|u_{\omega,i}\|^2_{\check{\Theta}_0} + C\sum_i |u_{\omega,i}|^2_{\check{\Theta}_0}.\end{aligned}$$

On the other hand, we obtain from inequality (17)

$$\begin{aligned}\left(\|v_\omega\|^+_{\hat{\Theta}_0}\right)^2 &\leq C_\sharp\left(\sum_i e(\omega,i)^{-3/2}\|u_{\omega,i}\|_{\check{\Theta}_0}\right)^2 + C\left(\sum_i |u_{\omega,i}|_{\check{\Theta}_0}\right)^2\\ &\leq C_\sharp \mathbf{m}(f,t)\sum_i \|u_{\omega,i}\|^2_{\check{\Theta}_0} + C\sum_i |u_{\omega,i}|^2_{\check{\Theta}_0}\end{aligned}$$

where we used the Schwarz inequality, equation (13) and the simple fact $e(\omega,i)^{-1} \leq \mathbf{m}(f,t)$ in the second inequality. We therefore obtain, for the operations (ii) and (iii),

$$\left(\|v_\omega\|_{\hat{\Theta}_0}\right)^2 \leq C_\sharp \mathbf{m}(f,t)\sum_i \|u_{\omega,i}\|^2_{\check{\Theta}_0} + C\sum_i |u_{\omega,i}|^2_{\check{\Theta}_0}.$$

Letting S be the identity map in Lemma 2.4, we see that $|u_{\omega,i}|_{\check{\Theta}_0} \leq C|u|_{\Theta_0}$ and $|v_{\omega,i}|_{\hat{\Theta}_0} \leq C|v_{\omega,i}|_{\hat{\Theta}(\omega,i)}$. These and inequality (15) give

$$|v_\omega|_{\hat{\Theta}_0} \leq C\sum_{1\leq i\leq I(\omega)} |v_{\omega,i}|_{\Theta(\omega,i)} \leq C\sum_{1\leq i\leq I(\omega)} |u_{\omega,i}|_{\check{\Theta}_0} \leq C|u|_{\Theta_0}.$$

We can now conclude the Lasota–Yorke type inequality in Lemma 2.6 from the estimates on the operations (i)–(iv) given above.

3. *Proof of Theorem 1.2*

The following proof of Theorem 1.2 is a modification of the argument by Tsujii [**18**].

3.1. *Notations.*

We recall some notation from Tsujii [**18**]. Let $\mathcal{A} = \{1, 2, \ldots, \ell\}$ and let $\mathcal{A}^n$ be the space of words of length n on $\mathcal{A}$. For a word $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$ and an integer $0 \le p \le n$, let $[\mathbf{a}]_p = (a_i)_{i=1}^p$. For $0 \le p \le n$, we define the equivalence relation $\sim_p$ on $\mathcal{A}^n$ so that $\mathbf{a} \sim_p \mathbf{b}$ if and only if $[\mathbf{a}]_p = [\mathbf{b}]_p$.

Let $\mathcal{P}$ be the partition of S^1 into the intervals $\mathcal{P}(k) = [(k-1)/\ell, k/\ell]$ for $k \in \mathcal{A}$. Then $\mathcal{P} = \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$ is the partition into the intervals

$$\mathcal{P}(\mathbf{a}) = \bigcap_{i=0}^{n-1} \tau^{-i}(\mathcal{P}(a_{n-i})), \quad \mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n.$$

Let $x_{\mathbf{a}}$ be the left end point of the interval $\mathcal{P}(\mathbf{a})$.

Remark 3.1. Notice that $\mathbf{a}$ is the *inverse* of the itinerary of the points in $\mathcal{P}(\mathbf{a})$.

For a point $x \in S^1$ and $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$, $\mathbf{a}(x)$ denotes the unique point $y \in \mathcal{P}(\mathbf{a})$ such that $\tau^n(y) = x$. For a C^r function $f \in C^r(S^1)$, $x \in S^1$ and $\mathbf{b} \in \mathcal{A}^n$, we put

$$s(x, \mathbf{b}; f) := f^{(n)}(\mathbf{b}(x)) = \sum_{i=1}^{n} f([\mathbf{b}]_i(x)).$$

Then we have

$$\frac{ds}{dx}(x, \mathbf{b}; f) = \sum_{i=1}^{n} \ell^{-i} \frac{df}{dx}([\mathbf{b}]_i(x)).$$

We will identify the unit circle S^1 with the lower boundary $S^1 \times \{0\}$ of X_f. If $f^{(n)}(\mathbf{b}(x)) \le t < f^{(n+1)}(\mathbf{b}(x))$, the image of the horizontal tangent vector $(1, 0)$ at $\mathbf{b}(x) \in S^1 \times \{0\}$ by the mapping T_f^t has slope $(ds/dx)(x, \mathbf{b}; f)$ and hence

$$(DT_f^t)_{\mathbf{b}(x)}(\mathbf{C}_f) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \;\middle|\; \left| \eta - \frac{ds}{dx}(x, \mathbf{b}; f) \cdot \xi \right| \le \ell^{-n} \theta_f |\xi| \right\}.$$

For $K > 1$, let $C_+^r(S^1; K)$ be the set $f \in C_+^r(S^1)$ such that $K^{-1} < f(x) < K$ for $x \in S^1$ and $\|f\|_{C^r} < K$. By virtue of Theorem 1.4, it is sufficient for the proof of Theorem 1.2 to show that, for each $\rho > 1$ and $K > 0$, the condition

$$\mathbf{m}(f) \le \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1} \tag{18}$$

holds for functions f in an open and dense subset of $C_+^r(S^1; K)$. We will prove this claim in the following. We henceforth fix an arbitrary $\rho > 1$ and $K > 0$. Note that for $f \in C_+^r(S^1; K)$, we have that

$$\theta_f \le \theta_K := K/(\gamma_0 \ell - 1), \quad \ell^{1/K} \le \lambda_{\min}(\mathbf{T}_f) \le \ell^K$$

and also that

$$\left| \frac{d^2 s}{dx^2}(x, \mathbf{b}; f) \right| = \left| \sum_{i=1}^{n} \ell^{-2i} \frac{d^2 f}{dx^2}([\mathbf{b}]_i(x)) \right| \le \frac{\ell^{-2} K}{1 - \ell^{-2}} \le \theta_K \tag{19}$$

for $x \in S^1$ and $\mathbf{b} \in \mathcal{A}^n$.

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3.2. *Some consequences of the condition* $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$. In this subsection, we see what kind of singular situation occurs if condition (18) does not hold. Fix $1 < \gamma < \ell$ such that $\gamma^K < \rho$.

PROPOSITION 3.2. *If* $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ *for* $f \in C^r_+(S^1; K)$*, then, for any* $n \geq 1$*, there exist* $\mathbf{c} \in \mathcal{A}^n$ *and* $B \subset \mathcal{A}^n$ *with* $\#B \geq \gamma^n$ *such that*

$$\left|\frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{a}; f) - \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}; f)\right| \leq 8\theta_K \cdot \ell^{-n} \quad \textit{for all } \mathbf{a}, \mathbf{b} \in B.$$

Proof. Take $\gamma < \bar{\gamma} < 1$ so close to γ that $\bar{\gamma}^K < \rho$. Then take $1 < \lambda < \lambda_{\min}(\mathbf{T}_f)$ so close to $\lambda_{\min}(\mathbf{T}_f)$ that $\bar{\gamma}^K \lambda_{\min}(\mathbf{T}_f) < \rho\lambda$. From the assumption, we can take an arbitrarily large $t \geq 0$, $z \in X_f$, $w \in T^{-t}(z)$ and $Z \subset T^{-t}(z)$ such that

$$(DT^t)_\zeta(\mathbf{C}_f) \cap (DT^t)_w(\mathbf{C}_f) \neq \{0\} \quad \text{for } \zeta \in Z \tag{20}$$

and

$$\sum_{\zeta \in Z} \frac{1}{E(\zeta, t; f)} \geq \rho^t \cdot \lambda_{\min}(\mathbf{T}_f)^{-t}. \tag{21}$$

Additionally, we may and do assume that $z \in S^1 \times \{0\}$. Set

$$m = \min\{n(x, s + t; f) \mid (x, s) \in Z\}.$$

(Recall the definition of $n(x, t; f)$ in §1.) Then we have

$$K^{-1}t < m < Kt \quad \text{and} \quad \ell^m > \lambda^t, \tag{22}$$

provided that t is sufficiently large. (To see the second inequality, note that we have $\ell^m \sim \lambda_{\min}(\mathbf{T}_f)^t$ for large t.)

For each $\zeta = (x, s) \in Z$, let $I(\zeta) \in \mathcal{A}^{n(x,s+t;f)}$ be the word such that $\zeta \in \mathcal{P}(I(\zeta))$. Set $A = \{[I(\zeta)]_m \mid \zeta \in Z\} \subset \mathcal{A}^m$. Then, applying formula (2), we can see that

$$\#A \cdot \ell^{-m} \geq \sum_{\zeta \in Z} \frac{1}{E(\zeta, t; f)}.$$

Combining this inequality with equation (21), we see that

$$\#A \geq \ell^m \rho^t \lambda_{\min}(\mathbf{T}_f)^{-t} > (\lambda\rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1})^t > \bar{\gamma}^{Kt} > \bar{\gamma}^m. \tag{23}$$

Note that condition (20) implies

$$\left|\frac{ds}{dx}(z, \mathbf{b}; f) - \frac{ds}{dx}(z, \mathbf{b}'; f)\right| \leq 4\theta_K \cdot \ell^{-m} \quad \text{for all } \mathbf{b}, \mathbf{b}' \in A. \tag{24}$$

We now start to consider an arbitrary integer $n \geq 1$ in the claim. Taking large t in the beginning, we may and do assume that the integer m above is much larger than n. For each $0 \leq k \leq m$, we split A into equivalence classes with respect to $\sim_k$ and let $A_k \subset A$ be one of those equivalence classes with maximum cardinality. Then the cardinality $q(k)$ of A_k is a decreasing sequence with respect to k. Further, we have $q(0) \geq \bar{\gamma}^m$ by equation (23) and also $q(m) = 1$ obviously. Therefore, we can find an integer $0 \leq m' \leq m - n$ such that $q(m' + n) \leq \gamma^{-n} q(m')$, provided that we took sufficiently large t in the beginning. We fix

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such an integer m' and let $A' \subset \mathcal{A}^{m-m'}$ be the set of words that are obtained by removing the first common m' letters (say $\mathbf{c}$) from the words in $A_{m'}$, and put $x = \mathbf{c}(z)$. It then follows from equation (24) that

$$\left| \frac{ds}{dx}(x, \mathbf{b}; f) - \frac{ds}{dx}(x, \mathbf{b}'; f) \right| \le 4\theta_K \cdot \ell^{-(m-m')} \le 4\theta_K \cdot \ell^{-n} \quad \text{for all } \mathbf{b}, \mathbf{b}' \in A'.$$

Set $B = \{[\mathbf{a}]_n \mid \mathbf{a} \in A'\}$. From the condition $q(m'+n) \le \gamma^{-n} q(m')$ in the choice of m', we have that $\#B \ge \gamma^n$. Also, since

$$\left| \frac{ds}{dx}(x, [\mathbf{a}]_n; f) - \frac{ds}{dx}(x, \mathbf{a}; f) \right| \le \theta_K \cdot \ell^{-n} \quad \text{for } \mathbf{a} \in \mathcal{A}^{m-m'},$$

we see that

$$\left| \frac{ds}{dx}(x, \mathbf{b}; f) - \frac{ds}{dx}(x, \mathbf{b}'; f) \right| \le 6\theta_K \cdot \ell^{-n} \quad \text{for all } \mathbf{b}, \mathbf{b}' \in B.$$

For $\mathbf{b}, \mathbf{c} \in \mathcal{A}^n$ and $f \in C^r_+(S^1; K)$, the variation of $(ds/dx)(\cdot, \mathbf{b}; f)$ on the interval $\mathcal{P}(\mathbf{c})$ is bounded by $\theta_K \ell^{-n}$, in view of equation (19). Therefore, translating the point x to the point $x_{\mathbf{c}}$, we obtain the conclusion of the proposition. □

To state the next proposition, we set up some constants. First take real numbers α and β such that $1 < \beta < \alpha < \gamma$ and then take positive integers p and ν such that

$$\beta^{-p}\ell^2 < 1 \quad \text{and} \quad (\nu+1)(p+1)\alpha^{-\nu} < 1.$$

We put

$$\delta = \frac{\log \gamma - \log \alpha}{\log \ell - \log \alpha} \in (0, 1).$$

Then we choose an integer $N > \nu$ such that

$$\ell^\nu \alpha^n < \gamma^n \quad \text{for } n \ge N$$

and

$$\ell^{-\nu}(\gamma/\beta)^{n'}(1 - (\nu+1)(p+1)\alpha^{-\nu}) \ge 1 \quad \text{for } n' \ge \delta N.$$

PROPOSITION 3.3. *If* $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ *for* $f \in C^r_+(S^1; K)$, *then for any* $n \ge N$, *there exist an integer* $\delta n \le n' \le n$, *a word* $\mathbf{d} \in \mathcal{A}^{n'}$ *and mutually disjoint subsets* $B_i \subset \mathcal{A}^{n'}$ *for* $1 \le i \le (\nu+1)(p+1)$ *such that:*

(a) *we have*

$$\left| \frac{ds}{dx}(x_{\mathbf{d}}, \mathbf{b}; f) - \frac{ds}{dx}(x_{\mathbf{d}}, \mathbf{b}'; f) \right| \le 10\theta_K \cdot \ell^{-n'}$$

for all $\mathbf{b}, \mathbf{b}' \in \bigcup_{i=1}^{(\nu+1)(p+1)} B_i$;

(b) $\#B_i \ge \beta^{n'}$ *for* $1 \le i \le (\nu+1)(p+1)$; *and*

(c) $[\mathbf{a}]_\nu = [\mathbf{b}]_\nu$ *for* $\mathbf{a} \in B_i$ *and* $\mathbf{b} \in B_j$ *if and only if* $i = j$.

Proof. Let $n \ge N$ and let $B \subset \mathcal{A}^n$ be the subset in the conclusion of Proposition 3.2. For $0 \le k \le [n/\nu]$, we split B into equivalence classes with respect to $\sim_{k\nu}$ and let $B_k \subset B$ be one of those equivalence classes with maximum cardinality. Then the cardinality $q(k)$

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of B_k is decreasing with respect to k and satisfies $q(0) = \#B \geq \gamma^n$ and $q([n/\nu]) \leq \ell^\nu < \gamma^n \alpha^{-[n/\nu]\nu}$, where the last inequality follows from the first condition in the choice of N.

Let k_0 be the smallest integer $1 \leq k \leq [n/\nu]$ such that $q(k) < \gamma^n \alpha^{-k\nu}$. By this choice of k_0, we have

$$q(k_0) < \alpha^{-\nu} q(k_0 - 1) \quad \text{and} \quad q(k_0 - 1) \geq \gamma^n \alpha^{-(k_0-1)\nu}. \tag{25}$$

Put $n' = n - (k_0 - 1)\nu$. Since $q(k) \leq \ell^{n-\nu\cdot k}$ obviously, we have

$$\ell^{n'} = \ell^{n-(k_0-1)\nu} \geq q(k_0 - 1) \geq \gamma^n \alpha^{n'-n} \quad \text{or} \quad n' \geq \frac{\log \gamma - \log \alpha}{\log \ell - \log \alpha} \cdot n = \delta n.$$

Let $B'_i \subset B_{k_0-1}$, $1 \leq i \leq \ell^\nu$, be the equivalence classes in B_{k_0-1} with respect to the relation $\sim_{k_0}$, arranged in decreasing order of cardinality. (Note that some of the B'_i may be empty.) Then we have a simple inequality

$$\begin{aligned}\min_{1 \leq i \leq (\nu+1)(p+1)} \#B'_i &\geq \frac{q(k_0 - 1) - (\nu + 1)(p + 1)q(k_0)}{\ell^\nu} \\ &\geq \ell^{-\nu} \gamma^n \alpha^{-(k_0-1)\nu}(1 - (\nu + 1)(p + 1)\alpha^{-\nu}) \geq \beta^{n'}\end{aligned}$$

where the second inequality follows from equation (25) and the latter from the second condition in the choice of N. Finally, let $B_i \subset \mathcal{A}^{n'}$ for $1 \leq i \leq (\nu + 1)(p + 1)$ be the subset of words that are obtained by removing the first common $(k_0 - 1)\nu$ letters (say $\mathbf{c}'$) from the words in B'_i. Then the conditions (b) and (c) hold. From the condition on the subset B in Proposition 3.2, we have

$$\left| \frac{ds}{dx}(x_{\mathbf{cc}'}, \mathbf{b};\ f) - \frac{ds}{dx}(x_{\mathbf{cc}'}, \mathbf{b}';\ f) \right| \leq 8\theta_K \cdot \ell^{-n'} \quad \text{for all } \mathbf{b}, \mathbf{b}' \in \bigcup_{i=0}^{(\nu+1)(p+1)} B_i.$$

Take $\mathbf{d} \in \mathcal{A}^{n'}$ such that $x_{\mathbf{cc}'} \in \mathcal{P}(\mathbf{d})$; $\mathbf{c}$ is as defined in Proposition 3.2. Then condition (a) holds because the variations of the functions $(ds/dx)(\cdot, \mathbf{a};\ f)$ for $\mathbf{a} \in \mathcal{A}^{n'}$ on $\mathcal{P}(\mathbf{d})$ are bounded by $\theta_K \ell^{-n'}$, in view of equation (19). □

3.3. *Generic perturbations.* We will show that the consequences of the condition $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ given in Proposition 3.3 hold only for a very small set of $f \in C^r_+(S^1;\ K)$. For this purpose, we next consider perturbations of the function f.

For $f \in C^r_+(S^1;\ K)$ and $\varphi_i \in C^\infty(S^1)$, $1 \leq i \leq m$, we consider the family

$$f_{\mathbf{t}}(x) = f(x) + \sum_{i=1}^{m} t_i \cdot \varphi_i(x) \tag{26}$$

with parameter $\mathbf{t} = (t_i)_{i=1}^m \in \mathbb{R}^m$. For a point $x \in S^1$ and a finite subset $\sigma = \{\mathbf{b}_i\}_{0 \leq i \leq p}$ of $\mathcal{A}^n$, let $G_{x,\sigma} : \mathbb{R}^m \to \mathbb{R}^p$ be the affine map defined by

$$G_{x,\sigma}(\mathbf{t}) = \left(\frac{ds}{dx}(x, \mathbf{b}_i;\ f_{\mathbf{t}}) - \frac{ds}{dx}(x, \mathbf{b}_0;\ f_{\mathbf{t}}) \right)_{i=1}^{p}.$$

Note that $G_{x,\sigma}(\mathbf{t})$ is independent of f in (26). For an affine map $A : \mathbb{R}^m \to \mathbb{R}^p$, let $\mathrm{Jac}(A)$ be the Jacobian of $DA|_{\ker(DA)^\perp}$, the restriction of the linear part DA to the orthogonal complement of its kernel when A is surjective, and set $\mathrm{Jac}(A) = 0$ otherwise. In other words, $\mathrm{Jac}(A)$ is the maximum among the Jacobians of the restrictions of DA to p-dimensional subspaces in $\mathbb{R}^m$. The following is a slight variation of Tsujii [**18**, Proposition 16].

PROPOSITION 3.4. *We can choose functions $\varphi_i \in C^\infty(S^1)$, $1 \le i \le m$, such that for any $x \in S^1$ and any subsets $A = \{\mathbf{a}_i\}_{1\le i\le(\nu+1)(p+1)}$ of $\mathcal{A}^\nu$, there exists a subset $A' = \{\mathbf{a}'_i\}_{0\le i\le p}$ of A such that we have $\mathrm{Jac}(G_{x,\sigma}) \ge 1$ whenever a subset $\sigma = \{\mathbf{b}_i\}_{0\le i\le p}$ of $\mathcal{A}^n$ with $n \ge \nu$ satisfies $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \le i \le p$.*

The proof of Proposition 3.4 is similar to that of Tsujii [**18**, Proposition 16]. For completeness, we give the proof in the last subsection.

3.4. *The end of the proof.* For $n \ge \nu$, $\mathbf{c} \in \mathcal{A}^n$ and $\sigma = (\mathbf{b}_i)_{i=0}^p \in (\mathcal{A}^n)^{p+1}$, let $Y(n, \mathbf{c}, \sigma)$ be the set of functions $f \in C^r_+(S^1; K)$ such that

$$\left|\frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}_i; f) - \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}_0; f)\right| \le 10\,\theta_K \cdot \ell^{-n} \quad \text{for all } 1 \le i \le p.$$

Note that $Y(n, \mathbf{c}, \sigma)$ is a closed subset in $C^r_+(S^1; K)$.

For $n \ge \nu$, let $Y(n)$ be the set of functions $f \in C^r_+(S^1; K)$ that belongs to $Y(n, \mathbf{c}, \sigma)$ for more than $[\beta^{n(p+1)}]$ combinations of $(\mathbf{c}, \sigma) \subset \mathcal{A}^n \times (\mathcal{A}^n)^{p+1}$ satisfying $\mathrm{Jac}(G_{x_{\mathbf{c}},\sigma}) \ge 1$. Let $Y_*(n) = \bigcup_{n'=[\delta n]}^n Y(n')$. Then $Y(n)$ and $Y_*(n)$ are also closed subsets in $C^r_+(S^1; K)$. Proposition 3.3 implies that, if $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$, then f belongs to the closed subset $\bigcap_{n\ge N} Y_*(n)$. To finish the proof of the theorem, we show that the complement of $\bigcap_{n\ge N} Y_*(n)$ is dense in $C^r_+(S^1; K)$.

Take a function $f \in C^r_+(S^1; K)$ arbitrarily and consider the family (26) with $\varphi_i \in C^\infty(S^1)$, $1 \le i \le m$, in Proposition 3.4. Take $\epsilon > 0$ so small that $f_{\mathbf{t}} \in C^r_+(S^1; K)$ for all $\mathbf{t} \in [-\epsilon, \epsilon]^m$. Let $X(n, \mathbf{c}, \sigma)$, $X(n)$ and $X_*(n)$ be the set of parameters $\mathbf{t} \in [-\epsilon, \epsilon]^m$ such that $f_{\mathbf{t}} \in Y(n, \mathbf{c}, \sigma)$, $f_{\mathbf{t}} \in Y(n)$ and $f_{\mathbf{t}} \in Y_*(n)$, respectively. From the definition of Jacobian in the previous subsection, we have $\mathrm{Leb}(X(n, \mathbf{c}, \sigma)) \le C\ell^{-np}$ for some constant $C > 0$ that depends on θ_K, m and ϵ. Therefore, taking the number of combinations of $(\mathbf{c}, \sigma)$ into consideration, we obtain

$$\mathrm{Leb}(X(n)) \le \frac{C\ell^{-np} \times \ell^n \times \ell^{(p+1)n}}{\beta^{(p+1)n}} < C(\beta^{-p}\ell^2)^n.$$

As we chose p such that $\beta^{-p}\ell^2 < 1$, we have $\mathrm{Leb}\big(\bigcap_{n\ge N} X_*(n)\big) = 0$ and hence the complement of $\bigcap_{n\ge N} Y_*(n)$ in $C^r_+(S^1; K)$ is dense.

Remark 3.5. The proof above shows also that the condition $\mathbf{m}(f) \le \lambda_{\min}(\mathbf{T}_f)^{-1}$ holds for a prevalent subset of $f \in C^r_+(S^1)$ in a measure-theoretical sense [**10, 19**].

3.5. *The proof of Proposition 3.4.* To prove Proposition 3.4, it is enough to show the following localized version of the claim.

PROPOSITION 3.6. *For each $y \in S^1$, we can choose functions $\varphi_{y,i} \in C^\infty(S^1)$ for $1 \le i \le \ell^\nu$ and a neighborhood U_y of y such that, for any point $x \in U_y$ and any subsets*

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$A = \{\mathbf{a}_i\}_{1\le i\le(\nu+1)(p+1)}$ of $\mathcal{A}^\nu$, there exists a subset $A' = \{\mathbf{a}'_i\}_{0\le i\le p}$ of A such that we have that $\mathrm{Jac}(G_{x,\sigma}) \ge 1$ whenever a subset $\sigma = \{\mathbf{b}_i\}_{0\le i\le p}$ of $\mathcal{A}^n$ with $n \ge \nu$ satisfies $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \le i \le p$.

In fact, once we have Proposition 3.6, we can take a finite subset $\{y(j)\}_{j=1}^J$ in S^1 so that the neighborhoods $U_{y(j)}$ in Proposition 3.6 cover S^1 and, letting $\{\varphi_i\}_{j=1}^m$ be the union of $\{\varphi_{y(j),i}\}_{i=1}^{\ell^\nu}$ for $1 \le j \le J$ in the corresponding conclusions of Proposition 3.6, we obtain Proposition 3.4.

Proof of Proposition 3.6. Take a point $y \in S^1$ arbitrarily. For $\mathbf{a}, \mathbf{b} \in \mathcal{A}^\nu$, we write $\mathbf{a} \prec \mathbf{b}$ if $\tau^q(\mathbf{b}(y)) = \mathbf{a}(y)$ for some $q \ge 0$. By simple combinatorial argument, we can show that this is a partial order on $\mathcal{A}^\nu$ and that, for each $\mathbf{a} \in \mathcal{A}^\nu$, there exists at most $(\nu + 1)$ elements $\mathbf{b} \in \mathcal{A}^\nu$ such that $\mathbf{b} \prec \mathbf{a}$. (See the proof of Tsujii [**18**, Proposition 16].)

For $0 < \epsilon < 1/2$ and $\mathbf{a} \in \mathcal{A}^\nu$, let $U(\epsilon)$ be the ϵ-neighborhood of y and $U_{\mathbf{a}}(\epsilon)$ the connected component of $\tau^{-\nu}(U(\epsilon))$ that contains $\mathbf{a}(y)$. We consider an integer $\mu > \nu$ that will be specified later. We then choose $\epsilon_0 > 0$ so small that $\tau^i(U_{\mathbf{b}}(\epsilon_0)) \cap U_{\mathbf{a}}(\epsilon_0) \ne \emptyset$ for some $1 \le i \le \mu$ only if $\mathbf{a} \prec \mathbf{b}$. Take functions $\varphi_{\mathbf{a}} \in C^\infty(S^1)$ for $\mathbf{a} \in \mathcal{A}^\nu$ supported on $U_{\mathbf{a}}(\epsilon_0)$ such that

$$\frac{d}{dx}\varphi_{\mathbf{a}}(y) = \ell^\nu \quad \text{on } U_{\mathbf{a}}(\epsilon_0/3) \quad \text{and} \quad \left|\frac{d}{dx}\varphi_{\mathbf{a}}(y)\right| < 2\ell^\nu \quad \text{on } S^1.$$

Finally, let $\varphi_{y,i}$, $1 \le i \le \ell^\nu$, be a rearrangement of $\varphi_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{A}^\nu$ and let $U_y = U(\epsilon_0/3)$.

We show that the conclusion of the proposition holds for the neighborhood U_y and the functions $\varphi_{y,i}$, $1 \le i \le \ell^\nu$, provided that the integer μ is sufficiently large. Consider the family (26) with $\varphi_i = \varphi_{y,i}$ and $m = \ell^\nu$ and suppose that a subset $A = \{\mathbf{a}_i\}_{1\le i\le(\nu+1)(p+1)}$ of $\mathcal{A}^\nu$ is given. From the property of the partial order $\prec$ on $\mathcal{A}^\nu$ mentioned above, we can choose a subset $A' = \{\mathbf{a}'_i\}_{0\le i\le p}$ of A that consists of maximal elements in A with respect to $\prec$. Let $\sigma = \{\mathbf{b}_i\}_{0\le i\le p}$ be a subset of $\mathcal{A}^n$ with $n \ge \nu$ such that $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \le i \le p$. For $\mathbf{b} \in \mathcal{A}^n$ and $x \in U_y$, we set

$$h_1(x, \mathbf{b}; \mathbf{t}) = \sum_{j=1}^{\min\{n,\mu\}} \ell^{-j} \frac{d}{dx} f_{\mathbf{t}}([\mathbf{b}]_j(x))$$

and

$$h_2(x, \mathbf{b}; \mathbf{t}) = \sum_{j=\min\{n,\mu\}+1}^{n} \ell^{-j} \frac{d}{dx} f_{\mathbf{t}}([\mathbf{b}]_j(x)),$$

so that

$$\frac{ds}{dx}(x, \mathbf{b};\ f_{\mathbf{t}}) = h_1(x, \mathbf{b}; \mathbf{t}) + h_2(x, \mathbf{b}; \mathbf{t}).$$

Accordingly, we decompose the affine map $G_{x,\sigma}$ into

$$G^{(1)}_{x,\sigma}(\mathbf{t}) = (h_1(x, \mathbf{b}_i; \mathbf{t}) - h_1(x, \mathbf{b}_0; \mathbf{t}))_{i=1,2,\ldots,p} : \mathbb{R}^{\ell^\nu} \to \mathbb{R}^p$$

and

$$G^{(2)}_{x,\sigma}(\mathbf{t}) = (h_2(x, \mathbf{b}_i; \mathbf{t}) - h_2(x, \mathbf{b}_0; \mathbf{t}))_{i=1,2,\ldots,p} : \mathbb{R}^{\ell^\nu} \to \mathbb{R}^p.$$

Let $\xi : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, \ell^\nu\}$ be the correspondence such that $\mathbf{a}'_i(y) \in \operatorname{supp}(\varphi_{y,\xi(i)})$ for $1 \le i \le p$, and consider the subspace of $\mathbb{R}^{\ell^\nu}$,

$$E = \{\mathbf{t} = (t_j)_{j=1}^{\ell^\nu} \in \mathbb{R}^{\ell^\nu} \mid t_j \neq 0 \text{ only if } j = \xi(i) \text{ for some } 1 \le i \le p\},$$

which is naturally identified with $\mathbb{R}^p$. Take any point $x \in U_y$ and let $L^{(1)}$ and $L^{(2)}$ be the matrices that represent the linear part of the affine mappings $G^{(1)}_{x,\sigma} : E \to \mathbb{R}^p$ and $G^{(2)}_{x,\sigma} : E \to \mathbb{R}^p$, respectively. As a consequence of the choice of $\mathbf{a}'_i$, we can see that $L^{(1)}$ is the identity matrix of size p while all the entries $L^{(2)}$ are bounded by $2\ell^{-\mu+\nu}(1 - \ell^{-1})^{-1}$. Therefore, if we take sufficiently large μ, it holds that

$$\operatorname{Jac}(DG_{x,\sigma}) \ge \operatorname{Jac}(DG_{x,\sigma}|_E) \ge 1/2.$$

Multiplying each $\varphi_{y,i}$ by 2, we can replace $1/2$ by 1 on the right-hand side. □

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A. *Appendix. Proof of Theorem 1.4*

We show that the negation of the conditions (i), (ii) and (iii) are all equivalent. First we introduce two quantities $\mathbf{n}(f, t)$ and $\mathbf{n}(f)$, similar to $\mathbf{m}(f, t)$ and $\mathbf{m}(f)$, respectively, as follows. Put $\hat{\mathbf{C}}_f = \{(x, y) \in \mathbb{R}^2 \mid |y| \le 2\theta_f |x|\} \supset \mathbf{C}_f$. For $t \ge 0$, $z \in X_f$ and a one-dimensional subspace $L \subset \mathbb{R}^2$, we define

$$\mathbf{n}(f, t, z, L) = \sum{}^{*} \frac{1}{E(\zeta, t; f)} \le 1$$

where $\sum^*$ is the sum over $\zeta \in (T_f^t)^{-1}(z)$ such that $(DT_f^t)_\zeta(\hat{\mathbf{C}}_f) \supset L$. Then we set

$$\mathbf{n}(f, t) = \max_{z \in X_f} \max_{L \in \mathbb{R}\mathbf{P}^1} \mathbf{n}(f, t, z, L)$$

and

$$\mathbf{n}(f) = \limsup_{t \to \infty} \mathbf{n}(f, t)^{1/t}.$$

Note that $\mathbf{n}(f, t)$ is *submultiplicative* with respect to t: $\mathbf{n}(f, t + s) \le \mathbf{n}(f, t) \cdot \mathbf{n}(f, s)$. In this point, the quantity $\mathbf{n}(f, t)$ is better than $\mathbf{m}(f, t)$. In particular, the limit in the definition of $\mathbf{n}(f)$ is actually exact.

We show that $\mathbf{m}(f) = 1$ implies $\mathbf{n}(f) = 1$. For this purpose, it is sufficient to prove the claim that

$$\mathbf{m}(f, s) \le \mathbf{n}(f, t) \quad \text{for any } t \ge 0 \text{ and } s = (b/a)t + b > t$$

where

$$a = \min_{x \in S^1} f(x) \quad \text{and} \quad b = \max_{x \in S^1} f(x).$$

Consider a point $z \in X_f$ and take $w \in T_f^{-s}(z)$. If

$$(DT_f^s)_\zeta(\mathbf{C}_f) \cap (DT_f^s)_w(\mathbf{C}_f) \neq \{0\} \tag{A1}$$

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for a point $\zeta \in T_f^{-s}(z)$, then we have

$$(DT_f^t)_{\zeta'}(\hat{\mathbf{C}}_f) \supset L := (DT_f^s)_w(\mathbb{R} \times \{0\}) \quad \text{for } \zeta' = T_f^{s-t}(\zeta) \in T_f^{-t}(z). \tag{A2}$$

Indeed, this follows from the fact that the differences between the slope of L and those of boundary lines of $(DT_f^s)_w(\mathbf{C}_f)$ are not greater than $\ell^{-[s/b]}\theta_f$, while the differences between the slopes of the boundary lines of $(DT_f^t)_{\zeta'}(\hat{\mathbf{C}}_f)$ and those of the boundary lines of $(DT_f^t)_{\zeta'}(\mathbf{C}_f)$ are greater than $\ell^{-[t/a]-1}\theta_f = \ell^{-[s/b]}\theta_f$. Hence, in view of equation (2), we have

$$\sum_{\zeta:\zeta \pitchfork w} \frac{1}{E(\zeta, s;\, f)} \le \sum{}^{*} \frac{1}{E(\zeta, t;\, f)}$$

where $\sum_{\zeta:\zeta\pitchfork w}$ denotes the sum over $\zeta \in T_f^{-s}(z)$ satisfying equation (A1) and $\sum^*$ denotes the same sum as that in the definition of $\mathbf{n}(f, t, z, L)$. Clearly, this implies the claim above.

We next show that f is cohomologous to a constant function if $\mathbf{n}(f) = 1$. Suppose $\mathbf{n}(f) = 1$. By the submultiplicative property of $\mathbf{n}(f, t)$, we have $\mathbf{n}(f, t) = 1$ for all $t \ge 0$. We can therefore take sequences of real numbers $t_n \ge 0$, points $z_n \in X_f$ and one-dimensional subspaces $L_n \in \mathbb{R}\mathbf{P}^1$ for $n \ge 1$ such that $\mathbf{n}(f, t_n, z_n, L_n) = 1$ for all $n \ge 1$ and, as $n \to \infty$:

- $t_n \to \infty$;
- z_n converges to some $z_\infty \in X_f$; and
- L_n converges to some L_∞ (in $\mathbb{R}\mathbf{P}^1$).

The condition $\mathbf{n}(f, t_n, z_n, L_n) = 1$ and equation (2) imply that the cone $(DT_f^{-t_n})_w(\hat{\mathbf{C}}_f)$ contains L_n for all the points $w \in T_f^{-t_n}(z_n)$. Note the unstable subspace (or the tangent space of the unstable manifold) for a backward orbit $(w(t))_{t\le 0}$ contained in $(DT_f^{-t})_{w(t)}(\mathbf{C}_f)$ for any $t \le 0$. Thus, by continuity, we see that the unstable subspaces for *all* backward orbits of z_∞ coincide with each other (and with L_∞). Moreover, such a property holds not only for the point z_∞ but for all the points in X_f because the set of points with such a property is closed and completely invariant with respect to the flow $\mathbf{T}_f$.

For $x \in S^1$, let $\psi(x)$ be the slope of the (unique) unstable subspace at $(x, 0) \in X_f$. Invariance of the unstable subspaces implies that we have

$$\psi(\tau(x)) = (f'(x) + \psi(x))/\ell \quad \text{for all } x \in S^1. \tag{A3}$$

Inductive use of this equality yields

$$\psi(x) = \sum_{n\ge 1} \sum_{\tau^n(y)=x} \ell^{-2n} f'(y) \quad \text{for all } x \in S^1, \tag{A4}$$

where the right-hand side converges in C^{r-1} sense. Since we have $\int_{S^1} \psi(x) = 0$ from equation (A4), the function

$$\varphi(x) = \int_0^x \psi(y)\, dy$$

is well defined and C^r on S^1. It follows from equation (A3) that

$$\varphi(\tau(x)) = \varphi(x) + f(x) - c \quad \text{for some constant } c. \tag{A5}$$

That is, f is cohomologous to a constant function.

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It is easy to show that $\mathbf{T}_f$ is not weakly mixing if f is cohomologous to a constant function. Suppose that $f \in C^r_+(S^1)$ is cohomologous to a constant function, that is, f satisfies equation (A5) for some $\varphi \in C^r(S^1)$ and $c \in \mathbb{R}$. By integrating both sides over S^1, we see that $c = \int_{S^1} f(x)\, dx > 0$. Define

$$\Phi(x, s) = \exp((2\pi i/c)(\varphi(x) + s)) \quad \text{for } (x, s) \in X_f.$$

Then $\Phi \circ T_f^t = e^{(2\pi i/c)t}\Phi$ for $t \geq 0$. Therefore, T_f^t is not weakly mixing.

To finish the proof, we show that $\mathbf{m}(f) = 1$ if the semi-flow $\mathbf{T}^t_f$ is not weakly mixing. Suppose that $\mathbf{T}^t_f$ is not weakly mixing. Then we can find a real number $a \neq 0$ and an L^2 function Φ on X_f such that $\Phi \circ T_f^t = e^{iat}\Phi$ for $t \geq 0$. Equivalently, there exists an L^2 function φ on S^1 such that $\Phi(x, s) = e^{-ias}\varphi(x)$ and $\varphi(\tau(x)) = e^{iaf(x)}\varphi(x)$ for $x \in S^1$ and $t \geq 0$. Actually, the last equality implies that φ is a C^r function and so is Φ. (For the proof of this fact, we refer to Parry and Pollicott [**14**, Proposition 4.2], for instance. Replace the symbolic dynamical system $\sigma : X^+ \to X^+$ and the space of Hölder functions on X^+ in the proof by $\tau : S^1 \to S^1$ and $C^r(S^1)$, respectively.) Let $L(z)$ be the null line of the differential $D_z\Phi$. Then this line field is invariant with respect to the semi-flow $\mathbf{T}_f$ and not tangent to the flow direction. Hence $L(z)$ is contained in the cone $(DT_f^{-t})_{w(t)}(\mathbf{C}_f)$ for any backward orbit $\{w(t)\}_{t\leq 0}$ of z and any $t \leq 0$. This and equation (2) imply that $\mathbf{m}(f, t) = 1$ for any $t \geq 0$ and hence that $\mathbf{m}(f) = 1$.

B. *Appendix. Proof of Lemma 2.4*

Let $\Gamma = \mathbb{Z}_+ \times \{+, -\}$, $c(+) = 1$ and $c(-) = 0$. Below, we write C_0 for constants that do not depend on S, h, Θ or Θ' and C for constants that may depend on them. Take an integer $\mu = \mu(S)$ such that

$$2^{-\mu+6}\|\xi\| \leq \|(DS_x)^{\mathrm{tr}}(\xi)\| \leq 2^{\mu-6}\|\xi\| \quad \text{for any } x \in K \text{ and any } \xi \in \mathbb{R}^2.$$

Let $\nu \leq \mu - 6$ be an integer such that

$$2^{\nu-1} < \Lambda(S, \Theta', K) \leq 2^{\nu}.$$

So we have

$$\|DS_x^{\mathrm{tr}}(\xi)\| \leq 2^{\nu}\|\xi\| \quad \text{if } x \in K \text{ and } (DS_x)^{\mathrm{tr}}(\xi) \notin \mathbf{C}'_-.$$

We write $(m, \tau) \hookrightarrow (n, \sigma)$ if either:

- $(\tau, \sigma) = (+, +)$ and $m - \mu \leq n \leq \max\{0, m + \nu + 6\}$; or
- $(\tau, \sigma) \in \{(-, -), (+, -)\}$ and $m - \mu \leq n \leq m + \mu$.

We write $(m, \tau) \not\hookrightarrow (n, \sigma)$ otherwise.

Consider a function $u \in C^r(R)$ and set $v := Lu$. For $(n, \sigma), (m, \tau) \in \Gamma$, we define

$$v_{n,\sigma}^{m,\tau} = \psi_{\Theta',n,\sigma}(D)L(u_{\Theta,m,\tau}),$$

so that $v_{\Theta',n,\sigma} = \sum_{(m,\tau)\in\Gamma} v_{n,\sigma}^{m,\tau}$. By Parseval's identity, we obtain

$$\sum_{(n,\sigma)\in\Gamma} \|v_{n,\sigma}^{m,\tau}\|_{L^2}^2 \leq \|L(u_{\Theta,m,\tau})\|_{L^2}^2 \leq C_0\gamma(S)^{-1}\|h\|_{L^\infty}^2\|u_{\Theta,m,\tau}\|_{L^2}^2. \tag{B1}$$

We also have the following estimate, but will postpone the proof for a while.

LEMMA B.1. *If* $(m,\tau) \not\hookrightarrow (n,\sigma)$, *we have*

$$\|v_{n,\sigma}^{m,\tau}\|_{L^2} \le C2^{-(r-1)\max\{m,n\}}\|u_{\Theta,m,\tau}\|_{L^2}. \tag{B2}$$

Remark B.2. If S is an affine map in the lemma above, the Fourier transform of $L(u_{\Theta,m,\tau})$ is supported on $DS^{\mathrm{tr}}(\mathrm{supp}(\psi_{\Theta,m,\tau}))$, which does not meet $\mathrm{supp}(\psi_{\Theta',n,\sigma})$ by the assumption $(m,\tau) \not\hookrightarrow (n,\sigma)$, and hence the assertion holds trivially with $v_{n,\sigma}^{m,\tau} = 0$. To prove the lemma above, we will estimate some oscillatory integrals using smoothness of S.

We first show the assertion that $|v|_{\Theta'} \le C|u|_\Theta$ for some constant C, which implies that L extends boundedly to $L : W_\dagger(R;\Theta) \to W_\dagger(R;\Theta')$. By definition, we have

$$|v|_{\Theta'}^2 \le 2\sum_{(n,\sigma)\in\Gamma} 2^{2(c(\sigma)-\epsilon)n}\left(\left\|\sum_{(m,\tau)\hookrightarrow(n,\sigma)} v_{n,\sigma}^{m,\tau}\right\|_{L^2}^2 + \left\|\sum_{(m,\tau)\not\hookrightarrow(n,\sigma)} v_{n,\sigma}^{m,\tau}\right\|_{L^2}^2\right)$$

where $\sum_{(m,\tau)\hookrightarrow(n,\sigma)}$ (respectively, $\sum_{(m,\tau)\not\hookrightarrow(n,\sigma)}$) denotes the sum over $(m,\tau)\in\Gamma$ such that $(m,\tau)\hookrightarrow(n,\sigma)$ (respectively, $(m,\tau)\not\hookrightarrow(n,\sigma)$). Since the relation $(m,\tau)\hookrightarrow(n,\sigma)$ holds only if $c(\sigma)\le c(\tau)$ and $|m-n|<\mu$, it holds that

$$\begin{aligned}\sum_{(n,\sigma)\in\Gamma}\left\|\sum_{(m,\tau)\hookrightarrow(n,\sigma)} 2^{(c(\sigma)-\epsilon)n} v_{n,\sigma}^{m,\tau}\right\|_{L^2}^2 &\le C\sum_{(n,\sigma)\in\Gamma}\sum_{(m,\tau)\in\Gamma} 2^{2(c(\tau)-\epsilon)m}\|v_{n,\sigma}^{m,\tau}\|_{L^2}^2\\ &\le C\sum_{(m,\tau)\in\Gamma} 2^{2(c(\tau)-\epsilon)m}\|u_{\Theta,m,\tau}\|_{L^2}^2 \le C|u|_\Theta^2\end{aligned}$$

where the second inequality follows from equation (B1). It also follows from Lemma B.1 and the Schwarz inequality that

$$\begin{aligned}&\sum_{(n,\sigma)\in\Gamma}\left\|\sum_{(m,\tau)\not\hookrightarrow(n,\sigma)} 2^{(c(\sigma)-\epsilon)n} v_{n,\sigma}^{m,\tau}\right\|_{L^2}^2\\ &\le \sum_{(n,\sigma)\in\Gamma}\left\|\sum_{(m,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)m-(r-1-\epsilon)\max\{n,m\}}\cdot 2^{(r-1)\max\{n,m\}+(c(\tau)-\epsilon)m} v_{n,\sigma}^{m,\tau}\right\|_{L^2}^2\\ &\le \sum_{(n,\sigma)\in\Gamma}\left(\sum_{(m,\tau)\in\Gamma} 2^{2c(\sigma)n-2c(\tau)m-2(r-1-\epsilon)\max\{n,m\}}\right)\left(\sum_{(m,\tau)\in\Gamma} 2^{2(c(\tau)-\epsilon)m}\|u_{\Theta,m,\tau}\|_{L^2}^2\right)\\ &\le C|u|_\Theta^2.\end{aligned} \tag{B3}$$

Thus we obtain $|v|_{\Theta'} \le C|u|_\Theta$ for $u\in C^r(R)$ and hence for $u\in W_\dagger(R;\Theta)$.

We next prove equations (10) and (11). The inequality (10) is easy to see:

$$(\|v\|_{\Theta'}^-)^2 \le \|v\|_{L^2}^2 \le \gamma(S)^{-1}\|h\|_{L^\infty}^2\|u\|_{L^2}^2 \le \gamma(S)^{-1}\|h\|_{L^\infty}^2\|u\|_\Theta^2.$$

To prove equation (11), we begin by writing the left-hand side as

$$(\|v\|_{\Theta'}^+)^2 = \|\psi_{\Theta',0,+}(D)v\|_{L^2}^2 + \sum_{n\ge1} 2^{2n}\|\psi_{\Theta',n,+}(D)v\|_{L^2}^2.$$

The first term on the right-hand side is bounded by $|v|_{\Theta'}^2$ and hence by $C|u|_\Theta^2$. The sum on the right-hand side is bounded by

$$2\cdot\left(\sum_{n\ge1}\left\|\sum_{(m,\tau)\hookrightarrow(n,+)} 2^n v_{n,+}^{m,\tau}\right\|_{L^2}^2 + \sum_{n\ge1}\left\|\sum_{(m,\tau)\not\hookrightarrow(n,+)} 2^n v_{n,+}^{m,\tau}\right\|_{L^2}^2\right).$$

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By the Schwarz inequality, we have

$$\left\| \sum_{(m,+)\hookrightarrow(n,+)} 2^n v_{n,+}^{m,+} \right\|_{L^2}^2 \le \left(\sum_{(m,+)\hookrightarrow(n,+)} 2^{2(n-m)} \right) \left(\sum_{(m,+)\hookrightarrow(n,+)} 2^{2m} \| v_{n,+}^{m,+} \|_{L^2}^2 \right)$$

where $\sum_{(m,+)\hookrightarrow(n,+)}$ denotes the sum over $m \ge 0$ such that $(m,+) \hookrightarrow (n,+)$. Note that we have $(m,\tau) \hookrightarrow (n,+)$ for $n \ge 1$ only if $\tau = +$ and $n \le m + \nu + 6$. We can therefore see, by using equation (B1), that

$$\sum_{n\ge1} \left\| \sum_{(m,\tau)\hookrightarrow(n,+)} 2^n v_{n,+}^{m,\tau} \right\|_{L^2}^2 \le C_0 \cdot 2^{2\nu} \gamma(S)^{-1} \|h\|_{L^\infty}^2 \|u\|_{\Theta}^2.$$

On the other hand, by using Lemma B.1 and the Schwarz inequality, we can show that

$$\sum_{n\ge0} \left\| \sum_{(m,\tau)\not\hookrightarrow(n,+)} 2^n v_{n,+}^{m,\tau} \right\|_{L^2}^2 < C|u|_{\Theta}^2$$

in a similar manner as equation (B3). We therefore obtain equation (11). Obviously, equations (10) and (11) imply equation (12) and hence L extends boundedly to $L : W_*(R;\Theta) \to W_*(R;\Theta')$. Finally, we complete the proof by proving Lemma B.1.

Proof of Lemma B.1. Since K is compact, we can take closed cones $\widetilde{\mathbf{C}}_+ \Subset \mathbf{C}_+$ and $\widetilde{\mathbf{C}}_- \Subset \mathbf{C}_-$ such that

$$(DS_\zeta)^{\mathrm{tr}}(\mathbb{R}^d \setminus \widetilde{\mathbf{C}}_+) \Subset \mathbf{C}'_- \quad \text{for } \zeta \in K.$$

Let $\tilde{\varphi}_+, \tilde{\varphi}_- : S^1 \to [0,1]$ be C^∞ functions satisfying

$$\tilde{\varphi}_+(\xi) = \begin{cases} 1 & \text{if } \xi \notin S^1 \cap \mathbf{C}_-, \\ 0 & \text{if } \xi \in S^1 \cap \widetilde{\mathbf{C}}_-, \end{cases} \qquad \tilde{\varphi}_-(\xi) = \begin{cases} 1 & \text{if } \xi \notin S^1 \cap \mathbf{C}_+, \\ 0 & \text{if } \xi \in S^1 \cap \widetilde{\mathbf{C}}_+. \end{cases}$$

Recall the function χ and define $\tilde{\psi}_n(\xi) = \chi(2^{-n-1}|\xi|) - \chi(2^{-n+2}|\xi|)$ for $n \ge 1$ and

$$\tilde{\psi}_{\Theta,n,\sigma}(\xi) = \begin{cases} \tilde{\psi}_n(\xi)\tilde{\varphi}_\sigma(\xi/|\xi|) & \text{if } n \ge 1, \\ \chi(2^{-1}|\xi|) & \text{if } n = 0 \end{cases}$$

for $(n,\sigma) \in \Gamma$. Then we have $\tilde{\psi}_{\Theta,n,\sigma}(\xi) = 1$ if $\xi \in \mathrm{supp}(\psi_{\Theta,n,\sigma})$. From the definition of the relation $\hookrightarrow$, there exists a constant $L > 1$, which may depend on S, such that if $(m,\tau) \not\hookrightarrow (n,\sigma)$ and $\max\{m,n\} \ge L$, it holds that

$$d(\mathrm{supp}(\psi_{\Theta',n,\sigma}), (DS_\zeta)^{\mathrm{tr}}(\mathrm{supp}(\tilde{\psi}_{\Theta,m,\tau}))) \ge L^{-1} \cdot 2^{\max\{n,m\}} \quad \text{for } \zeta \in K. \tag{B4}$$

In the case where $\max\{m,n\} < L$, it is easy to see that equation (B2) holds with the constant C depending on L. Thus we assume $\max\{m,n\} \ge L$ in the following.

We consider the operator $S_{n,\sigma}^{m,\tau}$ defined by

$$S_{n,\sigma}^{m,\tau} = \psi_{\Theta',n,\sigma}(D) \circ L \circ \tilde{\psi}_{\Theta,m,\tau}(D).$$

Then we have $v_{n,\sigma}^{m,\tau} = S_{n,\sigma}^{m,\tau} u_{\Theta,m,\tau}$ since $\tilde{\psi}_{\Theta,m,\tau}(D)(u_{\Theta,m,\tau}) = u_{\Theta,m,\tau}$. We may rewrite this operator $S_{n,\sigma}^{m,\tau}$ as

$$(S_{n,\sigma}^{m,\tau} u)(x) = (2\pi)^{-4} \int V_{n,\sigma}^{m,\tau}(x,y) \cdot u \circ S(y) \cdot |\det DS(y)|\, dy,$$

where

$$V_{n,\sigma}^{m,\tau}(x,y) = \int e^{i(x-w)\xi + i(S(w)-S(y))\eta} h(w)\psi_{\Theta',n,\sigma}(\xi)\tilde{\psi}_{\Theta,m,\tau}(\eta)\,dw\,d\xi\,d\eta. \tag{B5}$$

Since we have $\|u \circ S(y) \cdot |\det DS(y)|\|_{L^2} \le C\|u\|_{L^2}$, the inequality (B2) follows if the operator norm of the integral operator

$$H_{n,\sigma}^{m,\tau} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad H_{n,\sigma}^{m,\tau} v(x) = \int V_{n,\sigma}^{m,\tau}(x,y)v(y)\,dy$$

is bounded by $C \cdot 2^{-(r-1)\max\{n,m\}}$.

Apply the following formula of integration by parts for $(r-1)$ times in equation (B5),

$$\int e^{if(w)} g(w)\,dw = i \cdot \int e^{if(w)} \cdot \sum_{k=1}^{2} \partial_{w_k}\left(\frac{\partial_{w_k} f(w) \cdot g(w)}{\sum_{j=1}^{2}(\partial_{w_j} f(w))^2}\right) dw$$

where $w = (w_k)_{k=1}^2 \in \mathbb{R}^2$. We then obtain the expression

$$V_{n,\sigma}^{m,\tau}(x,y) = \int e^{i(x-w)\xi + i(S(w)-S(y))\eta} F(\xi,\eta,w)\psi_{\Theta',n,\sigma}(\xi)\tilde{\psi}_{\Theta,m,\tau}(\eta)\,dw\,d\xi\,d\eta$$

where $F(\xi,\eta,w)$ is continuous in w and C^∞ in ξ and η. Note that $F(\xi,\eta,w) = 0$ if $w \notin K$. From equation (B4), there is a constant $C_{\alpha\beta}$ for multi-indices α and β, such that

$$|\partial_\xi^\alpha \partial_\eta^\beta F(\xi,\eta,w)| \le C_{\alpha\beta} \cdot 2^{-n|\alpha| - m|\beta| - (r-1)\max\{n,m\}} \tag{B6}$$

for $w \in \mathbb{R}^2$, $\xi \in \operatorname{supp}(\psi_{\Theta',n,\sigma})$ and $\eta \in \operatorname{supp}(\tilde{\psi}_{\Theta,m,\tau})$. For $n \ge 0$ and $m \ge 0$, we set

$$G_{nm}(\xi,\eta,w) = F(2^n\xi, 2^m\eta, w)\psi_{\Theta',n,\sigma}(2^n\xi)\tilde{\psi}_{\Theta,m,\tau}(2^m\eta).$$

By changes of variable, we can rewrite $V_{n,\sigma}^{m,\tau}(x,y)$ as

$$\int 2^{2n+2m}(\mathcal{F}_{\xi\eta}^{-1} G_{nm})(2^n(x-w), 2^m(S(w)-S(y)), w)\,dw \tag{B7}$$

where $\mathcal{F}_{\xi\eta}^{-1}$ is the inverse Fourier transform with respect to the variables ξ and η. From equation (B6), there exists a constant $C_{\alpha\beta}$ for any multi-indices α and β such that

$$|\partial_\xi^\alpha \partial_\eta^\beta G_{nm}|_{L^\infty} \le C_{\alpha\beta} 2^{-(r-1)\max\{n,m\}}.$$

This implies that there exists a constant C such that

$$|\mathcal{F}_{\xi\eta}^{-1} G_{nm}(x,y,w)| \le C 2^{-(r-1)\max\{n,m\}}(1+|x|^2)^{-2}(1+|y|^2)^{-2}.$$

Applying this inequality in expression (B7) for $V_{n,\sigma}^{m,\tau}(x,y)$, we obtain the required estimate for $H_{n,\sigma}^{m,\tau}$ from Young's inequality. □

C. *Appendix. Proof of Lemma 2.3*

We prove inequality (8). For inequality (9), we refer to Baladi and Tsujii [**4**, Lemma 7.1]. Recall the argument in the proof of Lemma 2.4 in Appendix B, setting $S = \text{id}$. Notice

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that the assumptions of Lemma 2.4 then hold since we assume $\Theta' < \Theta$. Set $v_i = g_i \cdot u$ for $1 \le i \le I$. Then we have

$$\sum_{i=1}^{I} (\|v_i\|_{\Theta'}^{-})^2 \le \sum_{i=1}^{I} \|v_i\|_{L^2}^2 \le \|u\|_{L^2}^2 \le C\|u\|_{\Theta}^2. \tag{C1}$$

We have proved in the proof of Lemma 2.4 that

$$\sum_{n\ge 0} 2^{2n} \Bigg\| \sum_{(m,\tau)\not\hookrightarrow(n,+)} \psi_{\Theta',n,+}(D)(g_i u_{\Theta,m,\tau}) \Bigg\|_{L^2}^2 \le C|u|_{\Theta}^2.$$

Since we can and do put $\mu = 6$ in the setting of $S = \mathrm{id}$, the relation $(m, \tau) \hookrightarrow (n, +)$ holds only if $|m - n| \le 6$. Hence we have, by the Schwarz inequality,

$$\begin{aligned}
&\sum_{i=1}^{I} \sum_{n\ge 0} 2^{2n} \Bigg\| \sum_{(m,\tau)\hookrightarrow(n,+)} \psi_{\Theta',n,+}(D)(g_i u_{\Theta,m,\tau}) \Bigg\|_{L^2}^2 \\
&\quad\le 13 \cdot \sum_{i=1}^{I} \sum_{n\ge 0} \sum_{m:|m-n|\le 6} 2^{2n} \|\psi_{\Theta',n,+}(D)(g_i u_{\Theta,m,+})\|_{L^2}^2 \\
&\quad\le 13 \cdot 2^{12} \cdot \sum_{m\ge 0} \sum_{i} 2^{2m} \|g_i u_{\Theta,m,+}\|_{L^2}^2 \le C_0 \|u\|_{\Theta}^2.
\end{aligned}$$

We therefore obtain $\sum_{i=1}^{I} (\|v_i\|_{\Theta'}^{+})^2 \le C_0\|u\|_{\Theta}^2 + C|u|_{\Theta}^2$, which together with equation (C1) yields equation (8).

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