Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th May 2020 and solutions will be published in the July 2020 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2020.1 **(Paul Stephenson)**

Show analytically that

$$e^{\frac{1}{e}} > \sqrt{2}.$$

Problem 2020.2 **(Paul Stephenson)**

For a polygon with vertices on a grid of points with integer coordinates, Pick's theorem says that the area of the polygon is

$$i + \frac{b}{2} - 1.$$

Where i is the number of points in the interior and b is the number of points on the boundary. Find the analogous expression for the volume of a cuboid in a three-dimensional lattice with i interior grid points, f grid points lying in a face, e grid points lying on an edge and v (= 8) grid points at vertices.

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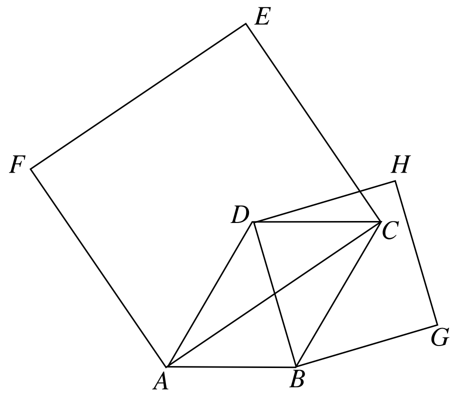
Solutions to 2019.5 and 2019.6

Both problems were solved by Patrick Luo (King's College London Mathematics School), Kunal Kumar (Tagore Sr. Secondary School), James Tan (Queen Elizabeth's School), Otto Arends Page (King's College London Mathematics School) and William Armstong (Bishop Wordsworth School). Problem 2019.5 was solved by Nathan Burn (Bishop Wordsworth School), Julian Charlton Lee (King's College London Mathematics School), Ben Gardner (Tonbridge School) and Pingchuan Zhu (St. Pauls School). Problem 2019.6 was solved by Suvir Rathore (Queen Elizabeth's School), Deepak Sinhmar (Hindu College, University of Delhi) and Student A (Panakkapasambie).

***Problem 2019.5* (Dao Thanh Oai)**

Let $ABCD$ be a parallelogram with A and B fixed points in a plane. Quadrilaterals $ACEF$ and $BGHD$ are squares. Show that the midpoint of FG is fixed when the line segment CD is translated in the plane.



Solution

Good solutions to this problem required specific points in the plane to be defined carefully using Cartesian coordinates or vectors. Without loss of generality A can lie at the origin with B on the positive x-axis such that $A = (0, 0), B = (a, 0), D = (p, q)$ and $C = (p + a, q)$ where $a, c, b \in \mathbb{R}$.

James, William, Patrick, Ben and Ping noted that the point G is a 90° clockwise rotation of point D about B and that point F is a 90° anti-clockwise rotation of C about A. The coordinates of F and G can then be found.

The point G can be found by the following transformation of point D in order: translating point B by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$, rotating by 90° clockwise about the origin followed by a translation by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$. A rotation about the origin 90° clockwise is equivalent to the mapping $(x, y) \to (y, -x)$.

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Starting from the point D, the transformed points are as follows

$$(p, q) \to (p - a, q) \to (q, -(p - a)) \to (q + a, -(p - a)),$$

giving $G = (q + a, a - p)$.

A rotation about the origin 90° anti-clockwise is given by $(x, y) \to (-y, x)$. Therefore, the coordinates of F are given by $(-q, p + a)$.

Finally, the midpoint of the line $\left(\frac{1}{2}\left(-q + (q + a)\right), \frac{1}{2}\left((p + a) + (a - p)\right)\right) = \left(\frac{1}{2}a, a\right)$ which is only dependent on a. Thus, the midpoint of FG is fixed when the line segment CD is translated in the plane.

Most students took a similar approach to that above and William, Julian and Pingchuan used matrices to define the transformations used.

Problem 2019.6 **(Chris Starr)**

For n odd, prove that

$$\left(\binom{n}{2} + 2\binom{n}{4} + 2^2\binom{n}{6} + \ldots\right)^2 + \left(\binom{n}{0} + \binom{n}{2} + 2\binom{n}{4} + 2^2\binom{n}{6} + \ldots\right)^2 = \left(\binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \ldots\right)^2.$$

Solution

There were a wide range of different approaches to this problem. Patrick, Kunal and William all took a similar approach and were very impressive solutions. James proved the result by induction. The simplest and most elegant methods followed a similar approach to the following solution:

Let $A = (1 + \sqrt{2})^n = \binom{n}{0} + \sqrt{2}\binom{n}{1} + 2\binom{n}{2} + 2\sqrt{2}\binom{n}{3} + \ldots$

and $B = (1 - \sqrt{2})^n = \binom{n}{0} - \sqrt{2}\binom{n}{1} + 2\binom{n}{2} - 2\sqrt{2}\binom{n}{3} + \ldots$.

$$\text{LHS} = \left(\frac{A + B - 2}{4}\right)^2 + \left(\frac{A + B + 2}{4}\right)^2 = \frac{A^2 + B^2 + 2AB + 4}{8},$$

$$\text{RHS} = \left(\frac{A - B}{2\sqrt{2}}\right)^2 + \frac{A^2 + B^2 - 2AB}{8}.$$

For n odd, $AB = (1 + \sqrt{2})^n(1 - \sqrt{2})^n = (-1)^n = -1$.

Therefore, LHS = RHS for n odd.

The first prize of £25 is awarded to William Armstrong. The second prize of £20 is awarded to James Tan.

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LEWIS ROBERTS

Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.