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Constancy results for special families of projections

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Abstract

Let $\{\mathbb{V} = V \times \mathbb{R}^l : V \in G(n-l, m-l)\}$ be the family of m-dimensional subspaces of $\mathbb{R}^n$ containing $\{0\} \times \mathbb{R}^l$, and let $\pi_{\mathbb{V}} : \mathbb{R}^n \to \mathbb{V}$ be the orthogonal projection onto $\mathbb{V}$. We prove that the mapping $V \mapsto \operatorname{Dim} \pi_{\mathbb{V}}(B)$ is almost surely constant for any analytic set $B \subset \mathbb{R}^n$, where Dim denotes either Hausdorff or packing dimension.

1. *Introduction*

Unlike Hausdorff dimension, packing dimension is not generally preserved by orthogonal projections. In 1994, M. Järvenpää exhibited in her PhD thesis [**4**] a compact set $K \subset \mathbb{R}^n$ of packing dimension $\dim_{\mathrm{P}} K =: s \leqslant m$, such that the projections of K onto every m-dimensional subspace in $\mathbb{R}^n$ have packing dimension strictly smaller than s. Three years later, K. Falconer and J. Howroyd [**2**] discovered a curious phenomenon: the packing dimension of the projections is almost surely constant – only this constant need not be s.

In this paper, we aim for similar results in a context different from Falconer and Howroyd's. We consider some (particular) subfamilies of the family of all orthogonal projections from $\mathbb{R}^n$ to m-dimensional subspaces – the simplest case covered being the projections onto all 'vertical' planes in $\mathbb{R}^3$. It is obvious that, in general, these subfamilies of projections preserve neither Hausdorff- nor packing dimension. We address the constancy questions in Theorems 1·3 and 1·4 by proving that 'maximal behavior is typical behavior' for the dimension of projections. Such results do not follow from the classical projection theorems of Marstrand, Kaufman and Mattila, even if one takes into account the refined versions with exceptional sets, see [**8**]. We should also mention that our techniques are quite different from the ones developed in [**2**].

As far as we know, constancy issues have not been studied previously for 'small' families of projections, that is, families with a parameter set smaller than the whole Grassmannian $G(n, m)$. However, such families have received some attention quite recently. In [**5**], concluding the work started in [**6**], E. Järvenpää, M. Järvenpää and T. Keleti provide a complete answer to the following question: given a general 'small' non-degenerate family of projections in $\mathbb{R}^n$, how much can the dimension of a set $B \subset \mathbb{R}^n$ (or a measure) drop under these projections? We emphasize that our families of projections are nowhere as general as the ones studied in [**5**]. The reason is simple: it is not clear to us, what is the greatest generality in which constancy results – such as the ones below – can be proven. At any rate, they are not true for all families considered in [**5**]: for instance, it is easy to find one-parameter families of projections onto planes in $\mathbb{R}^3$, for which there is no hope of constancy of any kind.

It is time to introduce the particular families of projections we will be concerned with. They are projections onto m-planes in $\mathbb{R}^n$, $2 \leqslant m < n$, parameterized by the Grassmannian $G(n-l, m-l)$ for some $0 < l < m$. Since

$$\dim_{\mathrm{H}} G(n-l, m-l) = (m-l)(n-m) < m(n-m) = \dim_{\mathrm{H}} G(n, m),$$

such families are 'small' in the sense introduced above. Write $\mathbb{V} = V \times \mathbb{R}^l$ for the m-dimensional subspace of $\mathbb{R}^n$ containing $\{0\} \times \mathbb{R}^l$, where V is an element of $G(n-l, m-l)$. We are interested (only) in the orthogonal projections $\pi_{\mathbb{V}} \colon \mathbb{R}^n \to \mathbb{V}$. The simplest case is obtained with $n = 3$, $m = 2$ and $l = 1$: then the mappings $\pi_{\mathbb{V}}$ are the orthogonal projections onto the 'vertical' planes in $\mathbb{R}^3$, that is, the planes containing the z-axis $\{0\} \times \mathbb{R}$.

We write $B_{\mathbb{V}} = \pi_{\mathbb{V}}(B)$. We will also make use of the projections onto the $(m-l)$-dimensional subspaces V; we will denote by π_V both the orthogonal projection $\mathbb{R}^{n-l} \to V$ and $\mathbb{R}^n \to V \times \{0\}$. We write $B_V = \pi_V(B)$, and denote by $\gamma_{n,m}$ the natural $O(n)$ invariant measure on the Grassmanian $G(n, m)$, see [**11**, 3·9]. Furthermore, $G(n, m)$ will be endowed with the metric d_π given by

$$d_\pi(V, W) = \|\pi_V - \pi_W\|, \quad V, W \in G(n, m), \tag{1·1}$$

where $\|\cdot\|$ denotes the operator norm. Below and above, $\dim_{\mathrm{H}}$ refers to Hausdorff dimension, whereas $\dim_{\mathrm{P}}$ refers to packing dimension and $\overline{\dim}_{\mathrm{B}}$ denotes the upper box dimension.

Before stating our main results on constancy, let us observe as Proposition 1·2 that for sets B with small enough dimension, it is possible to give an almost sure formula for $\dim_{\mathrm{H}} B_{\mathbb{V}}$ in terms of $\dim_{\mathrm{H}} B$. Perhaps surprisingly, the proposition is not a corollary of the bounds in [**5**], as they are not sharp for our particular families of projections. This only testifies that our projections have a very special form – and the proof of Proposition 1·2 heavily relies on this fact. Let us clarify the point with an example: according to the proposition below, the Hausdorff dimension of every 1-dimensional set is almost surely preserved under the 2-dimensional family of projections onto 2-dimensional planes in $\mathbb{R}^4$ which contain $\{0\} \times \mathbb{R}$. However, it is not true that the dimension of such sets is preserved under *arbitrary* non-degenerate 2-dimensional families of projections from $\mathbb{R}^4$ to 2-dimensional planes. Consider, for instance, the family of projections associated to 2-planes contained in $\mathbb{R}^3 \times \{0\}$. For these projections, the set $B = \{0\} \times \mathbb{R}$ is projected to a point for all considered directions, and so the dimension can drop from one to zero.

PROPOSITION 1·2. *Let $B \subset \mathbb{R}^n$ be an analytic set with* $\dim_{\mathrm{H}} B \leqslant m-l$. *Then* $\dim_{\mathrm{H}} B_{\mathbb{V}} = \dim_{\mathrm{H}} B$ *for $\gamma_{n-l,m-l}$ almost every $V \in G(n-l, m-l)$.*

For sets B of dimension bigger than $m-l$, it is no longer possible to give an almost sure formula for $\dim_{\mathrm{H}} B_{\mathbb{V}}$ in terms of $\dim_{\mathrm{H}} B$. Instead, we have the following constancy results.

THEOREM 1·3. *Let $B \subset \mathbb{R}^n$ be an analytic set, and write*

$$\mathfrak{m}_{\mathrm{H}} := \sup\{\dim_{\mathrm{H}} B_{\mathbb{V}} : V \in G(n-l, m-l)\}.$$

Then, $\dim_{\mathrm{H}} B_{\mathbb{V}} = \mathfrak{m}_{\mathrm{H}}$ *for $\gamma_{n-l,m-l}$ almost every $V \in G(n-l, m-l)$.*

THEOREM 1·4. *Let $B \subset \mathbb{R}^n$ be a bounded analytic set. Write*

$$\mathfrak{m}_{\mathrm{B}} := \sup\{\overline{\dim}_{\mathrm{B}} B_{\mathbb{V}} : V \in G(n-l, m-l)\}, \ \ \mathfrak{m}_{\mathrm{P}} := \sup\{\dim_{\mathrm{P}} B_{\mathbb{V}} : V \in G(n-l, m-l)\}.$$

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Then, the sets

$$E_{\mathrm{B}} := \{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B_{\mathbb{V}} \neq \mathfrak{m}_{\mathrm{B}}\},$$
$$E_{\mathrm{P}} := \{V \in G(n-l, m-l) : \dim_{\mathrm{P}} B_{\mathbb{V}} \neq \mathfrak{m}_{\mathrm{P}}\}$$

are meagre and have $\gamma_{n-l,m-l}$ measure zero. The statement concerning packing dimension holds for unbounded sets as well.

Remark 1·5. A routine argument shows that it is sufficient to prove Theorems 1·3 and 1·4 for bounded sets. Thus, we may and will only consider bounded sets in the sequel.

Throughout the paper we write $a \lesssim b$, if $a \leqslant Cb$ for some constant $C \geqslant 1$. Should we wish to emphasize that C depends on some parameter p, we may write $a \lesssim_p b$. The chain $a \lesssim b \lesssim a$ is abbreviated to $a \asymp b$. If d is a metric on a space X, we denote by $B_d(x, r)$ the closed ball with center $x \in X$ and radius r; the subscript d is dropped, if the metric is obvious from the context.

2. *Proof for the Hausdorff dimension*

Proof of Proposition 1·2. Let $0 < s < t < \dim_{\mathrm{H}} B \leqslant m - l$. By Frostman's lemma there exists a non-trivial finite measure μ with support in B, which satisfies the growth condition $\mu(B(x, r)) \leqslant r^t$ for $x \in \mathbb{R}^n$ and $r > 0$. It follows that the associated s-energy is finite,

$$I_s(\mu) := \int\int |x-y|^{-s} d\mu(x) d\mu(y) < \infty.$$

By the definition of the push-forward measure $\pi_{\mathbb{V}\sharp}\mu$ and Fubini's theorem,

$$\int_{G(n-l,m-l)} I_s(\pi_{\mathbb{V}\sharp}\mu)\, d\gamma_{n-l,m-l}(V)$$
$$= \int_B \int_B \int_{G(n-l,m-l)} |\pi_{\mathbb{V}}(x) - \pi_{\mathbb{V}}(y)|^{-s}\, d\gamma_{n-l,m-l}(V) d\mu(x) d\mu(y).$$

Now if $(x, y) \in B \times B$, $x \neq y$, is such that

$$\sum_{i=n-l+1}^{n} (x_i - y_i)^2 \leqslant \sum_{i=1}^{n-l} (x_i - y_i)^2,$$

we have

$$\int_{G(n-l,m-l)} |\pi_{\mathbb{V}}(x-y)|^{-s}\, d\gamma_{n-l,m-l}(V) \leqslant \int_{G(n-l,m-l)} |\pi_V(\pi(x) - \pi(y))|^{-s}\, d\gamma_{n-l,m-l}(V),$$
$$\lesssim |\pi(x) - \pi(y)|^{-s} \asymp |x-y|^{-s}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-l}$ is defined by $\pi(x) = (x_1, \ldots, x_{n-l})$. Here the second inequality follows from [**11**, Corollary 3·12].

On the other hand, if $(x, y) \in B \times B$ is such that

$$\sum_{i=1}^{n-l} (x_i - y_i)^2 < \sum_{i=n-l+1}^{n} (x_i - y_i)^2,$$

the pointwise estimate

$$|\pi_{\mathbb{V}}(x) - \pi_{\mathbb{V}}(y)|^{-s} \leqslant \left(\sum_{i=n-l+1}^{n} (x_i - y_i)^2 \right)^{-s/2} \asymp |x-y|^{-s}.$$

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holds. Together, these observations yield

$$\int_{G(n-l,m-l)} I_s(\pi_{\mathbb{V}\sharp}\mu)\, d\gamma_{n-l,m-l}(V) \lesssim \int_B \int_B |x-y|^{-s}\, d\mu(x)d\mu(y) = I_s(\mu) < \infty$$

and thus $\dim_{\mathrm{H}} B_{\mathbb{V}} \geqslant s$ for almost every $V \in G(n-l, m-l)$. The result follows.

The method of bounding energy integrals used in the proof of Proposition 1·2 cannot be applied to derive information on the Hausdorff dimension of projections of sets of dimension $s > m-l$; the problem is that integrals of the form

$$\int_{G(n-l,m-l)} |\pi_V(x) - \pi_V(y)|^{-s}\, d\gamma_{n-l,m-l}(V)$$

can be infinite in that case, which means that the average of the energies $I_s(\pi_{\mathbb{V}\sharp}\mu)$ over the planes $\mathbb{V}$ may easily be infinite as well. This is natural, recalling that the projections of $\operatorname{spt}\mu$ can have dimension strictly smaller than $\operatorname{spt}\mu$ for all planes $\mathbb{V}$. Consequently, we need to devise a new quantity to replace $I_s(\pi_{\mathbb{V}\sharp}\mu)$, which (a) is bounded in size so that that it integrates over the planes $\mathbb{V}$, yet (b) contains all vital information on the dimension of $\operatorname{spt}\pi_{\mathbb{V}\sharp}\mu$. The trick is to discretise μ on a scale $\delta > 0$, thus turning μ into an L^2-function μ_δ. Then, projecting μ_δ – instead of μ – onto the planes $\mathbb{V}$ results in a family of L^2-functions, denoted by $(\mu_\delta)_{\mathbb{V}}$. It turns out that the L^2-norms $\|(\mu_\delta)_{\mathbb{V}}\|_2$, for various $\delta > 0$, provide a substitute for $I_s(\pi_{\mathbb{V}\sharp}\mu)$ satisfying both requirements (a) and (b).

In contrast with many classical proofs related to projection phenomena, the measure μ we consider is not simply a Frostman measure supported on the set $B \subset \mathbb{R}^n$ we are projecting. Rather, μ is an abstract pull-back of a Frostman measure ν supported on one of the projections of B, namely the one with the (essentially) largest dimension. The key observation in the proof is that if the norms $\|\nu_\delta\|_2$ satisfy certain growth estimates, then the same estimates automatically transfer to the norms $\|(\mu_\delta)_{\mathbb{V}}\|_2$, for almost all planes $\mathbb{V}$. Having related these growth estimates to the dimensions of $\operatorname{spt}(\mu_\delta)_{\mathbb{V}}$, this translates into our claim that almost all projections of $\operatorname{spt}\mu$ have dimension at least $\dim \operatorname{spt}\nu$.

In our first lemma, we make precise the idea of discretising a measure μ on a scale $\delta > 0$, and relate the growth rate of $\|\mu_\delta\|_2$, as $\delta \searrow 0$, to the dimension of $\operatorname{spt}\mu$.

LEMMA 2·1. *Let μ be a finite measure on $\mathbb{R}^m$, and let $(\psi_{\delta_j})_{j\in\mathbb{N}}$ be a collection of smooth functions of the form*

$$\psi_{\delta_j}(x) = \delta_j^{-m}\psi(x/\delta_j),$$

where ψ is a fixed non-negative compactly supported smooth function, not equal to zero, and $\delta_j = 2^{-j}$. Suppose that the growth of the L^2-norms of the convolutions $\mu_{\delta_j} := \mu * \psi_{\delta_j}$ is bounded as follows:

$$\|\mu_{\delta_j}\|_2^2 \lesssim \delta_j^{s-m} \quad \text{for some } 0 < s < m. \tag{2·2}$$

Then $\dim_{\mathrm{H}} \operatorname{spt}\mu \geqslant s$.

Proof. Parseval's theorem and (2·2) give

$$\int_{\mathbb{R}^m} |\hat{\mu}(x)|^2 |\hat{\psi}(\delta_j x)|^2\, dx = \int_{\mathbb{R}^m} |\widehat{\mu * \psi_{\delta_j}}(x)|^2\, dx = \|\mu_{\delta_j}\|_2^2 \lesssim \delta_j^{s-m}.$$

Next, observe that there exists a constant $c > 0$ such that $|\hat{\psi}(\delta_j x)|^2 \geqslant c$ for $|x| \leqslant c\delta_j^{-1}$. Let

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$0 < r < s$. The r-energy of μ can be expressed through the Fourier transform $\hat{\mu}$, see for instance [**11**, lemma 12·12]. Then,

$$\begin{aligned} I_r(\mu) \asymp \int_{\mathbb{R}^m} |\hat{\mu}(x)|^2 |x|^{r-m}\, dx &\lesssim 1 + \sum_{j=1}^{\infty} 2^{j(r-m)} \int_{B(0,c2^j)} |\hat{\mu}(x)|^2\, dx \\ &\lesssim 1 + \sum_{j=1}^{\infty} 2^{j(r-s)} 2^{j(s-m)} \int |\hat{\mu}(x)|^2 |\hat{\psi}(2^{-j}x)|^2\, dx \\ &\lesssim 1 + \sum_{j=1}^{\infty} 2^{j(r-s)} < \infty, \end{aligned}$$

which means that $I_r(\mu) < \infty$, and so $\dim_{\mathrm{H}} \operatorname{spt} \mu \geqslant r$.

LEMMA 2·3. *Let $\mathcal{D}_\delta$ be a partition of $\mathbb{R}^d$ into dyadic cubes of side-length δ; thus, $\mathcal{D}_1 := \{\Pi_{i=1}^d [m_i, m_i + 1) : m_i \in \mathbb{Z}\}$, and $\mathcal{D}_\delta := \{\delta Q : Q \in \mathcal{D}_1\}$. Suppose that ν is a measure on $\mathbb{R}^d$ of the form*

$$\nu = \sum_{Q \in \mathcal{D}_\delta} c_Q \mathcal{L}^d \llcorner_Q, \qquad c_Q \geqslant 0,$$

where $\mathcal{L}^d \llcorner_Q$ denotes the restriction of the Lebesgue measure to Q. Then

$$I_s(\nu) \lesssim \delta^{t-s} I_t(\nu)$$

for all t, s with $0 < t \leqslant s < d$. The implicit constants depend on d, s and t, but not on $\delta > 0$ or the particular choice of ν, as long as it is of the form indicated above.

Proof. Define the relation $\sim$ on $\mathcal{D}_\delta \times \mathcal{D}_\delta$ by

$$Q \sim Q' \qquad \Longleftrightarrow \qquad \overline{Q} \cap \overline{Q'} \neq \emptyset.$$

If $x \in \mathbb{R}^d$, let $Q_x \in \mathcal{D}_\delta$ be the unique cube containing x. For $x, y \in \mathbb{R}^n$, we write $x \sim y$, if $Q_x \sim Q_y$. Then,

$$I_s(\nu) = \iint_{\{(x,y):x\sim y\}} |x-y|^{-s}\, d\nu x d\nu y + \iint_{\{(x,y):x\nsim y\}} |x-y|^{-s}\, d\nu x d\nu y.$$

For the second term, it suffices to note that $x \nsim y$ implies $|x - y| \geqslant \delta$, whence $|x - y|^{-s} \leqslant \delta^{t-s}|x - y|^{-t}$. To estimate the first term, write

$$\begin{aligned} \iint_{\{(x,y):x\sim y\}} |x-y|^{-s}\, d\nu x d\nu y &= \sum_{Q\sim Q'} c_Q c_{Q'} \int_Q \int_{Q'} |x-y|^{-s}\, dx dy \\ &\leqslant \sum_{Q\sim Q'} c_Q c_{Q'} \int_Q \int_{B(y,c(d)\delta)} |x-y|^{-s}\, dx dy \\ &\asymp \delta^{2d-s} \sum_{Q\sim Q'} c_Q c_{Q'}, \end{aligned}$$

where the constant $c(d)$ is chosen large enough so that for $Q' \sim Q$ and $y \in Q$, we have

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$Q' \subseteq B(y, c(d)\delta)$. Here we have used

$$\int_{B(y,c(d)\delta)} |x-y|^{-s}\,dx = \int_{B(0,c(d)\delta)} |x|^{-s}\,dx = \int_0^{c(d)\delta}\int_{S^{d-1}} r^{-s}r^{d-1}\,d\sigma^{d-1}dr$$
$$= \frac{(c(d)\delta)^{d-s}}{d-s}\int_{S^{d-1}} d\sigma^{d-1}$$

and $\int_Q dy = \delta^d$.

To bound the sum $\sum_{Q\sim Q'} c_Q c_{Q'}$, note that if $Q \in \mathcal{D}_\delta$ is fixed, it has only a finite number $N(d)$ of 'neighbours' $Q' \in \mathcal{D}_\delta$. In particular,

$$\sum_{Q\in\mathcal{D}_\delta}\left(\max_{Q\sim Q'} c_{Q'}\right)^2 \leqslant N(d)\sum_{Q\in\mathcal{D}_\delta} c_Q^2$$

and thus,

$$\sum_{Q\sim Q'} c_Q c_{Q'} = \sum_{Q\in\mathcal{D}_\delta} c_Q \sum_{Q'\in\mathcal{D}_\delta : Q'\sim Q} c_{Q'} \lesssim \sum_{Q\in\mathcal{D}_\delta}\left(c_Q\cdot\max_{Q'\sim Q} c_{Q'}\right) \lesssim \sum_{Q\in\mathcal{D}_\delta} c_Q^2,$$

which implies that

$$I_s(\nu) \lesssim \delta^{2d-s}\sum_{Q\in\mathcal{D}_\delta} c_Q^2 + \delta^{t-s}I_t(\nu). \qquad (2\cdot4)$$

Let us next bound the t-energy $I_t(\nu)$ from below. If $Q \in \mathcal{D}_\delta$, let Q^o be the cube which is concentric with Q but has only half the side-length. Then, if $x \in Q^o$, we have $B(x, \delta/c(d)) \subset Q$ for large enough $c(d)$, not necessarily the same as above, and this shows that

$$I_t(\nu) \geqslant \sum_{Q\in\mathcal{D}_\delta} c_Q^2 \int_{Q^o}\int_{B(y,\delta/c(d))} |x-y|^{-t}\,dxdy \asymp \delta^{2d-t}\sum_{Q\in\mathcal{D}_\delta} c_Q^2,$$

by a similar integration in spherical coordinates as before. It now follows from (2·4) that

$$I_s(\nu) \lesssim \delta^{t-s}\left(\delta^{2d-t}\sum_{Q\in\mathcal{D}_\delta} c_Q^2\right) + \delta^{t-s}I_t(\nu) \lesssim \delta^{t-s}I_t(\nu),$$

as claimed.

Proof of Theorem 1·3. Let $B \subset \mathbb{R}^n$ be an analytic set. Recall that

$$\mathfrak{m}_{\mathrm{H}} = \sup\{\dim_{\mathrm{H}} B_{V\times\mathbb{R}^l} : V \in G(n-l, m-l)\} \leqslant m,$$

and we intend to prove that $\dim_{\mathrm{H}} B_{\mathbb{V}} = \mathfrak{m}_{\mathrm{H}}$ almost surely. To this end, we may assume that $\mathfrak{m}_{\mathrm{H}} > 0$. Let $0 < \sigma < \mathfrak{m}_{\mathrm{H}}$ and find a subspace $V_0 \in G(n-l, m-l)$ such that $\dim_{\mathrm{H}} B_{V_0\times\mathbb{R}^l} > \sigma$. We will identify all the subspaces $\mathbb{V} = V\times\mathbb{R}^l$ with $\mathbb{R}^m$, so that $B_{\mathbb{V}} \subset \mathbb{R}^m$, and the projections $\pi_{\mathbb{V}} = \pi_{V\times\mathbb{R}^l}$, $V \in G(n-l, m-l)$, will all be $\mathbb{R}^m$-valued. Let Ψ be a non-negative radial symmetric smooth function on $\mathbb{R}^n$, satisfying

$$\chi_{B(0,1)} \leqslant \Psi \leqslant \chi_{B(0,2)}. \qquad (2\cdot5)$$

Then, for any $V \in G(n-l, m-l)$, the projection $\Psi_{\mathbb{V}}$ of Ψ to $\mathbb{R}^m$, defined by

$$\Psi_{\mathbb{V}}(x) = \int_{\pi_{\mathbb{V}}^{-1}\{x\}} \Psi\, d\mathcal{H}^{n-m},$$

is a non-negative compactly supported smooth function on $\mathbb{R}^m$, not identically equal to zero.

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Since Ψ is radial symmetric, the projections $\Psi_{\mathbb{V}}$ are independent of $\mathbb{V}$; to emphasise this, we write $\psi := \Psi_{\mathbb{V}}$. The plan of the proof is to use Lemma 2·1 as follows. We will find a finite Borel measure μ supported on the analytic B such that the growth estimate

$$\|(\mu_{V\times\mathbb{R}^l})_{\delta_j}\|_2^2 \lesssim \delta_j^{s-m}, \qquad 0 \leqslant s < \sigma, \tag{2·6}$$

holds for $\gamma_{n-l,m-l}$ almost every $V \in G(n-l, m-l)$, where $\mu_{V\times\mathbb{R}^l} = \mu_{\mathbb{V}} = \pi_{\mathbb{V}\sharp}\mu$, and $\delta_j = 2^{-j}$. As in Lemma 2·1, the measure $(\mu_{\mathbb{V}})_{\delta_j}$ is defined to be the convolution $\mu_{\mathbb{V}} * \psi_{\delta_j}$, where $\psi_{\delta_j}(x) = \delta_j^{-m}\psi(x/\delta_j)$. According to Lemma 2·1, establishing (2·6) will complete the proof of Theorem 1·3.

Before defining the measure μ, let us make one observation to simplify the proof of (2·6).

LEMMA 2·7. *Let μ be a finite Borel measure on $\mathbb{R}^n$. Then for all $\delta > 0$ and $\mathbb{V} = V \times \mathbb{R}^l$, $V \in G(n-l, m-l)$, we have*

$$(\mu_{\mathbb{V}})_\delta = (\mu_\delta)_{\mathbb{V}}, \tag{2·8}$$

where $\mu_\delta := \mu * \Psi_\delta$ and $(\mu_{\mathbb{V}})_\delta = \mu_{\mathbb{V}} * \psi_\delta$ with $\psi = \Psi_{\mathbb{V}}$.

Proof. Writing $\Psi_\delta(x) := \delta^{-n}\Psi(x/\delta)$, and denoting the transpose of the projection $\pi_{\mathbb{V}}$ by $\pi_{\mathbb{V}}^{\mathrm{T}} : \mathbb{R}^m \to \mathbb{R}^n$, we have the identity

$$\begin{aligned}\widehat{(\mu_{\mathbb{V}})_\delta}(x) &= \widehat{\mu_{\mathbb{V}} * \psi_\delta}(x) = \widehat{\mu_{\mathbb{V}}}(x)\widehat{\psi_\delta}(x)\\ &= \hat{\mu}(\pi_{\mathbb{V}}^{\mathrm{T}}(x))\widehat{\Psi}(\pi_{\mathbb{V}}^{\mathrm{T}}(\delta x)) = \hat{\mu}(\pi_{\mathbb{V}}^{\mathrm{T}}(x))\widehat{\Psi}(\delta \cdot \pi_{\mathbb{V}}^{\mathrm{T}}(x))\\ &= \hat{\mu}(\pi_{\mathbb{V}}^{\mathrm{T}}(x))\widehat{\Psi_\delta}(\pi_{\mathbb{V}}^{\mathrm{T}}(x)) = \widehat{\mu * \Psi_\delta}(\pi_{\mathbb{V}}^{\mathrm{T}}(x)) = \widehat{(\mu * \Psi_\delta)_{\mathbb{V}}}(x)\end{aligned}$$

for all $x \in \mathbb{R}^m$.

So, the order of discretising and projecting can be interchanged, and, in particular, $\|(\mu_{\mathbb{V}})_\delta\|_2 = \|(\mu_\delta)_{\mathbb{V}}\|_2$ for all $V \in G(n-l, m-l)$ and $\delta > 0$. But, in order to apply Lemma 2·3, we will need something more. Let $\mathcal{D}_\delta$ be the collection of dyadic cubes of side-length $\delta > 0$ in $\mathbb{R}^n$, as defined above for $d = n$. If μ is any finite Borel measure on $\mathbb{R}^n$, set

$$\mu^\delta := \sum_{Q\in\mathcal{D}_\delta} \frac{\mu(Q)}{\delta^n}\chi_Q.$$

Eventually, we will control the L^2-norms in (2·6) by estimating the L^2-norms of the projections $(\mu^\delta)_{\mathbb{V}}$. This is reasonable thanks to the following lemma.

LEMMA 2·9. *If μ is any finite Borel measure on $\mathbb{R}^n$, we have*

$$\|(\mu_{\mathbb{V}})_\delta\|_2 \lesssim \|(\mu^\delta)_{\mathbb{V}}\|_2 \lesssim \|(\mu_{\mathbb{V}})_{c\delta}\|_2 \tag{2·10}$$

for any $\delta > 0$ and $V \in G(n-l, m-l)$. The implicit constants in (2·10) *only depend on n and the choice of the function Ψ, as in* (2·5)*, and the constant c depends only on the dimension n.*

Proof. Fix $V \in G(n-l, m-l)$ and $\delta > 0$. Using (2·5), we first make an estimate in $\mathbb{R}^n$:

$$\begin{aligned}\mu_\delta(x) &\lesssim \delta^{-n}\int_{B(x,2\delta)} d\mu y \leqslant \delta^{-n} \sum_{\substack{Q\in\mathcal{D}_\delta\\ Q\cap B(x,2\delta)\neq\emptyset}} \mu(Q)\\ &\leqslant \delta^{-n}\int_{B(x,c(n)\delta)} \sum_{Q\in\mathcal{D}_\delta} \frac{\mu(Q)}{\delta^n}\chi_Q(y)\,dy \lesssim (\mu^\delta)_{c(n)\delta}(x),\end{aligned}$$

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where $c(n)$ is large enough that $Q \cap B(x, 2\delta) \neq \emptyset$ implies $Q \subset B(x, c(n)\delta)$. Here $(\mu^\delta)_{c(n)\delta} = \mu^\delta * \Psi_{c(n)\delta}$, as usual. Applying the previous estimate and (2·8) twice we obtain

$$(\mu_{\mathbb{V}})_\delta = (\mu_\delta)_{\mathbb{V}} \lesssim ((\mu^\delta)_{c(n)\delta})_{\mathbb{V}} = ((\mu^\delta)_{\mathbb{V}})_{c(n)\delta},$$

where $((\mu^\delta)_{\mathbb{V}})_{c(n)\delta} = (\mu^\delta)_{\mathbb{V}} * \psi_{c(n)\delta}$, as before. Now it suffices to note that the convolution of any function $f \in L^1(\mathbb{R}^m)$ with ψ_δ is controlled by a constant times the Hardy–Littlewood maximal function Mf, and the constant can be chosen to be independent of f. Indeed,

$$|f * \psi_\delta(x)| \leqslant \int_{\mathbb{R}^m} |\psi_\delta(x-y)||f(y)|dy \leqslant \frac{1}{\delta^m} \int_{B(x,2\delta)} |f(y)|dy$$
$$\asymp \frac{1}{\mathcal{L}^m(B(x,2\delta))} \int_{B(x,2\delta)} |f(y)|dy \leqslant Mf(x).$$

Applying this to $(\mu^\delta)_{\mathbb{V}}$, it follows that

$$\|(\mu_{\mathbb{V}})_\delta\|_2 \lesssim \|((\mu^\delta)_{\mathbb{V}})_{c(n)\delta}\|_2 \lesssim \|M(\mu^\delta)_{\mathbb{V}}\|_2 \lesssim \|(\mu^\delta)_{\mathbb{V}}\|_2,$$

where the upper bound is simply the boundedness of operator M in L^2.

For the converse inequality let us observe that (2·5) guarantees the existence of a constant c which depends only on the dimension n such that

$$c^{-n}\mu^\delta(x) \leqslant (c\delta)^{-n}\mu(B(x, c\delta)) \leqslant \mu_{c\delta}(x),$$

so that $\|(\mu^\delta)_{\mathbb{V}}\|_2 \lesssim \|(\mu_{c\delta})_{\mathbb{V}}\|_2 = \|(\mu_{\mathbb{V}})_{c\delta}\|_2$ as desired.

Next, we will define the measure μ, for which (2·6) will be verified. At the beginning of the proof, we found a special subspace $V_0 \in G(n-l, m-l)$ such that $\dim_{\mathrm{H}} B_{\mathbb{V}_0} > \sigma$. Since $B_{\mathbb{V}_0} \subset \mathbb{R}^m$ is an analytic set, we may use Frostman's lemma to find a non-trivial finite Borel measure $\mu_{\mathbb{V}_0}$, supported on $B_{\mathbb{V}_0}$ and satisfying $I_\sigma(\mu_{\mathbb{V}_0}) < \infty$. We may then 'pull back' the measure $\mu_{\mathbb{V}_0}$ inside the set B using the following result of A. Lubin from 1974.

LEMMA 2·11. ([**9**, Corollary 6]). *Let X, Y be analytic subsets of complete separable metric spaces, and let $f\colon X \to Y$ be a Borel function. Then, if ν is a measure supported on $f(X) \subset Y$, there exists a Borel measure μ on X such that $f_\sharp\mu = \nu$.*

We apply the lemma with $X = B$, $Y = B_{\mathbb{V}_0}$ and $\nu = \mu_{\mathbb{V}_0}$ to obtain a measure μ, supported on B, and such that

$$\pi_{\mathbb{V}_0\sharp}\mu = \mu_{\mathbb{V}_0}.$$

For $V \in G(n-l, m-l)$, we write $\mu_{\mathbb{V}} := \pi_{V\times\mathbb{R}^l\sharp}\mu$; clearly, the two definitions of $\mu_{\mathbb{V}_0}$ coincide. For $V = V_0$, we have the estimate

$$\|(\mu^\delta)_{\mathbb{V}_0}\|_2^2 \lesssim \|(\mu_{\mathbb{V}_0})_{c\delta}\|_2^2 \asymp \int_{\mathbb{R}^m} |\widehat{\mu_{\mathbb{V}_0}}(x)|^2|\hat{\psi}(c\delta x)|^2\,dx$$
$$\lesssim (c\delta)^{\sigma-m}\int_{\mathbb{R}^m} |\widehat{\mu_{\mathbb{V}_0}}(x)|^2|x|^{\sigma-m}\,dx \lesssim \delta^{\sigma-m},$$

using the rapid decay bound $|\widehat{\psi}(y)| \lesssim |y|^{(\sigma-m)/2}$ for $y \in \mathbb{R}^m$ (note that ψ is a Schwartz function) and the finiteness of the σ-energy of $\mu_{\mathbb{V}_0}$.

So, we have (2·6) for $V = V_0$. Using **only this information**, we intend to prove (2·6) for $\gamma_{n-l,m-l}$ almost all directions $V \in G(n-l, m-l)$.

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Given $h \in \mathbb{R}^l$, let $\mu_h^\delta \colon \mathbb{R}^{n-l} \to [0, \infty)$ be the function $\mu_h^\delta(x) = \mu^\delta(x, h)$. Recalling the definition of μ^δ, it is clear that the functions – or measures – μ_h^δ have precisely the form of the measure ν in Lemma 2·3 for $d = n - l$. In particular,

$$I_{m-l}(\mu_h^\delta) \lesssim_{t,l,m,n} \delta^{t-(m-l)} I_t(\mu_h^\delta), \qquad 0 < t \leqslant m - l. \tag{2·12}$$

If $V \in G(n-l, m-l)$, write $(\mu_h^\delta)_V$ for the orthogonal projection of μ_h^δ onto the subspace $V \subset \mathbb{R}^{n-l}$. Before making the final estimates, we need to record the following upper bound for the energy in terms of the L^2-norm.

LEMMA 2·13. *Let ν be a positive finite compactly supported Borel measure on $\mathbb{R}^{m-l}$, which is also an L^2-function. Then,*

$$I_t(\nu) \lesssim_{l,m,t} \|\nu\|_2^2, \quad 0 < t < m - l, \tag{2·14}$$

Proof. The Fourier transform of the finite positive measure ν is a positive definite function, so it satisfies the pointwise estimate $|\hat{\nu}(x)| \leqslant \hat{\nu}(0)$ for $x \in \mathbb{R}^{m-l}$, see for instance [**1**, p. 198, p. 220]. Using this we obtain

$$\begin{aligned} I_t(\nu) &\asymp_{l,m,t} \int_{\mathbb{R}^{m-l}} |\hat{\nu}(x)|^2 |x|^{t-(m-l)}\, dx \\ &\leqslant |\hat{\nu}(0)|^2 \int_{B(0,1)} |x|^{t-(m-l)} dx + \int_{\mathbb{R}^{m-l} \setminus B(0,1)} |\hat{\nu}(x)|^2\, dx \\ &\lesssim_{l,m,t} \|\nu\|_1^2 + \|\hat{\nu}\|_2^2 \lesssim \|\nu\|_2^2, \end{aligned}$$

as claimed.

We will also apply the estimate

$$\int_{G(n-l,m-l)} \|\nu_V\|_2^2\, d\gamma_{n-l,m-l}(V) \lesssim I_{m-l}(\nu), \tag{2·15}$$

valid for any finite measure ν on $\mathbb{R}^{n-l}$ with $I_{m-l}(\nu) < \infty$. This can be seen for instance from [**10**, Theorem 3.1].

Let $0 < t < m - l$. Combining (2·15), (2·12) and (2·14) we have

$$\begin{aligned} \int_{G(n-l,m-l)} \|(\mu_{\mathbb{V}})_\delta\|_2^2\, d\gamma_{n-l,m-l}(V) &\lesssim \int_{G(n-l,m-l)} \int_{\mathbb{R}^l} \|(\mu_h^\delta)_V\|_2^2\, dh\, d\gamma_{n-l,m-l}(V) \\ &= \int_{\mathbb{R}^l} \int_{G(n-l,m-l)} \|(\mu_h^\delta)_V\|_2^2\, d\gamma_{n-l,m-l}(V)\, dh \\ &\lesssim \int_{\mathbb{R}^l} I_{m-l}(\mu_h^\delta)\, dh \\ &\lesssim_t \delta^{t-(m-l)} \int_{\mathbb{R}^l} I_t(\mu_h^\delta)\, dh \\ &\leqslant \delta^{t-(m-l)} \int_{\mathbb{R}^l} I_t((\mu_h^\delta)_{V_0})\, dh \\ &\lesssim_t \delta^{t-(m-l)} \int_{\mathbb{R}^l} \|(\mu_h^\delta)_{V_0}\|_2^2\, dh \\ &= \delta^{t-(m-l)} \|(\mu^\delta)_{\mathbb{V}_0}\|_2^2 \lesssim \delta^{t-(m-l)+\sigma-m}. \end{aligned}$$

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If $s < \sigma$, we may choose $t < m - l$ so close to $m - l$ that $s < t - (m - l) + \sigma$. By Chebyshev's inequality

$$\gamma_{n-l,m-l}(\{V \in G(n-l,m-l) : \|(\mu_{\mathbb{V}})_\delta\|_2^2 \geqslant \delta^{s-m}\}) \leqslant \frac{1}{\delta^{s-m}} \int_{G(n-l,m-l)} \|(\mu_{\mathbb{V}})_\delta\|_2^2 \, d\gamma_{n-l,m-l}(V)$$

we obtain

$$\gamma_{n-l,m-l}(\{V \in G(n-l,m-l) : \|(\mu_{\mathbb{V}})_\delta\|_2^2 \geqslant \delta^{s-m}\}) \lesssim \delta^{t-(m-l)+\sigma-s}.$$

For $\delta_j = 2^{-j}$, $j \in \mathbb{N}$, combining this estimate with the easier Borel–Cantelli lemma shows that

$$\gamma_{n-l,m-l}\left(\bigcap_{p=1}^{\infty} \bigcup_{j \geqslant p} \{V \in G(n-l,m-l) : \|(\mu_{\mathbb{V}})_{\delta_j}\|_2^2 \geqslant \delta_j^{s-m}\}\right) = 0$$

and thus that the inequality $\|(\mu_{\mathbb{V}})_{2^{-j}}\|_2^2 \geqslant 2^{j(m-s)}$ can hold **infinitely often** only for a set of V's of $\gamma_{n-l,m-l}$ measure zero. For the rest of the subspaces V, we have $\|(\mu_{\mathbb{V}})_{2^{-j}}\|_2^2 \lesssim_V 2^{j(m-s)}$ for $j \in \mathbb{N}$, and, according to Lemma 2·1, this implies $\dim_{\mathrm{H}} B_{\mathbb{V}} \geqslant s$ for every such $V \in G(n-l,m-l)$.

3. *Proofs for upper box and packing dimensions*

A quick word on notation before we begin. If $E \subset \mathbb{R}^n$ is a bounded set and $\delta > 0$, we denote by $N(E,\delta)$ the least number of (closed) balls of radius δ required to cover E. The upper and lower box dimensions (Minkowski dimensions) of E are defined by

$$\underline{\dim}_{\mathrm{B}} E := \liminf_{\delta \to 0} \frac{\log N(E,\delta)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} E := \limsup_{\delta \to 0} \frac{\log N(E,\delta)}{-\log \delta}.$$

Analogous definitions can be made for totally bounded sets in metric spaces, for instance, we will use the concept of box dimensions on the Grassmanian.

The packing dimension of a set $E \subset \mathbb{R}^n$ is defined as

$$\dim_{\mathrm{P}} E := \inf \left\{ \sup_j \overline{\dim}_{\mathrm{B}} F_j : \; E \subset \bigcup_{j \in \mathbb{N}} F_j \right\}.$$

Theorem 1·4 contains statements concerning both upper box and packing dimension; accordingly, our proof divides into two parts. However, it turns out that the assertions for packing dimension easily reduce to their analogues for upper box dimension (via Lemma 3·12), so all the main ingredients of the proof are contained in the first part.

Let us briefly explain these ingredients in the lowest-dimensional interesting case, namely when $n = 3$, $m = 2$ and $l = 1$. Thus, we are considering projections in $\mathbb{R}^3$ onto the 'vertical' 2-dimensional subspaces (containing the z-axis). The key observation is, in fact, a result concerning planar sets and their projections onto one-dimensional subspaces. Fix $\delta > 0$, and let $K \subset \mathbb{R}^2$ be a bounded set. Suppose that for **some** one-dimensional subspace $L \subset \mathbb{R}^2$ the projection of K onto L contains $N \in \mathbb{N}$ δ-separated points. Then, the conclusion is that for 'almost' every one-dimensional subspace in $\mathbb{R}^2$ the projection of K contains $\gtrsim N$ δ-separated points (where the correct interpretation of $\gtrsim$ slightly differs from our normal usage).

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How do we use this observation for sets in $\mathbb{R}^3$? To begin with, we slice our bounded set $B \subset \mathbb{R}^3$ into disjoint horizontal pieces B_H of height δ. These pieces are 'planar enough' for the observation above to be applied. Namely, a moment's thought reveals that, at scale δ, the projections of the horizontal pieces onto the vertical subspaces $\mathbb{V} \subset \mathbb{R}^3$ resemble projections of certain planar sets onto one-dimensional subspaces. In particular, if $N(\pi_{\mathbb{V}_0}(B_H), \delta) = N \in \mathbb{N}$ for **some** vertical subspace $\mathbb{V}_0 \subset \mathbb{R}^3$, then the inequality $N(\pi_{\mathbb{V}}(B_H), \delta) \gtrsim N$ holds, in a suitable sense, for 'almost' every vertical subspace $\mathbb{V} \subset \mathbb{R}^3$. Since $N(\pi_{\mathbb{V}}(B), \delta)$ is roughly the sum of the numbers $N(\pi_{\mathbb{V}}(B_H), \delta)$ over all the horizontal pieces B_H, this property of the sets B_H transfers easily to the same property for the entire set B: if $N(\pi_{\mathbb{V}_0}(B), \delta) = N$ for **some** vertical subspace $\mathbb{V}_0 \subset \mathbb{R}^3$, then $N(\pi_{\mathbb{V}}(B), \delta) \gtrsim N$ for 'almost' every vertical subspace $\mathbb{V} \subset \mathbb{R}^3$. The assertion of Theorem 1·4 for upper box dimension follows immediately.

We begin with an estimate for the volumes of balls on the Grassmannian. In all likelihood, the proposition is well known, but we were unable to find a direct reference. Consequently, we chose to include a proof in Appendix A.

PROPOSITION 3·1. *Let $0 < m < n$. Then there exist constants $0 < c < C < \infty$ and $\delta_0 > 0$ such that*

$$c\delta^{m(n-m)} \leqslant \gamma_{n,m}(B(V, \delta)) \leqslant C\delta^{m(n-m)}$$

for all $V \in G(n, m)$ and all $0 < \delta < \delta_0$. Here the ball $B(V, \delta)$ is defined using the projection distance $d_\pi(V, W) = \|\pi_V - \pi_W\|$.

A set $E \subset G(n, m)$ is said to be δ-*separated* if $d_\pi(V, W) \geqslant \delta$ for any distinct elements $V, W \in E$.

Definition 3·2 ((δ, k)-sets). Let $C \subset B(0, 1) \subset \mathbb{R}^n$ be a finite set. We say that a δ-separated set C is a (δ, k)-*set*, if

$$\operatorname{card}[B(x, r) \cap C] \lesssim \left(\frac{r}{\delta}\right)^k$$

for every ball $B(x, r) \subset \mathbb{R}^n$ with radius $r \geqslant \delta$.

The following proposition is a generalization of [**12**, proposition 4·10] to higher dimensions. Essentially, the result is a discrete version of the Marstrand–Kaufman–Mattila projection theorem.

LEMMA 3·3. *Let $0 < \delta < 1$ and let $C \subset B(0, 1) \subset \mathbb{R}^n$ be a (δ, m)-set with $N \in \mathbb{N}$ points. Let $\tau > 0$, and let $E \subset G(n, m)$ be a δ-separated collection of subspaces such that*

$$N(C_V, \delta) \leqslant \delta^\tau N, \qquad \textit{for all } V \in E.$$

Then card $E \lesssim \delta^{\tau-(n-m)m} \cdot \log(1/\delta)$.

Proof. Let $\mathcal{D}_\delta$ be a partition of V into m-dimensional dyadic cubes of side-length $\delta > 0$. For a given subspace $V \in E$ we consider the 'tubes'

$$\mathcal{T}_V := \{T = \pi_V^{-1}(Q) : Q \in \mathcal{D}_\delta\},$$

and we define the relation

$$x \sim_V y \iff x, y \in T \in \mathcal{T}_V.$$

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We define an energy $\mathcal{E}$ by

$$\mathcal{E} := \sum_{V \in E} \text{card}\{(x, y) \in C \times C : x \sim_V y\}.$$

Writing

$$\mathcal{E}' := \sum_{V \in E} \text{card}\{(x, y) \in C \times C : x \sim_V y, x \neq y\},$$

we find that $\mathcal{E} = \mathcal{E}' + N \cdot \text{card}E$, and our goal is to show that $\mathcal{E} \lesssim \delta^{-(n-m)m} \cdot N \cdot \log(1/\delta)$. Proposition 3·1 implies that $\text{card}E \lesssim \delta^{-(n-m)m}$ since E is a δ-separated subset of $G(n, m)$. So, it remains to establish the desired upper bound for $\mathcal{E}'$.

Let us first observe that, for $x \neq y$,

$$\text{card}\{V \in E : x \sim_V y\} \lesssim \frac{\delta^{m(1-(n-m))}}{|x-y|^m}. \tag{3·4}$$

Namely, if V is any subspace such that $x \sim_V y$, then

$$B(V, \delta) \subset \{V : |\pi_V(x-y)| \leqslant \beta\delta\}$$

for some constant β depending only on m and n (here we also use the inclusion $C \subset B(0, 1)$). On the other hand, we have the measure bound, see [**11**, Lemma 3.11],

$$\gamma_{n,m}(\{V \in G(n, m) : |\pi_V(x-y)| \leqslant \beta\delta\}) \lesssim \left(\frac{\delta}{|x-y|}\right)^m.$$

Now (3·4) follows, since the set E is δ-separated and, according to Proposition 3·1, we have $\gamma_{n,m}(B(V, \delta)) \asymp \delta^{m(n-m)}$.

Using (3·4),

$$\begin{aligned}
\mathcal{E}' &= \sum_{x \in C} \sum_{j:\delta \leqslant 2^j \leqslant 1} \sum_{\substack{y \in C \\ 2^j \leqslant |x-y| < 2^{j+1}}} \text{card}\{V \in E : x \sim_V y\} \\
&\lesssim \sum_{x \in C} \sum_{j:\delta \leqslant 2^j \leqslant 1} \sum_{\substack{y \in C \\ 2^j \leqslant |x-y| < 2^{j+1}}} |x-y|^{-m} \delta^{m(1-(n-m))} \\
&\lesssim \sum_{x \in C} \sum_{j:\delta \leqslant 2^j \leqslant 1} \text{card}[C \cap B(x, 2^{j+1})] \cdot 2^{-jm} \delta^{m(1-(n-m))} \\
&\lesssim \sum_{x \in C} \sum_{j:\delta \leqslant 2^j \leqslant 1} \left(\frac{2^{j+1}}{\delta}\right)^m 2^{-jm} \delta^{m(1-(n-m))} \\
&= \sum_{x \in C} \sum_{j:\delta \leqslant 2^j \leqslant 1} \delta^{-(n-m)m} \asymp \delta^{-(n-m)m} \cdot N \cdot \log\left(\frac{1}{\delta}\right).
\end{aligned}$$

The asserted bound for $\text{card}E$ follows, once we have found an appropriate lower bound for $\mathcal{E}$. We may assume that $\delta^\tau N \geqslant 1$. The assumption $N(C_V, \delta) \leqslant \delta^\tau N$ guarantees that C

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can be covered by $K \lesssim \delta^{\tau} N$ tubes $T_1, \ldots, T_K \in \mathcal{T}_V$, which yields

$$\begin{aligned}\operatorname{card}\{(x, y) \in C \times C : x \sim_V y\} &= \sum_{j=1}^{K} \operatorname{card}\{(x, y) \in C \times C : x, y \in T_j\} \\ &= \sum_{j=1}^{K} \operatorname{card}[C \cap T_j]^2 \\ &\geqslant \frac{1}{K}\left(\sum_{j=1}^{K} \operatorname{card}[C \cap T_j]\right)^2 \\ &\gtrsim \delta^{-\tau} \cdot N^{-1} \cdot (\operatorname{card} C)^2 = \delta^{-\tau} \cdot N.\end{aligned}$$

Combing the upper and lower bounds for $\mathcal{E}$, we find

$$\delta^{-\tau} \cdot N \cdot \operatorname{card} E \lesssim \mathcal{E} \lesssim \delta^{-(n-m)m} \cdot N \cdot \log\left(\frac{1}{\delta}\right),$$

which is the desired result.

The following reformulation of the lemma will be used later (applied to the Grassmanian $G(n-l, m-l)$ instead of $G(n, m)$).

COROLLARY 3·5. *Let $C \subset B(0, 1) \subset \mathbb{R}^n$ be a (δ, m)-set with $N \in \mathbb{N}$ points. Then, if $E \subset G(n, m)$ is any δ-separated set with* card $E \geqslant \delta^{-\beta}$ *elements, we have*

$$\frac{1}{\operatorname{card} E} \sum_{V \in E} N(C_V, \delta) \gtrsim_{\tau} \delta^{(n-m)m-\tau} N, \qquad \tau < \beta.$$

Proof. According to Lemma 3·3, the set E contains at most

$$\lesssim \delta^{((n-m)m-\tau)-(n-m)m} \cdot \log(1/\delta) = \delta^{-\tau} \cdot \log(1/\delta)$$

subspaces V such that $N(C_V, \delta) \leqslant \delta^{(n-m)m-\tau} N$. Since $\tau < \beta$, the proportion of such subspaces in E is close to zero for small δ, and the claim follows.

3·1. *Proof of Theorem* 1·4 *for upper box dimension*

We are now ready to prove Theorem 1·4 for upper box dimension. The assumption on the analyticity of the set B will only be required later, in the proof for packing dimension. For the time being, we assume that $B \subset B(0, 1) \subset \mathbb{R}^n$ is an arbitrary set with

$$\mathfrak{m}_{\mathrm{B}} = \sup\{\overline{\dim}_{\mathrm{B}} B_{\mathbb{V}} : V \in G(n-l, m-l)\} > 0.$$

We will show for $0 \leqslant \sigma \leqslant \mathfrak{m}_{\mathrm{B}}$ that

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B_{\mathbb{V}} < \sigma\} \leqslant \max\{0, (n-m)(m-l) + \sigma - \mathfrak{m}_{\mathrm{B}}\}, \quad (3·6)$$

where $\underline{\dim}_{\mathrm{MB}}$ denotes the *modified lower box dimension*

$$\underline{\dim}_{\mathrm{MB}} E := \inf\left\{\sup_j \underline{\dim}_{\mathrm{B}} F_j : E \subset \bigcup_{j \in \mathbb{N}} F_j\right\}.$$

Recall that $G(n-l, m-l)$ is endowed with a metric so that $\dim_{\mathrm{H}} G(n-l, m-l) = (n-m)(m-l)$ and the $(n-m)(m-l)$-dimensional Hausdorff measure coincides with $\gamma_{n-l,m-l}$

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up to a positive and finite multiplicative constant. It is clear that sets $E \subset G(n-l, m-l)$ with

$$\underline{\dim}_{\mathrm{MB}} E < (n-m)(m-l)$$

are meager, i.e., countable unions of nowhere dense sets, and have $\gamma_{n-l,m-l}$ measure zero, so (3·6) will imply the upper box dimension part of Theorem 1·4.

As before, let $\mathcal{D}_\delta$ stand for the collection of dyadic cubes in $\mathbb{R}^n$ of side-length $\delta > 0$. Write

$$B^\delta := \bigcup \{Q \in \mathcal{D}_\delta : B \cap Q \neq \emptyset\}, \quad \delta > 0,$$

It is easy to check that

$$N((B^\delta)_{\mathbb{V}}, \delta) \asymp N(B_{\mathbb{V}}, \delta)$$

for any $V \in G(n-l, m-l)$ and $\delta > 0$. This shows that $\limsup_{\delta \to 0} \log N(B_{\mathbb{V}}, \delta)/-\log \delta = \limsup_{\delta \to 0} \log N((B^\delta)_{\mathbb{V}}, \delta)/-\log \delta$ and thus

$$\{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B_{\mathbb{V}} < \sigma\} \subset \bigcup_{i \in \mathbb{N}} \bigcap_{\delta \in (0,1/i)} \{V \in G(n-l, m-l) : N((B^\delta)_{\mathbb{V}}, \delta) \leqslant \delta^{-\sigma}\}.$$

Hence, by definition of $\underline{\dim}_{\mathrm{MB}}$, the bound (3·6) would follow from

$$\sup_i \underline{\dim}_{\mathrm{B}} E_i \leqslant \max\{0, (n-m)(m-l) + \sigma - \mathfrak{m}_{\mathrm{B}}\}, \qquad 0 \leqslant \sigma \leqslant \mathfrak{m}_{\mathrm{B}}, \tag{3·7}$$

where $E_i = \bigcap_{\delta \in (0,1/i)} \{V \in G(n-l, m-l) : N((B^\delta)_{\mathbb{V}}, \delta) \leqslant \delta^{-\sigma}\}$. We will now prove (3·7). Fix $i \in \mathbb{N}$ and write $E := E_i$. Given $\sigma < \sigma' < \mathfrak{m}_{\mathrm{B}}$, we may find a direction $V_0 \in G(n-l, m-l)$ and a sequence $(\delta_j)_{j \in \mathbb{N}}$ such that $\delta_j \searrow 0$, and $N((B^{\delta_j})_{\mathbb{V}_0}, \delta_j) \geqslant \delta_j^{-\sigma'}$.

3·1·1. *Decomposition into sets essentially in* $\mathbb{R}^{n-l}$

Let

$$\mathcal{H}_\delta := \{H = \mathbb{R}^{n-l} \times \Pi_{i=1}^{l} [k_i \delta, (k_i + 1)\delta) : (k_1, \ldots, k_l) \in \mathbb{Z}^l\}$$

and set $B^{\delta,H} := B^\delta \cap H$. Thus

$$B^\delta = \bigcup_{H \in \mathcal{H}_\delta} B^{\delta,H}.$$

In particular,

$$N((B^\delta)_{\mathbb{V}}, \delta) \asymp \sum_{H \in \mathcal{H}_\delta} N((B^{\delta,H})_{\mathbb{V}}, \delta) \tag{3·8}$$

for $V \in G(n-l, m-l)$ and $\delta > 0$. For our purposes, the sets $B^{\delta,H}$ are essentially sets in $\mathbb{R}^{n-l}$ in the following sense: for each $H \in \mathcal{H}_\delta$, there exists a set $P^{\delta,H} \subset \mathbb{R}^{n-l}$ and an l-tuple $(k_1, \ldots, k_l) \in \mathbb{Z}^l$ such that $B^{\delta,H} = P^{\delta,H} \times \Pi_{i=1}^{l} [k_i \delta, (k_i + 1)\delta)$. Moreover, the projection properties of the sets $B^{\delta,H}$ and $P^{\delta,H}$ are equivalent in the sense that $N((P^{\delta,H})_V, \delta) \asymp N((B^{\delta,H})_{\mathbb{V}}, \delta)$ for $V \in G(n-l, m-l)$ and $\delta > 0$, where $(P^{\delta,H})_V$ is the orthogonal projection of $P^{\delta,H}$ onto the $(m-l)$-dimensional subspace $V \subset \mathbb{R}^{n-l}$.

3·1·2. *Finding* $(\delta, m-l)$*-sets*

In order to apply Corollary 3·5 to $G(n-l, m-l)$, we need to extract some $(\delta, m-l)$-sets. Recall the special direction $V_0 \in G(n-l, m-l)$ with the property that $N((B^\delta)_{\mathbb{V}_0}, \delta_j) \geqslant \delta_j^{-\sigma'}$

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for every $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ and write $\delta := \delta_j$. For $H \in \mathcal{H}_\delta$, we set

$$M_H := N((B^{\delta,H})_{\mathbb{V}_0}, \delta) \asymp N((P^{\delta,H})_{V_0}, \delta).$$

Then $\sum_{H \in \mathcal{H}^\delta} M_H \gtrsim \delta^{-\sigma'}$ according to (3·8). As before, let $\mathcal{T}_{V_0}$ be a partition of $\mathbb{R}^{n-l}$ into tubes perpendicular to V_0. To be precise, let $\mathcal{D}_\delta$ be a partition of V_0 into dyadic cubes and then consider

$$\mathcal{T}_{V_0} := \{T = \pi_{V_0}^{-1}(Q) : \ Q \in \mathcal{D}_\delta\}.$$

Since $N((P^{\delta,H})_{V_0}, \delta) \asymp M_H$, we may find $K \asymp M_H$ tubes $T_1, \ldots, T_K \in \mathcal{T}_{V_0}$ such that each tube T_k, $k = 1, \ldots, K$, contains a point $x_k \in P^{\delta,H}$ and such that the set $C^{\delta,H} := \{x_1, \ldots, x_K\}$ is δ-separated. Moreover, since any ball $B(x,r) \subset \mathbb{R}^{n-l}$ of radius $r \geqslant \delta$ intersects no more than $\lesssim (r/\delta)^{m-l}$ tubes in $\mathcal{T}_{V_0}$, we may infer that the set $C^{\delta,H}$ is a $(\delta, m-l)$-set containing card $C^{\delta,H} \asymp M_H$ elements.

3·1·3. *Concluding the proof for upper box dimension*

Write $\delta = \delta_j$, where δ_j is as before. We apply Corollary 3·5 to the sets $C^{\delta,H}$, for every $H \in \mathcal{H}_\delta$. Let $E \subset G(n-l, m-l)$ be any δ-separated set of cardinality card $E \geqslant \delta^{-\beta}$, for some $\beta > 0$. Then,

$$\frac{1}{\text{card } E} \sum_{V \in E} N((B^\delta)_{\mathbb{V}}, \delta) \gtrsim \sum_{H \in \mathcal{H}_\delta} \left(\frac{1}{\text{card } E} \sum_{V \in E} N((C^{\delta,H})_V, \delta) \right)$$
$$\gtrsim \delta^{(n-m)(m-l)-\tau} \cdot \sum_{H \in \mathcal{H}_\delta} M_H \gtrsim \delta^{(n-m)(m-l)-\tau-\sigma'}, \quad \tau < \beta.$$

What does this mean? Recall that $\sigma < \sigma' < \mathfrak{m}_{\mathrm{B}}$. If

$$\beta > \max\{0, (n-m)(m-l) + \sigma - \sigma'\},$$

we may apply the previous estimate with some

$$\tau > \max\{0, (n-m)(m-l) + \sigma - \sigma'\}$$

to obtain the inequality

$$\frac{1}{\text{card } E} \sum_{V \in E} N((B^\delta)_{\mathbb{V}}, \delta) > \delta^{-\sigma},$$

at least for $\delta = \delta_j$ small enough.

This implies that

$$E \not\subset \{V : N((B^\delta)_{\mathbb{V}}, \delta) \leqslant \delta^{-\sigma}\}.$$

Thus, for small enough $\delta = \delta_j$, the maximum cardinality of a δ-separated subset of $\{V : N((B^\delta)_{\mathbb{V}}, \delta) \leqslant \delta^{-\sigma}\}$ is less than $\delta^{-\beta}$, for any

$$\beta > \max\{0, (n-m)(m-l) + \sigma - \sigma'\}.$$

Since $\sigma' < \mathfrak{m}_{\mathrm{B}}$ was arbitrary, this yields (3·7) and completes the proof of Theorem 1·4 for upper box dimension.

3·2. *Proof of Theorem* 1·4 *for packing dimension*

Let $B \subset \mathbb{R}^n$ be a bounded analytic set, and assume that

$$\mathfrak{m}_{\mathrm{P}} := \sup\{\dim_{\mathrm{P}} B_{\mathbb{V}} : V \in G(n-l, m-l)\} > 0.$$

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As in the case of upper box dimension, it suffices to prove for $0 \leqslant \sigma \leqslant \mathfrak{m}_{\mathrm{P}}$ that

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \dim_{\mathrm{P}} B_{\mathbb{V}} < \sigma\} \leqslant \max\{0, (n-m)(m-l) + \sigma - \mathfrak{m}_{\mathrm{P}}\}. \quad (3\cdot9)$$

Suppose that (3·9) fails. Then, we may find numbers $0 < \sigma < \sigma' < \mathfrak{m}_{\mathrm{P}}$ such that

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \dim_{\mathrm{P}} B_{\mathbb{V}} < \sigma\} > \max\{0, (n-m)(m-l) + \sigma - \sigma'\}. \quad (3\cdot10)$$

Choose $V_0 \in G(n-l, m-l)$ such that $\dim_{\mathrm{P}} B_{\mathbb{V}_0} > \sigma'$. Since $B_{\mathbb{V}_0} \subset \mathbb{R}^m$ is analytic, a result of Joyce and Preiss [**7**] permits us to find a compact set $K_{\mathbb{V}_0} \subset B_{\mathbb{V}_0}$ with positive and finite σ'-dimensional packing measure; $0 < \mathcal{P}^{\sigma'}(K_{\mathbb{V}_0}) < \infty$. Next, we apply the 'pull-back lemma' by Lubin to find a Borel measure μ supported on B, with the property that

$$\pi_{\mathbb{V}_0\sharp}\mu = \mathcal{P}^{\sigma'} \llcorner K_{\mathbb{V}_0}. \quad (3\cdot11)$$

Now $B^0 := \operatorname{spt}\mu \subset B$ is a μ-measurable set with $\mu(B^0) > 0$, and (3·10) holds, by monotonicity, with B replaced by B^0. We quote a lemma from [**12**].

LEMMA 3·12. (*Adapted from* [**12**, Lemma 4·5]). *Let μ be a Borel regular measure on $\mathbb{R}^n$, and let $\beta, \sigma > 0$. Assume that $B^0 \subset \mathbb{R}^n$ is μ-measurable with $0 < \mu(B^0) < \infty$, and*

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \dim_{\mathrm{P}} B^0_{\mathbb{V}} < \sigma\} > \beta.$$

Then, there exists a μ-measurable set $B' \subset B^0$ with $\mu(B') > 0$ such that

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B'_{\mathbb{V}} < \sigma\} > \beta.$$

The corresponding lemma in [**12**] only concerns projections of planar sets, but the proof works verbatim in the situation above. We intend to apply the lemma to the measure μ constructed above and the set $B^0 = \operatorname{spt}\mu$. Strictly speaking, the abstract 'pull-back lemma' from [**9**] does not tell us that the measure μ is Borel **regular**. However, inspecting the proof of [**12**, Lemma 4·5], the regularity of the measure is only used to guarantee the existence of compact sets $K \subset B^0 \cap \operatorname{spt}\mu$ with positive μ-measure. Fortunately, the existence of such sets is clear in our situation, since here $B^0 = \operatorname{spt}\mu$ is closed to begin with.

Applying Lemma 3·12 to the measure μ constructed above, we find a set B' in B^0 such that $\mu(B') > 0$, and

$$\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B'_{\mathbb{V}} < \sigma\} > \max\{0, (n-m)(m-l) + \sigma - \sigma'\}. \quad (3\cdot13)$$

However, we may infer from (3·11) that

$$\mathcal{P}^{\sigma'}(B'_{\mathbb{V}_0}) = \mu(\pi_{\mathbb{V}_0}^{-1}(B'_{\mathbb{V}_0})) \geqslant \mu(B') > 0,$$

and, in particular,

$$\mathfrak{m}'_{\mathrm{B}} := \sup\{\overline{\dim}_{\mathrm{B}} B'_{\mathbb{V}} : V \in G(n-l, m-l)\} \geqslant \overline{\dim}_{\mathrm{B}} B'_{\mathbb{V}_0} \geqslant \sigma'.$$

Now it follows from the upper box dimension part of the proof, namely the estimate (3·6), that

$$\begin{aligned}\underline{\dim}_{\mathrm{MB}}\{V \in G(n-l, m-l) : \overline{\dim}_{\mathrm{B}} B'_{\mathbb{V}} < \sigma\} &\leqslant \max\{0, (n-m)(m-l) + \sigma - \mathfrak{m}'_{\mathrm{B}}\} \\ &\leqslant \max\{0, (n-m)(m-l) + \sigma - \sigma'\}.\end{aligned}$$

This contradicts (3·13) and concludes the proof of Theorem 1·4 for bounded sets, and, according to Remark 1·5, for all sets.

Appendix A. Volumes of balls on the Grassmannian

In this final section, we prove Proposition 3·1. We start with a geometric lemma.

LEMMA A 1. *Let $V, W \in G(n,m)$. Then there exist orthonormal bases $\{v_1, \ldots, v_m\} \subset V$ and $\{w_1, \ldots, w_m\} \subset W$ such that*

$$|v_i - w_i| \lesssim \|\pi_V - \pi_W\|, \qquad 1 \leqslant i \leqslant m.$$

Proof. Write $\epsilon := \|\pi_V - \pi_W\|$. Choose some orthonormal bases for V and W, and form the $(n \times m)$-matrices Q_V and Q_W with the basis vectors as columns. Then $Q_V^{\mathrm{T}} = \pi_V$, and Q_V maps $\mathbb{R}^m$ isometrically onto V, as follows from

$$|x| = |\,\mathrm{Id}\,x| = |Q_V^{\mathrm{T}} Q_V x| \leqslant |Q_V x| \leqslant |x|, \qquad x \in \mathbb{R}^m.$$

Similar statements hold for Q_W. Consider the $(m \times m)$-matrix $M := Q_V^{\mathrm{T}} Q_W$. If $\epsilon < 1$, as we may assume, M is nonsingular; otherwise one finds a unit vector $x \in \ker M$, and then $|\pi_V(Q_W)x - \pi_W(Q_W)x| = 1 > \epsilon$. We perform the singular value decomposition (SVD) for M:

$$M = O_1 \Sigma O_2^{\mathrm{T}}.$$

Here $O_1, O_2 \in O(m)$, since $\det M \neq 0$, and Σ is a diagonal $(m \times m)$-matrix with non-negative entries, namely the *singular values* of M. We first aim to bound the singular values from below. Let $x \in \mathbb{R}^m$ be an arbitrary unit vector. Since $\|\pi_V - \pi_W\| = \epsilon$, we have

$$|Mx| = |\pi_V(Q_W x)| \geqslant |\pi_W(Q_W x)| - |\pi_V(Q_W x) - \pi_W(Q_W x)| \geqslant 1 - \epsilon,$$

using the fact that $Q_W x$ is a unit vector on W. Now, fix $1 \leqslant j \leqslant m$ and choose the unit vector $x \in \mathbb{R}^m$ so that $O_2^{\mathrm{T}} x$ equals the j^{th} standard basis vector e_j. Then

$$1 - \epsilon \leqslant |Mx| = |O_1 \Sigma O_2^{\mathrm{T}} x| = |\sigma_j O_1 e_j| = \sigma_j,$$

where σ_j is the j^{th} diagonal element in Σ – the j^{th} singular value. In conclusion, all the singular values σ_j satisfy $\sigma_j \geqslant 1 - \epsilon$. Now we are prepared to construct the bases. The SVD implies that

$$[Q_V O_1]^{\mathrm{T}} [Q_W O_2] = \Sigma.$$

We simply observe that the columns of the $(n \times m)$-matrices $Q_V O_1$ and $Q_W O_2$ form orthonormal bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_m\}$ for the subspaces V and W, respectively. Moreover, the inner product of any pair (v_i, w_j) satisfies

$$v_i \cdot w_j = \sigma_j \delta_{ij} \geqslant (1 - \epsilon)\delta_{ij}.$$

This means that the angles between the vectors v_i and w_i, $1 \leqslant i \leqslant m$, are $\lesssim \epsilon$, and the rest follows by simple trigonometry.

The measure $\gamma_{n,m}$ is $O(n)$-invariant, as follows immediately from the construction, see [**11**, section 3·9]. By $O(n)$-invariance, we of course mean that

$$\gamma_{n,m}(B(V, \delta)) = \gamma_{n,m}(B(OV, \delta))$$

for any m-plane $V \in G(n,m)$, any transformation $O \in O(n)$, and any $\delta > 0$. Since for any pair of m-planes $V, W \in G(n,m)$ we may find $O \in O(n)$ with $OV = W$, this allows us to make the following reduction: **in order to prove Proposition 3·1, it suffices to find an**

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m-plane $V \in G(n,m)$ with

$$0 < \liminf_{\delta\to 0} \frac{\gamma_{n,m}(B(V,\delta))}{\delta^{m(n-m)}} \leqslant \limsup_{\delta\to 0} \frac{\gamma_{n,m}(B(V,\delta))}{\delta^{m(n-m)}} < \infty. \tag{A 2}$$

Proof of (A 2) Unfortunately, we are not able to prove (A 2) for the measure $\gamma_{n,m}$ directly. Instead, the strategy will be roughly to (i) interpret the Grassmannian $G(n,m)$ as an $m(n-m)$-dimensional smooth submanifold of some Euclidean space, (ii) conclude that the natural Hausdorff measure on the submanifold satisfies a condition analogous to (A 2), and finally (iii) show that the measure $\gamma_{n,m}$ is equivalent to the said Hausdorff measure.

Step (i) involves the space $\bigwedge_m \mathbb{R}^n$ of all m-vectors over $\mathbb{R}^n$. For an introduction to the space $\bigwedge_m \mathbb{R}^n$, see [**13**, part I, chapter I]. We will mainly need to know that $\bigwedge_m \mathbb{R}^n$ is an $\binom{n}{m}$-dimensional vector space and can be endowed with a natural inner product, see [**13**, section 1·1·12]; we denote by $\|\cdot\|_m$ the norm induced by this inner product. The vectors in $\bigwedge_m \mathbb{R}^n$ can be expressed as linear combinations of *simple m-vectors* of the form $v_1 \wedge \cdots \wedge v_m$ where $v_1,\ldots,v_m \in \mathbb{R}^m$ and $\wedge$ is the wedge product. The subset

$$G := \{\mathbf{w} : \mathbf{w} \text{ is a simple } m\text{-vector, and } \|\mathbf{w}\|_m = 1\}$$

is a compact smooth $m(n-m)$-dimensional submanifold of $\bigwedge_m \mathbb{R}^n$, as shown in [**3**, section 3·2·28]. In particular, if we consider the $m(n-m)$-dimensional Hausdorff measure $\mathcal{H}^{m(n-m)}$ living on $G \subset \bigwedge_m \mathbb{R}^n$ – defined using the norm $\|\cdot\|_m$ – we may conclude that there exists a simple m-vector $\mathbf{w}_0 \in G$ such that

$$\lim_{\delta\to 0} \frac{\mathcal{H}^{m(n-m)}(B(\mathbf{w}_0,\delta))}{\delta^{m(n-m)}} = \kappa > 0. \tag{A 3}$$

Steps (i) and (ii) are now behind us; it only remains to relate G to $G(n,m)$. Consider a pair of vectors $\{-\mathbf{v},\mathbf{v}\} \subset G$. Since $\mathbf{v}$ is simple and $\mathbf{v} \neq 0$, we know that $\mathbf{v} = v_1 \wedge \cdots \wedge v_m$ for some linearly independent vectors $v_1,\ldots,v_m \in \mathbb{R}^n$. Hence, the set $\{v_1,\ldots,v_m\}$ spans a subspace $V \in G(n,m)$. We now consider the mapping $T\colon G \to G(n,m)$, defined by $T(\{-\mathbf{v},\mathbf{v}\}) = V$. Our first claims are that T **is 2-to-1 and surjective**. Let $V \in G(n,m)$, and consider the subspace L_V of $\bigwedge_m \mathbb{R}^n$ spanned by the simple m-vectors $v_1 \wedge \cdots \wedge v_m$ with $v_j \in V$ for $1 \leqslant j \leqslant m$. Since $L_V \cong \wedge_m \mathbb{R}^m$, we infer that $\dim_{\mathrm{H}} L_V = \binom{m}{m} = 1$. So, L_V is a one-dimensional subspace of $\bigwedge_m \mathbb{R}^n$, and, in particular, $G \cap L_V = \{-\mathbf{v},\mathbf{v}\}$ for some vector $\mathbf{v} \in G$. In other words,

$$T^{-1}(V) = \{-\mathbf{v},\mathbf{v}\},$$

just as we wanted. This observation allows us to push forward the metric from G to $G(n,m)$ by setting

$$d(V,W) := \operatorname{dist}(T^{-1}(V), T^{-1}(W)) = \min\{\|\mathbf{v}-\mathbf{w}\|_m, \|\mathbf{v}+\mathbf{w}\|_m\}, \tag{A 4}$$

provided that $T\mathbf{v} = V$ and $T\mathbf{w} = W$. Of course, dist refers to the distance with respect to $\|\cdot\|_m$. Verifying the triangle inequality for d is an easy case chase using the right hand side of (A 4). The upshot is that we may now use d to define an $m(n-m)$-dimensional Hausdorff measure $\mathcal{H}_d^{m(n-m)}$ on $G(n,m)$. We now relate $\mathcal{H}^{m(n-m)}$-densities on G to $\mathcal{H}_d^{m(n-m)}$-densities on $G(n,m)$. In fact, we have

$$\limsup_{\delta\to 0} \frac{\mathcal{H}^{m(n-m)}(B(\mathbf{v},\delta))}{\delta^{m(n-m)}} = \limsup_{\delta\to 0} \frac{\mathcal{H}_d^{m(n-m)}(B_d(T\mathbf{v},\delta))}{\delta^{m(n-m)}},$$

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and the same equation holds with lim sup replaced by lim inf. The proof is simple: given $\mathbf{v}_0 \in G$, we can find a $\|\cdot\|_m$-neighbourhood U of $\mathbf{v}_0$ in G so small that

$$\mathrm{dist}(\{-\mathbf{v}, \mathbf{v}\}, \{-\mathbf{w}, \mathbf{w}\}) = \|\mathbf{v} - \mathbf{w}\|_m$$

for all vectors $\mathbf{v}, \mathbf{w} \in U$. Then, the restriction $T|U : U \to (T(U), d)$ is an isometry, and

$$\mathcal{H}^{m(n-m)}(B(\mathbf{v}_0, \delta)) = \mathcal{H}_d^{m(n-m)}(T[B(\mathbf{v}_0, \delta)]) = \mathcal{H}_d^{m(n-m)}(B_d(T\mathbf{v}_0, \delta))$$

for small enough $\delta > 0$. Recalling (A 3), we have now proven that

$$\lim_{\delta \to 0} \frac{\mathcal{H}_d^{m(n-m)}(B_d(W_0, \delta))}{\delta^{m(n-m)}} = \kappa > 0 \tag{A 5}$$

with $W_0 = T\mathbf{w}_0$.

We next need to relate $\mathcal{H}_d^{m(n-m)}$ to $\gamma_{n,m}$. We first consider another Hausdorff measure on $G(n, m)$, namely $\mathcal{H}_\pi^{m(n-m)}$. The letter π refers to the projection metric $d_\pi(V, W) = \|\pi_V - \pi_W\|$ on $G(n, m)$. Our aim is to prove that the measures $\mathcal{H}_d^{m(n-m)}$ and $\mathcal{H}_\pi^{m(n-m)}$ are equivalent. To this end, it suffices to demonstrate the bilipschitz-equivalence of the metrics d and d_π:

$$cd(V, W) \leqslant d_\pi(V, W) \leqslant Cd(V, W), \qquad V, W \in G(n, m), \tag{A 6}$$

for some positive and finite constants c and C. To prove the rightmost inequality, we use the second estimate in [**13**, section 1·1·15(7)], namely that if $V, W \in G(n, m)$, and $\mathbf{v}, \mathbf{w} \in G$ are m-vectors with $T\mathbf{v} = V$ and $T\mathbf{w} = W$, then

$$|v - \pi_W v| \leqslant \|\mathbf{v} - \mathbf{w}\|_m$$

for all unit vectors $v \in V$. Since also $T(-\mathbf{w}) = W$, it follows that

$$d_\pi(V, W) \asymp \sup_{|v|=1} |v - \pi_W v| \leqslant \min\{\|\mathbf{v} - \mathbf{w}\|_m, \|\mathbf{v} + \mathbf{w}\|_m\} = d(V, W).$$

To prove the leftmost inequality in (A 6), we fix $V, W \in G(n, m)$ and use Lemma A 1 to find such orthonormal bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_m\}$ for V and W such that $|v_i - w_i| \lesssim d_\pi(V, W)$ for $1 \leqslant i \leqslant m$. Then, we use inequality [**13**, section 1·12·17] to conclude that

$$d(V, W) \leqslant \|v_1 \wedge \cdots \wedge v_m - w_1 \wedge \cdots \wedge w_m\|_m \lesssim m d_\pi(V, W).$$

This completes the proof of (A 6), and shows that $\mathcal{H}_d^{m(n-m)}(B) \asymp \mathcal{H}_\pi^{m(n-m)}(B)$ for any ball $B \subset G(n, m)$ (in either metric). From (A 5), we may now infer that

$$0 < \liminf_{\delta \to 0} \frac{\mathcal{H}_\pi^{m(n-m)}(B_\pi(W_0, \delta))}{\delta^{m(n-m)}} \leqslant \limsup_{\delta \to 0} \frac{\mathcal{H}_\pi^{m(n-m)}(B_\pi(W_0, \delta))}{\delta^{m(n-m)}} < \infty. \tag{A 7}$$

Finally, we observe that $\mathcal{H}_\pi^{m(n-m)}$ is a finite $O(n)$-invariant measure on $G(n, m)$. The finiteness part follows from the equivalence of $\mathcal{H}_\pi^{m(n-m)}$ with $\mathcal{H}_d^{m(n-m)}$, combined with the finiteness of the $\mathcal{H}^{m(n-m)}$-measure of the manifold G; in fact, the exact $\mathcal{H}^{m(n-m)}$-measure of G is computed at the end of [**3**, 3·2·28]. The $O(n)$-invariance was precisely the reason why we introduced the measure $\mathcal{H}_\pi^{m(n-m)}$: the metric d_π is $O(n)$-invariant, so all the corresponding Hausdorff measures are automatically $O(n)$-invariant. Now $\mathcal{H}_\pi^{m(n-m)}$ and $\gamma_{n,m}$ are both $O(n)$-invariant – hence uniformly distributed – measures on $G(n, m)$, and it follows from [**11**, theorem 3·4] that $\gamma_{n,m} = \beta \mathcal{H}_\pi^{m(n-m)}$ for some finite constant $\beta > 0$. We infer that (A 7) gives (A 2).

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