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ON TWO PROBLEMS CONCERNING NEUTRAL POLYVERBAL OPERATIONS ON GROUPS

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1. **Introduction.** In his paper [2] O. N. Golovin introduced the notion of a neutral polyverbal operation on groups, of which Moran's verbal operations [4] and Gruenberg's and Šmelkin's operations [3, 5] are special cases. The free and direct multiplication we shall call here trivial operations. It is known [5, 4] that Mal'cev's postulate does not hold for nontrivial verbal operations but the postulate of associativity does. For nontrivial Gruenberg's and Šmelkin's operations the situation is the reverse [5, 1].

More than 25 years ago A. I. Mal'cev posed a question about the existence of non-trivial regular operations for which both postulates hold. In [2] O. N. Golovin raised the question whether the associative neutral polyverbal operations form a lattice.

We will show here that these two problems concerning neutral polyverbal operations cannot simultaneously have affirmative solutions.

2. **Notation and statement of result.** Let $X=\prod_{i=1}^{\infty} {}^* X_i$ be the free product of free groups X_i of countable ranks. The *Cartesian subgroup* [4] of X will be denoted by C. According to [2], the normal subgroup, W, of X is called *polyverbal* if it is invariant with respect to permutation of factors X_i and with respect to endomorphisms of X, which are results of endomorphisms of the free factors X_i, $i=1, 2, \ldots$. The polyverbal subgroup is called *neutral* if it is contained in the Cartesian subgroup. Let $G=\prod_{i\in I}^* G_i$ be the free product of some set of groups G_i, $i \in I$. A homomorphism from X to G is said to be *regular* if the image of each free factor X_j is contained in some G_i, $i \in I$, and nontrivial images of different X_j are contained in different G_i. For the free product $G=\prod_{i\in I}^* G_i$ we define the *polyverbal subgroup* $W(G)$ as the subgroup generated by the images of all elements of W under all regular homomorphisms from X to G. If W is any neutral polyverbal (n.p. for short) subgroup then the n.p. W-product of the set G_i, $i \in I$ of groups is defined as

$$\prod_{i\in I}^{W} G_i = \left(\prod_{i\in I}^{*} G_i\right) \Big/ W(G).$$

We shall say that a W-operation satisfies *Mal'cev's postulate* if the subgroups $A_i \subseteq G_i$, $i \in I$ generate in $\prod_{i\in I}^{W} G_i$ the subgroup $\prod_{i\in I}^{W} A_i$. The intersection of any verbal subgroup [4] of X and the Cartesian subgroup gives us an example of a n.p. subgroup which defines the verbal operation [4].

By $W(G)_v$ we shall denote the least verbal subgroup in G which includes $W(G)$. For X we shall write $W(X)_v = W_v$. For A, $B \subseteq G$ denote by $[A, B]^G$ the normal closure of the commutator subgroup of A and B. By **U** we denote the variety determined by the verbal subgroup U.

We speak of the "lattice of n.p. operations", with the *join* of a W_1- and a W_2-operation defined as the $W_1 \cap W_2$-operation, and their *meet* defined as the $W_1 \cdot W_2$-operation.

We shall use the following theorems:

THEOREM 1 (*follows from* [5], *lemma* 3). *If Mal'cev's postulate holds for a nontrivial n.p. W-operation, then the W-product of two infinite cyclic groups is different from their direct product.*

THEOREM 2 ([1], *theorem* 10). *If a n.p. W-operation is associative then we have*

$$W_v \cap C = \prod_{i \neq j} [W(X_i)_v, X_j]^X \cdot W.$$

Now we state our result.

THEOREM. *If the associative n.p. operations form a lattice then none of them satisfies Mal'cev's postulate.*

3. **Lemmas and proof of the Theorem.**

LEMMA 1. *If a nontrivial n.p. W-operation coincides with the verbal $W_v \cap C$-operation on some variety* **U** *such that $U \subset W_v$, then Mal'cev's postulate does not hold for this W-operation.*

Proof. Because of the inclusion $U \subset W_v$, there exists a non-unit element $w \in W_v$ such that $w \notin U$. Let A and B be free groups of the variety **U**. Let a and b be non-unit verbal values of the word w in A and B respectively. By our assumption, the W-product of the groups A and B coincides with their verbal $W_v \cap C$-product. It follows that

$$W(A * B) = W(A * B)_v \cap [A, B]^{A*B}.$$

We can see now that the images of the elements a and b commute in the W-product of the groups A and B, because

$$[a, b] \in W(A * B)_v \cap [A, B]^{A*B} = W(A * B).$$

This means that in the W-product of the groups A and B, the subgroup generated by the images of the subgroups $\{a\}$ and $\{b\}$ is their direct product. According to Theorem 1 this implies that Mal'cev's postulate does not hold for our W-operation.

LEMMA 2. *If the n.p. subgroups W and $\overline{W} = [W_v, X] \cap W$ define associative operations, then the W-operation coincides with the verbal $W_v \cap C$-operation on some variety* **U**, *where $U \subset W_v$.*

Proof. Let $U=\overline{W}_v$ be the least verbal subgroup containing $\overline{W}$. Because of the obvious inclusions

$$[W_v, X] \supseteq \overline{W} \supseteq [W, X]$$

we can see that $U=[W_v, X]$.

By Theorem 2 we can now give a necessary condition of associativity for W- and $\overline{W}$-operations:

$$W_v \cap C = \prod_{i \neq j} [W(X_i)_v, X_j]^X \cdot W;$$

$$U \cap C = \prod_{i \neq j} [\overline{W}(X_i)_v, X_j]^X \cdot \overline{W}.$$

As $[W(X_i)_v, X_j] \subseteq [W_v, X] \cap C = U \cap C$ and $\overline{W} \subseteq W$ we now have

$$W_v \cap C \subseteq \prod_{i \neq j} [\overline{W}(X_i)_v, X_j]^X W.$$

Because of the obvious validity of the converse inclusion and the equality $\overline{W}(X_i)_v = U(X_i)_v$ we have

$$W_v \cap C = \prod_{i \neq j} [U(X_i)_v, X_j]^X W,$$

from which it follows that the W- and $W_v \cap C$-operation coincide on the variety **U**, with $U \subset W_v$.

Proof of the Theorem. Let W be any polyverbal subgroup, then the $(W_v, V) \cap C$-operation is associative because it is a verbal operation.

If now the associative n.p. operations form a lattice, then, for any associative n.p. W-operation, the corresponding $\overline{W}$-operation (as in Lemma 2) will also be associative, as a join of two associative operations, because

$$\overline{W} = ([W_v, X] \cap C) \cap W.$$

Now, by Lemmas 2 and 1, Mal'cev's postulate does not hold for the W-operation.

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