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Equivariant cohomology of torus orbifolds

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Abstract. We calculate the integral equivariant cohomology, in terms of generators and relations, of locally standard torus orbifolds whose odd degree ordinary cohomology vanishes. We begin by studying GKM-orbifolds, which are more general, before specializing to half-dimensional torus actions.

1 Introduction

The interest in the integral cohomology of orbifolds stems from the subtleties that appear when in comparison with their manifold counterparts. The main, and first, example of this is *weighted projective space*. It can easily be seen that its rational cohomology is the same as that of ordinary projective space. Kawasaki [Kaw73] proved, surprisingly, that integrally their cohomologies are also additively isomorphic but that they have different product structures. Weighted projective spaces, when thought of as toric varieties, admit a natural half-dimensional compact torus action whose equivariant cohomology was studied in [BFR09]. A similar phenomenon is observed whereby the rational equivariant cohomology is the same as for ordinary projective space, but the integral equivariant cohomology distinguishes them. In this paper, we consider a much wider class of orbifolds with torus action where this phenomenon persists, namely GKM-orbifolds and torus orbifolds, and calculate their integral equivariant cohomology.

Goresky et al. [GKM98] showed that a lot could be said about the equivariant cohomology of a wide range of spaces with compact torus action by considering a combinatorial approximation of the space. More specifically, if a space X with compact torus T^k-action is *equivariantly formal* (which is implied if $H^{odd}(X) = 0$), then each closed one-dimensional orbit is a copy of a two-sphere that is rotated according to some element of $\mathrm{Hom}(T^k, S^1)$. They proved that the equivariant cohomology $H^*_{T^k}(X;\mathbb{R})$ (note the real coefficients) is encoded in the one-skeleton, the union of the zero- and one-dimensional orbits. It is isomorphic to the algebra of *piecewise polynomials* on the one-skeleton, i.e., the algebra given by attaching to each fixed point

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of X elements of the polynomial algebra $H^*(BT^k;\mathbb{R})$, so that if two fixed points belong to a two-sphere, their polynomials agree modulo the element of $\mathrm{Hom}(T^k, S^1)$ attached to that sphere.

The one-skeleton can be modeled on a *labeled graph*, and Guillemin and Zara [GZ01] abstract this idea, studying these graphs as objects of interest in their own right, and coin the terms GKM-manifold, GKM-orbifold, and GKM-graph. The algebra of piecewise polynomials on the one-skeleton described beforehand is extended to the abstract labeled graphs and is often called the *equivariant cohomology of the graph*. A similar technique can be used for the equivariant cohomology of GKM-manifolds with integer coefficients where the polynomials used now lie in $H^*(BT^k;\mathbb{Z})$.

Torus manifolds, first studied in [HM03] and consequently in [MP06, MMP07], can be considered as a special case of GKM-manifolds where the the torus acts effectively with exactly half the dimension of the manifold. This has the added bonus that we can explicitly give generators and relations for the integral equivariant cohomology. Examples of torus manifolds include toric manifolds (defined as compact nonsingular toric varieties), quasitoric manifolds, and even-dimensional spheres.

The integral equivariant cohomology of smooth toric varieties is known to be given by the *face ring* of the corresponding fan and, for quasitoric manifolds, the face ring of the quotient simple polytope. For the wider class of torus manifolds, it was proved by Masuda and Panov [MP06] that the integral equivariant cohomology is isomorphic to the face ring of an appropriate simplicial poset. In [MMP07], they use *Thom classes* of the associated *torus graph* to explicitly give the generators and relations for the integral equivariant cohomology of the graph. This is isomorphic to the integral equivariant cohomology of the torus manifold if its ordinary cohomology vanishes in all odd degrees.

When we move from manifolds to orbifolds, the picture slightly changes. In the case of singular toric varieties, including toric varieties having orbifold singularities, it was proved in [BFR09, Fra10] that its equivariant cohomology with integer coefficients is given by the ring $PP[\Sigma]$ of piecewise polynomials on its fan Σ if its ordinary odd degree cohomology vanishes. Note in particular that the ordinary cohomology of a toric manifold is concentrated in even degrees; hence, its equivariant cohomology is isomorphic to $PP[\Sigma]$, which is isomorphic to the face ring of the fan in this case. However, this is not true for orbifolds in general. In [BSS17, BNSS19, KMZ17], several verifiable conditions for toric orbifolds to have vanishing odd degree cohomology are studied, and the authors of [BSS17] show that, under the condition of vanishing odd degree cohomology, the equivariant cohomology of a projective toric orbifold can be realized as a subring of the usual face ring of the fan that satisfies an *integrality condition*.

GKM orbifolds, as first defined in [GZ01], are orbifold versions of GKM-manifolds. The closed one-dimensional orbits are now *spindles* (see Example 2.2). Examples of GKM-orbifolds include weighted Grassmanians [AM15, CR02] and weighted flag varieties [QS11]. In Section 2, we define an abstract orbifold GKM-graph, similar to [GZ01], which is in some sense a rational version of the manifold case. After showing how a GKM-orbifold produces such a graph, we then define the integral cohomology of an orbifold GKM-graph $H_T^*(\Gamma,\alpha)$ and prove that the integral equivariant cohomol-

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ogy of the GKM-orbifold is isomorphic to $H_T^*(\Gamma, \alpha)$ when its odd degree cohomology vanishes (among other conditions).

In Section 3, as in the manifold case, we concentrate on half-dimensional torus actions. We work entirely combinatorially in this section by initially considering the orbifold analogue of a torus graph. We then define the *weighted face ring* of an orbifold torus graph, which is a polynomial ring given in terms of generators and relations using integral linear combinations of *rational Thom classes*, and show that this weighted face ring is isomorphic to the equivariant cohomology of the orbifold torus graph.

We move back toward geometry in Section 4, where we consider *locally standard* GKM-orbifolds. This leads to objects known as *torus orbifolds* (see [GGKRW18, KMZ17]), orbifolds with a half-dimensional torus action, whose quotient space Q is a *manifold with faces*. Each such orbifold can be characterized by labeling the codimension-one faces of Q which is *dual* to the the corresponding torus graph. From this characteriation, there is a canonical method to build a torus orbifold using the *quotient construction*, which we show reproduces the original torus orbifold. Using results from the previous sections, we give the integral equivariant cohomology of this torus orbifold when its odd degree cohomology vanishes.

In Section 5, we give an explicit formula for the integral equivariant cohomology of all four-dimensional torus orbifolds for which there is a complete obstruction to the vanishing of the odd degree cohomology.

Throughout this paper, cohomology is taken with integral coefficients, unless stated otherwise. As is natural, we make the usual identification $\mathrm{Hom}(S^1, T^k) = H_2(BT^k) \cong \mathbb{Z}^k$ and write its standard integral basis as $\{\varepsilon_1, \dots, \varepsilon_k\}$, where ε_i is the inclusion into the ith copy of S^1 in $T^k := S^1 \times \cdots \times S^1$. The dual of this notion gives us the identification $\mathrm{Hom}(T^k, S^1) = H^2(BT^k) \cong \mathbb{Z}^k$ with its dual integral basis $\{\varepsilon_1^*, \dots, \varepsilon_k^*\}$, where the elements now relate to projections.

One other way to consider the identification $\mathrm{Hom}(T^k, S^1) = H^2(BT^k)$ is to consider elements of $\mathrm{Hom}(T^k, S^1)$ as complex one-dimensional T^k-representations. These can be thought of as T^k-equivariant complex line bundles over a point, and taking the equivariant first Chern classes of these bundles produces the identification. As T^k is abelian, the one-dimensional complex representations form all of the irreducible representations.

2 GKM-orbifolds

We begin with a brief introduction of torus actions on orbifolds. We refer to [LT97, Section 2] and [GGKRW18, Section 2] for more details regarding Lie group actions on orbifolds.

Let $\mathcal{X} = (X, \mathcal{U})$ be a $2n$-dimensional orbifold, where $\mathcal{U}$ denotes an orbifold atlas on the underlying topological space X. To be more precise, $\mathcal{U}$ consists of the maximal collection of orbifold charts

$$\left\{(\tilde{U}, G, \phi\colon \tilde{U} \longrightarrow U \subseteq X)\right\}$$

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which covers X, where $\tilde{U}$ is an open subset of $\mathbb{R}^{2n}$, G is a finite subgroup of $O(2n)$, and ϕ is a G-equivariant map which induces a homeomorphism from $\tilde{U}/G$ to an open subset U of X.

Suppose that there is an action of a k-dimensional compact torus T^k on X, for some $k \leq n$, with nonempty fixed point set. Then, we may consider a T^k invariant neighborhood U_p of a fixed point p in X. Each element $t \in T^k$ induces a smooth map $L_t \colon U_p \to U_p$, which yields a smooth lift $\tilde{L}_t \colon \tilde{U}_p \to \tilde{U}_p$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{U}_p & \xrightarrow{\tilde{L}_t} & \tilde{U}_p \\ \downarrow{\scriptstyle /G_p} & & \downarrow{\scriptstyle /G_p} \\ U_p & \xrightarrow{L_t} & U_p. \end{array}$$

Notice that $(\tilde{U}_p, G_p, \tilde{U}_p \xrightarrow{/G_p} U_p)$ forms an orbifold chart, and G_p is called the *local group* around p.

Now, we consider the tangent space $T_p\tilde{U} (:= T_p\tilde{U}_p)$ of $\tilde{U}_p$ at p and the tangential representation with respect to the action of T^k on $\tilde{U}_p$. Since T^k is abelian, the tangent space $T_p\tilde{U}$ is decomposed into complex one-dimensional representations

$$T_p\tilde{U} \cong \bigoplus_{i=1}^{n} V(\alpha_{i,p}), \tag{2.1}$$

where $V(\alpha_{i,p})$ denotes a complex one-dimensional T^k-representation with weight $\alpha_{i,p} \in \mathrm{Hom}(T^k, S^1)$.

Definition 2.1 A closed $2n$-dimensional-oriented orbifold $\mathcal{X} = (X, \mathcal{U})$ with an action of a k-dimensional torus T^k, with $k \leq n$, is called a *GKM-orbifold* if

(1) there are finitely many T^k-fixed points and connected components of one-dimensional orbits and
(2) the weights $\{\alpha_{1,p}, \ldots, \alpha_{n,p}\} \subset \mathrm{Hom}(T^k, S^1)$ of the tangential representation at each fixed point $p \in X$, as in (2.1), are pairwise linearly independent.

Remark 2.1 Although Guillemin and Zara [GZ01] require a T^k-invariant almost complex structure for their GKM-manifolds and GKM-orbifolds, we shall not assume the existence of an invariant almost complex structure. Also see Remark 2.4.

Example 2.2 (Spindle) Consider the quotient space

$$S^2(m,n) := S^3/S^1\langle m,n\rangle,$$

where we identity two points (z_1, z_2) and (z_1', z_2') in $S^3 \subset \mathbb{C}^2$ if

$$(z_1', z_2') = (t^m z_1, t^n z_2) \tag{2.2}$$

for some positive integers m, n, and $t \in S^1$. We denote by $[z_1 : z_2]$ the equivalence class coming from (2.2). Although $S^2(m,n)$ is equipped with an orbifold structure induced from the equivalence relation, its underlying topological space is homeomorphic to S^2.

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Hence, we may think of the S^1-action on $S^2(m,n)$ being induced from the standard $T^2/S^1\langle m,n\rangle \cong S^1$-action on S^2, whose fixed point set consists of two isolated points that are connected by a connected component of one-dimensional orbits. We refer to [GZ01, Example 1.2.1] and [DKS19].

The two assumptions in Definition 2.1 lead us to understand the set of one-dimensional orbits as a union of two-spheres which are connected only by fixed points. This brings about a graph by associating the set of fixed points and the set of one-dimensional orbits with the set of vertices and the set of edges, respectively.

Furthermore, the tangential representation (2.1) together with the order of the local group around a fixed point produces a certain labeling on each edge. To be more precise, the orientability of $\mathcal{X}$ allows the local group G_p to be taken as a finite subgroup of $SO(2n)$, and by [GGKRW18, Proposition 2.8], there exists a Lie group $\tilde{T}$ such that $\tilde{T}$ acts on $\tilde{U}_p$ and $\tilde{T}$ is an extension of T^k by G_p. We can also say that G_p commutes with every connected subgroup of $\tilde{T}$ using [GGKRW18, Corollary 2.9]. Since the identity component $\tilde{T}^o$ (which as a connected, compact, abelian Lie group is therefore a torus) of $\tilde{T}$ has a pairwise linear independent representation $\mathbb{C}^n \simeq \bigoplus_{i=1}^n V(\varepsilon_i^*)$, any two-dimensional complex representation $\varepsilon_i^* \times \varepsilon_j^* : \tilde{T}^o \to U(2)$ does not commute with the elements of U(2) except in the case of a maximal torus $T^2 \simeq \mathrm{im}(\varepsilon_i^* \times \varepsilon_j^*)$. This implies that the centralizer of $\tilde{T}^o$ in $SO(2n)$ coincides with a maximal torus and that G_p must be a subgroup of this. Therefore, $G_{i,p}$, the projection of G_p onto its ith coordinate, is a subgroup of $SO(2)$. Here, we notice that the order $|G_p|$ of G_p is a multiple of $|G_{i,p}|$, for each $i = 1, \ldots, n$.

To get a labeling on each edge, which encodes the given orbifold structure, we consider the following composition:

$$T^k \xrightarrow{\alpha_{i,p}} S^1 \xrightarrow{\xi_{i,p}} S^1/G_{i,p}, \tag{2.3}$$

where $\alpha_{i,p} \in \mathrm{Hom}(T^k, S^1)$ and $\xi_{i,p}(e^{2\pi i x}) = [e^{2\pi i x}]$. Identifying $\mathrm{Hom}(T^k, S^1)$ with $H^2(BT^k)$, the composition $\xi_{i,p} \circ \alpha_{i,p}$ in (2.3) defines an element in $H^2(BT^k;\mathbb{Q})$.

Motivated by this geometric interpretation, we give the following definition of an *abstract orbifold GKM-graph*.

Definition 2.2 For two positive integers k and n, with $k \le n$, an *orbifold GKM-graph* is a triple (Γ, α, θ) defined as follows:

(1) Γ is an n-valent graph with the set $\mathcal{V}(\Gamma)$ of vertices and the set $\mathcal{E}(\Gamma)$ of oriented edges.
(2) $\alpha\colon \mathcal{E}(\Gamma) \to H^2(BT^k;\mathbb{Q})$ is a map such that
   (a) the set of vectors $\alpha(\mathcal{E}_p(\Gamma))$ are pairwise linearly independent for every $p \in \mathcal{V}(\Gamma)$, where $\mathcal{E}_p(\Gamma)$ is the set of outgoing edges from p and
   (b) for an oriented edge $e \in \mathcal{E}(\Gamma)$, there are positive integers r_e and $r_{\bar{e}}$ such that $r_e\alpha(e) = \pm r_{\bar{e}}\alpha(\bar{e}) \in H^2(BT^k)$, where $\bar{e}$ denotes the edge e with the reversed orientation.

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(3) θ is a collection of bijections

$$\theta_e : \mathcal{E}_{i(e)}(\Gamma) \longrightarrow \mathcal{E}_{t(e)}(\Gamma),$$

such that $c_{e,e'}(\alpha(\theta_e(e')) - \alpha(e')) \equiv 0 \bmod r_e\alpha(e) (= \pm r_{\bar{e}}\alpha(\bar{e})) \in H^2(BT^k)$, for some $c_{e,e'} \in \mathbb{Z}\backslash\{0\}$, where $i(e)$ is the initial vertex and $t(e)$ is the terminal vertex of $e \in \mathcal{E}(\Gamma)$.

The function α and the collection θ are called an *axial function* and a *connection* on Γ, respectively.

Given a GKM-orbifold $\mathcal{X}$, we immediately obtain a labelled graph (Γ, α). We can choose a connection θ on Γ, α in a similar way to [GZ01]: let $e \in \mathcal{E}(\Gamma)$, with $p = i(e)$ and $q = t(e)$, and $\mathcal{E}_p(\Gamma) = \{e_1^p = e, e_2^p, \dots, e_n^p\}$ and $\mathcal{E}_q(\Gamma) = \{e_1^q = \bar{e}, e_2^q, \dots, e_n^q\}$. Then, the restriction to S_e (the spindle associated to e) of the tangent orbibundle to $\mathcal{X}$ splits equivariantly as a sum of line orbibundles $\bigoplus_{i=1}^n L_i$, and we can relabel the elements in $\mathcal{E}_p(\Gamma)$ and $\mathcal{E}_q(\Gamma)$, so that $L_i|_p = T_pX_{e_i^p}$ and $L_i|_q = T_qX_{e_i^q}$. From this, we get the identification $e_i^p \mapsto e_i^q$ which defines a connection on Γ, α and produces an orbifold GKM-graph. The integer c_{e,e_i^p} can then be seen to be the Chern number of the vector bundle $\varphi^* L_i$ where $\varphi : \mathbb{C}P^1 \to S_e = S^2(m,n)$ (for some $m, n \in \mathbb{Z}\backslash 0$) is defined by $\varphi[z_1 : z_2] = [z_1^m : z_2^n]$.

Remark 2.3 If we choose r_e and $r_{\bar{e}}$ as the minimal integers such that $r_e\alpha(e) = \pm r_{\bar{e}}\alpha(\bar{e})$, then a multiple of r_e coincides with the order of the local group G_p of an orbifold chart around $p = i(e)$. In particular, if a fixed point $p \in X^T \cong \mathcal{V}(\Gamma)$ is a smooth point, i.e., an orbifold chart around p has the trivial local group, then we may choose r_e to be +1 for each $e \in \mathcal{E}_p(\Gamma)$.

Remark 2.4 An *almost complex structure* on $\mathcal{X}$ is an endomorphism $J : T\mathcal{X} \to T\mathcal{X}$ satisfying $J^2 = -id$. If a GKM-orbifold is equipped with an T^k-invariant almost complex structure J, then the orientation induced from J forces the associated axial function to satisfy $r_e\alpha(e) = -r_{\bar{e}}\alpha(\bar{e}) \in H^2(BT^k)$, as in [GZ01, Definition 2.1.1].

Example 2.5 (Spindles) Here, we calculate two orbifold GKM-graphs (Γ, α, θ) for a spindle $S^2(m,n)$ with respect to two different S^1-actions on it.

(1) If we suppose that the pair (m,n) are coprime integers, then there exist $a, b \in \mathbb{Z}$ such that $mb - na = 1$. We can then write the S^1-action on the spindle $S^2(m,n)$ in Example 2.2 as

$$t \cdot [z_1 : z_2] = [t^a z_1 : t^b z_2],$$

for a choice of a and b as above, since $t \mapsto [t^a : t^b]$ defines an isomorphism $S^1 \to T^2/S^1\langle m,n\rangle$. Note that this action is effective and has two fixed points $[1:0]$ and $[0:1]$. The graph Γ is given by the two vertices connected by an edge, and the connection θ is the only one. To obtain the axial function α, we see that

$$t \cdot [1 : z_2] = [t^a : t^b z_2] = [(t^{-\frac{a}{m}})^m t^a : (t^{-\frac{a}{m}})^n t^b z_2] = [1 : t^{\frac{-an+bm}{m}} z_2] = [1 : t^{\frac{1}{m}} z_2],$$

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Figure 1: The GKM-graph for an effective spindle.

Figure 2: The GKM-graph for a spindle with diagonal action.

where the second equality holds because of the relation (2.2). Therefore, the weight of the orbifold tangential representation $T_{[1:0]}U = V(\alpha_1)/\mathbb{Z}_m$ around the fixed point $[1:0]$ is given by $\frac{1}{m}\varepsilon^* \in H^2(BS^1;\mathbb{Q})$, where we denote by ε^* the standard basis of $H^2(BS^1)$. Hence, we have $\alpha(e) = \frac{1}{m}\varepsilon^*$ for the edge e emanating from the vertex corresponding to $[1:0]$. A similar computation for the other fixed point $[0:1]$ induces an orbifold GKM-graph for the S^1-action on a spindle $S^2(m,n)$ as described by Figure 1.

(2) An S^1-action on a spindle $S^2(m,n)$ could be given by a diagonal action

$$t \cdot [z_1 : z_2] = [tz_1 : tz_2],$$

where now the action may not be effective. As in the previous example, Γ is given by the two vertices connected by an edge and θ is uniquely determined. We now calculate the weight of the orbifold tangential representation around the fixed point $[1:0]$:

$$t \cdot [1 : z_2] = [t : tz_2] = [1 : t^{\frac{m-n}{m}} z_2]$$

and in a similar fashion for the fixed point $[0:1]$. Figure 2 now gives the orbifold GKM-graph for this action.

Remark 2.6 For more on the spindles $S^2(m,n)$, including describing effective actions when (m,n) are not a pair of coprime integers, see [DKS19].

The next example exhibits a $2n$-dimensional GKM-orbifold equipped with a k-dimensional torus action where $k < n$.

Example 2.7 (Weighted projective space) Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^4$ given by

$$t \cdot (z_1, z_2, z_3, z_4) = (z_1, tz_2, tz_3, tz_4)$$

for some $t \in \mathbb{C}^*$ and $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. It induces an action of $\mathbb{C}^*$ on $\wedge^2\mathbb{C}^4$ as follows

$$t \cdot (z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}) = (tz_{12}, tz_{13}, tz_{14}, t^2z_{23}, t^2z_{24}, t^2z_{34}), \tag{2.4}$$

where we write $z_{ij} := z_i \wedge z_j$ for simplicity. The action defined in (2.4) gives us a weighted projective space $\mathbb{P}_{(1,1,1,2,2,2)}(\wedge^2\mathbb{C}^4)$. Notice that the standard T^4-action on $\mathbb{C}^4$ induces a T^4-action on $\wedge^2\mathbb{C}^4$. Moreover, the circle subgroup $\{(1,t,t,t) \mid t \in S^1\}$

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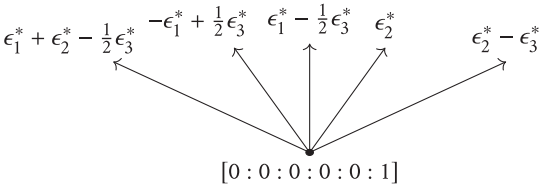

Figure 3: Orbifold GKM-graph around $[0:0:0:0:0:1] \in \mathbb{P}_{(1,1,1,2,2,2)}(\wedge^2 \mathbb{C}^4)$.

of T^4 acts trivially on $\mathbb{P}_{(1,1,1,2,2,2)}(\wedge^2\mathbb{C}^4)$. Now, it is straightforward to see that $\mathbb{P}_{(1,1,1,2,2,2)}(\wedge^2\mathbb{C}^4)$ is a GKM-orbifold with respect to the residual T^4/S^1-action. Here, we consider the following short exact sequence to identify T^4/S^1 with T^3:

$$1 \longrightarrow S^1 \xrightarrow{\varpi} T^4 \xrightarrow{\lambda} T^3 \longrightarrow 1, \tag{2.5}$$

where $\varpi(t) = (1, t, t, t)$ and λ is defined appropriately so that $\ker(\lambda) = \mathrm{im}(\varpi)$, for instance $\lambda(t_1, t_2, t_3, t_4) = (t_1, t_2 t_4^{-1}, t_3 t_4^{-1})$. Choose a right splitting

$$\rho\colon T^3 \longrightarrow T^4$$

of (2.5) defined by $\rho(r_1, r_2, r_3) = (r_1, r_2, r_3, 1)$, which leads us to identity T^3 with $\mathrm{coker}(\varpi)$. Now, we compute the orbifold tangential representation around a fixed point $[0:0:0:0:0:1]$ as follows:

$$\begin{aligned}(r_1, r_2, r_3) \cdot & [z_{12} : z_{13} : z_{14} : z_{23} : z_{24} : 1] \\ &= [r_1 r_2 z_{12} : r_1 r_3 z_{13} : r_1 z_{14} : r_2 r_3 z_{23} : r_2 z_{24} : r_3] \\ &= [r_1 r_2 r_3^{-\frac{1}{2}} z_{12} : r_1 r_3^{\frac{1}{2}} z_{13} : r_1 r_3^{-\frac{1}{2}} z_{14} : r_2 z_{23} : r_2 r_3^{-1} z_{24} : 1]\end{aligned}$$

It shows us that

$$T_{[0,0,0,0,0,1]}U \cong \bigoplus_{i=1}^{5} V_{\alpha_i}/G_i,$$

where each one-dimensional orbifold vector bundle representation V_{α_i}/G_i is determined by

$$\left(\varepsilon_1^* + \varepsilon_2^* - \frac{1}{2}\varepsilon_3^*, \varepsilon_1^* + \frac{1}{2}\varepsilon_3^*, \varepsilon_1^* - \frac{1}{2}\varepsilon_3^*, \varepsilon_2^*, \varepsilon_2^* - \varepsilon_3^*\right) \in \bigoplus_{i=1}^{5} H^2(BT^3; \mathbb{Q}),$$

respectively. A similar computation for the other fixed points yields the corresponding orbifold GKM-graph which is a complete graph with six vertices. Figure 3 shows a part of this orbifold GKM-graph around the vertex corresponding to $[0, 0, 0, 0, 0, 1]$.

Next, we introduce an algebraic object obtained from a given orbifold GKM-graph. The definition is similar to the one defined in [GZ01, Section 1.7], but we emphasize that we use $\mathbb{Z}$ as the coefficient ring.

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Definition 2.3 Given an orbifold GKM-graph (Γ, α, θ), we define the *equivariant cohomology* $H^*_{T^k}(\Gamma, \alpha)$ *of the graph* (Γ, α, θ) as follows:

$$H^*_{T^k}(\Gamma, \alpha) = \left\{ f : \mathcal{V}(\Gamma) \longrightarrow H^*(BT^k) \mid f(i(e)) \equiv f(t(e)) \bmod \tilde{r}_e \alpha(e) \right\},$$

where $\tilde{r}_e$ is the smallest positive integer satisfying the condition (2)-(b) of Definition 2.2.

We notice that $H^*_{T^k}(\Gamma, \alpha)$ has a natural graded ring structure given by vertex-wise addition and multiplication, and the ith degree is defined by

$$H^i_{T^k}(\Gamma, \alpha) = \left\{ f : \mathcal{V}(\Gamma) \longrightarrow H^i(BT^k) \mid f(i(e)) \equiv f(t(e)) \bmod \tilde{r}_e \alpha(e) \right\}.$$

It is easy to check that $H^*_{T^k}(\Gamma, \alpha) := \bigoplus_{i \geq 0} H^i_{T^k}(\Gamma, \alpha)$. Moreover, it is equipped with a natural $H^*(BT^k)$-algebra structure.

Remark 2.8 If (Γ, α, θ) is defined from a smooth toric manifold [DJ91], then the equivariant cohomology $H^*_{T^k}(\Gamma, \alpha)$ is isomorphic to the face ring of the orbit space. We will discuss this further in Sections 3 and 4 in a wider category of spaces including all smooth toric manifolds.

The equivariant cohomology of (Γ, α, θ) can be defined over rational or real coefficients, which coincides with the definition of the cohomology ring of a graph in [GZ01, Section 1.7]:

$$H^*_{T^k}(\Gamma, \alpha; \mathbb{Q}) = \left\{ f : \mathcal{V}(\Gamma) \longrightarrow H^*(BT^k; \mathbb{Q}) \mid f(i(e)) \equiv f(t(e)) \bmod \alpha(e) \right\}.$$

It has a natural $H^*(BT^k; \mathbb{Q})$-algebra structure. We notice that $H^*_{T^k}(\Gamma, \alpha)$ is a subring of $H^*_{T^k}(\Gamma, \alpha; \mathbb{Q})$.

Theorem 2.9 *Let $\mathcal{X} = (X, \mathcal{U})$ be a GKM-orbifold with respect to an effective action of torus T, whose underlying topological space X is homotopic to a T-CW complex. Assume that all isotropy subgroups of T are connected and $H^{odd}(X) = 0$. Then, the equivariant cohomology ring $H^*_T(X)$ is isomorphic to $H^*_T(\Gamma, \alpha)$ as an $H^*(BT)$-algebra.*

Proof The proof is similar to [BFR09, Proposition 2.2] and [Fra10, Theorem 1.3]. The assumption $H^{odd}(X; \mathbb{Z}) = 0$ implies that the Serre spectral sequence of the fibration

$$X \longrightarrow ET \times_T X \longrightarrow BT$$

degenerates at the E_2-page. This is equivalent to the exactness of the Chang–Skjelbred sequence

$$0 \longrightarrow H^*_{T^k}(X) \xrightarrow{\iota^*} H^*_{T^k}(X_0) \xrightarrow{\delta} H^{*+1}_{T^k}(X_1, X_0) \longrightarrow \cdots, \tag{2.6}$$

over $\mathbb{Z}$-coefficients by the connectedness of the isotropy subgroups of T. We refer to [FP07, Theorem 1.1]. Here, X_0 and X_1 denote the set of fixed points and the union of all zero- and one-dimensional orbits in X, respectively. The map $\iota : X_0 \hookrightarrow X$ is the

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inclusion of fixed points into X. Since X_0 is finite, we have

$$H^*_{T^k}(X_0) \cong \bigoplus_{p\in X_0} H^*(BT^k).$$

Let X_e be the connected component of the fixed point set X^{T_e} corresponding to the edge e, where we identify $H^2(BT)$ with $\mathrm{Hom}(T, S^1)$, and $T_e := \ker(\tilde{r}_e\alpha(e))$, where $\tilde{r}_e$ is defined as in Definition 2.3. Then X_e can be regarded as a set of orbits of $T/T_e \cong S^1$, hence it is homeomorphic to a two-sphere S^2. Since every circle action on S^2 is isomorphic to the standard one, X_e has two fixed points $\{p, q\}$ which correspond to the vertices of e. Observe the following commutative diagram:

(2.7)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*_{T^k}(X_e) & \xrightarrow{\iota_e^*} & H^*_{T^k}(\{p,q\}) & \xrightarrow{\delta_e} & H^{*+1}_{T^k}(X_e,\{p,q\}) & \longrightarrow & \cdots \\
 & & \| & & \cong\uparrow & & & & \\
0 & \longrightarrow & H^*_{T^k}(X_e) & \longrightarrow & H^*_{T^k}(X_e\backslash\{q\})\oplus H^*_{T^k}(X_e\backslash\{p\}) & \xrightarrow{j_e} & H^*_{T^k}(X_e\backslash\{p,q\}) & \longrightarrow & \cdots \\
 & & & & \cong\uparrow & & \cong\uparrow & & \\
 & & & & H^*(BT^k)\oplus H^*(BT^k) & \xrightarrow{\tilde{j}_e} & H^*(BT_e) & &
\end{array}$$

where the first row is the long exact sequence of the pair $(X_e, \{p, q\})$, and the second row is the Mayer–Vietoris sequence, and the third row is induced from the inclusion $T_e \to T^k$. Moreover, $\tilde{r}_e\alpha_e$, as an element of $\mathrm{Hom}(T^k, S^1)$, induces a short exact sequence

$$1 \longrightarrow T_e \longrightarrow T^k \xrightarrow{\tilde{r}_e\alpha_e} S^1 \longrightarrow 1,$$

which allows us to identify $H^*(BT_e)$ with $H^*(BT^k)/\langle\tilde{r}_e\alpha(e)\rangle$.

Note that X_1 is the union of 2-spheres intersecting only at fixed points. Hence, a successive application of the relative Mayer–Vietoris sequence yields an isomorphism

$$H^*_{T^k}(X_1, X_0) \cong \bigoplus_{e\in\mathcal{E}} H^*_{T^k}(X_e, \{p, q\}).$$

Moreover, diagram (2.7) shows that $\ker\delta_e \cong \ker j_e$. Hence, by fixing an orientation on $\mathcal{E}(\Gamma)$, the kernel of the differential δ in (2.6) can be realized as

$$\begin{aligned}
\ker\delta &\cong \bigcap_{e\in\mathcal{E}(\Gamma)} \ker\delta_e \cong \bigcap_{e\in\mathcal{E}(\Gamma)} \ker j_e \\
&= \{f\colon \mathcal{V}(\Gamma) \longrightarrow H^*(BT^k) \mid f(i(e)) - f(t(e)) \equiv 0 \bmod (\tilde{r}_e\alpha(e))\}, \qquad (2.8)
\end{aligned}$$

where the equality in (2.8) holds because the map $\tilde{j}_e$ in (2.7) sends $(f(i(e)), f(t(e)))$ to $f(i(e)) - f(t(e))$. Notice that the exactness of (2.6) implies that (2.8) is isomorphic to image of monomorphism $\iota^*\colon H^*_{T^k}(X) \to H^*_{T^k}(X_0)$. Hence, the result follows. ∎

3 Orbifold torus graphs

In this section, we focus on the case when $k = n$ from the definition of an orbifold GKM-graph and from now on write $T := T^n$. All discussions in this section generalize

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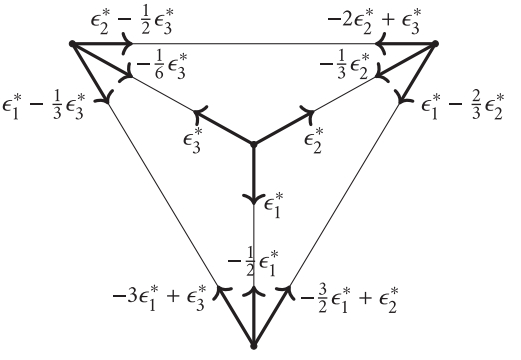

Figure 4: An orbifold torus graph of (Γ, α).

some of the ideas in [MMP07, Section 3] to the orbifold category. We begin by defining a particular class of orbifold GKM-graphs as follows:

Definition 3.1 An *abstract orbifold torus graph* is a pair (Γ, α) of an n-valent graph Γ and a function

$$\alpha\colon \mathcal{E}(\Gamma) \longrightarrow H^2(BT;\mathbb{Q})$$

called an *axial function* such that

(1) the set of vectors $\alpha(\mathcal{E}_p(\Gamma))$ is linearly independent for every $p \in \mathcal{V}(\Gamma)$ and
(2) for each edge $e \in \mathcal{E}(\Gamma)$, there are positive integers r_e and $r_{\overline{e}}$ such that $r_e\alpha(e) = \pm r_{\overline{e}}\alpha(\overline{e}) \in H^2(BT)$, where $\overline{e}$ denotes the edge e with the reversed orientation.

Remark 3.1 Although Definition 2.2 assumes the pairwise linear independency of α, here we further assume the linearly independency of α as in Definition (3.1)-(2). This makes the abstract orbifold torus graph of Definition 3.1 generalize the notion of a *torus graph* in [MMP07, Section 3] to the orbifold setup.

We notice that the first condition of Definition 3.1 determines a connection θ on (Γ, α) uniquely. This allows us to define a *face* of an abstract orbifold torus graph as follows: let Γ' be a d-valent subgraph of Γ, where $d < n$. Then

$$F := (\Gamma', \alpha|_{\mathcal{E}(\Gamma')}) \tag{3.1}$$

is called a *d-dimensional face* of (Γ, α, θ) if it is invariant under the uniquely determined connection θ. Figure 4 is an example of an orbifold torus graph. Considering the graph Γ as a one-skeleton of three-simplex Δ^3, one can see that faces of (Γ, α) are given by intersecting (Γ, α) with faces of Δ^3.

From an orbifold torus graph (Γ, α), we shall define an algebraic object $\mathbb{Z}[\Gamma, \alpha]$ which we call *a weighted face ring*. Let $\mathcal{F}^{(d)}$ be the set of all d-dimensional faces of (Γ, α) for $0 \le d \le n$, and $\mathcal{F} := \mathcal{F}^{(0)} \cup \cdots \cup \mathcal{F}^{(n)}$, the set of all faces of (Γ, α). Notice that $\mathcal{F}^{(n)} = \{\Gamma\}$, $\mathcal{F}^{(0)} = \mathcal{V}(\Gamma)$ and $\mathcal{F}^{(1)}$ is the set of edges of Γ ignoring the orientation.

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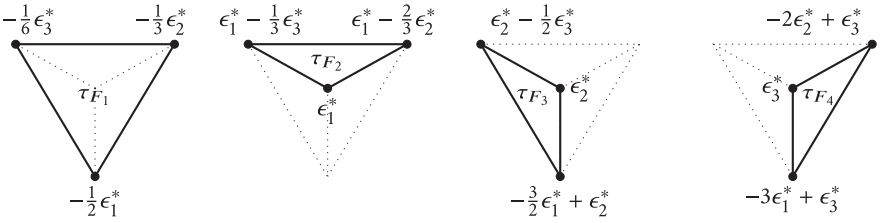

Figure 5: Rational Thom classes of degree 2.

Associating each face F with a formal generator x_F, with $\deg x_F = 2(n - \dim F)$, we obtain a polynomial ring $\mathbf{k}[x_F \mid F \in \mathcal{F}]$ for any commutative ring $\mathbf{k}$ with unit. We define $x_\Gamma = 1$ and $x_\varnothing = 0$ by convention. In this paper, we mainly focus on the case when $\mathbf{k}$ is $\mathbb{Q}$ or $\mathbb{Z}$. Now, we consider a ring homomorphism

$$\mu\colon \mathbb{Q}[x_F \mid F \in \mathcal{F}] \longrightarrow H_T^*(\Gamma, \alpha; \mathbb{Q}) \tag{3.2}$$

sending x_F to the element τ_F defined by

$$\tau_F(v) := \begin{cases} \prod_{\substack{i(e)=v \\ e \notin \Gamma'}} \alpha(e) & \text{if } v \in \mathcal{V}(\Gamma'); \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

see (3.1) for the relation between F and Γ'. We call τ_F the *rational Thom class* corresponding to $F \in \mathcal{F}$. See Figure 5 for rational Thom class of degree 2 for the graph (Γ, α) described in Figure 4.

Restricting μ of (3.1) to the ring $\mathbb{Z}[x_F \mid F \in \mathcal{F}]$ of polynomials with integral coefficients, we have the following subset $\mathscr{Z}_{\Gamma,\alpha}$ of $\mathbb{Z}[x_F \mid F \in \mathcal{F}]$:

$$\mathscr{Z}_{\Gamma,\alpha} := \left\{ f \in \mathbb{Z}[x_F \mid F \in \mathcal{F}] \,\middle|\, \forall v \in \mathcal{V}(\Gamma),\ \mu(f)(v) \in H^*(BT) \right\}. \tag{3.4}$$

Indeed, the collection (3.4) is closed under addition and multiplication induced from $\mathbb{Z}[x_F \mid F \in \mathcal{F}]$, hence it is a subring.

Remark 3.2 When (Γ, α) is a torus graph as defined in [MMP07], the image of the axial function α sits in $H^2(BT)$. Hence, in this case, $\mathscr{Z}_{\Gamma,\alpha} = \mathbb{Z}[x_F \mid F \in \mathcal{F}]$.

We notice that the combinatorics of Γ gives the relation

$$\tau_F \tau_E = \tau_{E \vee F} \sum_{G \in E \cap F} \tau_G, \tag{3.5}$$

where $E \vee F$ denotes the minimal face containing both E and F and $G \in E \cap F$ runs through all connected components. We refer to the proof of [MP06, Lemma 6.3]. Therefore, it is straightforward to see from the definition of $\mathscr{Z}_{\Gamma,\alpha}$ that

$$\left\{ x_F x_E - x_{E \vee F} \sum_{G \in E \cap F} x_G \mid E, F, G \in \mathcal{F} \right\} \tag{3.6}$$

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is a subset of $\mathscr{Z}_{\Gamma,\alpha}$ and defines an ideal of $\mathscr{Z}_{\Gamma,\alpha}$ because the image of each element in (3.5) via μ is identically zero. Now, we have the following definition of a *weighted face ring*.

Definition 3.2 The *weighted face ring* of an abstract torus orbifold graph (Γ, α) is the quotient ring

$$\mathbb{Z}[\Gamma, \alpha] := \mathscr{Z}_{\Gamma,\alpha}/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by elements of (3.6).

In the proof of [MMP07, Theorem 5.5], they consider a torus graph (Γ, α) and prove that the map μ in (3.2), with $\mathbb{Z}$-coefficients, factors through the map

$$\mathbb{Z}[x_F \mid F \in \mathcal{F}]/\mathcal{I} \longrightarrow H_T^*(\Gamma, \alpha) \tag{3.7}$$

and is indeed an isomorphism. In this case, the domain of (3.7) coincides with $\mathbb{Z}[\Gamma, \alpha]$ of Definition 3.2, see Remark 3.2.

If we work over rationals, the role of the integers $\tilde{r}_e$ in an abstract orbifold torus graph cancels out, which makes the theory the same as [MMP07, Theorem 5.5]. Hence, we apply the same argument to get the following lemma:

Lemma 3.3 *Given a torus orbifold graph* (Γ, α)*, there is a ring isomorphism between* $\mathbb{Q}[x_F \mid F \in \mathcal{F}]/\mathcal{I}$ *and* $H_T^*(\Gamma, \alpha; \mathbb{Q})$.

The next theorem extends the result of [MMP07] to the category of torus orbifolds.

Theorem 3.4 *There is a ring isomorphism between* $\mathbb{Z}[\Gamma, \alpha]$ *and* $H_T^*(\Gamma, \alpha)$.

Proof First, we consider a map

$$\nu\colon \mathscr{Z}_{\Gamma,\alpha} \longrightarrow H_T^*(\Gamma, \alpha),$$

which is the restriction of μ defined in (3.2) to $\mathscr{Z}_{\Gamma,\alpha}$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}[x_F \mid F \in \mathcal{F}] & \xrightarrow{\ \mu\ } & H_T^*(\Gamma, \alpha; \mathbb{Q}) \\ {\scriptstyle \iota_1}\uparrow & & \uparrow{\scriptstyle \iota_2} \\ \mathscr{Z}_{\Gamma,\alpha} & \xrightarrow{\ \nu\ } & H_T^*(\Gamma, \alpha). \end{array} \tag{3.8}$$

We prove the theorem by showing that ν is a surjective homomorphism with kernel $\mathcal{I}$.

We first show the surjectivity. Lemma 3.3 implies that $H_T^*(\Gamma, \alpha; \mathbb{Q})$ is generated by the rational Thom classes τ_F. Hence, any element in $H_T^*(\Gamma, \alpha)$ is an integral linear

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combination of rational Thom classes, say $g := \sum_{F\in\mathcal{F}} a_F\tau_F$ for some $a_F \in \mathbb{Z}$. Moreover, g satisfies

$$g(v) \in H^*(BT), \quad \text{for all } v \in \mathcal{V}(\Gamma), \tag{3.9}$$

by Definition 2.3, which means $\sum_{F\in\mathcal{F}} a_F x_F \in \mathbb{Q}[x_F \mid F \in \mathcal{F}]$ is indeed an element in $\mathscr{Z}_{\Gamma,\alpha}$ because of (3.4). Hence, the map ν is surjective.

Next, we show that $\ker \nu = \mathcal{I}$. Recall that the map μ factors through the map

$$\mathbb{Q}[x_F \mid F \in \mathcal{F}]/\mathcal{I} \longrightarrow H_T^*(\Gamma, \alpha; \mathbb{Q})$$

by Lemma 3.3. Hence, we have $\ker \mu = \mathcal{I}$. Moreover, the commutativity of (3.8) gives $\iota_2 \circ \nu = \mu \circ \iota_1$, which implies that $\ker \nu \subset \ker \mu = \mathcal{I}$. To show the reverse inclusion, take an element $h \in \ker \mu$. Since $\ker \mu = \mathcal{I} \subset \mathscr{Z}_{\Gamma,\alpha}$, we may assume that $h = \iota_1(h)$. Now, the commutativity of (3.8) together with the injectivity of ι_2 establishes $h \in \ker \nu$. ∎

Remark 3.5 For a calculation of the weighted face ring $\mathbb{Z}[\Gamma, \alpha]$, when Γ has exactly two vertices, see [DKS19].

4 Torus orbifolds

We now consider *torus orbifolds*, $2n$-dimensional GKM-orbifolds equipped with n-dimensional torus actions which may have finite kernels. Using the results from the previous sections, we calculate their integral equivariant cohomology ring, in terms of generators and relations, when their ordinary odd degree cohomology vanishes.

4.1 Locally standard torus orbifolds

We begin with defining a *locally standard* torus action on a torus orbifold. For each point $p \in X$, there exists an orbifold chart $(\tilde{U}, G, \phi\colon \tilde{U} \to U)$ of a T-invariant neighborhood U around p, an equivariant diffeomorphism ψ from an open subset W of $\mathbb{C}^n$ to $\tilde{U}$ and a surjective covering homomorphism $\xi\colon (S^1)^n \to T$ with $\ker \xi \cong G$ such that the following diagram commutes:

$$\begin{array}{ccccc} (S^1)^n \times W & & \xrightarrow{\xi\times(\phi\circ\psi)} & & T \times U \\ \downarrow & & & & \downarrow \\ W & \xrightarrow{\psi} & \tilde{U} & \xrightarrow{\phi} & U, \end{array}$$

where vertical maps represent torus actions on W and U, respectively. In particular, the action of $(S^1)^n$ on W is the standard one. Such an orbifold together with a preferred orientation on each T-invariant suborbifold of codimension 1 is called a *locally standard torus orbifold*. It is one of the immediate consequences of locally standard actions that the quotient U/T is diffeomorphic to an open subset of

$$W/(S^1)^n \cong \mathbb{R}^n_{\geq} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}. \tag{4.1}$$

Weighted projective spaces are typical examples of locally standard torus orbifolds. Here, we choose a particular half-dimensional torus actions. To be more precise, we

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consider a weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ as a quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by a weighted S^1-action given by

$$t \cdot (z_0, \dots, z_n) = (t^{a_0} z_0, \dots, t^{a_n} z_n)$$

with respect to some weight vector $(a_0, \dots, a_n) \in \mathbb{N}^{n+1}$. Then, the standard T^{n+1}-action on S^{2n+1} defines the residual action of an n-dimensional torus $T^{n+1}/S^1 \cong T^n$.

Example 4.1 Consider the weighted projective space $\mathbb{P}(1, 2, 3, 6)$, as a quotient space S^7/S^1, where S^1 acts on S^7 by

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, t^2 z_2, t^3 z_3, t^6 z_4).$$

The standard T^4-action on S^7 induces a residual T^4/S^1-action on $\mathbb{P}(1, 2, 3, 6)$. Consider the following short exact sequence

$$1 \longrightarrow S^1 \xrightarrow{\varpi} T^4 \xrightarrow{\lambda} T^3 \longrightarrow 1,$$

where $\varpi(t) = (t, t^2, t^3, t^6)$ and $\lambda(t_1, t_2, t_3, t_4) = (t_1^{-2} t_2, t_1^{-3} t_3, t_1^{-6} t_4)$, so that $\ker(\lambda) = \mathrm{im}(\varpi)$. Identifying $\mathrm{coker}(\varpi)$ with T^3 by taking a right splitting $\rho\colon T^3 \to T^4$ defined by $\rho(r_1, r_2, r_3) = (1, r_1, r_2, r_3)$, we calculate the orbifold tangential representations around each fixed point by a similar manner to Example 2.7. The corresponding orbifold GKM-graph coincides with the one described in Figure 4, which is indeed an orbifold torus graph.

We continue this section by introducing a combinatorial model for a torus orbifold, and end up with studying its equivariant cohomology ring with integer coefficients.

Remark 4.2 We refer to [GGKRW18] for an exposition of torus orbifolds. The authors also discuss GKM-graphs for torus orbifolds, but they adopt $H^2(BT)$ as the target space of the axial function. The resulting equivariant cohomologies of torus orbifolds that they calculate are then taken with rational coefficients.

4.2 Quotient construction

Given a $2n$-dimensional locally standard torus orbifold $\mathcal{X} = (X, \mathcal{U})$, we consider the orbit space $Q := X/T$. The local standardness implies that any point in Q has a neighborhood diffeomorphic to an open subset of (4.1). The orbifold atlas $\mathcal{U}$ leads each of these neighborhoods to fit together, so that the orbit space Q has the structure of a *manifold with faces*, [BP15, Definition 7.1.2]. The points in Q corresponding to zero-dimensional orbits, and points corresponding to $(n-1)$-dimensional orbits are called *vertices* and *facets*, respectively.

Let $\mathcal{F}(Q) := \{F_1, \dots, F_m\}$ be the set of facets of Q and $\pi\colon X \to Q$ be the orbit map. We choose elements

$$\{\lambda_1, \dots, \lambda_m\} \subset H_2(BT) \cong \mathrm{Hom}(S^1, T) \tag{4.2}$$

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such that $\lambda_i(S^1) \subset T$ is the isotropy subgroup of $\pi^{-1}(F_i)$, for $i = 1, \dots, m$. We call each $\pi^{-1}(F_i)$ a *characteristic suborbifold*. The local standardness of the torus action leads us to the next proposition.

Proposition 4.3 *Let v be a vertex of Q and $F_{i_1}, \dots, F_{i_n}$ facets containing v. Then, the set $\{\lambda_{i_1}, \dots, \lambda_{i_n}\} \subset H_2(BT)$ is linearly independent.*

Conversely, beginning with an n-dimensional manifold Q with faces, we consider a function

$$\lambda : \mathcal{F}(Q) \longrightarrow H_2(BT),$$

satisfying the condition that the vectors $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_k})\}$ are linearly independent whenever $F_{i_1} \cap \cdots \cap F_{i_k} \neq \varnothing$. We call such a function λ and a pair (Q, λ) a *characteristic function*, and a *characteristic pair*, respectively. Due to technical reasons, we always assume that Q is a CW-complex.

Given a characteristic pair (Q, λ), one can construct a torus orbifold $X(Q, \lambda)$ as follows:

$$X(Q, \lambda) := (Q \times T)/\sim, \tag{4.3}$$

where the equivalence relation $\sim$ is given by $(x, t) \sim (y, s)$ if and only if $x = y$ and $t^{-1}s \in T_{F(x)}$. Here, $F(x)$ denotes the face of Q containing x in its relative interior, and $T_{F(x)}$ is the torus generated by $\lambda(F_{i_1}), \dots, \lambda(F_{i_k})$, if $F(x) = F_{i_1} \cap \cdots \cap F_{i_k}$. We note that the CW-complex structure on Q gives a T-CW complex structure on $X(Q, \lambda)$.

The orbifold structure can be obtained in the same manner as described in [PS10, Section 2.1] for quasitoric orbifolds. We notice that $X(Q, \lambda)$ is equipped with an n-dimensional torus T action by multiplication on the second factor, and the orbit map is the projection of the first factor.

Given a point $x \in X(Q, \lambda)$, let $F(x)$ be the face of Q containing $\pi(x)$ in its relative interior. Then the isotropy subgroup of x is the torus $T_{F(x)}$, which yields the following proposition.

Proposition 4.4 *Let $X(Q, \lambda)$ be a torus orbifold associated to a characteristic pair. Then every isotropy subgroup is connected.*

The next theorem extends [MP06, Lemma 4.5] and [PS10, Lemma 2.2] to torus orbifolds, whose proof is similar to the proof of [DJ91, Proposition 1.8] where we note that $H^2(Q; \mathbb{Z}^n)$ is isomorphic to $[Q, BT]$, hence, the assumption for $H^2(Q; \mathbb{Z}^n)$ being trivial implies that any principal T-bundle over Q is trivial, which plays an important role in the proof.

Theorem 4.5 *Let $\mathcal{X} = (X, \mathcal{U})$ be a locally standard torus orbifold with T-action and Q the orbit space such that $H^2(Q; \mathbb{Z}^n)$ is trivial. Let $\lambda : \mathcal{F}(Q) \to \mathbb{Z}^n \cong H_2(BT)$ be the characteristic function defined by $\lambda(F_i) = \lambda_i$ as in (4.2). Then there is an*

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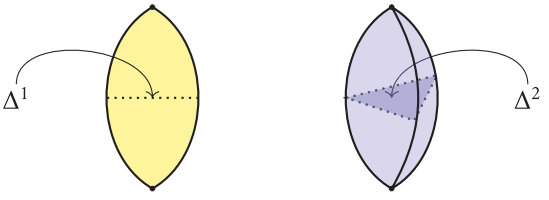

Figure 6: Suspensions of Δ^1 and Δ^2.

equivarianthomeomorphism f between X and $X(Q,\lambda)$ such that the following diagram commutes:

$$\begin{array}{ccc} X(Q,\lambda) & \xrightarrow{\ f\ } & X \\ & {\scriptstyle pr_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Q & \end{array}$$

where pr_1 is the projection onto the first factor and π is the orbit map.

Example 4.6 Let Q be the suspension of the $(n-1)$-simplex Δ^{n-1} for $n \geq 2$, see Figure 6 for the case when $n = 2$ and 3. Then there are exactly n facets, say $F_1, \ldots, F_n$, on Q. Consider a characteristic function

$$\lambda\colon \{F_1, \ldots, F_n\} \longrightarrow H_2(BT)$$

defined by $\lambda(F_j) := \sum_{i=1}^{n} a_{ij}\varepsilon_i \in H_2(BT)$. The integers $(a_{ij})_{1\leq i,j\leq n}$ form an $n \times n$ square matrix Λ which we regard as an automorphism on $\mathbb{R}^n$. Now the space

$$(\Delta^{n-1} \times T)/\sim, \tag{4.4}$$

where the equivalence relation $\sim$ is same as in (4.3), can be identified with a quotient of an odd sphere S^{2n-1} by the action of the finite group

$$\ker(\exp \Lambda\colon T \longrightarrow T).$$

The resulting space is known as an *orbifold lens space*, see [BSS17, Section 3.3]. Hence, the resulting torus orbifold $X(Q,\lambda)$ is exactly the suspension of an orbifold lens space. We refer to [DKS19] for the discussion on these spaces. In particular, if the determinant of Λ is ± 1, i.e. , the set $\{\lambda(F_1), \ldots, \lambda(F_n)\}$ forms a $\mathbb{Z}$-basis of $H_2(BT)$, then the resulting space (4.4) is homeomorphic to S^{2n-1}. Hence, the resulting torus orbifold $X(Q,\lambda)$ is homeomorphic to S^{2n}.

4.3 Equivariant cohomology ring of $X(Q,\lambda)$

Given a characteristic pair (Q,λ) for a torus orbifold, one can derive an orbifold GKM-graph as follows. Let

$$\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$$

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be the one-skeleton of Q, which is an n-valent graph. We define a function

$$\alpha\colon \mathcal{E}(\Gamma) \longrightarrow H^2(BT;\mathbb{Q})$$

by: if $e = F_{k_1} \cap \cdots \cap F_{k_{n-1}} \in \mathcal{E}(\Gamma)$ with the initial vertex $i(e) = e \cap F_{k_n}$, then $\alpha(e)$ is defined by the following system of equations

$$\begin{cases} \langle \alpha(e), \lambda(F_{k_1}) \rangle = \cdots = \langle \alpha(e), \lambda(F_{k_{n-1}}) \rangle = 0; \\ \langle \alpha(e), \lambda(F_{k_n}) \rangle = 1, \end{cases} \tag{4.5}$$

where $\langle , \rangle$ denotes the natural paring between cohomology and homology. In particular, the integers r_e (see Definition 2.2) are described below in the proof of Lemma 4.7.

Intrinsically, each characteristic vector represents the S^1-subgroup of T which acts trivially on characteristic suborbifold. Hence, each edge $e \in \mathcal{E}(\Gamma)$ represents an intersection of $n-1$ many characteristic suborbifolds. Indeed, this intersection is homeomorphic to a two-dimensional sphere and isomorphic to a spindle as an orbifold, fixed by a rank $n-1$ subgroup of T. The collection of equations in the first line of (4.5) contains the information of circle subgroups of T which acts trivially on an invariant two-sphere, and the second equation tells us the the residual S^1-action.

Lemma 4.7 *Let (Q, λ) be a characteristic pair associated to a torus orbifold. Then the function α defined by (4.5) is an axial function on Γ. In particular, the set $\{\alpha(e) \mid e \in \mathcal{E}_p(\Gamma), p \in \mathcal{V}(\Gamma)\}$ is linearly independent.*

Proof Let $\lambda(F_j) = \sum_{i=1}^n a_{ij}\varepsilon_i \in H_2(BT)$, for $j = 1, \ldots, m$. Identifying $H_2(BT)$ and $\mathbb{Z}^n$ by associating ε_i with the ith standard unit vector in $\mathbb{Z}^n$, we may write $\lambda(F_j) = (a_{1j}, \ldots, a_{nj}) \in \mathbb{Z}^n$. For each vertex v of Q with $v = F_{i_1} \cap \cdots \cap F_{i_n}$, we write

$$\Lambda_v := \begin{bmatrix} \lambda(F_{i_1})^t & \cdots & \lambda(F_{i_n})^t \end{bmatrix}. \tag{4.6}$$

Given an oriented edge $e = F_{k_1} \cap \cdots \cap F_{k_{n-1}}$ of Q, let $i(e) = e \cap F_{k_n}$ be the initial vertex and $t(e) = e \cap F_{k'_n}$ be the terminal vertex. Consider $\Lambda_{i(e)}$ and $\Lambda_{t(e)}$ with

$$r_e := \left|\det \Lambda_{i(e)}\right| \text{ and } r_{\tilde{e}} := \left|\det \Lambda_{t(e)}\right|,$$

respectively. Let $\alpha(e)_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}}$ be the coordinate expression of $\alpha(e)$ with respect to the basis $\{\varepsilon_1^*, \ldots, \varepsilon_n^*\} \subset H^2(BT)$. Then, the system of equations (4.5) implies that

$$\alpha(e)_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}} \cdot \Lambda_{i(e)} = (0, \ldots, 0, 1) \quad \text{and} \quad \alpha(\tilde{e})_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}} \cdot \Lambda_{t(e)} = (0, \ldots, 0, 1).$$

Hence, we have

$$\alpha(e)_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}} = (0, \ldots, 0, 1) \cdot \Lambda_{i(e)}^{-1} \quad \text{and} \quad \alpha(\tilde{e})_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}} = (0, \ldots, 0, 1) \cdot \Lambda_{t(e)}^{-1},$$

which are the last rows of $\Lambda_{i(e)}^{-1}$ and $\Lambda_{t(e)}^{-1}$, respectively. Moreover, since the first $n-1$ columns of $\Lambda_{i(e)}$ and $\Lambda_{t(e)}$ are identical, the last row of $\mathrm{adj}(\Lambda_{i(e)})$ agrees with the last row of $\mathrm{adj}(\Lambda_{t(e)})$. This implies that

$$\det \Lambda_{i(e)} \cdot \alpha(e)_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}} = \det \Lambda_{t(e)} \cdot \alpha(\tilde{e})_{\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}}.$$

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Hence, we get $r_e\alpha(e) = \pm r_{\bar{e}}\alpha(\bar{e})$ as desired.

The second assertion follows immediately from the linear independence of characteristic vectors around a vertex. ∎

When (Γ, α) is obtained from a characteristic pair (Q, λ), certain elements of weighted face ring $\mathbb{Z}[\Gamma, \alpha]$ can be immediately read off from (Q, λ).

Proposition 4.8

(1) *For each facet F of Q, let $\boxtimes_F := \operatorname{lcm}\{|\det \Lambda_v| \mid v \in \mathcal{V}(F)\}$, where Λ_v is the $n \times n$ square matrix as defined in (4.6). Then, $\boxtimes_F x_F$ is an element of $\mathcal{Z}_{\Gamma,\alpha}$.*

(2) *Let $\lambda(F_j) = \sum_{i=1}^{n} a_{ij}\varepsilon_i \in H_2(BT)$ for facets $F_1, \ldots, F_m$ of Q. Then*

$$\left\{ \sum_{j=1}^{m} a_{ij} x_{F_j} \;\middle|\; i = 1, \ldots, n \right\} \tag{4.7}$$

are elements of $\mathcal{Z}_{\Gamma,\alpha}$.

Proof The proof of (1) is straightforward from the definition of rational Thom classes, see (3.3). In order to prove (2), it is enough to show that the restriction of (4.7) to each vertex $v \in \mathcal{V}(\Gamma)$ is an element of $H^2(BT)$. If $v = F_{k_1} \cap \cdots \cap F_{k_n}$ for some facets $F_{k_1}, \ldots, F_{k_n}$ of Q, the definitions of μ and rational Thom classes (see (3.2) and (3.3), respectively) yield

$$\mu\left(\sum_{j=1}^{m} a_{ij} x_{F_j} \right)(v) = \sum_{\ell=1}^{n} a_{ik_\ell} \tau_{F_{k_\ell}}(v) = \sum_{\ell=1}^{n} a_{ik_\ell} \alpha(e_{k_\ell}),$$

where $e_{k_\ell} = F_{k_1} \cap \cdots \cap F_{k_{\ell-1}} \cap F_{k_{\ell+1}} \cap \cdots \cap F_{k_n}$ with initial vertex v.

The system of equations (4.5) together with a coordinate expression $\alpha(e_{k_\ell})_{\{\varepsilon_1^*, \ldots, \varepsilon_n^*\}}$ as in the proof of Lemma 4.7 allows us to continue the computation as follows:

$$\begin{aligned} \sum_{\ell=1}^{n} a_{ik_\ell} \alpha(e_{k_\ell}) &= \sum_{\ell=1}^{n} a_{ik_\ell} \cdot [0 \cdots 0 \underset{\ell\text{th}}{1} 0 \cdots 0] \cdot \Lambda_v^{-1} \\ &= [a_{ik_1} \cdots a_{ik_n}] \cdot \Lambda_v^{-1} \\ &= [0 \cdots 0 \underset{i\text{th}}{1} 0 \cdots 0] \in \mathbb{Z}^n. \end{aligned} \tag{4.8}$$

The final equality follows because $[a_{ik_1} \cdots a_{ik_n}]$ is the ith row of Λ_v.

Recall that we identify $H^2(BT)$ with $\mathbb{Z}^n$ via the basis $\{\varepsilon_1^*, \ldots, \varepsilon_n^*\}$. Hence, the vector $[0 \cdots 0\ 1\ 0 \cdots 0]$ in the last line of (4.8) corresponds to $\varepsilon_i^* \in H^2(BT)$, which establishes the claim. ∎

Remark 4.9 (1) We note that the computation (4.8) shows that, for each $i = 1, \ldots, n$, the image of elements in the set (4.7) via the map μ are the constant functions $\varepsilon_i^* : \mathcal{V}(\Gamma) \to H^2(BT)$ defined by $v \mapsto \varepsilon_i^*$, for every $v \in \mathcal{V}(\Gamma)$.

(2) If $\sum_{F \in \mathcal{F}(k)} c_F x_F$ is an element of $\mathcal{Z}_{\Gamma,\alpha}$, so is a constant multiple

$$r \cdot \sum_{F \in \mathcal{F}(k)} c_F x_F = \sum_{F \in \mathcal{F}(k)} r c_F x_F$$

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for some $r \in \mathbb{Z}\backslash\{0\}$. Hence, we may choose the vector $(\tilde{c}_F)_{F\in\mathcal{F}(k)}$ (up to sign) having minimal length in $((c_F)_{F\in\mathcal{F}(k)} \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^{|\mathcal{F}(k)|}$, such that

$$\sum_{F\in\mathcal{F}(k)} \tilde{c}_F x_F \in \mathscr{Z}_{\Gamma,\alpha}.$$

We call such elements *minimal Thom classes*.

(3) In [BFR09], the equivariant cohomology of weighted projective spaces with integral coefficients was calculated, and we observe that their *Courant functions* correspond to the minimal Thom classes associated to facets of the underlying simplex. Since all *global polynomials* appear in $\mathbb{Z}[\Gamma,\alpha]$, by Remark 4.9(1), all of the generators of their algebra have been covered.

(4) When $X(Q,\lambda)$ is a projective toric variety [CLS11, Ful93], namely Q is a lattice polytope and λ is defined to be primitive outward normal vectors of facets, then the set of linear elements in equation (3.4) coincides with the set $\mathrm{CDiv}_T(X(Q,\lambda))$ of T-invariant Cartier divisors of $X(Q,\lambda)$. We refer to [CLS11, Section 4.2] and [Ful93, Section 3.3] for more details.

Example 4.10 Consider a characteristic pair (Δ^3,λ) where characteristic function λ is defined by $\lambda(F_1) = (-2,-3,-6), \lambda(F_2) = (1,0,0), \lambda(F_3) = (0,1,0)$ and $\lambda(F_4) = (0,0,1)$ for facets F_i, where $1 \le i \le 4$. The induced orbifold torus graph (Γ,α) and the rational Thom classes of degree 2 are described in Figures 4 and 5, respectively. Proposition 4.8-(1) says that $6x_{F_i}$, $i = 1,\dots,4$, are elements of $\mathscr{Z}_{\Gamma,\alpha}$. However, the minimal Thom classes corresponding to facets are $6x_{F_1}$, $3x_{F_2}$, $2x_{F_3}$, and x_{F_4}. One can also see that $-2x_{F_1} + x_{F_2}$, $-3x_{F_1} + x_{F_3}$, and $-6x_{F_1} + x_{F_4}$ are elements of $\mathscr{Z}_{\Gamma,\alpha}$, whose images via μ are constant functions represented by ε_1^*, ε_2^*, and ε_3^*, respectively.

The main theorem of this section is as follows:

Theorem 4.11 *Let (Q,λ) be a characteristic pair and X the associated torus orbifold with $H^{odd}(X) = 0$. Then there is an isomorphism*

$$H_T^*(X) \cong \mathbb{Z}[\Gamma,\alpha]$$

of $H^*(BT)$-algebras, where (Γ,α) is the orbifold torus graph obtained from (Q,λ). Furthermore, if $H^*(X)$ is free over $\mathbb{Z}$, then $\mathbb{Z}[\Gamma,\alpha]$ is finitely generated.

Proof A CW-complex structure on Q defines a T-CW complex structure on X. Hence, the assumption $H^{odd}(X) = 0$, together with Proposition 4.4, leads us to apply Theorem 2.9. Finally, the first result follows from Theorem 3.4.

The proof of the second assertion is similar to the proof of [BFR09, Lemma 2.1]. If $H^*(X)$ is free over $\mathbb{Z}$ and vanishes in odd degrees, the Serre spectral sequence of the fibration

$$X \longrightarrow ET \times_T X \xrightarrow{\pi} BT$$

degenerates at the E_2-page, and there is a ring isomorphism $H_T^*(X) \cong H^*(X) \otimes H^*(BT)$ by the Leray–Hirsh theorem. Hence, the ring generators come from either $H^*(X)$ or $H^2(BT)$. Recall from Remark 4.9-(1) that the generators of $H^2(BT)$ are

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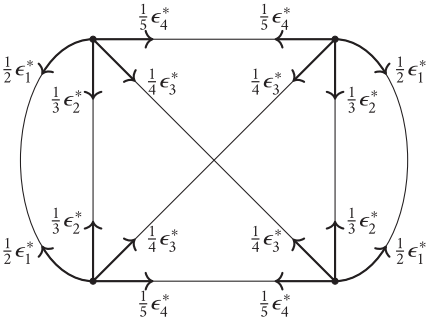

Figure 7: An orbifold torus graph without geometric origin.

already elements of $\mathscr{Z}_{\Gamma,\alpha}$ which are degree 2 elements of $\mathbb{Z}[\Gamma,\alpha]$. Moreover, the result of [FP07, Theorem 1.1] implies that the set of elements in $H_T^*(X)(\cong \mathbb{Z}[\Gamma,\alpha])$ with degree less than or equal to the dimension of X surjects onto

$$H^*(X) \cong H_T^*(X)/\mathrm{im}(\pi^* : H^{>0}(BT) \longrightarrow H_T^*(X)),$$

which establishes the assertion. ∎

From the algebraic point of view, it is not obvious that $\mathbb{Z}[\Gamma,\alpha]$ is finitely generated. The proof of the second assertion uses an application of geometry to answer this algebraic question.

Corollary 4.12 *Let X and (Γ,α) be as above. If $H^*(X)$ is free over $\mathbb{Z}$ and vanishes in all odd degrees, then $H^*(X)$ is isomorphic to $\mathbb{Z}[\Gamma,\alpha]/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by elements of* (4.7).

Remark 4.13 Note that not every orbifold torus graph, as defined in Section 3, originates from torus orbifolds as discussed in this section. For instance, the quotient of the rational graph equivariant cohomology $H_T^*(\Gamma,\alpha;\mathbb{Q})$ of the orbifold torus graph described in Figure 7 by $H^{>0}(BT;\mathbb{Q})$ fails Poincaré duality. Hence, results in [Sat56] imply that (Γ,α) cannot be induced from an orbifold whose odd degree cohomology vanishes.

5 An application: four-dimensional torus orbifolds

In this section, we consider the case when Q is an $m(\geq 2)$-gon, namely a two-dimensional manifold with faces having m vertices and m sides. We begin by setting up the following notation. See Figure 8, where the left picture gives the full description for the case when $m = 2$.

- $\mathcal{F}^{(1)} = \{F_1,\ldots,F_m\}$, the set of facets which are edges in this case.
- $\mathcal{F}^{(0)} = \{v_1,\ldots,v_m\}$, the set of vertices, where $v_k = F_k \cap F_{k+1}$.
- $\lambda : \mathcal{F}^{(1)} \to H_2(BT)$, a characteristic function and we write

$$\lambda(F_k) := a_k\varepsilon_1 + b_k\varepsilon_2, \quad \text{for some } a_k, b_k \in \mathbb{Z}.$$

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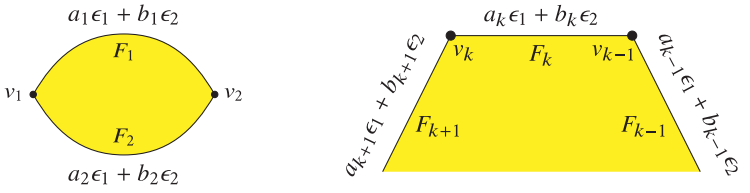

Figure 8: Facets and vertices of 2-gon and $m(\geq 3)$-gon.

- $D_k := \det \Lambda_{v_k} = \det[\lambda(F_k)^t\ \lambda(F_{k+1})^t] = a_k b_{k+1} - b_k a_{k+1}$.

Here, we identify the index $m+1$ with 1. Hence, $v_m = F_m \cap F_1$ and $D_m = a_m b_1 - b_m a_1$.

Following the result of [KMZ17, Corollary 4.1], $H^*(X)$ is concentrated in even degrees if and only if

$$\operatorname{span}_{\mathbb{Z}}\{a_k\varepsilon_1 + b_k\varepsilon_2 \mid k = 1, \dots, m\} = H_2(BT), \tag{5.1}$$

or equivalently,

$$\gcd\{D_1, \dots, D_m\} = 1, \tag{5.2}$$

see [BSS17, Example 4.4] and [BNSS19, Section 4.1] for a more general discussion.

Hence, we may apply Theorem 3.4 to four-dimensional torus orbifolds satisfying (5.1) or (5.2). Preceding the computation of equivariant cohomology, we first introduce a certain $GL_2(\mathbb{R})$-representation, which will play an important role in our calculation.

5.1 A $GL_2(\mathbb{R})$-representation

In this subsection, we identify $GL_2(\mathbb{R})$ with the general linear group $GL(H_2(BT^2;\mathbb{R}))$ by taking $\{\varepsilon_1, \varepsilon_2\}$ as a basis of $H_2(BT^2;\mathbb{R})$. Let $V_n := \bigoplus_{t=0}^{n} \mathbb{R} r^{n-t}s^t$ be a vector space of homogeneous polynomials of degree n with variables r and s. Then, V_n is isomorphic to $H^{2n}(BT^2;\mathbb{R})$ as a vector space by associating r, s with $\varepsilon_1^*, \varepsilon_2^* \in H^2(BT^2;\mathbb{R})$, respectively. Observe that V_n is a $GL_2(\mathbb{R})$-representation space with respect to the action $\Psi\colon GL_2(\mathbb{R}) \times V_n \to V_n$ defined by

$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, f(r,s)\right) = f(ar+bs, cr+ds),$$

which gives a group homomorphism

$$\Phi^{(n)}\colon GL_2(\mathbb{R}) \longrightarrow GL(V_n). \tag{5.3}$$

Note that $\Phi^{(1)}$ is the identity with respect to the basis $\{\varepsilon_1, \varepsilon_2\}$ of $H_2(BT^2;\mathbb{R})$ and its dual basis $\{\varepsilon_1^*, \varepsilon_2^*\}$ of $H^2(BT^2;\mathbb{R}) \cong V_1$.

Example 5.1 When $n = 2$, the ordered basis $\{r^2, rs, s^2\}$ of V_2 gives the following matrix presentation of (5.3)

$$\Phi^{(2)}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

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Indeed,

$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, r^2\right) = (ar+bs)^2 = a^2r^2 + 2abrs + b^2s^2;$$
$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, rs\right) = (ar+bs)(cr+ds) = acr^2 + (ad+bc)rs + bds^2;$$
$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, s^2\right) = (cr+ds)^2 = c^2r^2 + 2cdrs + d^2s^2.$$

5.2 Equivariant cohomology ring

Each pair of adjacent facets F_k, F_{k+1} defines a (2×2) matrix

$$\Lambda_k := \left[\begin{array}{c|c} \lambda(F_k) & \lambda(F_{k+1}) \end{array}\right] = \begin{bmatrix} a_k & a_{k+1} \\ b_k & b_{k+1} \end{bmatrix} \text{for } 1 \le k \le m-1;$$
$$\Lambda_m := \left[\begin{array}{c|c} \lambda(F_m) & \lambda(F_1) \end{array}\right] = \begin{bmatrix} a_m & a_1 \\ b_m & b_1 \end{bmatrix}.$$

Using the $(n+1)\times(n+1)$ matrix $\Phi^{(n)}(\Lambda_k)$, we define a square matrix $\tilde{\Lambda}_k^{(n)}$ of size $(nm \times nm)$ as follows:

$$\tilde{\Lambda}_1^{(n)} := \left[\begin{array}{c|c} \Phi^{(n)}(\Lambda_1) & \mathbf{0} \\ \hline \mathbf{0} & I_{n(m-1)-1} \end{array}\right];$$

$$\tilde{\Lambda}_k^{(n)} := \left[\begin{array}{c|c|c} I_{n(k-1)} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \Phi^{(n)}(\Lambda_k) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n(m-k)-1} \end{array}\right], \quad \text{for } 2 \le k \le m-1;$$

$$\tilde{\Lambda}_m^{(n)} := \left[\begin{array}{c|c|c} \mathbf{0} & I_{n(m-1)-1} & \mathbf{0} \\ \hline \mathfrak{s}_{n+1} & \mathbf{0} & \mathfrak{s}_1 \quad \cdots \quad \mathfrak{s}_n \end{array}\right],$$

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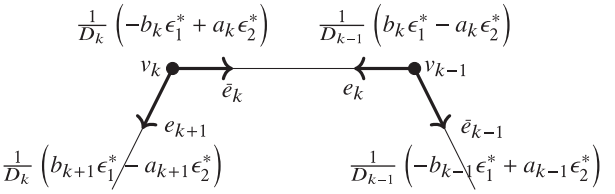

Figure 9: The orbifold torus graph.

where $\mathbf{0}$ and $\mathfrak{s}_i$ denote the zero matrix of size fitting in the block and the ith column of $\Phi^{(n)}(\Lambda_m)$, respectively. We define $\mathbb{L}_k^{\deg 2n}$ to be the sublattice of $\mathbb{Z}^{nm}$ spanned by the row vectors of $\tilde{\Lambda}_k^{(n)}$.

Theorem 5.2 *Let $X := X(Q,\lambda)$ be a four-dimensional torus orbifold satisfying condition (5.1) or (5.2). Then $H_T^*(X)$ is generated by the union of*

(i) $\left\{ \sum_{1\le i\le m} c_i x_{F_i} \;\middle|\; (c_1,\dots,c_m) \in \bigcap_{k=1}^m \mathbb{L}_k^{\deg 2} \right\}$ *and*

(ii) $\left\{ \sum_{\substack{t=0,1\\ 1\le i\le m}} c_{t,i} x_{F_i}^{2-t} x_{F_{i+1}}^{t} \;\middle|\; (c_{0,1}, c_{1,1}, c_{0,2}, c_{1,2}, \dots, c_{1,m}) \in \bigcap_{k=1}^m \mathbb{L}_k^{\deg 4} \right\}$.

Proof The orbifold torus graph (Γ,α) associated to (Q,λ) is given by

$$\mathcal{V}(\Gamma) = \mathcal{F}^{(0)} = \{v_1,\dots,v_m\} \quad \text{and} \quad \mathcal{E}(\Gamma) = \{e_k, \bar{e}_k \mid k = 1,\dots,m\},$$

where $e_k = F_k$ with initial vertex v_{k-1}. The system of equations (4.5) yields

$$\alpha(e_k) = \frac{1}{D_{k-1}}\left(b_k\varepsilon_1^* - a_k\varepsilon_2^*\right) \quad \text{and} \quad \alpha(\bar{e}_k) = \frac{1}{D_k}\left(-b_k\varepsilon_1^* + a_k\varepsilon_2^*\right),$$

which gives the associated orbifold torus graph (Γ,α), see Figure 9.

In the associated weighted ring $\mathbb{Z}[\Gamma,\alpha] = \mathscr{Z}_{\Gamma,\alpha}/\mathcal{I}$, the ideal $\mathcal{I}$ allows us to express an arbitrary element of degree $2n$ in $\mathbb{Z}[\Gamma,\alpha]$ by

$$\sum_{t=0}^{n-1}\sum_{i=1}^{m} c_{t,i}^{(n)} x_{F_i}^{n-t} x_{F_{i+1}}^{t}. \tag{5.4}$$

Recall from (3.4) that an element (5.4) is indeed an element of $\mathbb{Z}[\Gamma,\alpha]$ if and only if

$$\left(c_{0,k}^{(n)}\tau_{F_k}^n + c_{1,k}^{(n)}\tau_{F_k}^{n-1}\tau_{F_{k+1}} + \cdots + c_{n-1,k}^{(n)}\tau_{F_k}\tau_{F_{k+1}}^{n-1} + c_{0,k+1}^{(n)}\tau_{F_{k+1}}^n\right)\Big|_{v_k} \tag{5.5}$$

is an element of $H^{2n}(BT)$ for each vertex v_k, $k = 1,\dots,m$. Notice that the restrictions of the rational Thom classes τ_{F_k} and $\tau_{F_{k+1}}$ to v_k are given by

$$\tau_{F_k}\big|_{v_k} = \frac{1}{D_k}\left(b_{k+1}\varepsilon_1^* - a_{k+1}\varepsilon_2^*\right) \quad \text{and} \quad \tau_{F_{k+1}}\big|_{v_k} = \frac{1}{D_k}\left(-b_k\varepsilon_1^* + a_k\varepsilon_2^*\right), \tag{5.6}$$

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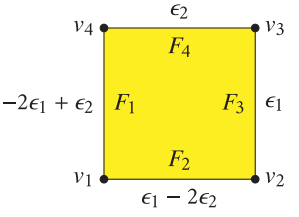

Figure 10: A characteristic pair.

respectively. Hence, by identifying

$$H^{2n}(BT) \cong \bigoplus_{t=0}^{n} \mathbb{Z}(\varepsilon_1^*)^{n-t}(\varepsilon_2^*)^t,$$

and plugging (5.6) into (5.5), we conclude that (5.5) is an element of $H^{2n}(BT)$ if and only if

$$(c_{0,k}^{(n)}, \dots, c_{n-1,k}^{(n)}, c_{0,k+1}^{(n)}) \cdot \Phi^{(n)}\left(\Lambda_k^{-1}\right) \in \mathbb{Z}^{n+1},$$

which is equivalent to saying that $(c_{0,k}^{(n)}, \dots, c_{n-1,k}^{(n)}, c_{0,k+1}^{(n)})$ is an element in the sublattice of $\mathbb{Z}^{n+1}$ spanned by the row vectors of $\Phi^{(n)}(\Lambda_k)$. It is again equivalent to say that

$$(c_{0,1}^{(n)}, \dots, c_{n-1,1}^{(n)}, \dots, c_{0,k}^{(n)}, \dots, c_{n-1,k}^{(n)}, \dots, c_{0,m}^{(n)}, \dots, c_{n-1,m}^{(n)}) \tag{5.7}$$

is an element of $\mathbb{L}_k^{\deg 2n}$, because there is no condition for entries in (5.7) except for $c_{0,k}^{(n)}, \dots, c_{n-1,k}^{(n)}$ and $c_{0,k+1}^{(n)}$. The same computation for all other vertices concludes that (5.7) has to be an element of $\bigcap_{k=1}^{m} \mathbb{L}_k^{\deg 2n}$.

Finally, the assumption (5.1) or (5.2) implies that $H^*(X)$ is free over $\mathbb{Z}$. Moreover, the second assertion of Theorem 4.11 concludes that $\mathbb{Z}[\Gamma, \alpha]$ is generated by elements of degree less than or equal to 4. In degree 2, (5.4) becomes $\sum_{i=1}^{m} c_{0,i}^{(1)} x_{F_i}$ and its coefficients satisfy $(c_{0,1}^{(1)}, \dots, c_{0,m}^{(1)}) \in \bigcap_{k=1}^{m} \mathbb{L}_k^{\deg 2}$, which coincides with (i) by writing $c_i := c_{0,i}^{(1)}$ for $i = 1, \dots, m$. In degree 4, an element (5.4) can be written by $\sum_{\substack{t=0,1 \\ 1 \le i \le m}} c_{t,i}^{(2)} x_{F_i}^{2-t} x_{F_i+1}^{t}$ such that $(c_{0,1}^{(2)}, c_{1,1}^{(2)}, c_{0,2}^{(2)}, c_{1,2}^{(2)}, \dots, c_{0,m}^{(2)}, c_{1,m}^{(2)}) \in \bigcap_{k=1}^{m} \mathbb{L}_k^{\deg 4}$. This also agrees with (ii) by writing $c_{t,i} := c_{t,i}^{(2)}$ for $t = 0, 1$ and $i = 1, \dots, m$. Hence, the proof is completed. ∎

We finish this paper by considering a specific example: the four-dimensional torus orbifold[1] associated to a Cartan matrix of type A, see Figure 10 for the characteristic pair (Q, λ). We refer the readers to [Blu15].

[1] It is also a *toric orbifold* in the sense of [DJ91, Section 7].

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Notice that (Q,λ) in Figure 10 satisfies (5.1) and (5.2). Hence the cohomology of the corresponding torus orbifold $X(Q,\lambda)$ is torsion free and concentrated in even degrees. In Example 5.3, we apply Theorem 4.11, in particular Theorem 5.2, to calculate $H_T^*(X(Q,\lambda))$. We can also apply Corollary 4.12 to obtain $H^*(X(Q,\lambda))$. In Example 5.6, we calculate explicit generators of $H^*(X(Q,\lambda))$ and their relations.

Example 5.3 (Equivariant cohomology) To apply Theorem 5.2, we begin with calculating matrices $\tilde{\Lambda}_k^{(n)}$ for $n = 1, 2$, and $1 \le k \le 4$. When $n = 1$, the homomorphism $\Phi^{(1)}: GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ is the identity. Hence, we have

$$\tilde{\Lambda}_1^{(1)} = \left[\begin{array}{cc|cc} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \tilde{\Lambda}_2^{(1)} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\tilde{\Lambda}_3^{(1)} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \tilde{\Lambda}_4^{(1)} = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right].$$

Now, we have sublattices $\mathbb{L}_k^{\deg 2}$ of $\mathbb{Z}^4$ generated by the row vectors of $\tilde{\Lambda}_k^{(1)}$ for each $1 \le k \le 4$, whose intersection $\bigcap_{k=1}^4 \mathbb{L}_k^{\deg 2}$ can be generated by

$$\{(-2,1,1,0),(1,-2,0,1),(0,0,2,0),(0,0,0,2)\}. \tag{5.8}$$

Hence, $H_T^2(X(Q,\lambda))$ is generated by

$$\begin{aligned} \zeta_1 &:= -2x_1 + x_2 + x_3, \\ \zeta_2 &:= x_1 - 2x_2 + x_4, \\ \zeta_3 &:= 2x_3, \\ \zeta_4 &:= 2x_4, \end{aligned} \tag{5.9}$$

where we denote $x_i := x_{F_i}$ for $1 \le i \le 4$ for simplicity. We notice that $\{\zeta_1, \zeta_2\}$ are elements described in (4.7).

Remark 5.4 An explicit example of (5.8) needs some tedious calculation, or one can use a computer program, for example the module of "Toric lattices" of SAGE, see [Nov10] and [S$^+$].

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When $n = 2$, using the computation in Example 5.1, we have the following four (8×8) matrices

$$\tilde{\Lambda}_1^{(2)} = \left[\begin{array}{ccc|c} 4 & -4 & 1 & \\ -2 & 5 & -2 & \mathbf{0} \\ 1 & -4 & 4 & \\ \hline & \mathbf{0} & & I_5 \end{array}\right], \quad \tilde{\Lambda}_2^{(2)} = \left[\begin{array}{c|ccc|c} I_2 & & \mathbf{0} & & \mathbf{0} \\ \hline & 1 & 2 & 1 & \\ \mathbf{0} & -2 & -2 & 0 & \mathbf{0} \\ & 4 & 0 & 0 & \\ \hline \mathbf{0} & & \mathbf{0} & & I_3 \end{array}\right],$$

$$\tilde{\Lambda}_3^{(2)} = \left[\begin{array}{c|ccc|c} I_4 & & \mathbf{0} & & \mathbf{0} \\ \hline & 1 & 0 & 0 & \\ \mathbf{0} & 0 & 1 & 0 & \mathbf{0} \\ & 0 & 0 & 1 & \\ \hline \mathbf{0} & & \mathbf{0} & & 1 \end{array}\right], \quad \tilde{\Lambda}_4^{(2)} = \left[\begin{array}{c|c|cc} \mathbf{0} & I_5 & \mathbf{0} & \\ \hline 4 & & 0 & 0 \\ -2 & \mathbf{0} & 0 & -2 \\ 1 & & 1 & -2 \end{array}\right].$$

Now, we have sublattices $\mathbb{L}_k^{\deg 4}$ of $\mathbb{Z}^8$ generated by row vectors of $\tilde{\Lambda}_k^{(2)}$ above for $1 \leq k \leq 4$, respectively. Finally, one can find generators of the intersection $\bigcap_{k=1}^4 \mathbb{L}_4^{\deg 4}$ as follows:

$$\{(1,2,1,0,3,0,1,2),(0,3,3,0,1,0,0,0),(0,0,9,0,3,0,0,0),(0,0,0,2,2,0,0,0),$$
$$(0,0,0,0,4,0,0,0),(0,0,0,0,0,1,0,0),(0,0,0,0,0,0,2,2),(0,0,0,0,0,0,0,4)\},$$

which means that $H_T^4(X(Q,\lambda))$ is generated by

$$\begin{aligned} \eta_1 &:= x_1^2 + 2x_1x_2 + x_2^2 + 3x_3^2 + x_4^2 + 2x_1x_4, \\ \eta_2 &:= 3x_1x_2 + 3x_2^2 + x_3^2, \\ \eta_3 &:= 9x_2^2 + 3x_3^2, \\ \eta_4 &:= 2x_2x_3 + 2x_3^2, \\ \eta_5 &:= 4x_3^2, \\ \eta_6 &:= x_3x_4, \\ \eta_7 &:= 2x_4^2 + 2x_1x_4, \\ \eta_8 &:= 4x_1x_4. \end{aligned} \tag{5.10}$$

Remark 5.5 One can also interpret generators $\zeta_1, \ldots, \zeta_4$ in (5.9) and $\eta_1, \ldots \eta_8$ in (5.10) as elements in $H_T^*(\Gamma, \alpha)$ via Theorem 3.4. Recall that the map ν in (3.8) sends x_{F_i} to τ_{F_i} for $i = 1, \ldots, 4$, which are illustrated in Figure 11. For instance,

$$\nu(\eta_8)(\nu) = 4\tau_{F_1}\tau_{F_4}(\nu) = \begin{cases} -2(\varepsilon_1^*)^2 - 4\varepsilon_1^*\varepsilon_2^*, & \text{if } \nu = F_1 \cap F_4; \\ 0, & \text{otherwise.} \end{cases}$$

One can immediately see that $\nu(\eta_8)$ above satisfies the condition for $H_T^*(\Gamma, \alpha)$ in Definition 2.3.

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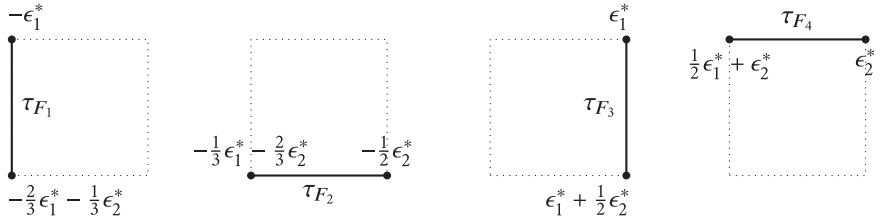

Figure 11: Rational Thom classes.

Example 5.6 (Singular cohomology) Recall that $H^*(X(Q,\lambda)) \cong \mathbb{Z}[\Gamma,\alpha]/\mathcal{J}$ by Corollary 4.12. Here, $\mathcal{J}$ is the ideal generated by $\{\zeta_1,\zeta_2\}$, see Example 5.3. To obtain the minimal number of generators and relations, we may choose $y := \zeta_3$ and $z := \zeta_4$ as generators of $H^2(X(Q,\lambda))$.

Moreover, one can see

$$\eta_1 = \eta_3 = \eta_4 = \eta_7 = 0, \tag{5.11}$$

$$-\eta_5 = \eta_8 = 2\eta_2 = 2\eta_6. \tag{5.12}$$

modulo two ideals $\mathcal{I}$ and $\mathcal{J}$. To see (5.11) and (5.12) explicitly, the following relation

$$x_1x_2 = 2x_1^2 = 2x_2^2. \tag{5.13}$$

may be helpful, which can be obtained by the following computation.

$$\begin{aligned}
x_2^2 &= (2x_1 - x_3)x_2 && (\text{by } \zeta_1 = -2x_1 + x_2 + x_3 = 0)\\
&= 2(2x_2 - x_4)x_2 - x_2x_3 && (\text{by } \zeta_2 = x_1 - 2x_2 + x_4 = 0)\\
&= 4x_2^2 - x_2x_3 && (\text{by } x_2x_4 = 0)\\
&= 4x_2^2 - x_2(2x_1 - x_2) && (\text{by } \zeta_1 = -2x_1 + x_2 + x_3 = 0)\\
&= 5x_2^2 - 2x_1x_2,
\end{aligned}$$

which implies that $2x_2^2 = x_1x_2$. Moreover,

$$\begin{aligned}
x_1x_2 &= x_1(2x_1 - x_3) && (\text{by } \zeta_1 = -2x_1 + x_2 + x_3 = 0)\\
&= 2x_1^2. && (\text{by } x_1x_3 = 0)
\end{aligned}$$

Combining these two, we have (5.13). Hence, for instance

$$\begin{aligned}
\eta_1 &= x_1^2 + 2x_1x_2 + x_2^2 + 3x_3^2 + x_4^2 + 2x_1x_4\\
&= 12x_1^2 - 10x_1x_2 + 8x_2^2\\
&= 0,
\end{aligned}$$

where the second and third equalities follow from the linear ideal $\mathcal{J}$ and (5.13), respectively.

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Thus, we may put $w := \eta_2 = \eta_6$ as a generator of $H^4(X(Q,\lambda))$. Combining w with degree 2 generators, we have more relations:

$$y^2 = z^2 = -2w,\ yz = 4w,\ yw = zw = 0$$

modulo the ideal $\mathcal{I} + \mathcal{J}$. Indeed, these can be verified by the following computation, where we use the ideal $\mathcal{I} + \mathcal{J}$ and (5.13) again.

$$\begin{aligned}
y^2 &= 4x_3^2 = 4(2x_1 - x_2)^2 = 4(4x_1^2 - 4x_1x_2 + x_2^2) = -12x_1^2,\\
z^2 &= 4x_4^2 = 4(-x_1 + 2x_2)^2) = 4(x_1^2 - 4x_1x_2 + 4x_2^2) = -12x_1^2,\\
w &= x_3x_4 = (2x_1 - x_2)(-x_1 + 2x_2) = 6x_1^2,\\
yz &= 4x_3x_4 = 4w,\\
yw &= 2x_3(x_3x_4) = 2x_3 \cdot 6x_1^2 = 0,\\
zw &= 2x_4(x_3x_4) = 2x_4 \cdot 6x_2^2 = 0.
\end{aligned}$$

Finally, we conclude that

$$H^*(X(Q,\lambda)) \cong \mathbb{Z}[y,z,w]/\mathcal{K}$$

where $\deg y = \deg z = 2$, $\deg w = 4$ and $\mathcal{K}$ is the ideal generated by

$$\{y^2 + 2w, z^2 + 2w, yz - 4w, yw, zw\}.$$

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