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CALABI–YAU DOUBLE COVERINGS OF FANO–ENRIQUES THREEFOLDS

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Abstract This note is a report on the observation that the Fano–Enriques threefolds with terminal cyclic quotient singularities admit Calabi–Yau threefolds as their double coverings. We calculate the invariants of those Calabi–Yau threefolds when the Picard number is one. It turns out that all of them are new examples.

Keywords: Fano–Enriques threefold; Calabi–Yau threefold; Fano threefold

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1. Introduction

The threefolds whose hyperplane sections are Enriques surfaces were studied by Fano in a famous paper [**5**]. The modern proofs for the results of [**5**] were given in [**3**]. Such varieties are always singular and their canonical divisors are not Cartier but numerically equivalent to Cartier divisors. We call such threefolds *Fano–Enriques threefolds* (see Definition 2.1). In this note, we consider Fano–Enriques threefolds whose singularities are terminal cyclic quotient ones. It is worth noting that any Fano–Enriques threefold with terminal singularities admits a $\mathbb{Q}$-smoothing to one with terminal cyclic quotient singularities [**10**]. The canonical coverings (which are double-coverings) of Fano–Enriques threefolds with terminal cyclic quotient singularities are smooth Fano threefolds [**1**, **13**]. Hence all the singular points of such Fano–Enriques threefolds are of type $\frac{1}{2}(1,1,1)$. Using the classification of smooth Fano threefolds, Bayle [**1**] and Sano [**13**] gave a classification of such threefolds. In this note, we observe that all those Fano–Enriques threefolds also admit some Calabi–Yau threefolds as their double covering, branched along some smooth surfaces and those singularities. A *Calabi–Yau threefold* Y is a compact Kähler manifold with trivial canonical class such that the intermediate cohomology groups of its structure sheaf are trivial ($h^1(Y,\mathcal{O}_Y) = h^2(Y,\mathcal{O}_Y) = 0$). We calculate the invariants of these Calabi–Yau double coverings when their Picard numbers are one (Table 1). It turns out

Table 1. *Invariants of Calabi–Yau double coverings.*

X_d	H_Y^3	$H_Y \cdot c_2(Y)$	$h^{1,1}(Y)$	$h^{1,2}(Y)$
X_1	4	28	1	45
X_2	8	32	1	33
X_3	2	20	1	37
X_4	4	28	1	45

that all the Calabi–Yau threefolds are new examples. Although a number of Calabi–Yau threefolds have been constructed, those with Picard number one are still quite rare. Note that they are primitive and play an important part in the moduli spaces of all Calabi–Yau threefolds [**7**].

2. Calabi–Yau double coverings

As the higher-dimensional algebraic geometry has been developed, the definition of Fano–Enriques threefolds has also evolved and has been generalized. We adopt the following version of the definition.

Definition 2.1. A three-dimensional normal projective variety W is called a Fano–Enriques threefold if W has canonical singularities, and $-K_W$ is not a Cartier divisor but numerically equivalent to an ample Cartier divisor H_W.

Prokhorov proved in [**11**] that the generic surface in the linear system $|H_W|$ is an Enriques surface with canonical singularities and that the Enriques surface is smooth if the singularities of W are isolated and $-K_W^3 \neq 2$. We refer to [**2**, **6**, **12**] for more systematic expositions of Fano–Enriques threefolds. In this note, we consider the case that W has only terminal cyclic quotient singularities. We summarize the properties of W [**1**, **3**, **5**, **13**].

(1) All the singularities of W are the type of $\frac{1}{2}(1,1,1)$.

(2) The number of singularities of W is eight.

(3) $-2K_W$ is linearly equivalent to $-2H_W$.

(4) There is a smooth Fano threefold that covers doubly W, branched only at the singularities of W.

Bayle [**1**] and Sano [**13**] gave a classification of smooth Fano threefolds that double-cover Fano–Enriques threefolds.

Let $\varphi : X \to W$ be the double covering, branched along the singularities of W. Then X is one of smooth Fano threefolds in [**13**, Theorem 1.1]. We want to find a Calabi–Yau threefold that double-covers W, using the following theorem, which is a special case of [**9**, Theorem 1.1].

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Theorem 2.2. *Let W be a projective three-dimensional variety with singularities of type $\frac{1}{2}(1,1,1)$ such that $h^1(W,\mathcal{O}_W) = h^2(W,\mathcal{O}_W) = 0$. Suppose that the linear system $|-2K_W|$ contains a smooth surface S. Then there is a Calabi–Yau threefold Y that is a double covering of W with the branch locus $S \cup \mathrm{Sing}(W)$.*

Let p_1, p_2 be any points of X. From the description of the Fano threefold Xs in [**13**, Theorem 1.1], one can find an effective divisor D from $|-K_X|$ such that D does not contain p_1, p_2. Let θ be the covering involution on X, i.e. the quotient $X/\langle\theta\rangle$ is W. Therefore, for any point $q \in W$, we can find an effective divisor D in the linear system $|-K_X|$ such that $D \cap \varphi^{-1}(\{q\}) = \emptyset$. Note that the effective divisor

$$\varphi(D) + \varphi(\theta(D))$$

belongs to the linear system $|-2K_W|$ and does not contain the point q. So the linear system $|-2K_W|$ is base-point free and we can find a smooth surface S from it. Hence, by Theorem 2.2, there is a Calabi–Yau threefold Y that covers doubly W, branched along S and singularities of W.

Since $H^i(X,\mathcal{O}_X) = 0$ and $H^i(W,\mathcal{O}_W) = 0$ for $i = 1, 2$, we have isomorphisms

$$H^2(X,\mathbb{Z}) \simeq \mathrm{Pic}(X), \qquad H^2(W,\mathbb{Z}) \simeq \mathrm{Pic}(W)$$

by the exponential sequences. Hence we can regard classes of Cartier divisors of X, W as elements of $H^2(X,\mathbb{Z})$ and $H^2(W,\mathbb{Z})$, respectively. Let r be the index of Fano threefold X (i.e. the largest integer r such that $-K_X = rH_X$ for some ample divisor H_X of X).

Now we calculate the invariants of Y. For a double covering with dimension higher than two, it is a non-trivial task to calculate the topological invariants even in the case where the base of the covering is smooth. In our case, S is an ample divisor of W, so it may be worth trying to apply the Lefschetz hyperplane theorem. However, W is not smooth, so the usual Lefschetz hyperplane theorem does not apply here. There are other versions of the Lefschetz hyperplane theorem for singular varieties, but they all require that $W - S$ is smooth, which is not true for our case. We prove a type of the Lefschetz hyperplane theorem for $S \subset W$. We say that an element α of an additive abelian group G is divisible by an integer k if $\alpha = k\alpha'$ for some element $\alpha' \in G$. α is said to be primitive if it is divisible by only ± 1. We denote the quotient of G by its torsion part as G_f.

Lemma 2.3. *The map $H^2(W,\mathbb{Q}) \to H^2(S,\mathbb{Q})$, induced by the inclusion $S \hookrightarrow W$, is injective and the image $H_W|_S$ in $H^2(S,\mathbb{Z})_f$ of $H_W \in H^2(W,\mathbb{Z})$ is divisible by r.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & W \\ \uparrow & & \uparrow \\ S_X & \xrightarrow{\ \varphi|_{S_X}\ } & S \end{array}$$

where $S_X = \varphi^{-1}(S)$ and the vertical maps are inclusions. Note that

$$\varphi|_{S_X} : S_X \to S$$

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is an unramified double covering and we have a pull-back

$$(\varphi|_{S_X})^* : H^2(S, \mathbb{Q}) \to H^2(S_X, \mathbb{Q}).$$

We have an induced commutative diagram:

$$\begin{array}{ccc} H^2(X, \mathbb{Q}) & \xleftarrow[\varphi^*]{} & H^2(W, \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^2(S_X, \mathbb{Q}) & \xleftarrow[(\varphi|_{S_X})^*]{} & H^2(S, \mathbb{Q}) \end{array}$$

Note that the pull-back $\varphi^* : H^2(W, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is injective. Since S_X is a smooth ample divisor of X, the map $H^2(X, \mathbb{Q}) \to H^2(S_X, \mathbb{Q})$ is injective by the Lefschetz hyperplane theorem. So we have the injectivity of the map

$$H^2(W, \mathbb{Q}) \to H^2(S, \mathbb{Q}).$$

Consider another commutative diagram:

$$\begin{array}{ccc} H^2(X, \mathbb{Z}) & \xleftarrow[\varphi^*]{} & H^2(W, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(S_X, \mathbb{Z})_f & \xleftarrow[(\varphi|_{S_X})^*]{} & H^2(S, \mathbb{Z})_f \end{array}$$

Note that $-K_X = rH_X$ and recall that θ is the covering involution on X. Then $\theta^*(-K_X) = -K_X$ in $H^2(X, \mathbb{Z})$. Since $H^2(X, \mathbb{Z})$ has no torsion, $\theta^*(H_X) = H_X$ in $H^2(X, \mathbb{Z})$ and so

$$(\theta|_{S_X})^*(H_X|_{S_X}) = \theta^*(H_X)|_{S_X} = H_X|_{S_X}$$

in $H^2(S_X, \mathbb{Z})_f$. Hence $h' := H_X|_{S_X}$ lies in the image of the map

$$H^2(S, \mathbb{Z})_f \to H^2(S_X, \mathbb{Z})_f.$$

Note that $\varphi^*(H_W) = -K_X = rH_X$. Hence

$$(\varphi|_{S_X})^*(H_W|_S) = \varphi^*(H_W)|_{S_X} = rH_X|_{S_X} = rh'$$

is divisible by r in $H^2(S_X, \mathbb{Z})_f$. Since the map

$$(\varphi|_{S_X})^* : H^2(S, \mathbb{Z})_f \to H^2(S_X, \mathbb{Z})_f$$

is injective, $H_W|_S$ is divisible by r in $H^2(S, \mathbb{Z})_f$. □

We note that H_W is primitive in $H^2(W, \mathbb{Z})$. By the above lemma, $H_W|_S$ is not primitive in $H^2(S, \mathbb{Z})_f$ when $r > 1$. This is different from what the usual Lefschetz hyperplane theorem expects for smooth threefolds.

Proposition 2.4. *We have*

$$h^2(Y) \leq h^2(X),$$

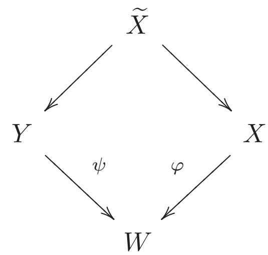
$$e(Y) = e(X) - 24 - 2(-K_X)^3$$

and

$$\psi^*(H_W) \cdot c_2(Y) = (-K_X)^3 + 24,$$

where $\psi : Y \to W$ *is the double covering in Theorem 2.2,* $e(Y)$ *is the topological Euler characteristic of* Y *and* $c_2(Y)$ *is the second Chern class of* Y*.*

Proof. Consider the following fibre product of two double covers:



Then it is easy to see that:

(a) $\widetilde{X} \to Y$ is an étale double cover, and

(b) $\widetilde{X} \to X$ is the double cover branched along a member $\widetilde{S}$ of $|-2K_X|$.

Using the fact that $\widetilde{S}$ is an ample divisor of $\widetilde{X}$, one can show that $h^2(\widetilde{X}) = h^2(X)$ [4]. Since $h^2(Y) \leq h^2(\widetilde{X})$, we have $h^2(Y) \leq h^2(X)$.

Note that $e(X) = 2e(W) - 8$ and $e(S_X) = 2e(S)$. Note also that $S_X \sim -2K_X$ and, by the Riemann–Roch theorem,

$$1 = \chi(X, \mathcal{O}_X) = \tfrac{1}{24} c_2(X) \cdot (-K_X).$$

By the adjunction formula, we have

$$e(S_X) = c_2(X) \cdot (-2K_X) + 4(-K_X)^3 = 48 + 4(-K_X)^3.$$

So

$$e(Y) = 2e(W) - e(S) - 8 = e(X) - \tfrac{1}{2} e(S_X) = e(X) - 24 - 2(-K_X)^3.$$

Note that $\psi^*(S) \sim 2S_Y$ and $S \sim 2H_W$, where $S_Y = \psi^{-1}(S)$. So $\psi^*(H_W) \cdot c_2(Y) = S_Y \cdot c_2(Y)$. By the adjunction formula,

$$\begin{aligned} S_Y \cdot c_2(Y) &= -S_Y^3 + c_2(S_Y) \\ &= -\psi^*(H_W)^3 + e(S_Y) \\ &= -2H_W^3 + e(S) \end{aligned}$$

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$$= -\varphi^*(H_W)^3 + \tfrac{1}{2}e(S_X)$$
$$= -(-K_X)^3 + 24 + 2(-K_X)^3 = (-K_X)^3 + 24.$$

Hence $\psi^*(H_W) \cdot c_2(Y) = (-K_X)^3 + 24$. □

We are interested in the case where the Calabi–Yau threefold Y has Picard number one. Hence we assume that X has Picard number one. There are four such families [**13**]:

- X_1: complete intersection of a quadric and a quartic in the weighed projective space $\mathbb{P}(1,1,1,1,1,2)$, $r_1 = 1$, $-K^3_{X_1} = 4$, $e(X_1) = -56$.
- X_2: complete intersection of three quadrics in $\mathbb{P}^6$, $r_2 = 1$, $-K^3_{X_2} = 8$, $e(X_2) = -24$.
- X_3: hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$, $r_3 = 2$, $-K^3_{X_3} = 16$, $e(X_3) = -16$.
- X_4: compete intersection of two quadrics in $\mathbb{P}^5$, $r_4 = 2$, $-K^3_{X_4} = 32$, $e(X_4) = 0$.

Theorem 2.5. *Suppose that X has Picard number one. Then W and Y have Picard number one,*

$$H^3_Y = \frac{1}{r^3}(-K^3_X)$$

and

$$H_Y \cdot c_2(Y) = \frac{1}{r}((-K_X)^3 + 24),$$

where H_Y is an ample generator of $\mathrm{Pic}(Y)$.

Proof. By Proposition 2.4,

$$1 \le h^2(W) \le h^2(Y) \le h^2(X) = 1,$$

so W and Y have Picard number one. Since Y has Picard number one, $\psi^*(H_W) = kH'_Y$ for some ample generator H'_Y of $\mathrm{Pic}(Y)$ ($\simeq H^2(Y,\mathbb{Z})$) and a positive integer k. Note that $H_Y - H'_Y$ is a torsion element and that H'_Y is primitive in $H^2(Y,\mathbb{Z})$. We also note that S_Y is a smooth ample divisor of Y. By the Lefschetz hyperplane theorem, $H'_Y|_{S_Y}$ is primitive in $H^2(S_Y,\mathbb{Z})_f$. By Lemma 2.3, $H_W|_S$ is divisible by r in $H^2(S,\mathbb{Z})_f$. So its image $(\psi|_{S_Y})^*(H_W|_S)$ in $H^2(S_Y,\mathbb{Z})_f$ is divisible by r. Note that

$$(\psi|_{S_Y})^*(H_W|_S) = \psi^*(H_W)|_{S_Y} = k(H'_Y|_{S_Y}).$$

So k is divisible by r. Let $k = lr$ for some positive integer l. We will show that $l = 1$. Note that

$$H^3_Y = {H'_Y}^3 = \frac{1}{k^3}\psi^*(H_W)^3 = \frac{2}{k^3}H^3_W = \frac{1}{k^3}\varphi^*(H_W)^3 = \frac{1}{r^3l^3}(-K^3_X)$$

and

$$H_Y \cdot c_2(Y) = \frac{1}{k}\psi^*(H_W) \cdot c_2(Y) = \frac{1}{rl}((-K^3_X) + 24).$$

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For X_1, X_3 and X_4, the condition of H_Y^3 being a positive integer requires that $l = 1$. For X_2, by the Riemann–Roch theorem, we have

$$\chi(Y, H_Y) = \frac{H_Y^3}{6} + \frac{H_Y \cdot c_2(Y)}{12} = \frac{8}{6l^3} + \frac{32}{12l} = \frac{4+8l^2}{3l^3},$$

which should be an integer. So we also have $l = 1$ in this case. Therefore,

$$H_Y^3 = \frac{1}{r^3}(-K_X^3)$$

and

$$H_Y \cdot c_2(Y) = \frac{1}{r}((-K_X)^3 + 24). \qquad \square$$

By Proposition 2.4 and the relation $e(Y) = 2(h^{1,1}(Y) - h^{1,2}(Y))$, we can determine all the Hodge numbers of Y. We list the invariants of the Calabi–Yau threefold Ys in Table 1. It turns out that they are all new examples. See [8, Appendix I] for a list of known examples of Calabi–Yau threefolds of Picard number one.

Note that the invariants of Y_1 and those of Y_4 overlap. Consider the commutative diagram in the proof of Proposition 2.4. For X_1, the branch locus of $\widetilde{X}_1 \to X_1$ is a quadric section, thus $\widetilde{X}_1$ is a $(2,2,4)$-weighted complete intersection of $\mathbb{P}(1,1,1,1,1,1,2)$. For X_4, the branch locus of $\widetilde{X}_4 \to X_4$ is a quartic section, thus $\widetilde{X}_4$ is also a $(2,2,4)$-weighted complete intersection of $\mathbb{P}(1,1,1,1,1,1,2)$. Therefore, $\widetilde{X}_1$ and $\widetilde{X}_4$ are in the same family. Since Y_1 and Y_4 are étale $\mathbb{Z}_2$-quotients of $\widetilde{X}_1$ and $\widetilde{X}_4$, respectively, they have the same invariants.

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