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REMARKS TO THE UNIQUENESS PROBLEM OF MEROMORPHIC MAPS INTO $P^N(C)$, IV

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§1. Introduction

Let $H_1, H_2, \cdots, H_{N+2}$ be hyperplanes in $P^N(C)$ located in general position and $\nu_1, \nu_2, \cdots, \nu_{N+2}$ divisors on C^n. We consider the set $\mathscr{F}(H_i, \nu_i)$ of all non-degenerate meromorphic maps of C^n into $P^N(C)$ such that the pull-backs $\nu(f, H_i)$ of the divisors (H_i) on $P^N(C)$ by f are equal to ν_i for any $i = 1, 2, \cdots, N+2$. In the previous paper [6], the author showed that $\mathscr{F} := \mathscr{F}(H_i, \nu_i)$ cannot contain more than $N+1$ algebraically independent maps. Relating to this, the following theorem will be proved.

THEOREM. *The set $\mathscr{F}$ is finite.*

We give here an example which shows that the number $\sharp\mathscr{F}$ of elements in $\mathscr{F}$ is not less than $(N+1)!$. Take $N+1$ nowhere zero entire functions $h_1, \cdots, h_{N+1}$ such that $h_i/h_j \not\equiv$ const if $i \neq j$, and define

$$F := h_1 + h_2 + \cdots + h_{N+1}\,.$$

We consider hyperplanes

$$(1) \qquad \begin{aligned} H_i\ &: w_i = 0 \qquad (1 \leqq i \leqq N+1) \\ H_{N+2}&: w_1 + w_2 + \cdots + w_{N+1} = 0 \end{aligned}$$

in $P^N(C)$ and divisors

$$\begin{aligned} &\nu_i = 0 \qquad (1 \leqq i \leqq N+1) \\ &\nu_{N+2} := \nu_F \end{aligned}$$

on C^n, where $w_1 : w_2 : \cdots : w_{N+1}$ are homogeneous coordinates on $P^N(C)$ and ν_F denotes the divisor defined by the zero-multiplicity of F. Then, $\mathscr{F} := \mathscr{F}(H_i, \nu_i)$ contains

$$f^\sigma = h_{\sigma(1)} : h_{\sigma(2)} : \cdots : h_{\sigma(N+1)}$$

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for any permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & N+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(N+1) \end{pmatrix}$. Therefore, $\sharp\mathscr{F} \geqq (N+1)!$.

It is an interesting problem to ask if $\sharp\mathscr{F}$ is bounded from above by a constant depending only on N. But, the author cannot yet reply to it.

As an application of the above theorem, we shall show the following:

Let $f: C^n \to P^N(C)$ be a non-degenerate meromorphic map and $\gamma: C^n \to C^n$ a biholomorphic map. If $\nu(f, H_i)(\gamma(z)) = \nu(f, H_i)(z)$ for $N+2$ hyperplanes H_i $(1 \leqq i \leqq N+2)$ in general position, then there exists some positive integer j_0 such that $f \circ \gamma^{j_0} = f$, where $\gamma^{j_0} = \gamma \circ \gamma \circ \cdots \circ \gamma$ (j_0-times).

Here, we cannot always take $j_0 = 1$. Consider a holomorphic map

$$f(z) := e^{\sin (z/(N+1))} : e^{\sin ((z+2\pi)/(N+1))} : \cdots : e^{\sin ((z+2N\pi)/(N+1))}$$

of C into $P^N(C)$ and a biholomorphic map $\gamma: C \to C$ defined by $\gamma(z) = z + 2\pi (z \in C)$. For hyperplanes H_i $(1 \leqq i \leqq N+2)$ defined by (1), we see

$$\nu(f, H_i)(\gamma(z)) = \nu(f, H_i)(z) \qquad (1 \leqq i \leqq N+2),$$

but $f(z+2\pi) \neq f(z)$. In this case, we have to take $j_0 = N+1$.

In the proof of the above theorem, the classical theorem of E. Borel ([1]) plays an essential role. We can generalize it to the case that meromorphic functions of order less than one are taken as coefficients. By the similar arguments as in the proof of the above theorem, we shall give some results on relations between meromorphic functions of order less than one and meromorphic functions with γ-invariant zeros and poles for a biholomorphic map $\gamma: C^n \to C^n$. One of them includes the following result as a special case.

THEOREM. *Let $\varphi_1, \cdots, \varphi_p$ be meromorphic functions on C of order less than one and $g_1, \cdots, g_p$ meromorphic functions on C with $\nu_{g_i}(z+\omega) = \nu_{g_i}(z)$ for a non-zero constant ω. If $\sum_{i=1}^p \varphi_i g_i \equiv 0$ and $\sum_{i \in I} \varphi_i g_i \not\equiv 0$ for any proper subset I of $\{1, 2, \cdots, p\}$, then there exists some positive integer j_0 such that h_{i_1}/h_{i_2} is a periodic function with period $j_0\omega$ for any i_1 and i_2.*

By applying this, we shall generalize a recent result by Urabe-Yang in [11] and [12] which motivated the studies in this paper.

§ 2. Preliminaries

Let $\varphi(z)$ be a non-zero holomorphic function on a domain D in C^n. For each point $a = (a_1, \cdots, a_n) \in D$, we expand φ as a convergent series

$$\varphi(a_1 + u_1, \cdots, a_n + u_n) = \sum_{m=0}^{\infty} P_m(u_1, \cdots, u_n)$$

on a neighborhood of a, where P_m is a homogeneous polynomial of degree m or $P_m \equiv 0$. We define

$$\nu_\varphi(a) := \min\{m; P_m(u_1, \cdots, u_n) \not\equiv 0\}.$$

In case that φ is meromorphic, taking non-zero holomorphic functions φ_1 and φ_2 in a neighborhood of a such that $\varphi = \varphi_1/\varphi_2$ and

$$\operatorname{codim}\{\varphi_1 = \varphi_2 = 0\} \geqq 2,$$

we define $\nu_\varphi^0 = \nu_{\varphi_1}$, $\nu_\varphi^\infty = \nu_{\varphi_2}$ and $\nu_\varphi = \nu_{\varphi_1} - \nu_{\varphi_2}$, which are determined independently of the choices of φ_1 and φ_2. By definition, a divisor on D is an integer-valued function on D such that for any point $a \in D$ there is a non-zero meromorphic function φ with $\nu = \nu_\varphi$ on a neighborhood of a and the carrier of ν is an analytic set

$$|\nu| := \overline{\{z \in D; \nu(z) \neq 0\}} \cap D.$$

DEFINITION 2.1. Let ν be a divisor on C^n. Take a positive constant s arbitrarily. We define the *counting function* of ν by

$$N(r, \nu) := \begin{cases} \dfrac{1}{W}\displaystyle\int_s^r \frac{dt}{t^{2n-1}} \int_{|\nu| \cap \overline{B(t)}} \nu(z) v_{n-1}(z) & (r > s) \text{ if } n > 1 \\ \displaystyle\int_s^r \frac{1}{t}\Big(\sum_{|z| \leqq t} \nu(z)\Big) dt & (r > s) \text{ if } n = 1, \end{cases}$$

where

$$v_1 := \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$$

$$v_{n-1} := \frac{1}{(n-1)!} v_1 \wedge v_1 \wedge \cdots \wedge v_1 \qquad ((n-1)\text{-times})$$

$$W := \frac{\pi^{n-1}}{(n-1)!}$$

$$B(t) := \{z = (z_1, \cdots, z_n); \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 < t^2\}$$

and the integral over $|\nu| \cap \overline{B(t)}$ means that the integral over the manifold consisting of all regular points of $|\nu| \cap \overline{B(t)}$.

DEFINITION 2.2. Let φ be a non-zero meromorphic function on C^n. The *order function* of φ is defined by

$$T(r, \varphi) := N(r, \nu_\varphi^\infty) + \frac{1}{\Phi(r)} \int_{S(r)} \log^+ |\varphi(z)| \sigma_r(z) \qquad (r > s),$$

where $\log^+ x = \max(\log x, 0)$, $\Phi(r) = (2\pi^n/(n-1)!)r^{2n-1}$, $S(r) := \{z; \|z\| = r\}$ and σ_r denotes the area element of $S(r)$. In case $\varphi \equiv 0$, we define $T(r, \varphi) \equiv 0$. The order of φ is defined by

$$\rho(\varphi) := \limsup_{r \to \infty} \frac{\log^+ T(r, \varphi)}{\log r} (\leqq +\infty).$$

As in the case of meromorphic functions on C, we can prove

(2.3) *If φ is holomorphic, then*

$$\rho(\varphi) = \limsup_{r \to \infty} \frac{\log^+ \log^+ M(r, \varphi)}{\log r},$$

where

$$M(r, \varphi) := \max\{|\varphi(z)|; \|z\| = r\}.$$

For the proof, see W. Stoll [9].

(2.4) *Let φ_1 and φ_2 be non-zero meromorphic functions on C^n of order less than a positive number ρ. Then, $\varphi_1 + \varphi_2$, $\varphi_1 - \varphi_2$, $\varphi_1\varphi_2$ and φ_1/φ_2 are also of order less than ρ.*

In fact, we can find some positive constants M and ρ_0 with $0 < \rho_0 < \rho$ such that $T(r, \varphi_i) \leqq Mr^{\rho_0}$ $(i = 1, 2)$ for sufficiently large r. Putting $\psi := \varphi_1 \pm \varphi_2$, or $\varphi_1\varphi_2^{\pm 1}$, we have easily

$T(r, \psi) \leqq T(r, \varphi_1) + T(r, \varphi_2) + O(1) \leqq 2Mr^{\rho_0} + O(1)$

and so $\rho(\psi) \leqq \rho_0 < \rho$.

DEFINITION 2.5. Let ν be a divisor on C^n. We define the *order* of ν by

$$\rho(\nu) := \limsup_{r \to \infty} \frac{\log^+ N(r, \nu)}{\log r}.$$

Take a pure $(n-1)$-dimensional analytic set in C^n. We can define a divisor ν_V on C^n such that $|\nu_V| = V$ and $\nu_V(z) = 1$ for any regular point z of V. We call the order of ν_V the order of V.

(2.6) *Let φ be a non-constant meromorphic function on C^n. Then, for any $a \in C$*

$$N(r, \nu_{\varphi-a}^0) \leqq T(r, \varphi) + O(1).$$

For the proof, see H. Fujimoto [2], pp. 34–35.

(2.7) *For a divisor* ν *on* C^n *there exists a meromorphic function* φ *on* C^n *such that* $\nu_\varphi = \nu$ *and* $\rho(\varphi) \leqq \rho(\nu)$.

For the proof, see *W.* Stoll [9].

(2.8) *Let* φ *be a holomorphic function on* C^n *and* $\omega = (\omega_1, \cdots, \omega_n) \in C^n - \{0\}$. *We define a holomorphic function* φ_ω *on* C *by* $\varphi_\omega(z) := \varphi(z\omega)$, *where* $z\omega = (z\omega_1, \cdots, z\omega_n)$. *Then,* $\rho(\varphi_\omega) \leqq \rho(\varphi)$.

This is an immediate consequence of (2.3), because

$$M(r, \varphi_\omega) = \max_{\|z\|=r} |\varphi(z\omega)| \leqq M(\|\omega\| r, \varphi) .$$

(2.9) *If* $h(z)$ *is a nowhere zero non-constant holomorphic function on* C^n, *then* $\rho(h) \geqq 1$.

This is well-known for the case $n = 1$ (c.f., [8]). Let $n \geqq 2$. We can take a point $\omega \in C^n - \{0\}$ such that $h_\omega(z) := h(z\omega) \not\equiv$ const. By (2.8), we see

$$\rho(h) \geqq \rho(h_\omega) \geqq 1 .$$

We denote the set of all nowhere zero holomorphic functions on C^n by H^* and the set of all meromorphic functions of order less than one by Φ_0. And, for $h, h' \in H^*$, we mean by $h \sim h'$ and $h \not\sim h'$ that $h/h' \equiv$ const and $h/h' \not\equiv$ const respectively.

Now, we give a generalization of the classical theorem of E. Borel.

THEOREM 2.10. *Let* $h_1, \cdots, h_p \in H^*$ *and* $\varphi_1, \cdots, \varphi_p \in \Phi_0$. *If* $h_i \not\sim h_j$ *for any* i, j *with* $i \neq j$ *and*

$$\varphi_1 h_1 + \varphi_2 h_2 + \cdots + \varphi_p h_p \equiv 0 , \tag{2}$$

then

$$\varphi_1 \equiv \varphi_2 \equiv \cdots \equiv \varphi_p \equiv 0 .$$

Proof. This is a well-known fact if $n = 1$ (c.f., for example, [7], p. 100). Let us consider the case $n \geqq 2$. To prove Theorem 2.10 by induction on p, it suffices to show that at least one φ_i vanishes. Assume that $\varphi_i \not\equiv 0$ for any i. For a point $\omega \in C^n - \{0\}$, we define $(\varphi_i)_\omega(z) := \varphi_i(z\omega)$ and $(h_i)_\omega(z) := h_i(z\omega)$. We see easily

$$\bigcup_i \{\omega; (\varphi_i)_\omega \equiv 0\} \cup \bigcup_{i<j} \left\{\omega; \frac{(h_i)_\omega}{(h_j)_\omega} \equiv \frac{h_i(0)}{h_j(0)}\right\} \subsetneqq C^n - \{0\} .$$

Therefore, we can find some $\omega \in C^n - \{0\}$ such that $(\varphi_i)_\omega \not\equiv 0$ $(1 \leqq i \leqq p)$ and $(h_i)_\omega/(h_j)_\omega \not\equiv \text{const}\,(1 \leqq i < j \leqq p)$. The assumption (2) gives the identity

$$(\varphi_1)_\omega(h_1)_\omega + \cdots + (\varphi_p)_\omega(h_p)_\omega \equiv 0\,.$$

This contradicts Theorem 2.10 for the case $n = 1$. We have thus the desired result.

COROLLARY 2.11. *Let $h_1, \cdots, h_p \in \boldsymbol{H}^*$ and assume that*

$$h_1^{\ell_1}h_2^{\ell_2} \cdots h_p^{\ell_p} \not\equiv \mathit{const}$$

for any non-zero vector $(\ell_1, \ell_2, \cdots, \ell_p)$ of integers. If finitely many $\varphi_{\ell_1\cdots\ell_p} \in \Phi_0$ satisfy

$$\sum_{(\ell_1,\cdots,\ell_p)} \varphi_{\ell_1\cdots\ell_p} h_1^{\ell_1} \cdots h_p^{\ell_p} \equiv 0\,,$$

then $\varphi_{\ell_1\cdots\ell_p} \equiv 0$ for any $(\ell_1, \cdots, \ell_p)$.

Proof. Since $h_1^{\ell_1} \cdots h_p^{\ell_p} \in \boldsymbol{H}^*$ and

$$h_1^{\ell_1} \cdots h_p^{\ell_p} \not\sim h_1^{m_1} \cdots h_p^{m_p}$$

whenever $(\ell_1, \cdots, \ell_p) \neq (m_1, \cdots, m_p)$, Corollary 2.11 is a direct result of Theorem 2.10.

COROLLARY 2.12. *Let $h_1, \cdots, h_p \in \boldsymbol{H}^*$ and $\varphi_1, \cdots, \varphi_p \in \Phi_0$ satisfy the condition that $\varphi_i \not\equiv 0$ and*

$$\varphi_1 h_1 + \varphi_2 h_2 + \cdots + \varphi_p h_p \equiv 0\,.$$

Consider the partition of indices

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_a$$

such that, for any $i \in I_\alpha$ and $i' \in I_{\alpha'}$, $h_i \sim\cdot h_{i'}$ if $\alpha = \alpha'$, and $h_i \not\sim h_{i'}$ if $\alpha \neq \alpha'$. Then, for any α,

$$\sum_{i \in I_\alpha} \varphi_i h_i \equiv 0\,.$$

Proof. Taking an index $i_\alpha \in I_\alpha$ for each α, we define

$$\psi_\alpha := \sum_{i \in I_\alpha} \varphi_i h_i / h_{i_\alpha}\,.$$

Then, $\psi_\alpha \in \Phi_0$, $h_{i_\alpha} \not\sim h_{i_{\alpha'}}$ if $\alpha \neq \alpha'$, and $\sum_\alpha \psi_\alpha h_{i_\alpha} \equiv 0$. By Theorem 2.10, we have $\psi_1 \equiv \cdots \equiv \psi_a \equiv 0$. This gives Corollary 2.12.

§ 3. Basic lemmas

Take $h_{ij} \in H^*$ and $\varphi_{ij} \in \Phi_0$ with $\varphi_{ij} \not\equiv 0$, where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots$. Defining $f_{ij} := \varphi_{ij} h_{ij}$, we consider a matrix

$$\mathscr{M} := (f_{ij};\, i = 1, 2, \cdots, p,\, j = 1, 2, \cdots)$$

with p rows and countably many columns.

LEMMA 3.1. *If we perform the operations* (a) *changing the order of the indices* $i = 1, 2, \cdots, p$, (b) *replacing a suitable subsequence of the indices* j*'s by* $j = 1, 2, \cdots$ *and* (c) *multiplying each row and each column by a common element of* H^*, *then* $\mathscr{M} = \{f_{ij} := \varphi_{ij} h_{ij}\}$ *may be assumed to satisfy the conditions;*

(i) $h_{ij_1} \not\sim h_{ij_2}$ *if* $1 \leqq i \leqq r$ *and* $j_1 \neq j_2$,

(ii) $h_{ij} \equiv$ const *for any* j *if* $r + 1 \leqq i \leqq p$,

where $0 \leqq r < p$ *and* $r = 0$ *means that* $h_{ij} \equiv$ const *for any* i, j.

Proof. Dividing h_{ij} $(1 \leqq i \leqq p)$ by h_{pj}, we may assume $h_{pj} \equiv 1$ for each j. We consider the smallest integer r such that, after performing the operations (a) ~ (c), the condition (ii) is satisfied, where we may assume $0 < r < p$. Then, for any $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots$, there are only finitely many j' such that $h_{ij} \sim h_{ij'}$. Because, if not, we have some i_0 with $1 \leqq i_0 \leqq r$ and j_0 such that $h_{i_0 j_0} \sim h_{i_0 j}$ for infinitely many j. After performing suitable operations (a) ~ (c), we may assume $h_{rj} \equiv$ const, which contradicts the property of r. We can choose indices $j_1, j_2, \cdots$ such that $j_{\kappa-1} < j_\kappa$ and, for any $i = 1, 2, \cdots, r$,

$$h_{ij_\kappa} \not\sim h_{ij_1},\ h_{ij_\kappa} \not\sim h_{ij_2}, \cdots, h_{ij_\kappa} \not\sim h_{ij_{\kappa-1}}\,.$$

If we replace the indices $j = j_1, j_2, \cdots$ by $j = 1, 2, \cdots$, we obtain the conclusion of Lemma 3.1.

LEMMA 3.2. *Assume that* $\mathscr{M} = \{f_{ij} := \varphi_{ij} h_{ij}\}$ *satisfies the conclusion of Lemma* 3.1 *and, furthermore, for any* $j_1, \cdots, j_p$,

$$\det (f_{ij};\, i = 1, 2, \cdots, p,\, j = j_1, j_2, \cdots, j_p) \equiv 0. \tag{3}$$

If for any j *there exist indices* $j^*_{r+1}, \cdots, j^*_p$ *such that* $j < j^*_{r+1} < \cdots < j^*_p$ *and*

$$\det (f_{ij};\, i = r + 1, \cdots, p,\, j = j^*_{r+1}, \cdots, j^*_p) \not\equiv 0\,, \tag{4}$$

then

$$\det (f_{ij};\, i = 1, 2, \cdots, r,\, j = j_1, j_2, \cdots, j_r) \equiv 0$$

for any $j_1, j_2, \cdots, j_r$.

Proof. If $r = 0$, we have nothing to prove. Let $r > 0$. The set H^* can be regarded as a multiplicative group and includes $C^* := C - \{0\}$ as a subgroup. The factor group $G := H^*/C^*$ is a torsionfree abelian group. We denote the class in G containing an element $h \in H^*$ by $[h]$. We choose finitely many or countably many elements $\eta_1, \eta_2, \cdots, \eta_\tau, \cdots$ in H^* such that

(i) $[\eta_1], [\eta_2], \cdots, [\eta_\tau], \cdots$ are linearly independent over Z and

(ii) each h_{ij} can be represented as

$$h_{ij} = c_{ij}\eta_1^{\ell_{ij}^1}\eta_2^{\ell_{ij}^2} \cdots \eta_\tau^{\ell_{ij}^\tau} \cdots , \tag{5}$$

where $c_{ij} \in C^*$, $\ell_{ij}^\tau \in Z$ and $\ell_{ij}^\tau = 0$ except finitely many τ for each (i, j). Define

$$\ell_{ij} = (\ell_{ij}^1, \ell_{ij}^2, \cdots, \ell_{ij}^\tau, \cdots).$$

By the assumption, $\ell_{ij_1} \neq \ell_{ij_2}$ if $1 \leqq i \leqq r$ and $j_1 \neq j_2$, and $\ell_{ij} = 0$ for any j if $r + 1 \leqq i \leqq p$.

Now, assume

$$\det (f_{ij};\ 1 \leqq i \leqq r, j = j_1, \cdots, j_r) \not\equiv 0 \tag{6}$$

for some $j_1, \cdots, j_r$ with $j_1 < \cdots < j_r$. Then, we can prove

(3.3) *There exist indices* $j_{r+1}, \cdots, j_p$ *with* $j_{r+1} < \cdots < j_p$ *such that, for any* $s = r + 1, \cdots, p$,

(A) $\operatorname{rank} (f_{ij};\ i = r + 1, \cdots, p,\ j = j_{r+1}, \cdots, j_s) = s - r$,

(B) $\ell_{ij_s} \neq \ell_{\sigma_1 j_1} + \cdots + \ell_{\sigma_{s-1} j_{s-1}} - (\ell_{\tau_1 j_1} + \cdots + \ell_{\tau_{s-1} j_{s-1}})$

whenever $1 \leqq i \leqq r$ *and* $\sigma_1, \cdots, \sigma_{s-1}, \tau_1, \cdots, \tau_{s-1} \in \{1, 2, \cdots, p\}$.

To see this, we first choose j_{r+1} with $j_r < j_{r+1}$ such that

$$(f_{r+1 j_{r+1}}, \cdots, f_{p j_{r+1}}) \not\equiv (0, \cdots, 0).$$

Let $j_{r+1}, \cdots, j_{s-1}$ be chosen so that $j_{r+1} < \cdots < j_{s-1}$ and they satisfy the conditions (A) and (B). By $\mathfrak{m}$ we denote the field of all meromorphic functions on C^n. If we set $f_j = {}^t(f_{r+1 j}, \cdots, f_{pj}) \in \mathfrak{m}^{p-r}$, then $f_{j_{r+1}}, \cdots, f_{j_{s-1}}$ are linearly independent over $\mathfrak{m}$. Therefore, there are at most $s - r - 1$ linearly independent elements g's in $\mathfrak{m}^{p-r}$ such that

$$\operatorname{rank} (f_{j_{r+1}}, \cdots, f_{j_{s-1}}, g) \leqq s - r - 1.$$

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On the other hand, for any j, there are indices $j^*_{r+1}, \cdots, j^*_p$ with $j < j^*_{r+1} < \cdots < j^*_p$ satisfying the condition (4). We can choose an index j_s among $j^*_{r+1}, \cdots, j^*_p$ such that

$$\mathrm{rank}\,(f_{j_{r+1}}, \cdots, f_{j_{s-1}}, f_{j_s}) = s - r\,.$$

Accordingly, there are infinitely many j_s's satisfying the condition (A). Next, let us examine the condition (B). The set

$$\{\ell_{\sigma_1 j_1} + \cdots + \ell_{\sigma_{s-1} j_{s-1}} - (\ell_{\tau_1 j_1} + \cdots + \ell_{\tau_{s-1} j_{s-1}});\\ 1 \leqq \sigma_1, \cdots, \sigma_{s-1}, \tau_1, \cdots, \tau_{s-1} \leqq p\}$$

is finite. Since $\ell_{i j_1} \neq \ell_{i j_2}$ if $1 \leqq i \leqq r$ and $j_1 \neq j_2$, there are only finitely many j_s's such that

$$\ell_{i j_s} = \ell_{\sigma_1 j_1} + \cdots + \ell_{\sigma_{s-1} j_{s-1}} - (\ell_{\tau_1 j_1} + \cdots + \ell_{\tau_{s-1} j_{s-1}})$$

for some $i \in \{1, 2, \cdots, r\}$ and $\sigma_1, \cdots, \sigma_{s-1}, \tau_1, \cdots, \tau_{s-1} \in \{1, 2, \cdots, p\}$. Consequently, we can find infinitely many j_s's satisfying the conditions (A) and (B). And, we have the desired indices $j_{r+1}, \cdots, j_p$ inductively.

Now we go back to the proof of Lemma 3.2. Let $j_1, \cdots, j_p$ satisfy the conditions (6) and (A), (B) of (3.3). We denote by S_p the symmetric group of all permutations of p letters $1, 2, \cdots, p$ and set

$$S_p^{(1)} := \left\{\sigma = \begin{pmatrix} 1\,2 & \cdots & p \\ \sigma_1\sigma_2 & \cdots & \sigma_p \end{pmatrix};\ 1 \leqq \sigma_i \leqq r \text{ for } i = 1, 2, \cdots, r\right\}$$
$$S_p^{(2)} := S_p - S_p^{(1)}\,.$$

The assumption (3) may be rewritten

$$(7)\qquad \sum_{\sigma \in S_p^{(1)}} \psi_\sigma h_\sigma + \sum_{\sigma \in S_p^{(2)}} \psi_\sigma h_\sigma \equiv \sum_{\sigma \in S_p} \psi_\sigma h_\sigma \equiv 0\,,$$

where, for $\sigma = \begin{pmatrix} 1\,2 & \cdots & p \\ \sigma_1\sigma_2 & \cdots & \sigma_p \end{pmatrix} \in S_p$,

$$\psi_\sigma := \mathrm{sgn}\,(\sigma)\varphi_{\sigma_1 j_1}\varphi_{\sigma_2 j_2} \cdots \varphi_{\sigma_p j_p} \in \Phi_0$$
$$h_\sigma := h_{\sigma_1 j_1} h_{\sigma_2 j_2} \cdots h_{\sigma_p j_p} \in H^*\,.$$

We shall show $h_\sigma \not\sim h_\tau$ whenever $\sigma \in S_p^{(1)}$ and $\tau \in S_p^{(2)}$. On the contrary, suppose $h_\sigma \sim h_\tau$ for some $\sigma \in S_p^{(1)}$ and $\tau \in S_p^{(2)}$. By substituting (5) and observing the exponents, we get

$$\ell_{\sigma_1 j_1} + \cdots + \ell_{\sigma_p j_p} = \ell_{\tau_1 j_1} + \cdots + \ell_{\tau_p j_p}\,.$$

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By definition, $\{\sigma_{r+1}, \cdots, \sigma_p\} = \{r+1, \cdots, p\}$, and $\{\tau_{r+1}, \cdots, \tau_p\} \neq \{r+1, \cdots, p\}$. Choose index s with $r+1 \leqq s \leqq p$ such that $\tau_s \notin \{r+1, \cdots, p\}$ and $\tau_{s+1}, \cdots, \tau_p \in \{r+1, \cdots, p\}$. Since $\ell_{ij} = 0$ for any j and $i = r+1, \cdots, p$,

$$\ell_{\tau_s j_s} = \ell_{\sigma_1 j_1} + \cdots + \ell_{\sigma_{s-1} j_{s-1}} - (\ell_{\tau_1 j_1} + \cdots + \ell_{\tau_{s-1} j_{s-1}}) .$$

This contradicts the condition (B) of (3.3). We now apply Corollary (2.12) to the indentity (7). From the above shown fact, we can conclude

$$\sum_{\sigma \in S_p^{(1)}} \psi_\sigma h_\sigma = 0 .$$

On the other hand,

$$\begin{aligned} &\sum_{\sigma \in S_p^{(1)}} \psi_\sigma h_\sigma \\ &= \Bigg(\sum_{\sigma = \binom{1\ 2\ \cdots r}{\sigma_1 \sigma_2 \cdots \sigma_r}} \operatorname{sgn}(\sigma) f_{\sigma_1 j_1} \cdots f_{\sigma_r j_r} \Bigg) \times \Bigg(\sum_{\tau = \binom{r+1 \cdots p}{\tau_{r+1} \cdots \tau_p}} \operatorname{sgn}(\tau) f_{\tau_{r+1} j_{r+1}} \cdots f_{\tau_p j_p} \Bigg) \\ &= \det(f_{ij};\, i = 1, 2, \cdots, r,\ j = j_1, \cdots, j_r) \\ &\qquad \times \det(f_{ij};\, i = r+1, \cdots, p, j = j_{r+1}, \cdots, j_p) . \end{aligned}$$

This does not vanish because of (6) and the conclusion of (3.3). The proof of Lemma 3.2 is completed.

LEMMA 3.4. *As in Lemma* 3.2, *suppose that* $\mathscr{M} = \{f_{ij}\}$ *satisfies the condition* (3). *Then, after performing the operations* (b) *and* (c) *of Lemma* 3.1, *we can find indices* $i_1, \cdots, i_m$ *with* $1 \leqq i_1 < \cdots < i_m \leqq p$ *such that*

$$h_{ij} \equiv \text{const}$$

for $i = i_1, \cdots, i_m$ *and* $j = 1, 2, \cdots,$ *and*

$$\det(f_{ij};\, i = i_1, \cdots, i_m,\ j = j_1, \cdots, j_m) \equiv 0$$

for any $j_1, j_2, \cdots, j_m$.

Proof. This is shown by induction on p. If $p = 2$, the conclusion is trivial. Suppose that Lemma 3.4 is true for the case $\leqq p-1$. We may assume that $\mathscr{M}$ satisfies the conditions (i) and (ii) of Lemma 3.1. If the assumption of Lemma 3.2 is satisfied, then we can apply the induction hypothesis to functions f_{ij} for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots$ and so obtain the desired conclusion. Otherwise, there is some j_0 such that

$$\det(f_{ij};\, i = r+1, \cdots, p,\ j = j_{r+1}, \cdots, j_p) \equiv 0$$

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for any $j_{r+1}, \cdots, j_p$ larger than j_0. If we replace $j_0+1, j_0+2, \cdots$ by $1, 2, \cdots$ and set $i_1 = r+1, \cdots, i_m = p$, we have also the desired conclusion.

§ 4. The main theorem

Firstly, we shall recall some notation and terminologies. Let f be a meromorphic map of C^n into $P^N(C)$ which is non-degenerate, that is, the image of f is not included in any hyperplane in $P^N(C)$. For arbitrarily fixed homogeneous coordinates $w_1: w_2: \cdots: w_{N+1}$, f has a reduced representation

$$f = f_1: f_2: \cdots: f_{N+1},$$

where $f_1, \cdots, f_{N+1}$ are holomorphic on C^n and satisfy the condition

$$\operatorname{codim}\{f_1 = f_2 = \cdots = f_{N+1} = 0\} \geqq 2.$$

Take a hyperplane

$$H: a^1 w_1 + a^2 w_2 + \cdots + a^{N+1} w_{N+1} = 0$$

in $P^N(C)$. Regarding it as a divisor on $P^N(C)$, we define its pull-back $\nu(f, H)$ by

$$\nu(f, H)(z) = \nu_F(z) \qquad (z \in C^n)$$

with a holomorphic function

$$F: = a^1 f_1 + a^2 f_2 + \cdots + a^{N+1} f_{N+1}.$$

Now, we consider hyperplanes $H_1, H_2, \cdots, H_{N+2}$ in $P^N(C)$ located in general position and divisors $\nu_1, \nu_2, \cdots, \nu_{N+2}$ on C^n. As is stated in § 1, we denote by $\mathscr{F} := \mathscr{F}(H_i, \nu_i)$ the set of all non-degenerate meromorphic maps of C^n into $P^N(C)$ such that $\nu(f, H_i) = \nu_i$ for $i = 1, 2, \cdots, N+2$. The main Theorem is the following.

THEOREM 4.1. *The set $\mathscr{F}$ contains at most finitely many maps.*

For the proof, we identify $P^N(C)$ with the subspace

$$\{w_1 + w_2 + \cdots + w_{N+2} = 0\}$$

in $P^{N+1}(C)$, where $w_1: \cdots: w_{N+2}$ are homogeneous coordinates on $P^{N+1}(C)$. Moreover, by a suitable change of coordinates, we may assume

$$H_i = \{w_i = 0\} \cap P^N(C) \qquad (1 \leqq i \leqq N+2).$$

Suppose that $\mathscr{F}$ contains infinitely many mutually distinct maps $f^1, f^2, \cdots, f^j, \cdots$. Using the above coordinates, we take a reduced representation

$$f^j = f_1^j : f_2^j : \cdots : f_{N+2}^j$$

of each f^j. By (2.7) there exist entire functions k_i with $\nu_{k_i} = \nu_i$ for $i = 1, 2, \cdots, N+2$. Since $\nu(f^j, H_i) = \nu_i$,

$$h_{ij} := f_i^j / k_i \in \boldsymbol{H}^* .$$

They satisfy $\sum_{i=1}^{N+2} h_{ij} k_i \equiv 0$ for any $j = 1, 2, \cdots$.

LEMMA 4.2. *Let $h_{ij} \in \boldsymbol{H}^*$ $(1 \leqq i \leqq p,\ j = 1, 2, \cdots)$ and k_i $(1 \leqq i \leqq p)$ be non-zero entire functions satisfying*

$$\sum_{i=1}^{p} h_{ij} k_i = 0 \tag{8}$$

for any j and, furthermore,

$$\sum_{i \in I} h_{ij} k_i \not\equiv 0 \tag{9}$$

for any j and any proper subset I of $\{1, 2, \cdots, p\}$. Then, there exists a subsequence $\{j_1, j_2, \cdots\}$ of $\{1, 2, \cdots\}$ such that $h_{ij} \equiv$ const for any $i = 1, 2, \cdots, p$ and $j = j_1, j_2, \cdots$ after dividing each row and each column of (h_{ij}) by a common element of $\boldsymbol{H}^*$.

Proof. The proof is given by induction on p. If $p = 2$, we have easily Lemma 4.2 because $h_{1j_1}/h_{2j_1} = h_{1j_2}/h_{2j_2}$ for any j_1 and j_2. Suppose that Lemma 4.2 is true in the case $\leqq p-1$. Eliminating k_i from the identities (8), we get

$$\det(h_{ij};\ i = 1, 2, \cdots, p,\ j = j_1, j_2, \cdots, j_p) \equiv 0$$

for any $j_1, j_2, \cdots, j_p$. We now apply Lemma 3.4. After performing the operations (a) $\sim$ (c) of Lemma 3.1, it may be assumed that $h_{ij} \equiv$ const for any j and i with $r+1 \leqq i \leqq p$, and

$$\det(h_{ij};\ i = r+1, \cdots, p,\ j = j_{r+1}, \cdots, j_p) \equiv 0$$

for any $j_{r+1}, \cdots, j_p$, where $0 \leqq r \leqq p-1$. Suppose that $r > 0$. We may assume that $h_{1j_1} \not\sim h_{1j_2}$ for any j_1, j_2 with $j_1 \neq j_2$ by the same argument as in the proof of Lemma 3.1. When we multiply the i-th row of (h_{ij}) by a function in $\boldsymbol{H}^*$, (8) does not alter if we replace k_i by one divided by the same function. When we multiply the j-th column of (h_{ij}) by a function in H^*, (8) remains valid if (8) is replaced by one divided by the same function. Therefore, we may assume in the original identities (8) that $h_{1j_1} \not\sim h_{1j_2}$ if $j_1 \neq j_2$ and $h_{ij} \equiv \text{const}$ if $r+1 \leqq i \leqq p$.

Since

$$\operatorname{rank}(h_{ij};\ r+1 \leqq i \leqq p,\ j = 1, 2, \cdots) < p - r,$$

we can find a non-zero vector $(\lambda_{r+1}, \cdots, \lambda_p) \in C^{p-r}$ such that

$$\sum_{i=r+1}^{p} \lambda_i h_{ij} = 0 \qquad (j = 1, 2, \cdots).$$

Take a regular matrix $A = (a_{ij};\ r+1 \leqq i, j \leqq p)$ of order $p - r$ such that $a_{ip} = \lambda_i$ $(r+1 \leqq i \leqq p)$. Define functions $k^*_{r+1}, \cdots, k^*_p$ by the relations

$$k_i = \sum_{\ell=r+1}^{p} a_{i\ell} k^*_\ell \qquad (r+1 \leqq i \leqq p).$$

Then, (8) becomes

$$\sum_{i=1}^{r} h_{ij} k_i + \textstyle\sum_{\ell=r+1}^{p-1} h^*_{\ell j} k^*_\ell = 0,$$

where

$$h^*_{\ell j} := \sum_{i=r+1}^{p} a_{i\ell} h_{ij} \in C.$$

For convenience' sake, we set $k^*_i := k_i$ and $h^*_{ij} := h_{ij}$ for $i = 1, 2, \cdots, r$. After changing the indices j's suitably, we can take a subset I of $\{1, 2, \cdots, p-1\}$ such that $1 \in I$,

$$\sum_{i \in I} h^*_{ij} k^*_i \equiv 0$$

and for any proper subset I' of I and any $j = 1, 2, \cdots$,

$$\sum_{i \in I'} h^*_{ij} k^*_i \not\equiv 0.$$

By the assumption (9), there is some $i_0 \in I \cap \{r+1, \cdots, p\}$. Since $\sharp I \leqq p - 1$, by the induction hypothesis we see $h^*_{ij} \equiv \text{const}$ for $i \in I$ and $j = 1, 2, \cdots$ after suitable changes of indices and h^*_{ij}. This is a contradiction. Because, $h^*_{i_0 j} \equiv \text{const}$ for any j and $h^*_{1j_1} \not\sim h^*_{1j_2}$ for any j_1, j_2 with $j_1 \neq j_2$. Consequently, $r = 0$ and we have Lemma 4.2.

Proof of Theorem 4.1. As a consequence of Lemma 4.2, changing k_i suitably, taking a suitable subsequence of the indices j's and choosing a suitable reduced representation of each f^j, we may assume $h_{ij} \equiv \text{const}$ for any i, j, and particularly

$$h_{11} \equiv h_{21} \equiv \cdots \equiv h_{p1} \equiv 1,$$

where $p = N + 2$. Then

$$f^j = h_{1j}f_1^1 : h_{2j}f_2^1 : \cdots : h_{N+2j}f_{N+2}^1$$

for $j = 2, 3, \cdots$, which satisfy

$$h_{1j}f_1^1 + h_{2j}f_2^1 + \cdots + h_{N+2j}f_{N+2}^1 = 0.$$

By the assumption that f^1 is non-degenerate, we obtain

$$h_{1j} = h_{2j} = \cdots\cdot = h_{N+2j}.$$

This shows that

$$f^1 = f^2 = \cdots,$$

which is absurd. We have thus Theorem 4.1.

THEOREM 4.3. *Let $\gamma: C^n \to C^n$ be a biholomorphic map and $f: C^n \to P^N(C)$ a non-degenerate meromorphic map. If there exist hyperplanes $H_1, \cdots, H_{N+2}$ in general position such that $\nu(f, H_i) \circ \gamma = \nu(f, H_i)$ $(1 \leqq i \leqq N+2)$, then $f \circ \gamma^{j_0} = f$ for some positive integer j_0.*

Proof. Consider

$$\mathscr{F} := \mathscr{F}(H_1, \cdots, H_{N+2}, \nu(f, H_1), \cdots, \nu(f, H_{N+2})).$$

Obviously, the assumption implies that $f \circ \gamma^j \in \mathscr{F}$ for any positive integer j. Since $\# \mathscr{F} < \infty$, $f \circ \gamma^{j_1} = f \circ \gamma^{j_2}$ for some j_1, j_2 with $j_1 < j_2$. Then, $f \circ \gamma^{j_0} = f$ for $j_0 := j_2 - j_1$.

§ 5. Meromorphic functions of semi-invariant type

Let $\gamma: C^n \to C^n$ be a biholomorphic map and Φ a family of meromorphic functions on C^n.

DEFINITION 5.1. We call Φ a *γ-admissible family* if it satisfies the following conditions;

(i) Φ is a field which includes C,

(ii) any $\varphi \in \Phi$ is of order less than one,

(iii) Φ is γ-invariant, namely, $\varphi \circ \gamma \in \Phi$ whenever $\varphi \in \Phi$,

(iv) if $\varphi \circ \gamma^j = c\varphi$ for some $\varphi \in \Phi$, $c \in C$ and a positive integer j, then $\varphi \equiv$ const.

EXAMPLE 5.2. 1°. The field C of all constant functions is obviously a γ-admissible family for any biholomorphic map γ: $C^n \to C^n$.

2°. Let us consider a linear map $\gamma(z) = Az + B$, where A is a regular matrix of order n and $B \in C^n$. If there is no pure $(n-1)$-dimensional analytic set V in C^n which is of order less than one and γ^{j_0}-invariant for some positive integer j_0, then the field Φ_0 of all meromorphic functions of order less than one is a γ-admissible family. In fact, by (2.4) Φ_0 is a field and obviously satisfies the conditions (i) ~ (iii). We now suppose that $\varphi \circ \gamma^{j_0} = c\varphi$ for some nonconstant $\varphi \in \Phi_0$, $c \in C^*$ and a positive integer j_0. Then, $V := |\nu_\varphi^0| \cup |\nu_\varphi^\infty|$ is a γ^{j_0}-invariant analytic set which is not empty because of (2.9). And, V is of order less than one by (2.6), which contradicts the assumption. Therefore, Φ_0 satisfies also the condition (iv).

In the case of $n = 1$, the map γ defined by $\gamma(z) = z + \omega$ for some $\omega \in C^*$ has the above-mentioned property. For, if a discrete set V is γ^{j_0}-invariant and contains a point z_0, we have also $z_0 + jj_0\omega \in V$ for $j = 1, 2, \cdots$. Then, there exists a positive constant c such that

$$\sharp\{z \in V;\ |z| \leqq t\} \geqq ct$$

for a sufficiently large t, and so

$$N(r, \nu_V) \geqq cr$$

for a sufficiently large r. The set V is not of order less than one.

In the following, γ denotes a biholomorphic map of C^n onto C^n itself and Φ denotes a γ-admissible family.

DEFINITION 5.3. A meromorphic function $F(z)$ on C is called to be of *(γ, Φ)-semi-invariant type* if it has a representation

$$F(z) = \varphi_1(z)g_1(z) + \cdots + \varphi_p(z)g_p(z) \tag{10}$$

with $\varphi_1, \cdots, \varphi_p \in \Phi$ and meromorphic functions $g_1, \cdots, g_p$ on C^n such that $g_i \circ \gamma = c_i g_i$ for some $c_i \in C$.

DEFINITION 5.4. A representation (10) is called a *reduced representation* if it satisfies the conditions;

(i) $F(z) \not\equiv \sum_{i \in I} \varphi_i g_i$ for any proper subset I of $\{1, 2, \cdots, p\}$,

(ii) whenever $c_{i_1} = c_{i_2} = \cdots = c_{i_m}$, $\{\varphi_{i_1}, \cdots, \varphi_{i_m}\}$ and $\{g_{i_1}, \cdots, g_{i_m}\}$ are both linearly independent over C.

(5.5) *any meromorphic function of* (γ, Φ)*-semi-invariant type has a reduced representation.*

Let $F(z)$ have a representation (10) with $\varphi_i \in \Phi$ and g_i such that $g_i \circ \gamma = c_i g_i$ for some $c_i \in C$. Changing indices, we may assume

$$c_1 = \cdots = c_{p_1},\ c_{p_1+1} = \cdots = c_{p_2}, \cdots, c_{p_{a-1}+1} = \cdots = c_{p_a}.$$

and $c_{p_\alpha} \neq c_{p_{\alpha'}}$ if $\alpha \neq \alpha'$, where $1 \leqq p_1 < \cdots < p_a = p$. For example, for indices $1, 2, \cdots, p_1$, it may be assumed that $\varphi_1, \cdots, \varphi_r$ $(1 \leqq r \leqq p_1)$ are linearly independent and

$$\varphi_i = \sum_{j=1}^{r} c_{ij}\varphi_j \qquad (r+1 \leqq i \leqq p_1)$$

for some $c_{ij} \in C$. Then, if we set

$$\tilde{g}_j := g_j + \sum_{i=r+1}^{p_1} c_{ij} g_i\,,$$

we see $\tilde{g}_j \circ \gamma = c_{p_1}\tilde{g}_j$ and

$$\sum_{i=1}^{p_1} \varphi_i g_i = \sum_{j=1}^{r} \varphi_j \tilde{g}_j\,.$$

Moreover, we may choose indices such that $\tilde{g}_1, \cdots, \tilde{g}_s$ are linearly independent and

$$\tilde{g}_j = \sum_{\ell=1}^{s} d_{\ell j}\tilde{g}_\ell \qquad (s+1 \leqq j \leqq r)$$

for some $d_{\ell j} \in C$. We have then

$$\sum_{i=1}^{p_1} \varphi_i g_i = \sum_{\ell=1}^{s} \tilde{\varphi}_\ell \tilde{g}_\ell\,,$$

where $\tilde{\varphi}_\ell := \varphi_\ell + \sum_{j=s+1}^{r} d_{\ell j}\varphi_j \in \Phi$ and $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_s$ are linearly independent. By the same reason, $\sum_{i=p_{\alpha-1}+1}^{p_\alpha} \varphi_i g_i$ has a reduced representation for each α, whence we conclude (5.5).

THEOREM 5.6. *Let* $F(z)$ *have two reduced representations*

$$F(z) = \sum_{i=1}^{p} \varphi_i f_i = \sum_{j=1}^{q} \psi_j g_j\,,$$

where $\varphi_i \in \Phi$, $\psi_j \in \Phi$, $f_i \circ \gamma = c_i f_i$ *and* $g_j \circ \gamma = d_j g_j$ *for some* $c_i, d_j \in C$. *Then,*

$p=q$ and, after a suitable change of indices, we can find a partition of indices

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_a$$

satisfying the conditions that, for each $\alpha = 1, 2, \cdots, a$,

(i) $c_i = c_{i'}$ *and* $d_i = d_{i'}$ *if* $i, i' \in I_\alpha$,

(ii) $\sum_{i\in I_\alpha} \varphi_i f_i = \sum_{i\in I_\alpha} \psi_i g_i$,

(iii) *there is a regular matrix* $C^\alpha = (c^\alpha_{ij}; i, j \in I_\alpha)$ *such that*

$$g_j = \sum_{i\in I_\alpha} c^\alpha_{ij} f_i, \quad \varphi_i = \sum_{j\in I_\alpha} c^\alpha_{ij} \psi_j .$$

For the proof, we need some lemmas.

LEMMA 5.7. *Let $\varphi_1, \cdots, \varphi_p, g_1, \cdots, g_p$ be non-zero meromorphic functions on C^n such that $\varphi_i \in \Phi$ and $g_i \circ \gamma = c_i g_i$ for some $c_i \in C$. If*

$$\det((\varphi_i \circ \gamma^{j-1})(g_i \circ \gamma^{j-1});\ \ 1 \leqq i, j \leqq p) \equiv 0\,, \tag{11}$$

then $c_{i_1} = c_{i_2} = \cdots = c_{i_m}$ and $\varphi_{i_1}, \varphi_{i_2}, \cdots, \varphi_{i_m}$ are linearly dependent for some $i_1, \cdots, i_m$ with $1 \leqq i_1 < \cdots < i_m \leqq p$.

Proof. This is shown by induction on p. If $p = 2$, $\varphi_1 \circ \gamma / \varphi_2 \circ \gamma = (c_2/c_1)(\varphi_1/\varphi_2)$ and $\varphi_1/\varphi_2 \in \Phi$ by (11). By Definition 5.1, (iv), $\varphi_1/\varphi_2 \equiv$ const and $c_1 = c_2$, which gives Lemma 5.7. Suppose that Lemma 5.7 is valid in the case $\leqq p-1$. For brevity's sake, we define $f_{ij}: = (\varphi_i \circ \gamma^{j-1})(g_i \circ \gamma^{j-1})$. For each $j = p-1, p-2, \cdots$ we subtract the j-th column multiplied by f_{pj+1} from the $(j+1)$-th column multiplied by f_{pj} in order. Consequently we obtain

$$\det(f_{pj} f_{ij+1} - f_{pj+1} f_{ij};\ 1 \leqq i, j \leqq p-1) \equiv 0\,. \tag{12}$$

Define

$$\tilde\varphi_i : = c_i \varphi_p (\varphi_i \circ \gamma) - c_p \varphi_i (\varphi_p \circ \gamma)$$
$$\tilde g_i : = g_i g_p$$

for $i = 1, 2, \cdots, p-1$. Then, $\tilde\varphi_i \in \Phi$, $\tilde g_i \circ \gamma = c_i c_p \tilde g_i$ and

$$\det((\tilde\varphi_i \circ \gamma^{j-1})(\tilde g_i \circ \gamma^{j-1});\ 1 \leqq i, j \leqq p-1) \equiv 0\,.$$

If $\tilde\varphi_i \tilde g_i = f_{p1} f_{i2} - f_{p2} f_{i1} \equiv 0$, we have easily $c_i = c_p$, $\varphi_i/\varphi_p \equiv$ const and so the conclusion of Lemma 5.7. We may assume $\tilde\varphi_i \tilde g_i \not\equiv 0$ for any i. We now apply the induction hypothesis to functions $\tilde\varphi_i$, $\tilde g_i$. There are indices $i_1, \cdots,$

i_{m-1} with $1 \leqq i_1 < \cdots < i_{m-1} \leqq p-1$ such that $c_{i_1} = \cdots = c_{i_{m-1}}$ and

$$\sum_\ell a_\ell \tilde{\varphi}_{i_\ell} = \varphi_p\Big(\sum_\ell a_\ell c_{i_\ell}(\varphi_{i_\ell} \circ \gamma)\Big) - c_p(\varphi_p \circ \gamma)\Big(\sum_\ell a_\ell \varphi_{i_\ell}\Big) = 0$$

for some non-zero vector $(a_1, \cdots, a_{m-1})$. This implies that a function $\psi := \sum_\ell a_\ell \varphi_{i_\ell}/\varphi_p$ in Φ satisfies $\psi \circ \gamma = (c_p/c_{i_1})\psi$, where we may assume $\psi \not\equiv 0$. By Definition 5.1, (iv), $\psi \equiv$const and $c_{i_1} = c_p$. Consequently, $c_{i_1} = \cdots = c_{i_{m-1}} = c_p$ and $\varphi_{i_1}, \cdots, \varphi_{i_{m-1}}, \varphi_p$ are linearly dependent. The proof is completed.

LEMMA 5.8. *Let $\varphi_1, \cdots, \varphi_p, g_1, \cdots, g_p$ be functions as in Lemma 5.7 and assume that*

(13) $$\varphi_1 g_1 + \varphi_2 g_2 + \cdots + \varphi_p g_p \equiv 0\,.$$

Consider the partition of indices

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_a$$

such that, for any $i \in I_\alpha$ and $i' \in I_{\alpha'}$, $c_i = c_{i'}$ if $\alpha = \alpha'$, and $c_i \neq c_{i'}$ if $\alpha \neq \alpha'$. Then, for any α,

(i) $\sum_{i \in I_\alpha} \varphi_i g_i \equiv 0$,

(ii) $\{\varphi_i;\, i \in I_\alpha\}$ *are linearly dependent.*

Proof. By (13), we have

$$(\varphi_1 \circ \gamma^{j-1})(g_1 \circ \gamma^{j-1}) + \cdots + (\varphi_p \circ \gamma^{j-1})(g_p \circ \gamma^{j-1}) \equiv 0$$

for $j = 1, 2, \cdots, p$. Therefore,

$$\det\,((\varphi_i \circ \gamma^{j-1})(g_i \circ \gamma^{j-1});\quad 1 \leqq i,\, j \leqq p) \equiv 0\,.$$

By Lemma 5.7, $\{\varphi_i;\, i \in I_\alpha\}$ are linearly dependent for some α. This shows that (ii) is a consequence of (i). To prove (i), it suffices to get an absurd conclusion under the assumption that

$$F_\alpha := \sum_{i \in I_\alpha} \varphi_i g_i \not\equiv 0$$

for any α. Take a reduced representation

$$F_\alpha(z) = \sum_{i \in \tilde{I}_\alpha} \tilde{\varphi}_i(z)\tilde{g}_i(z)\,,$$

where $\tilde{I}_\alpha \subseteqq I_\alpha$, $\tilde{\varphi}_i \in \Phi$ and $\tilde{g}_i \circ \gamma = c_{i_\alpha}\tilde{g}_i$ for some $i_\alpha \in I_\alpha$. Then, the identity

$$\sum_{\alpha=1}^{a} \sum_{i \in \tilde{I}_\alpha} \tilde{\varphi}_i \tilde{g}_i = 0$$

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contradicts Lemma 5.7. We have thus Lemma 5.8.

LEMMA 5.9. *Let* $\varphi_1, \cdots, \varphi_p \in \Phi$ *and* $g_1, \cdots, g_p$ *be meromorphic functions on* C^n *such that* $g_i \circ \gamma = c_i g_i$ *for some* $c_i \in C$ *and*

$$\varphi_1 g_1 + \varphi_2 g_2 + \cdots + \varphi_p g_p \equiv 0 .$$

If $\varphi_1, \cdots, \varphi_p$ *are linearly independent, then* $g_1 \equiv \cdots \equiv g_p \equiv 0$.

Proof. If $g_{i_0} \not\equiv 0$ for some i_0, $\{\varphi_i;\, c_i = c_{i_0}\}$ are linearly dependent by Lemma 5.8. This contradicts the assumption. We have thus the conclusion of Lemma 5.9.

Proof of Theorem 5.6. Take the partitions of indices

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_a$$
$$\{1, 2, \cdots, q\} = J_1 \cup J_2 \cup \cdots \cup J_b$$

such that, for $i \in I_\alpha$, $i' \in I_{\alpha'}$, $j \in J_\beta$, $j' \in J_{\beta'}$, we have $c_i = c_{i'}$, $d_j = d_{j'}$ if $\alpha = \alpha'$, $\beta = \beta'$, and $c_i \neq c_{i'}$, $d_j \neq d_{j'}$ if $\alpha \neq \alpha'$, $\beta \neq \beta'$. Define

$$F_\alpha := \sum_{i \in I_\alpha} \varphi_i f_i ,$$
$$G_\beta := \sum_{j \in I_\beta} \psi_j g_j ,$$

which do not vanish by Definition 5.4, (i). Apply Lemma 5.8 to the identity

$$\sum_{i=1}^{p} \varphi_i f_i - \sum_{j=1}^{q} \psi_j g_j \equiv 0 .$$

We see easily $a = b$ and $F_\alpha \equiv G_\alpha$ $(1 \leqq \alpha \leqq a)$ after a suitable change of indices. This gives (i) and (ii) of Theorem 5.6.

To prove (iii), we may assume $a = b = 1$ and so $c_1 = \cdots = c_p = d_1 = \cdots = d_q$. Since $\psi_1, \cdots, \psi_q$ are linearly independent, we can choose indices such that $\psi_1, \cdots, \psi_q, \varphi_1, \cdots, \varphi_r$ are linearly independent and

$$\varphi_i = \sum_{j=1}^{q} c_{ij} \psi_j + \sum_{j=1}^{r} d_{ij} \varphi_j \qquad (\mathrm{r} + 1 \leqq i \leqq p) , \tag{14}$$

where $0 \leqq r \leqq p$ and c_{ij}, $d_{ij} \in C$. Then

$$\sum_{j=1}^{q} \psi_j \Big(g_j - \sum_{i=r+1}^{p} c_{ij} f_i \Big) - \sum_{j=1}^{r} \varphi_j \Big(f_j + \sum_{i=r+1}^{p} d_{ij} f_i \Big) = 0 .$$

It follows from Lemma 5.9 that

$$g_j = \sum_{i=r+1}^{p} c_{ij} f_i \qquad (1 \leqq j \leqq q) ,$$

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$$f_j + \sum_{i=r+1}^{p} d_{ij} f_i = 0 \qquad (1 \leqq j \leqq r).$$

We note here the case $r \geqq 1$ is impossible because $f_1, \cdots, f_p$ are linearly independent. Therefore, (14) becomes

$$\varphi_i = \sum_{j=1}^{q} c_{ij} \psi_j .$$

The similar argument is available if we exchange the roles of f_i's and g_j's. We can conclude that $p = q$ and $C = (c_{ij})$ is a regular matrix.

§6. Meromorphic functions with γ-invariant zeros and poles

In this section, γ denotes a biholomorphic map of C^n onto C^n itself and Φ denotes a γ-admissible family. For non-zero meromorphic functions g_1 and g_2 on C^n, we mean by notation $g_1 \underset{\gamma}{\sim} g_2$ that g_1/g_2 is γ^{j_0}-invariant, namely, $g_1 \circ \gamma^{j_0}/g_2 \circ \gamma^{j_0} = g_1/g_2$ for some positive integer j_0.

THEOREM 6.1. *Let $\varphi_1, \cdots, \varphi_p \in \Phi$ and $g_1, \cdots, g_p$ be non-zero meromorphic functions with $\nu_{g_i} \circ \gamma = \nu_{g_i}$. If*

$$\varphi_1 g_1 + \varphi_2 g_2 + \cdots + \varphi_p g_p \equiv 0 \tag{15}$$

and $\sum_{i \in I} \varphi_i g_i \not\equiv 0$ for any proper subset I of $\{1, 2, \cdots, p\}$, then

$$g_1 \underset{\gamma}{\sim} g_2 \underset{\gamma}{\sim} \cdots \underset{\gamma}{\sim} g_p .$$

For the proof, we give

LEMMA 6.2. *Let $\varphi_1, \cdots, \varphi_p \in \Phi$ and $g_1, \cdots, g_p$ be meromorphic functions such that $\varphi_i \not\equiv 0$, $g_i \not\equiv 0$ and $\nu_{g_i} \circ \gamma = \nu_{g_i}$. If*

$$\det\left((\varphi_i \circ \gamma^{j-1})(g_i \circ \gamma^{j-1});\ i = 1, 2, \cdots, p,\ j = j_1, \cdots, j_p\right) \equiv 0 \tag{16}$$

for any $j_1, j_2, \cdots, j_p$, then

$$g_{i_1} \underset{\gamma}{\sim} g_{i_2} \underset{\gamma}{\sim} \cdots \underset{\gamma}{\sim} g_{i_m}, \qquad m \geqq 2$$

and $\varphi_{i_1}, \cdots, \varphi_{i_m}$ are linearly dependent over C for some indices $i_1, \cdots, i_m$ with $1 \leqq i_1 < i_2 < \cdots < i_m \leqq p$.

Proof. We prove this by induction on p. If $p = 2$, we have

$$\frac{\varphi_1}{\varphi_2} \frac{\varphi_2 \circ \gamma}{\varphi_1 \circ \gamma} = \frac{g_1 \circ \gamma}{g_1} \frac{g_2}{g_2 \circ \gamma} .$$

This is reduced to a constant c because the left side is in Φ and the right side is in H^* by the assumption. Therefore,

$$\frac{\varphi_1 \circ \gamma}{\varphi_2 \circ \gamma} = c\frac{\varphi_1}{\varphi_2}.$$

And, $\varphi_1/\varphi_2 \equiv$ const and $c = 1$ by virtue of Definition 5.1, (iv). This implies the γ-invariance of g_1/g_2.

Suppose that Lemma 6.2 is true in the case $\leqq p-1$. Changing indices, we may assume that

(α) if $r+1 \leqq i_1 < i_2 \leqq p$, then $g_{i_1} \circ \gamma^j / g_{i_2} \circ \gamma^j = c g_{i_1}/g_{i_2}$ for some positive integer j and a constant c,

(β) if $1 \leqq i_1 \leqq r$ and $r+1 \leqq i_2 \leqq p$, then there is no constant c with such a property for any j.

Moreover, replacing γ^j by γ, we may take $j = 1$ in (α). On the other hand, (16) remains valid if we divide the j-th column of the matrix

$$((\varphi_i \circ \gamma^{j-1})(g_i \circ \gamma^{j-1});\ i = 1, 2, \cdots, p,\ j = 1, 2, \cdots)$$

by $g_p \circ \gamma^{j-1}$. Replacing g_i/g_p by g_i, we may assume that $g_p \equiv 1$ and so $g_{r+1}, \cdots, g_p$ satisfies $g_i \circ \gamma = c_i g_i$ for some c_i. Define

$$h_{ij} := \frac{g_i \circ \gamma^{j-1}}{g_i}$$

for any $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots$. Then, $h_{ij} \equiv$ const for any j if $r+1 \leqq i \leqq p$. Moreover, $h_{ij_1}/h_{ij_2} \not\equiv$ const if $1 \leqq i \leqq r$ and $j_1 < j_2$. For, if not, $g_i \circ \gamma^{j_2 - j_1}/g_i \equiv$ const, which contradicts the above condition (β). Divide the i-th row of (16) by g_i. We have then

$$\det((\varphi_i \circ \gamma^{j-1})h_{ij};\ i = 1, 2, \cdots, p,\ j = j_1, \cdots, j_p) \equiv 0$$

for any $j_1, \cdots, j_p$.

We now assume that the conclusion of Lemma 6.2 is false, namely, $\varphi_{i_1}, \cdots, \varphi_{i_m}$ are linearly independent whenever

$$g_{i_1} \underset{\gamma}{\sim} g_{i_2} \underset{\gamma}{\sim} \cdots \underset{\gamma}{\sim} g_{i_m}.$$

Give a positive integer j arbitrarily and define

$$j^*_{r+1} := j,\ j^*_{r+2} := 2j, \cdots, j^*_p := (p-r)j.$$

And, apply Lemma 5.7 to γ^j, $\varphi_i \circ \gamma^j$ and $g_i \circ \gamma^j$ $(r+1 \leqq i \leqq p)$ instead of γ, φ_i and g_i respectively. As its consequence, we see

$$\det((\varphi_i\circ\gamma^{kj})(g_i\circ\gamma^{kj});\ i=r+1,\cdots,p,\ k=1,\cdots,p-r)\not\equiv 0\,,$$

whence

$$\det((\varphi_i\circ\gamma^{j-1})h_{ij};\ i=r+1,\cdots,p,\ j=j^*_{r+1},\cdots,j^*_p)\not\equiv 0\,.$$

This shows that a matrix $\mathscr{M}=((\varphi_i\circ\gamma^{j-1})h_{ij})$ satisfies all assumptions of Lemma 3.2. We can conclude

$$\det((\varphi_i\circ\gamma^{j-1})(g_i\circ\gamma^{j-1});\ i=1,2,\cdots,r,\ j=j_1,\cdots,j_r)\equiv 0$$

for any $j_1,\cdots,j_r$. On the other hand, by the assumption the conclusion of Lemma 5.7 does not occur. This is a contradiction. We have thus Lemma 6.2.

Proof of Theorem 6.1. According to (15), we have

$$(\varphi_1\circ\gamma^{j-1})(g_1\circ\gamma^{j-1})+\cdots+(\varphi_p\circ\gamma^{j-1})(g_p\circ\gamma^{j-1})\equiv 0$$

for any $j=1,2,\cdots$. Therefore,

$$\det((\varphi_i\circ\gamma^{j-1})(g_i\circ\gamma^{j-1});\ i=1,\cdots,p,\ j=j_1,\cdots,j_p)\equiv 0$$

for any $j_1,\cdots,j_p$. If $p=2$, the desired conclusion is a direct result of Lemma 6.2. Now, suppose that Theorem 6.1 is true in the case $\leqq p-1$ and false in the case p. Changing indices, we may assume

$$g_{r+1}\underset{\gamma}{\sim}g_{r+2}\underset{\gamma}{\sim}\cdots\underset{\gamma}{\sim}g_p\underset{\gamma}{\not\sim}g_1$$

and $\varphi_{r+1},\cdots,\varphi_p$ are linearly dependent over $\boldsymbol{C}$ by the help of Lemma 6.2, where $1\leqq r<p$. Replacing $g_ig_p^{-1}$ by g_i and γ^j by γ for a suitable positive integer j, each g_i with $r+1\leqq i\leqq p$ may be assumed to be γ-invariant. Moreover, we may write

$$\varphi_p=c_{r+1}\varphi_{r+1}+\cdots+c_{p-1}\varphi_{p-1}$$

with some constants $c_{r+1},\cdots,c_{p-1}$. Define

$$\begin{aligned}\tilde g_i:&=g_i && (1\leqq i\leqq r)\\ \tilde g_i:&=g_i+c_ig_p && (r+1\leqq i\leqq p-1)\,.\end{aligned}$$

Then, $\tilde g_{r+1},\cdots,\tilde g_{p-1}$ are γ-invariant and (15) is rewritten as

$$\sum_{i=1}^{p-1}\varphi_i\tilde g_i=0\,.$$

Take a subset I of $\{1,2,\cdots,p-1\}$ which is minimal among subsets with the property that $1\in I$ and

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$$\sum_{i \in I} \varphi_i \tilde{g}_i \equiv 0 . \tag{17}$$

By the assumption, $I \not\subseteq \{1, 2, \cdots, r\}$ and so I contains some i_0 in $\{r+1, \cdots, p-1\}$. Since $\sharp I \leqq p-1$, we can conclude form (17)

$$g_1 \underset{\gamma}{\sim} g_{i_0} \underset{\gamma}{\sim} g_{r+1} \underset{\gamma}{\sim} \cdots \underset{\gamma}{\sim} g_p$$

by the induction hypothesis. This is a contradiction. Theorem 6.1 is true in the case p too. Consequently, we have Theorem 6.1.

COROLLARY 6.3. *Let $\varphi_1, \cdots, \varphi_p$ be non-zero functions in Φ and $g_1, \cdots, g_p$ non-zero meromorphic functions on C^n with $\nu_{g_i} \circ \gamma = \nu_{g_i}$ satisfying*

$$\varphi_1 g_1 + \varphi_2 g_2 + \cdots + \varphi_p g_p \equiv 0 .$$

Then, there exists a partition of indices

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_a$$

such that, for any α,

$$\sum_{i \in I_\alpha} \varphi_i g_i \equiv 0$$

and $g_i \underset{\gamma}{\sim} g_{i'}$ if $i, i' \in I_\alpha$.

Proof. It suffices to take a partition

$$\{1, 2, \cdots, p\} = I_1 \cup \cdots \cup I_a$$

such that, for any α, $\sum_{i \in I_\alpha} \varphi_i g_i \equiv 0$ and $\sum_{i \in I'_\alpha} \varphi_i g_i \not\equiv 0$ whenever $I'_\alpha \subsetneqq I_\alpha$. By Theorem 6.1, we see easily $g_i \underset{\gamma}{\sim} g_{i'}$ for any $i, i' \in I_\alpha$.

§7. Generalizations of Urabe-Yang's results

In this section, we restrict ourselves to the study of meromorphic functions on C. As in §§ 2 and 3 we denote by Φ_0 the set of all meromorphic functions of order less than one. We consider a biholomorphic functions of order less than one. We consider a biholomorphic map $\gamma_\omega: C \to C$ defined by $\gamma_\omega(z) = z + \omega$ for a constant $\omega \in C^*$.

We first give the following generalization of a result in [12].

Let us consider meromorphic functions

$$\begin{aligned} F &:= \varphi_0 + \sum_{i=1}^{p} \varphi_i f_i \\ G &:= \psi_0 + \sum_{j=1}^{q} \psi_j g_j \end{aligned} \tag{18}$$

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satisfying the conditions:

(i) $\varphi_0, \varphi_1, \cdots, \varphi_p, \psi_0, \psi_1, \cdots, \psi_q \in \Phi_0$ and $f_1, \cdots, f_p, g_1, \cdots, g_q$ are nonzero holomorphic functions on C such that $f_i \circ \gamma_{\omega_1} = f_i$ and $g_j \circ \gamma_{\omega_2} = g_j$ for some $\omega_1, \omega_2 \in C^*$,

(ii) (18) are both reduced representations when they are regarded as meromorphic functions of $(\gamma_{\omega_1}, \Phi_0)$- and $(\gamma_{\omega_2}, \Phi_0)$-semi-invariant type respectively,

(iii) ψ_0 does not belong to the set $\{\varphi_i, \psi_1, \cdots, \psi_q\}_C$ of all linear combinations of $\varphi_i, \psi_1, \cdots, \psi_q$ with constant coefficients for any $i = 1, 2, \cdots, p$,

(iv) $\min\{\nu_{\varphi_0}, \nu_{\varphi_1}, \cdots, \nu_{\varphi_p}\} = \min\{\nu_{\psi_0}, \nu_{\psi_1}, \cdots, \nu_{\psi_q}\}$.

THEOREM 7.1. *If $\nu_F - \nu_G$ is of order less than one, then ω_1/ω_2 is a rational number and $F(z) = cG(z)$ for some $c \in C^*$.*

Remark. In the special case where $p = q = 1$ and $\psi_1 = \varphi_1 = 1$, Theorem 7.1 is Theorem 1 in [12].

For the proof of Theorem 7.1, we need

LEMMA 7.2. *Let $\varphi_1, \cdots, \varphi_p, \psi_1, \cdots, \psi_q \in \Phi_0$ and $f_1, \cdots, f_p, g_1, \cdots, g_q$ be holomorphic functions on C such that $f_i \circ \gamma_{\omega_1} = f_i$ and $g_j \circ \gamma_{\omega_2} = g_j$ for some $\omega_1, \omega_2 \in C^*$. If*

$$\sum_{i=1}^{p} \varphi_i f_i = \sum_{j=1}^{q} \psi_j g_j =: F(z)$$

and $F(z)$ is not of order less than one, then ω_1/ω_2 is a rational number.

Proof. It may be assumed that $\sum_{i=1}^{p} \varphi_i f_i$ and $\sum_{j=1}^{q} \psi_j g_j$ are both reduced representations of $F(z)$ when they are regarded as meromorphic functions of $(\gamma_{\omega_1}, \Phi_0)$- and $(\gamma_{\omega_2}, \Phi_0)$-semi-invariant type.

We first study the case $q = 1$. Without loss of generality, we may assume $\psi_1 \equiv 1$ and so $F \circ \gamma_{\omega_2} = F$. We have then

$$\sum_{i=1}^{p} (\varphi_i \circ \gamma_{\omega_2})(f_i \circ \gamma_{\omega_2}) = \sum_{i=1}^{p} \varphi_i f_i. \tag{19}$$

We note that f_i and $f_i \circ \gamma_{\omega_2}$ are γ_{ω_1}-invariant. We can regard both sides of (19) as reduced representations of a meromorphic function of $(\gamma_{\omega_1}, \Phi_0)$-semi-invariant type. By the help of Theorem 5.6, (iii), we can find a regular matrix $C = (c_{ij})$ such that

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(20) $$\varphi_i(z+\omega_2)=\sum_{j=1}^{p} c_{ij}\varphi_j(z)$$

$$f_j(z)=\sum_{i=1}^{p} c_{ij} f_i(z+\omega_2)\,.$$

Then, by the classical theorem of Jordan, if we take a regular linear transformation

$$\tilde{\varphi}_k=\sum_{\ell=1}^{p} r_{k\ell}\varphi_\ell$$

suitably, (20) is reduced to the relations

$$\tilde{\varphi}_i(z+\omega_2)=\sum_{j=1}^{p} d_{ij}\tilde{\varphi}_j$$

such that $\lambda_i:=d_{ii}\neq 0$, $\varepsilon_i:=d_{ii+1}$ is equal either to 0 or to 1, and $d_{ij}=0$ if $i\geqq j+1$ or $j>i+1$, where $\lambda_i=\lambda_{i+1}$ if $\varepsilon_i=1$. Particularly, we see $\tilde{\varphi}_p(z+\omega_2)=\lambda_p\tilde{\varphi}_p$. As is shown in Example 5.2, 2°, it follows that $\tilde{\varphi}_p\equiv\text{const}$ and $\lambda_p=1$. If $\varepsilon_i=0$ for some i $(1\leqq i\leqq p-1)$, we see also $\tilde{\varphi}_i(z+\omega_2)=\lambda_i\tilde{\varphi}_i(z)$ and hence $\tilde{\varphi}_i\equiv\text{const}$. This is a contradiction because $\tilde{\varphi}_1,\cdots,\tilde{\varphi}_p$ are linearly independent. Therefore, $\varepsilon_1=\cdots=\varepsilon_{p-1}=1$ and so $\lambda_1=\lambda_2=\cdots=\lambda_p=1$. Define

$${}_{\ell}(z)=\sum_{k=1}^{p} r_{k\ell} f_k(z)\,.$$

Then,

$$\tilde{f}_j(z)=\sum_{i=1}^{p} d_{ij}\tilde{f}_i(z+\omega_2)\,.$$

In particular, $\tilde{f}_1(z)=\tilde{f}_1(z+\omega_2)$ and $\tilde{f}_2(z)=\tilde{f}_2(z+\omega_2)+\tilde{f}_1(z+\omega_2)$. Now, assume that ω_1/ω_2 is not a rational number. Since $\tilde{f}_1(z)$ is a periodic holomorphic function with period ω_1 and simultaneously ω_2, it must be a constant. Then,

$$\tilde{f}_2'(z)=\tilde{f}_2'(z+\omega_2)\,.$$

$\tilde{f}_2'$ is also periodic with period ω_1 and ω_2. Hence, $\tilde{f}_2'(z)\equiv\text{const}=:c$. We can write

$$\tilde{f}_2(z)=cz+d\,,$$

where $d\in \boldsymbol{C}$. On the other hand, $\tilde{f}_2(z+\omega_1)=\tilde{f}_2(z)$. We conclude $c=0$ and so $\tilde{f}_2(z)\equiv\text{const}$. This is absurd because $f_1,\cdots,f_p$ are linearly independent. Consequently, ω_1/ω_2 is a rational number.

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Now, we shall prove Lemma 7.2 in the general case. By the assumption,

$$\sum_{i=1}^{p} \varphi_i(z + k\omega_2)f_i(z + k\omega_2) = \sum_{j=1}^{q} \psi_j(z + k\omega_2)g_j(z)$$

for any $k = 0, 1, 2, \cdots$. Since $\psi_1, \cdots, \psi_q$ are assumed to be linearly independent, we have

$$\det(\psi_j(z + (k-1)\omega_2);\ 1 \leqq j,\ k \leqq q) \not\equiv 0$$

as a result of Lemma 5.7. Choosing $\chi^j_{ik} \in \Phi_0$ suitably, we get

$$g_j(z) = \sum_{i\,k} \chi^j_{ik}(z)f_i(z + (k-1)\omega_2)\,.$$

Since $g_j(z)$ and $f_i(z + (k-1)\omega_2)$ are periodic with period ω_2 and ω_1 respectively, by applying Lemma 7.2 with $q = 1$, we conclude that ω_1/ω_2 is a rational number.

Proof of Theorem 7.1. By the assumption, we can write

$$F(z) = h(z)\varphi(z)G(z) \tag{21}$$

with $h \in H^*$ and $\varphi \in \Phi_0$. Substituting $z + k\omega_1$ for z in this identity, for each $k = 0, 1, \cdots, p$ we have

$$\begin{aligned}\varphi_0(z + k\omega_1) &+ \sum_{i=1}^{p} \varphi_i(z + k\omega_1)f_i(z) \\ &= h(z + k\omega_1)\varphi(z + k\omega_1)(\psi_0(z + k\omega_1) + \sum_{j=1}^{q} \psi_j(z + k\omega_1)g_j(z + k\omega_1))\,,\end{aligned}$$

both sides of which we denote by $\chi_k(z)$. Eliminating $f_1, \cdots, f_p$, we obtain

$$\begin{aligned}\Phi(z) :&= \det(\varphi_0(z + k\omega_1), \varphi_1(z + k\omega_1), \cdots, \varphi_p(z + k\omega_1);\ 0 \leqq k \leqq p) \\ &= \det(\chi_k(z), \varphi_1(z + k\omega_1), \cdots, \varphi_p(z + k\omega_1);\ 0 \leqq k \leqq p)\,,\end{aligned}$$

the right side of which we may rewrite

$$\Phi(z) = \sum_{0 \leqq k \leqq p, 0 \leqq \ell \leqq q} \tilde{\varphi}_{k\ell}(z)h(z + k\omega_1)g_\ell(z + k\omega_1)$$

where $g_0 \equiv 1$ and $\tilde{\varphi}_{k\ell} \in \Phi_0$. Since $\varphi_0, \varphi_1, \cdots, \varphi_p$ are linearly independent, $\Phi(z) \not\equiv 0$ by Lemma 5.7. Then, as is easily seen by Corollary 6.3, we can find some k_0 and ℓ_0 such that $1 \underset{\gamma_{\omega_2}}{\sim} h(z + k_0\omega_1)g_{\ell_0}(z + k_0\omega_1)$, and so $h(z + k_0\omega_1)g_{\ell_0}(z + k_0\omega_1)$ is periodic with period $j_0\omega_2$ for a positive integer j_0. Therefore, $h(z)$ itself is periodic with period $j_0\omega_2$. In view of (21) and Lemma

7.2, we can conclude that ω_1/ω_2 is a rational number. Then, f_i, g_j and h are all periodic with period $\omega := k_1\omega_1 = k_2\omega_2$ for some non-zero integers k_1, k_2. We may regard both sides of the identity

$$\varphi_0 + \sum_{i=1}^{p} \varphi_i f_i = \varphi\psi_0 h + \sum_{j=1}^{q} \varphi\psi_j h g_j$$

as two reduced representations of a meromorphic function $F(z)$ of (γ_ω, Φ_0)-semi-invariant type. According to Theorem 5.6, (iii), $p = q$ and there is a regular matrix $C = (c_{ij})$ such that

$$\varphi_i = \sum_{j=0}^{p} c_{ij}\varphi\psi_j \qquad (0 \leqq i \leqq p) \tag{22}$$

$$g_j h = \sum_{i=0}^{p} c_{ij} f_i \qquad (0 \leqq j \leqq p)\,, \tag{23}$$

where $f_0 \equiv g_0 \equiv 1$.

We now take a function $\chi \in \Phi_0$ such that

$$\nu_\chi = \min(\nu_{\varphi_0}, \cdots, \nu_{\varphi_p}) = \min(\nu_{\psi_0}, \cdots, \nu_{\psi_q})$$

by the use of (2.7). Changing $\chi\varphi_i$ and $\chi\psi_j$ by φ_i and ψ_j respectively, we may assume that $\varphi_0, \cdots, \varphi_p, \psi_0, \cdots, \psi_q$ are all holomorphic and

$$\{\varphi_0 = \cdots = \varphi_p = 0\} = \{\psi_0 = \cdots = \psi_q = 0\} = \phi\,. \tag{24}$$

We next write $\varphi = \beta/\alpha$ with holomorphic functions $\alpha, \beta \in \Phi_0$ which have no common zero. Then, (22) becomes

$$\alpha\varphi_i = \sum_j c_{ij}\beta\psi_j\,.$$

If $\beta \not\equiv$ const, β has a zero $z_0 \in C$ because of (2.7). Then, $\alpha(z_0)\varphi_i(z_0) = 0$ for any i and so $\alpha(z_0) = 0$ by (24), which is absurd. We conclude $\beta \equiv$ const. Similarly, we see $\alpha \equiv$ const. We may assume $\varphi \equiv 1$. In (22), if $c_{i0} \neq 0$ for some i with $1 \leqq i \leqq p$, then $\psi_0 \in \{\varphi_i, \psi_1, \cdots, \psi_p\}_C$ which contradicts the assumption (iii). So, $c_{i0} = 0$ for $i = 1, 2, \cdots, p$. We conclude from (23)

$$h = g_0 h = c_{00} \equiv \text{const}\,.$$

This shows Theorem 7.1.

Let us consider two entire functions

$$F = \sum_{i=1}^{p} \varphi_i f_i$$

$$G = \sum_{j=1}^{q} \psi_j g_j\,,$$

where $\varphi_1, \cdots, \varphi_p, \psi_1, \cdots, \psi_q$ are entire functions of order less than one, and $f_1, \cdots, f_p$ and $g_1, \cdots, g_q$ are periodic entire functions with period ω_1 and ω_2 respectively.

COROLLARY 7.3. *Assume that* $\{z, \varphi_1, \cdots, \varphi_p\}$, $\{z, \psi_1, \cdots, \psi_q\}$, $\{1, f_1, \cdots, f_p\}$ *and* $\{1, g_1, \cdots, g_q\}$ *are all linearly independent, and* $z \notin \{\varphi_i, \psi_1, \cdots, \psi_q\}_C$, $z \notin \{\psi_j, \varphi_1, \cdots, \varphi_p\}_C$ *for any* i, j. *If the sets of all fixed points of* $F(z)$ *and* $G(z)$ *coincide with each other except a divisor of order less than one, then* ω_1/ω_2 *is a rational number and* $F(z) = G(z)$.

Proof. Define $\tilde{F}(z) = z - F(z)$ and $\tilde{G}(z) = z - G(z)$. If we set $\varphi_0(z) = \psi_0(z) = z$, they satisfy obviously the conditions (i) $\sim$ (iii). Moreover, (iv) is also satisfied. Because, if φ is a non-constant entire function of order less than one such that $\nu_z \geqq \nu_\varphi$, then we have necessarily $\varphi(z) = cz$ for some $c \in C^*$. On the other hand, the assumption implies that $\nu_{\tilde{F}} - \nu_{\tilde{G}}$ is of order less than one. Therefore, by Theorem 7.1, ω_1/ω_2 is a rational number and there is a constant c such that

$$z - \sum_{i=1}^{p} \varphi_i f_j = c\left(z - \sum_{j=1}^{q} \psi_i g_j\right),$$

so that

$$(c-1)z + \sum_{i=1}^{p} \varphi_i f_i = c\left(\sum_{j=1}^{q} \psi_j g_j\right).$$

Both sides are regarded as reduced representations of a meromorphic function of (γ_ω, Φ_0)-semi-invariant type for some $\omega \in C^*$, By Theorem 5.6, we have easily $c = 1$. This gives Corollary 7.3.

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