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Nonlinear interaction between a resonance-mode ($\mathbf{k}_{\parallel} = 0$) wave and energetic plasma particles

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Abstract. Nonlinear resonant interaction between energetic particles and quasi-electrostatic wave propagating perpendicularly to the ambient magnetic field in a homogeneous plasma is studied in detail, with the main focus on the wave amplification or attenuation caused by resonant interaction. As for $\mathbf{k}_{\parallel} \to 0$, resonance velocity determined from linear resonance conditions tends to infinity, the interaction under discussion is a completely nonlinear phenomenon. While the wave amplitude may increase or decrease depending on the wave parameters and integral characteristics of the energetic particle distribution function, an essential wave amplification may occur when the wave frequency is close to a multiple of cyclotron harmonic, and when the transversal energy of resonant particles is large enough.

1. Introduction

Resonance wave–particle interaction is one of the fundamental plasma phenomena with a great number of applications. The well-known condition for resonance interaction between a wave and a particle in magnetized plasma has the form

$$\omega - k_{\parallel} v_{\parallel} = n\Omega_c, \tag{1.1}$$

where ω and $k_{\parallel}$ are the wave frequency and parallel (along the ambient magnetic field) wave normal vector, respectively, $v_{\parallel}$ is the particle parallel velocity, Ω_c is the particle cyclotron frequency, and n is an integer. The number of works on the theory of resonance wave–particle interaction in plasma is huge, thus a serious account of them is hardly possible in an introduction to a paper, although this paper deals with one particular problem of this theory. Thus, we limit ourselves to several remarks necessary for our consideration.

As one can see from (1.1), the case $\mathbf{k}_{\parallel} = 0$ considered in the present paper is, in a sense, degenerated. In this case, at least in non-relativistic approximation, the resonance conditions do not depend on particle characteristics. Formally, for $\mathbf{k}_{\parallel} \to 0$ and $\omega \neq n\Omega_c$, the resonance velocity tends to infinity implying that the resonance wave–particle interaction vanishes. That is why it is usually assumed that transversely propagating waves are not subject to linear resonant instability.

It is well understood, of course, that the resonance conditions (1.1) are written in linear approximation, neglecting the particle velocity variation under the influence of the wave, and that the linear stage of wave damping (or growth in the case of instability) is only an initial stage. We mention the investigation of the nonlinear stage of Landau damping for Langmuir waves started by Mazitov (1965) and Al'tshul and Karpman (1965), and finalized by O'Neil (1965), who considered nonlinear dynamics of resonance particles in the wave field that led to an essential modification of the results predicted by linear theory. In these, as well as in many other investigations of the nonlinear stage of resonant instabilities, the nonlinear resonance region is considered to be centered on the resonance velocity determined by linear resonance conditions and, in this sense, the case $\mathbf{k}_\parallel = 0$ is definitely a particular one.

At the same time, nonlinear particle dynamics in the field of transversely propagating waves, both in homogeneous and in an inhomogeneous plasma, have been extensively studied by many authors (see, in particular, Karney (1978, 1979), Gell and Nakach (1980), Ryabova and Shklyar (1983), Shklyar (1986) and references therein). Those studies, however, were mostly directed at the investigation of regular and stochastic particle motion, in particular, of particle acceleration and heating in the field of an intense wave. The back influence of energetic particles upon transversely propagating wave has not yet been considered.

2. Basic equations

As is well known (see, for instance, Ginzburg and Rukhadze (1972)), resonance waves in plasma are quasi-electrostatic. Accordingly, we write the field of the wave propagating perpendicularly to the ambient magnetic field $\mathbf{B}_0$ directed along the z-axis in the form

$$\mathbf{E} = (E_x, 0, 0), \quad E_x = E\sin(kx - \omega t) \tag{2.1}$$

with constant wave number k and wave frequency ω. We study particle dynamics in the field of constant amplitude wave assuming that the back influence of particles upon the wave may be found by means of successive approximation method using energy conservation law. This approach called the approximation of a given field has been put forward by Mazitov (1965), Al'tshul and Karpman (1965), and O'Neil (1965).

The following consideration is applicable to the dynamics of energetic electrons in the field of an upper hybrid wave in the case when the electron plasma frequency is larger or much larger than electron cyclotron frequency, as well as to dynamics of energetic ions in the field of a lower hybrid wave. That is why we do not specify particle charge and mass denoting them by q and m, respectively. The equations of motion then take the form

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{E}}{m} + \frac{q}{mc}[\mathbf{v}\times\mathbf{B}_0]. \tag{2.2}$$

In the geometry under consideration, particles execute free motion in the z-direction, with $v_z = v_{z0}$, $z = z_0 + v_{z0}t$, where the subscript '0' denotes the initial values of the corresponding quantities. The x and y components of (2.2) describing the transversal motion of particles read

$$\frac{dx}{dt} = v_x, \quad \frac{dv_x}{dt} = \Omega_c v_y + \frac{qE}{m}\sin(kx - \omega t),$$
$$\frac{dy}{dt} = v_y, \quad \frac{dv_y}{dt} = -\Omega_c v_x, \tag{2.3}$$

where particle cyclotron frequency

$$\Omega_c = \frac{qB_0}{mc}$$

has been introduced.

Equations (2.3) have the obvious integral of motion

$$x_c \equiv x + \frac{v_y}{\Omega_c} = \text{constant} \tag{2.4}$$

which is nothing but the x-coordinate of the particle guiding center. Since, apart from that, the variable y does not enter the right-hand sides of (2.3), the equations for v_x and v_y, after excluding the variable x from their right-hand side with the help of (2.4), form a closed set of two equations. Those equations, however, do not have the Hamiltonian form. That is why it is more convenient to use, instead of v_x and v_y, two other variables, namely, the transversal adiabatic invariant μ and particle gyrophase φ connected with the quantities v_x and v_y by the relations

$$v_x = \sqrt{\frac{2\mu\Omega}{m}}\cos\varphi, \quad v_y = -\sqrt{\frac{2\mu\Omega}{m}}\sin\varphi\,\mathrm{sign}(q), \quad \Omega \equiv |\Omega_c|. \tag{2.5}$$

Relations (2.5) together with (2.3) permit us to obtain equations for φ and μ in a straightforward way resulting in

$$\frac{d\varphi}{dt} = \Omega - \frac{qEk}{m\Omega\lambda}\sin(kx_c + \lambda\sin\varphi - \omega t)\sin\varphi,$$
$$\frac{d\mu}{dt} = \frac{qE\lambda}{k}\sin(kx_c + \lambda\sin\varphi - \omega t)\cos\varphi, \tag{2.6}$$

where the parameter λ which has the meaning of dimensionless Larmour radius is expressed through the variable μ according to the relation

$$\lambda = k\sqrt{\frac{2\mu}{m\Omega}}. \tag{2.7}$$

It is easy to see that (2.6) may be derived from the Hamiltonian

$$H = \Omega\mu + \frac{qE}{k}\cos(kx_c + \lambda\sin\varphi - \omega t) \tag{2.8}$$

with μ and φ being the canonically conjugated momentum and phase. One should bear in mind that the quantity λ is the function of canonical momentum μ according to relation (2.7). Using the well-known expansion of the expression $\exp(\mathrm{i}\,\lambda\sin\varphi)$ in Bessel functions, one can rewrite the Hamiltonian (2.8) in the form

$$H = \Omega\mu + \frac{qE}{k}\sum_{n=-\infty}^{\infty} J_n(\lambda)\cos(kx_c + n\varphi - \omega t) \tag{2.9}$$

which reveals the possibility of resonance effects in wave–particle interaction. In zero-order approximation ($E \to 0$) the rate of phase variation in the nth term in

(2.9) is equal to $n\Omega-\omega$. If this quantity tends to zero, the corresponding term in the Hamiltonian gives rise to resonant variation of the particle momentum. We see that, in contrast to oblique wave propagation where resonance conditions define particle longitudinal velocity, in the case of strictly perpendicular propagation the resonance conditions $n\Omega-\omega = 0$ determine the wave frequency. For $(n\Omega-\omega) \to 0$, the influence of the wave field upon gyrophase variation should be taken into account to obtain a correct description of particle dynamics. Depending on the wave amplitude, two fundamentally different situations may arise. Deferring quantitative criteria for a moment, we describe those situations in a qualitative manner. If the wave amplitude is so large that the second term in the equation for phase (see (2.6)) is comparable to Ω, then the particle motion becomes stochastic. In the opposite case, the slowly varying term in the Hamiltonian is most important, although the influence of the wave field upon particle gyrophase should, of course, be taken into account. We only consider this case in this paper. The statement above concerning the origin of stochasticity is known as Chirikov's criterion of stochasticity (Chirikov 1979).

The development given above is not at all original and may be found in a number of papers (see, for instance, the papers cited in the last paragraph of Sec. 1). It should be considered as a reminder aimed at facilitating the following consideration and making it more consistent. To bring out the original part of the present work we mention that the previous studies of the case under discussion were limited to an investigation of the particle dynamics, while, to the best of the author's knowledge, the back influence of particles upon wave growth or damping has not yet been considered. The latter is the main focus of the present work.

3. Particle dynamics in the approximation of isolated resonance

As was mentioned above, we are interested in the case when the wave frequency is close to the nth cyclotron harmonic, i.e. $|\omega - n\Omega| \ll \Omega$, while the wave amplitude is small enough (see the quantitative criterion below). In this case we may retain in the Hamiltonian (2.9) only the resonance term proportional to $J_n(\lambda)$, which appears to be slowly varying, as well as the term $\Omega\mu$, of course, which describes particle unperturbed motion in the absence of the wave field. This approach is called the approximation of isolated resonance. The corresponding reduced Hamiltonian then takes the form

$$H_n(\varphi, \mu; t) = \Omega\mu + \frac{qE}{k} J_n(\lambda)\cos(kx_c + n\varphi - \omega t). \tag{3.1}$$

This Hamiltonian is the function of canonical variables (φ, μ) and depends explicitly on time. In order to get rid of this dependence, it is appropriate to choose the quantity

$$\zeta = kx_c + n\varphi - \omega t \tag{3.2}$$

as a new phase, retaining the momentum μ as the second dependent variable. The corresponding equations of motion that follow from (3.1) and (3.2):

$$\begin{aligned} \frac{d\zeta}{dt} &= (n\Omega - \omega) + \frac{nqE}{k}\frac{dJ_n[\lambda(\mu)]}{d\mu}\cos\zeta, \\ \frac{d\mu}{dt} &= \frac{nqE}{k}J_n(\lambda)\sin\zeta \end{aligned} \tag{3.3}$$

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also have canonical form and can be derived from the Hamiltonian

$$h(\zeta,\mu) = (n\Omega - \omega)\mu + \frac{nqE}{k} J_n(\lambda)\cos\zeta, \tag{3.4}$$

which does not depend explicitly on time and, thus, is the constant of the motion. We remind the reader that the quantity λ is a function of canonical momentum μ according to (2.7), which is explicitly indicated in the first equation in (3.3). However, in most cases we write simply λ, on the understanding that it does not lead to confusion. We see that in the approximation of isolated resonance, the problem is reduced to the investigation of a one-dimensional conservative system. To establish the frame of validity of this approximation we notice that the rate of gyrophase variation caused by the wave field is of the order $(qE/k)[dJ_n(\lambda)/d\mu]$. As was mentioned above, the approximation of isolated resonance is applicable when this quantity is much less than Ω, i.e.

$$\frac{qE}{k}\frac{dJ_n(\lambda)}{d\mu} \ll \Omega.$$

At the same time, nonlinear effects in the approximation of isolated resonance become important when

$$p \equiv \frac{nqE}{k(|n\Omega - \omega|)}\frac{dJ_n(\lambda)}{d\mu} \gtrsim 1. \tag{3.5}$$

Before proceeding to the analysis of particle dynamics, we change to dimensionless variables

$$t' = \Omega t, \quad \mu' = \frac{k^2\mu}{m\Omega}. \tag{3.6}$$

In new variables, the equations of motion take the form (omitting primes)

$$\begin{aligned} \frac{d\zeta}{dt} &= \alpha + n\beta\frac{dJ_n(\lambda)}{d\mu}\cos\zeta, \\ \frac{d\mu}{dt} &= n\beta J_n(\lambda)\sin\zeta, \end{aligned} \tag{3.7}$$

while the corresponding Hamiltonian reads

$$h(\zeta,\mu) = \alpha\mu + n\beta J_n(\lambda)\cos\zeta. \tag{3.8}$$

Dimensionless parameters which enter (3.7) and (3.8) are defined as follows:

$$\alpha = \frac{n\Omega - \omega}{\Omega}, \quad \beta = \frac{qEk}{m\Omega^2}, \quad \lambda = \sqrt{2\mu}. \tag{3.9}$$

To gain some insight into the character of particle motion, we turn to examination of particle trajectories in the phase space (ζ,μ) determined by the equation $h(\zeta,\mu) = \text{constant}$. The corresponding contour plots for various values of parameters are shown in Figs 1 and 2. As mentioned above, the character of particle motion is determined by the parameter p (see (3.5)), which can be written in dimensionless variables as

$$p = \frac{n\beta}{\alpha\lambda}\frac{dJ_n(\lambda)}{d\lambda}. \tag{3.10}$$

Although this parameter depends essentially on particle momentum, it is also indicated in the figures, when appropriate. Dependence of particle dynamics on the wave amplitude and, thus, on the nonlinear parameter p, for fixed values of n

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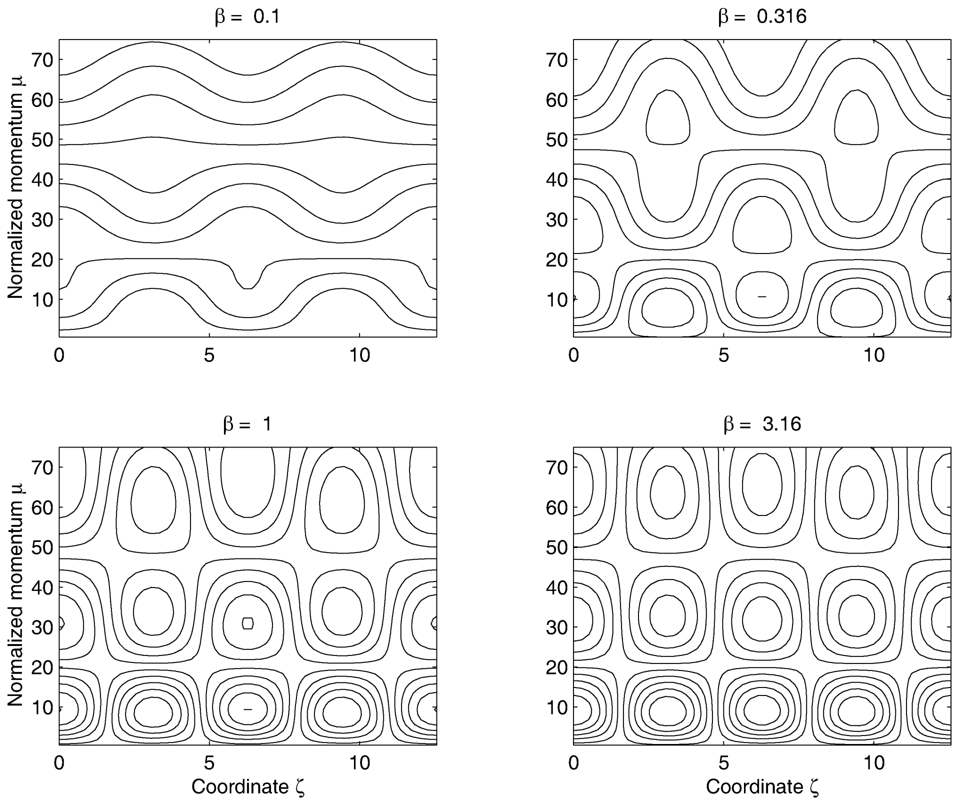

Figure 1. Phase space for $n = 3$, $\alpha = 0.0146$, and different values of the amplitude β (see (3.9) for the definition of dimensionless parameters).

and α is illustrated by Fig. 3. For small p, the phase changes almost linearly and, consequently, the curve of momentum variation in time resembles the line on the phase plane. With the increase of p, the phase trapping occurs, and the time of momentum oscillations essentially decreases.

The most striking feature of particle behavior in the field of transversely propagating wave is revealed in averaged (over initial phase) nonlinear particle dynamics. It appears that, for a given number n of the closest cyclotron harmonic, the sign of averaged momentum variation depends only on the initial momentum, and is independent of time, the wave frequency, or the wave amplitude in the nonlinear regime. This contrasts with the case of oblique propagation where the averaged variation of particle energy depends on details of the particle distribution function in the resonance region, i.e. at $v_\parallel \simeq (\omega - n\Omega_c)/k_\parallel$. The above-mentioned feature is illustrated in Figs 4–7 which exhibit averaged (over the initial phase) variations of particle momentum as functions of initial momentum itself, for various values of parameters indicated in the figure captions. In all figures, the smooth dashed curve depicts the value proportional to the quantity

$$J_n(\lambda)\frac{dJ_n(\lambda)}{d\lambda} \equiv \frac{\lambda}{2}\frac{dJ_n^2(\lambda)}{d\mu}. \tag{3.11}$$

In Fig. 4, the dimensionless time at which the momentum variations are displayed is equal to 151 which corresponds to the "number of linear periods' $\alpha t/2\pi \simeq 0.3$.

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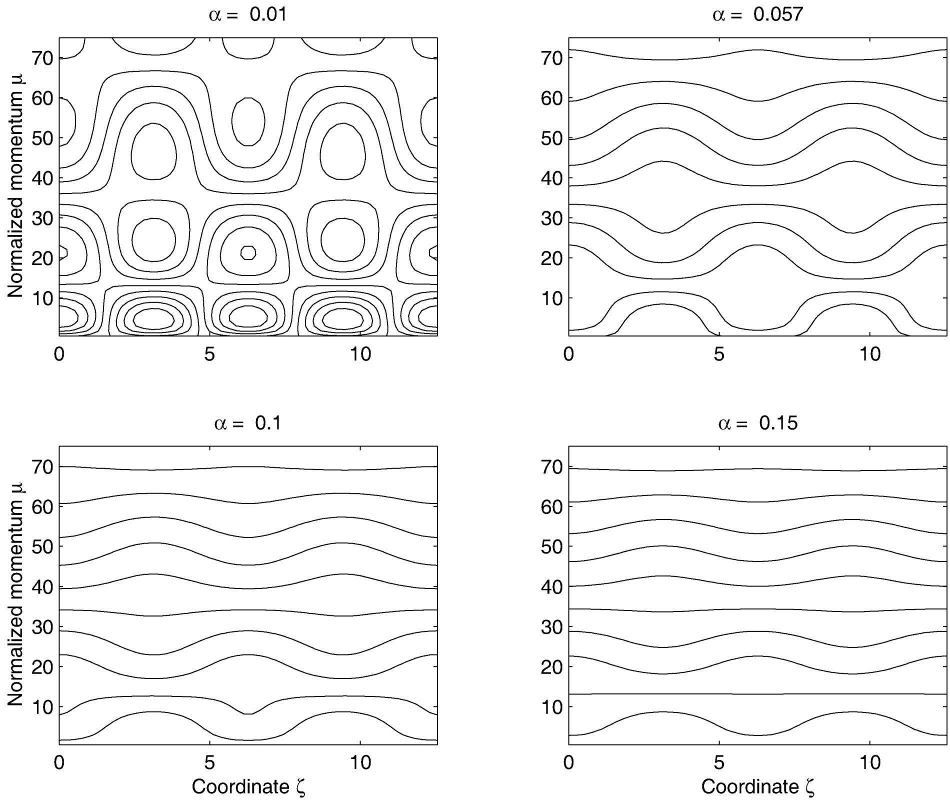

Figure 2. Phase space for $n = 2$, $\beta = 0.6$, and different values of α (i.e. of normalized frequency deviation from the nth cyclotron harmonic).

The averaged variations of particle momentum as functions of initial Larmor radius pictured at various moments of time are shown in Fig. 5. Increasing the amplitude by a factor of 10 (Fig. 6) makes the variations of particle momentum much larger and less smooth, but does not affect the feature under discussion: the averaged momentum variation as usual has the sign of the quantity (3.11). This feature remains in effect if we vary the frequency deviation from the closest harmonic provided that it remains small enough, which is illustrated by Fig. 7.

The feature of particle motion demonstrated above, namely, that the averaged (over initial phase) variation of particle momentum always has the sign of the quantity $dJ_n^2(\lambda_0)/d\lambda_0$ may be understood from the following consideration. Let us introduce, instead of the variables ζ, μ three other quantities ξ, η and μ, the first two defined as

$$\xi = J_n(\lambda)\cos\zeta, \quad \eta = J_n(\lambda)\sin\zeta, \tag{3.12}$$

where, as before, $\lambda = \sqrt{2\mu}$. These quantities obey the following equations

$$\begin{aligned} \frac{d\xi}{dt} &= -\alpha\eta, \\ \frac{d\eta}{dt} &= \alpha\xi + n\beta J_n(\lambda)\frac{dJ_n(\lambda)}{d\mu}, \\ \frac{d\mu}{dt} &= n\beta\eta, \end{aligned} \tag{3.13}$$

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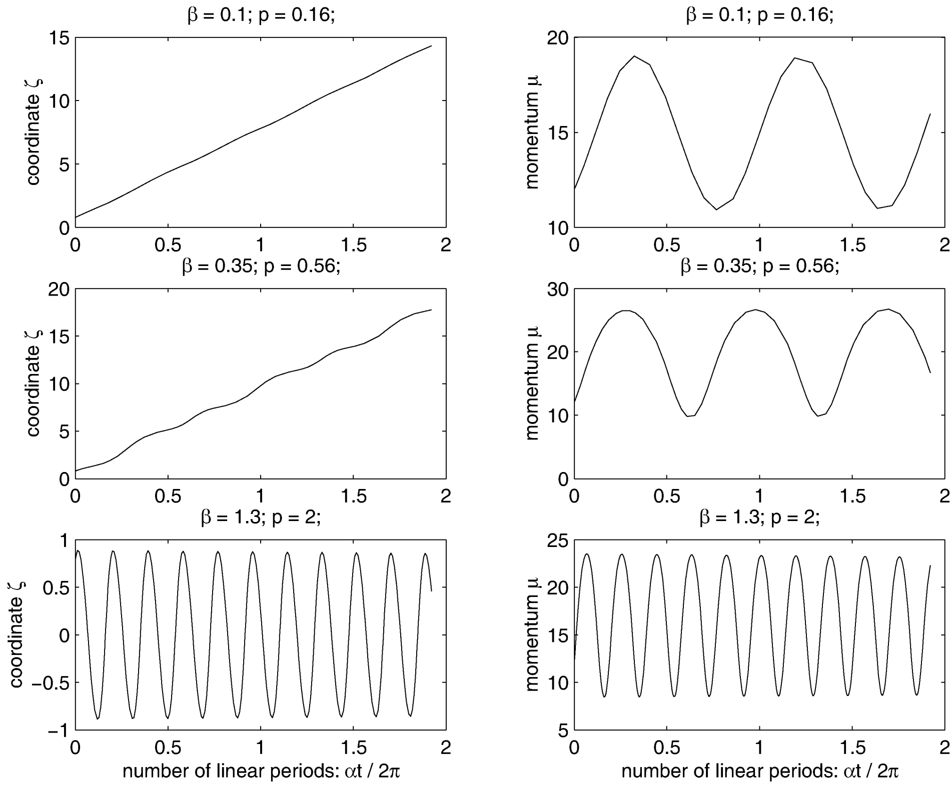

Figure 3. Variation of the particle phase (left column) and momentum (right column) as functions of time for $n = 4$, $\alpha = 0.0345$, and different values of dimensionless amplitude β.

that easily follow from definitions (3.12) and (3.7). The set of equations (3.13) has an obvious integral of motion $\alpha\mu + n\beta\xi$, which, of course, is nothing but (3.8) expressed through new variables. Excluding ξ from the set of equations (3.13) with the help of this integral we obtain the equations for μ, η

$$\frac{d\mu}{dt} = n\beta\eta, \quad \frac{d\eta}{dt} = -\frac{\alpha(\alpha\mu - h)}{n\beta} + \frac{n\beta}{2}\frac{dJ_n^2(\lambda)}{d\mu}, \tag{3.14}$$

which may be derived from the Hamiltonian

$$\mathscr{H}(\mu, \eta) = \frac{n\beta}{2}\eta^2 + \frac{(\alpha\mu - h)^2}{2n\beta} - \frac{n\beta}{2}J_n^2(\lambda) \quad (\lambda \equiv \sqrt{2\mu}), \tag{3.15}$$

where the quantities μ and η play the roles of canonically conjugated phase and momentum, respectively. The Hamiltonian (3.15), as well as (3.14), depends on h as a parameter, but does not depend explicitly on time and, thus, is a constant of the motion. Using the definition (3.8) one can easily show that $\mathscr{H} \equiv 0$, which is not surprising since the equations of motion under discussion have only one integral of motion. The advantage of using the Hamiltonian (3.15) consists in that it has the standard form of a sum of kinetic and potential energy. It describes the motion of a particle with effective mass equal to $(1/n\beta)$ in the potential

$$V(\mu) = \frac{(\alpha\mu - h)^2}{2n\beta} - \frac{n\beta}{2}J_n^2(\sqrt{2\mu}). \tag{3.16}$$

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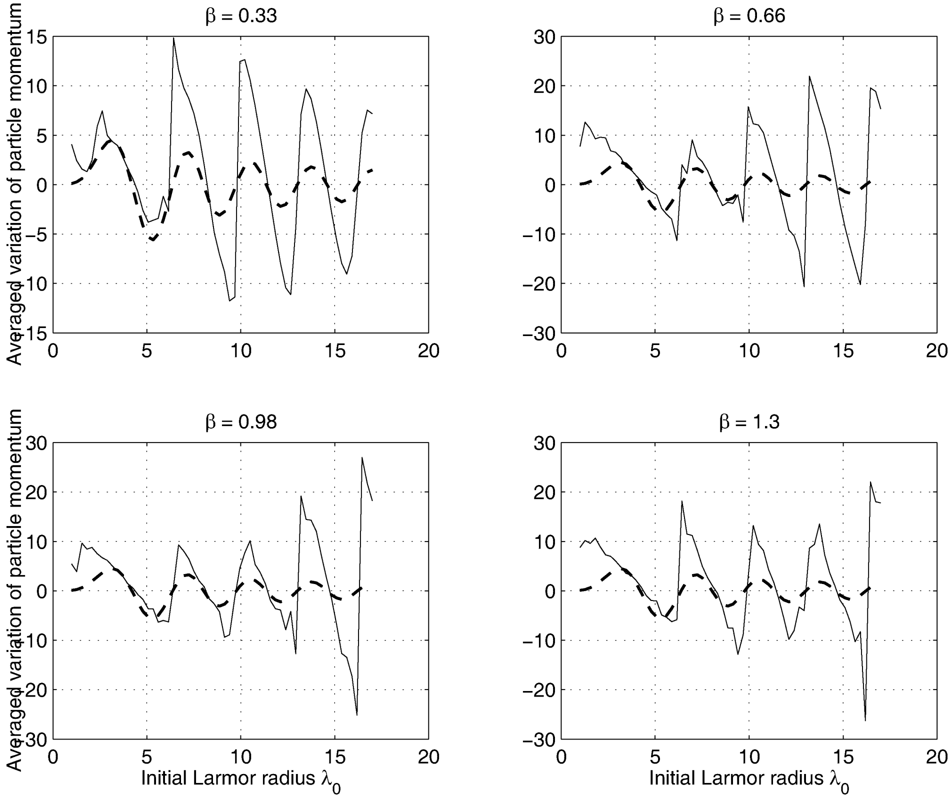

Figure 4. Averaged over initial phase variation of the particle momentum, $\overline{(\mu - \mu_0)}$, as a function of the initial Larmor radius $\lambda_0 \equiv \sqrt{2\mu_0}$ for various values of wave amplitude β and $n = 3$, $\alpha = 0.0123$.

Such a motion permits a direct interpretation. One should, however, keep in mind that the quantity μ plays the role of a coordinate in the Hamiltonian (3.15). The potential $V(\mu)$ is shown in Fig. 8 with a solid line.

Turning to the analysis of particle motion in the potential (3.16) we first compare the forces originating from the first and the second terms in (3.16), i.e. the first and the second terms in (3.14) for η. From definition (3.15) and the condition $\mathscr{H} = 0$ it follows that $|\alpha\mu - h| \lesssim |n\beta J_n(\lambda)|$. Using this inequality we find that the ratio of the first term in (3.14) to the second term is less than $1/p$, thus, in nonlinear regime, the second term in the potential (shown in Fig. 8 by the dashed curve) is the dominant term. Analysis of particle motion in the potential (3.16) permits us to understand why the average variation of particle momentum has the sign of the quantity (3.11). Qualitative explanation of this fact is the following. Since the particle motion represents oscillations in the potential which is close to symmetrical with respect to its minimum, the averaged over time coordinate μ coincides with the coordinate μ_m of the potential minimum. Under assumption of ergodicity, the same is true for the coordinate μ averaged over particle initial coordinate μ_0. Thus, for particles with initial coordinate larger than μ_m, their coordinate decreases on average and *vice versa*. We remind the reader that in the canonical representation

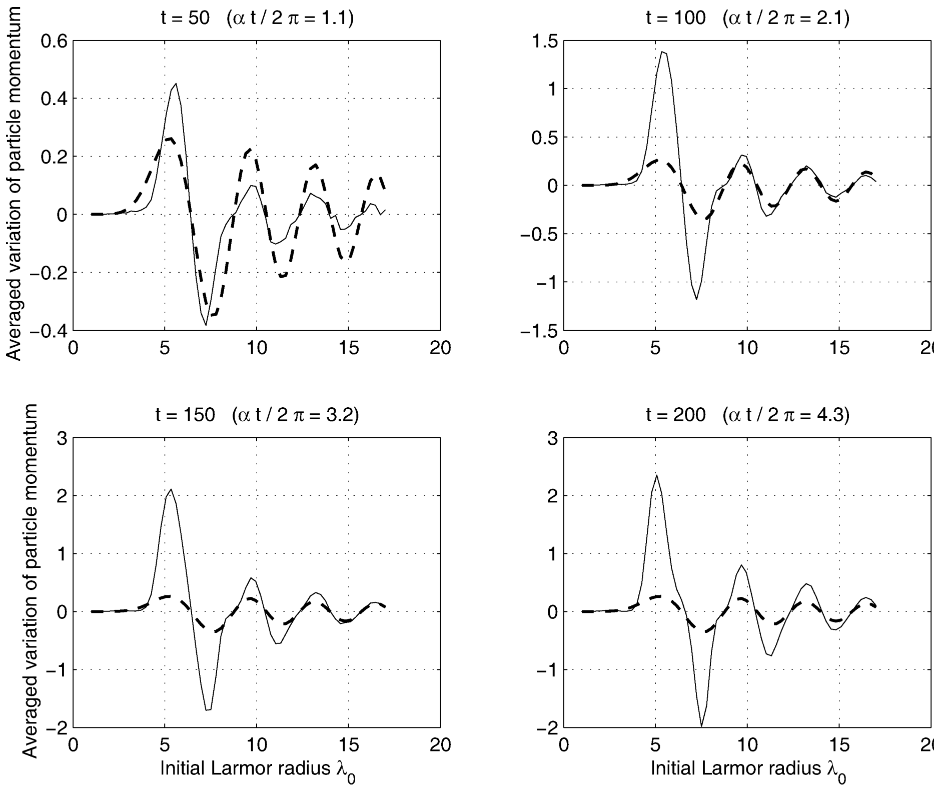

Figure 5. The averaged variation of the particle momentum, $\overline{(\mu - \mu_0)}$, as a function of the initial Larmor radius $\lambda_0 \equiv \sqrt{2\mu_0}$ depicted at various moments of time for the wave amplitude $\beta = 0.3$ and $n = 5$, $\alpha = 0.1347$.

(3.15), the variable μ, which is the normalized transversal adiabatic invariant, plays the role of a coordinate.

The quantity $J_n(\lambda)\, dJ_n(\lambda)/d\lambda$ is encountered in the linear theory of transversely propagating waves as a determining factor in non-resonant fluid-like instabilities (see, for example, Gurnett and Bhattacharjee (2005), and the original paper by Tataronis and Crawford (1970)). The case studied in the present paper is principally different. We consider nonlinear resonant wave–particle interaction which involves energetic particles in presence of 'cold' background particles supporting the wave propagation. In the case $\mathbf{k}_{\|} \to 0$ under discussion, wave–particle interaction that leads to energy exchange between wave and energetic particles and under certain conditions results in the wave growth is a completely nonlinear effect. Our consideration reveals the reason why the quantity mentioned above is the defining parameter in the energy exchange between the wave and energetic particles.

4. Back influence of energetic particles upon the wave

We proceed from the equation

$$\frac{dU}{dt} = -\langle \mathbf{j}_{\text{NL}} \cdot \mathbf{E} \rangle. \tag{4.1}$$

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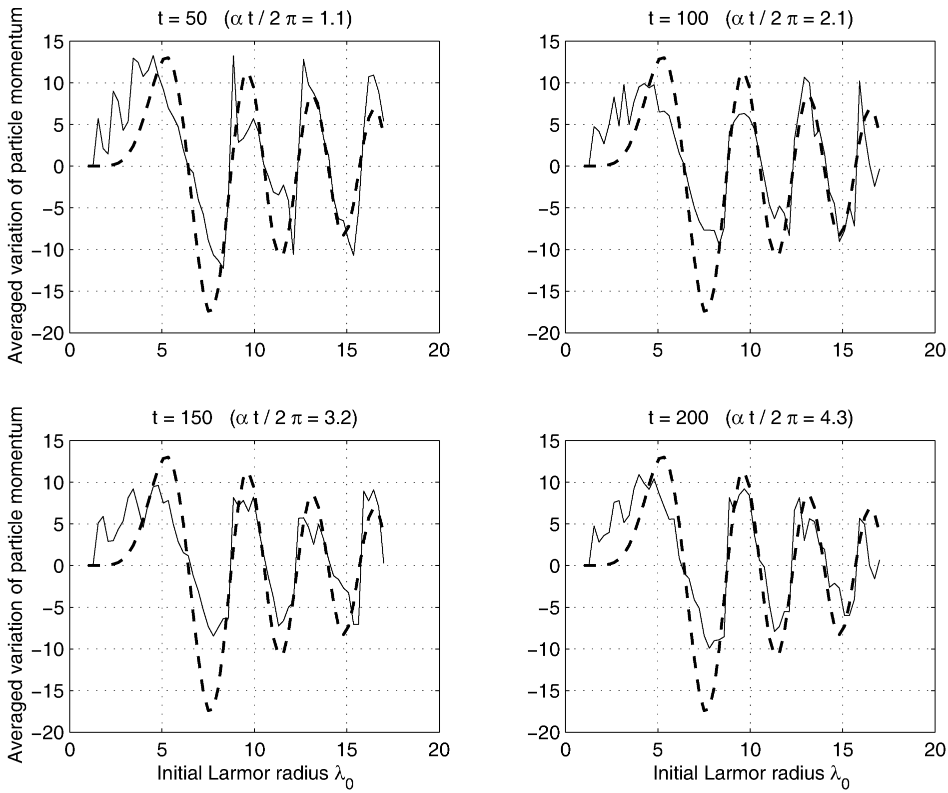

Figure 6. Averaged variation of the particle momentum for the same n and α as in Fig. 5, but for $\beta = 3$.

Here U is the wave energy density which includes the energy of the electromagnetic field and the oscillating energy of cold non-resonant particles. For the quasi-electrostatic wave under discussion, this quantity is determined by the relation (see, e.g., Shafranov (1967))

$$U = \frac{\omega}{16\pi}\frac{\partial \epsilon}{\partial \omega}|E|^2, \tag{4.2}$$

with $\epsilon = (k_i \varepsilon_{ij} k_j)/k^2$, where ε_{ij} is dielectric tensor. In the cold plasma approximation, the components of ε_{ij} are given by the well-known relations (see, for instance, Ginzburg and Rukhadze (1972)) that give ($\omega_{\rm pe}$, $\omega_{\rm pi}$ are the electron and ion plasma frequencies, respectively):

$$\epsilon = 1 - \frac{\omega_{\rm pe}^2}{(\omega^2 - \Omega_{\rm ce}^2)} \tag{4.3}$$

for upper hybrid resonance waves propagating perpendicularly to the ambient magnetic field, and

$$\epsilon \simeq 1 - \frac{\omega_{\rm pe}^2}{\Omega_{\rm ce}^2} - \frac{\omega_{\rm pi}^2}{\omega^2} \tag{4.4}$$

for transversely propagating lower hybrid resonance waves. We should note that in both cases $\omega(\partial\epsilon/\partial\omega) > 0$, thus, the waves have positive energy. This means (see (4.1)) that their amplitude increases if the energetic particles lose the energy and

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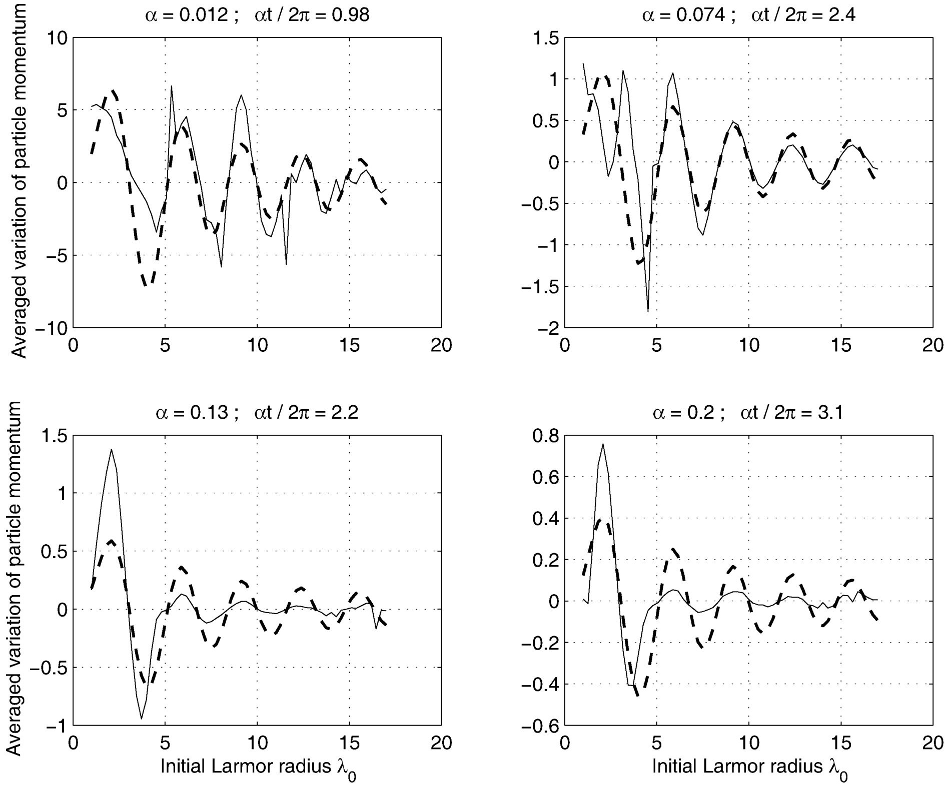

Figure 7. Averaged variation of the particle momentum as a function of the initial Larmor radius λ_0 for $n = 2$, $\beta = 0.3$, and various α.

vice versa. Also, for theses waves the group velocity $\mathbf{v}_g \to 0$, thus, the full derivative dU/dt in (4.1) is, in fact, equivalent to $\partial U/\partial t$. The quantity $\mathbf{j}_{\text{NL}}$ on the right-hand side of (4.1) is the nonlinear current of energetic particles, $\mathbf{E}$ is the wave electric field given by (2.1), and the angular brackets denote the averaging over the wave spatial period.

The aim of the following transformation is to express the quantity $\langle \mathbf{j}_{\text{NL}} \cdot \mathbf{E} \rangle$ through the averaged variation of particle momentum that we have studied before. Towards this aim, we first write this expression in the explicit form, returning for a moment to the initial dimensional variables

$$\langle \mathbf{j}_{\text{NL}} \cdot \mathbf{E} \rangle = \frac{qEk}{2\pi} \int \sin(kx - \omega t) F(t, x, \mathbf{v}_\perp) v_x \, dx \, d\mathbf{v}_\perp, \tag{4.5}$$

where $F(t, x, \mathbf{v}_\perp)$ is the particle distribution function integrated over longitudinal velocity v_z which obeys a Boltzman–Vlasov equation of the form

$$\frac{\partial F}{\partial t} + \mathbf{v}_\perp \frac{\partial F}{\partial \mathbf{r}_\perp} + \frac{q}{m} \left\{ \frac{1}{c} [\mathbf{v}_\perp \times \mathbf{B}_0] + \mathbf{E} \right\} \frac{\partial F}{\partial \mathbf{v}_\perp} = 0. \tag{4.6}$$

We proceed in the usual manner, namely, we multiply (4.6) by $mv_\perp^2/2\Omega \equiv \mu$ and integrate it over $dx\, d\mathbf{v}_\perp$. Integrating the last term by parts, and taking into account that the second term vanishes after averaging over x as the distribution F is a

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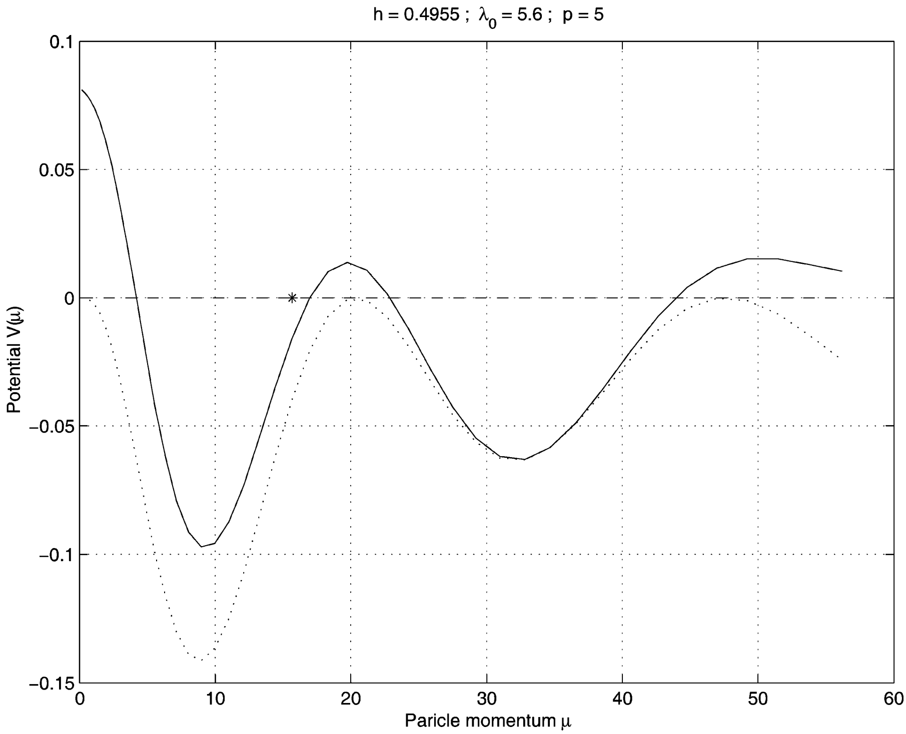

Figure 8. Effective potential with a particle transverse adiabatic invariant as a canonical coordinate. The initial position of the particle, marked in the figure with an asterisk, corresponds to a negative value of the quantity $dJ_n^2(\lambda_0)/d\lambda_0$.

periodic function of x and does not depend on y, we obtain

$$\int \mu \frac{\partial F(t, x, \mathbf{v}_\perp)}{\partial t}\, dx\, d\mathbf{v}_\perp - \frac{qE}{\Omega} \int \sin(kx - \omega t) F(t, x, \mathbf{v}_\perp) v_x\, dx\, d\mathbf{v}_\perp = 0. \qquad (4.7)$$

The second term in (4.7), up to the factor, is equal to the right-hand side of (4.5). Thus, from (4.7) and (4.5) we have

$$\langle \mathbf{j}_{\text{NL}} \cdot \mathbf{E} \rangle = \frac{k\Omega^2}{2\pi m} \frac{d}{dt} \int \mu F(t, x, \mu, \varphi)\, dx\, d\mu\, d\varphi. \qquad (4.8)$$

In relation (4.8), we have changed the variables from $(v_x,\ v_y)$ to canonical variables (μ, φ) according to (2.5), and took the derivative with respect to time out of the integral sign.

We now note that the equations of motion that we have considered are written in variables μ, ζ and, thus, the corresponding distribution is the function of the variables t, μ, and ζ, the latter quantity being expressed through the variables t, x, μ, φ according to the relation

$$\zeta = kx - \omega t + n\varphi - \lambda \sin \varphi, \qquad (4.9)$$

which follows from (3.2), (2.4), and (2.5). We then change the variables of integration in (4.8) from x, μ, φ to ζ, μ, φ. The Jacobian of this transformation is equal to $1/k$. Performing the integration with respect to φ and taking into account the fact that in the variables t, μ, φ, ζ the distribution function does not depend on φ, we obtain from (4.8)

$$\langle \mathbf{j}_{\rm NL} \cdot \mathbf{E} \rangle = \frac{\Omega^2}{m} \frac{d}{dt} \int \mu F(t, \zeta, \mu)\, d\zeta\, d\mu. \tag{4.10}$$

According to Liouville's theorem

$$F(t, \zeta, \mu)\, d\zeta\, d\mu = F_0[\zeta_0(t, \zeta, \mu), \mu_0(t, \zeta, \mu)]\, d\zeta_0\, d\mu_0, \tag{4.11}$$

where the subscript '0' denotes the quantities at $t = 0$, i.e. F_0 is the initial distribution function, ζ_0, μ_0 are particle initial phase and momentum expressed through the current values of ζ, μ, and t. Changing the variables in (4.10) from ζ, μ to ζ_0, μ_0 with the account of (4.11), substituting the result in (4.1), and carrying in the derivative with respect to time under the integral sign we obtain

$$\frac{dU}{dt} = -\frac{\Omega^2}{m} \int \frac{\partial \mu(t, \zeta_0, \mu_0)}{\partial t} F_0(\zeta_0, \mu_0)\, d\zeta_0\, d\mu_0. \tag{4.12}$$

It is natural to assume that the initial distribution function does not depend on ζ_0. Then, (4.12) gives, after integration with respect to time, the variation of wave energy density resulting from wave–particle interaction:

$$\Delta U = -\frac{2\pi\Omega^2}{m} \int \Delta\mu(\mu_0) F_0(\mu_0)\, d\mu_0, \tag{4.13}$$

where

$$\Delta\mu(\mu_0) = \langle \Delta\mu(\zeta_0, \mu_0) \rangle \equiv \frac{1}{2\pi} \int \Delta\mu(\zeta_0, \mu_0)\, d\zeta_0 \tag{4.14}$$

is the variation of particle momentum, averaged over the initial phase, as a function of the initial momentum, i.e. exactly the quantity which we have analyzed in detail in the previous section. Equation (4.12) is the quantitative expression of predetermined result: if the integral on the right-hand side of (4.12) is negative so that the energetic particles in total lose their energy, then the wave energy and, thus, the wave amplitude increases.

As we have seen above, the quantity $\Delta\mu(\mu_0)$ depends on the wave characteristics and particle momentum, thus, the value and the sign of the right-hand side in (4.12) strongly depend on the distribution function of energetic electrons. However, the magnitude of ΔU may be estimated using the most general characteristics of the distribution function and other parameters of the problem. To this aim, we write the distribution function of energetic electrons in the form

$$F_0(\mu) = \frac{n_{\rm R}}{v_{\rm T\perp}^2} f\left(\frac{v_\perp^2}{v_{\rm T\perp}^2}\right), \tag{4.15}$$

where $n_{\rm R}$ is the density of energetic particles, $v_{\rm T\perp}$ is their characteristic transversal velocity, and it is assumed that the magnitude and the characteristic scale of the function f are of the order of 1. Changing to dimensionless variables μ' and λ (see (3.6) and (3.9)), and again omitting prime, we obtain

$$\Delta U = -\frac{2\pi m \Omega^4 n_{\rm R}}{k^4 v_{\rm T\perp}^2} \int f\left(\frac{\lambda_0^2}{\lambda_{\rm T}^2}\right) \Delta\mu(\lambda_0) \lambda_0\, d\lambda_0, \tag{4.16}$$

where

$$\lambda_{\rm T} = \frac{k v_{\rm T\perp}}{\Omega}. \tag{4.17}$$

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As was mentioned above, the wave–particle interaction under discussion is only efficient under the condition $\lambda_{\rm T} > n \sim \omega/\Omega$, which we will assume. The integrand in (4.16) is the product of a sign changing function $\Delta\mu(\lambda_0)$ with the characteristic scale of variation $\sim \pi$, and a positively defined smooth function $f(\lambda_0^2/\lambda_{\rm T}^2)\lambda_0$ which tends to zero at $\lambda_0 \gg \lambda_{\rm T}$ and has a characteristic scale of variation $\sim \lambda_{\rm T}$. In the case when $\lambda_{\rm T} \gtrsim \pi$, the magnitude of the integral in (4.16) may be estimated as follows

$$\int f\left(\frac{\lambda_0^2}{\lambda_{\rm T}^2}\right)\Delta\mu(\lambda_0)\lambda_0\, d\lambda_0 \sim f(1)\Delta\mu(\lambda_{\rm T})\lambda_{\rm T} \sim \Delta\mu(\lambda_{\rm T})\lambda_{\rm T}. \tag{4.18}$$

Substituting this estimation into (4.16) we obtain

$$|\Delta U| \sim n_{\rm R} m v_{\rm T\perp}^2 \lambda_{\rm T}^{-3}\Delta\mu(\lambda_{\rm T}), \tag{4.19}$$

where $\lambda_{\rm T}$ is determined in (4.17), and it is assumed that $\lambda_{\rm T} > n \sim \omega/\Omega$. We see that a possible energy gain by transversely propagating quasi-electrostatic wave due to nonlinear resonant interaction with energetic particles decreases rapidly with an increasing number of the closest cyclotron harmonic.

Let us apply the results obtained above to one particular case, namely, consider possible amplification of an upper hybrid resonance (UHR) wave in the equatorial region of the magnetosphere at L-shell ~ 4 owing to nonlinear resonant interaction with energetic electrons. Intense UHR waves have been observed outside the plasmapause by Kurth et al. (1979). These waves have been suggested by Shklyar and Kliem (2006) as a possible driver of relativistic electron precipitation bursts registered by low-orbiting satellites (see, for example, Blake et al. (1996), Nakamura et al. (2000), and references therein). For estimations, we use the following figures for energetic electron density $n_{\rm R}$, their thermal velocity $v_{\rm T}$, electron plasma frequency $\omega_{\rm p}$, and electron cyclotron frequency $\omega_{\rm c}$:

$$n_{\rm R} \sim 1\ {\rm cm}^{-3}, \quad v_{\rm T} \sim 10^9\ {\rm cm\ s}^{-1}, \quad \omega_{\rm p} \sim 2.6\times10^5\ {\rm rad\ s}^{-1}, \quad \omega_{\rm c} \sim 8.6\times10^4\ {\rm rad\ s}^{-1}.$$

The value of electron plasma frequency given above corresponds to the density of cold electrons $n_{\rm c} \sim 21\ {\rm cm}^{-3}$. For these parameters, the upper hybrid resonance frequency, which defines the wave frequency ω, is about $3\omega_{\rm c}$, so that the number n of the closest cyclotron harmonic is equal to 3. For such a wave $\omega\partial\epsilon/\partial\omega \simeq 2$, and the wave energy $U \simeq |E|^2/8\pi$ (see (4.2)). Relation (4.19) then gives the following estimation of the maximum wave amplitude (in SI units)

$$|E| \sim 4.5\lambda_{\rm T}^{-3/2}[\Delta\mu(\lambda_{\rm T})]^{1/2}\ {\rm V\ m}^{-1}. \tag{4.20}$$

The quantity $\Delta\mu(\lambda_{\rm T})$ which enters into relation (4.20) is dimensionless and, according to numerical results of the previous section, is of the order of unity. At the same time, the quantity $\lambda_{\rm T}$ is most uncertain, as the magnitude of wave normal vector for quasi-resonant waves is badly defined. In a sense, the k value itself is determined from the condition that the corresponding wave has maximum amplification. The requirement $\lambda_{\rm T} > n \sim \omega/\Omega$ which sets the lower limit on the value of $\lambda_{\rm T}$ and, thus, on the value of k, remains in effect, of course. The estimation (4.20) is quite consistent with measurements by Kurth et al. (1979) who have reported the value of $|E| \sim (1-20)$ mV m^{-1}.

5. Concluding remarks

The nonlinear interaction between quasi-electrostatic wave propagating transversely to the ambient magnetic field and energetic particles is a particular case of wave–particle interaction since in linear approximation such an interaction vanishes for wave frequencies not equal to a multiple of cyclotron frequency. In the case of transverse propagation, the influence of the wave field upon the variation of particle gyrophase is of critical importance and, consequently, the interaction has a nonlinear nature. This interaction has been studied by several authors (see the references given in Sec. 1) with the main aim to study particle dynamics. The main focus of the present study is the variation of the wave field intensity caused by the interaction with energetic particles. The consideration is applicable to upper hybrid resonance wave interaction with energetic electrons, as well as to lower hybrid wave interaction with energetic ions.

A self-consistent consideration of wave–particle interaction which simultaneously describes both the evolution of the wave field and particle distribution function demands a solution of nonlinear set of Maxwell–Boltzmann equations where both the wave field and the distribution function are unknown, which, as a rule, requires computer simulations. The consideration in the present paper is based on the approximation of a given field (see Sec. 2 for references) which is not self-consistent by its nature, although it has proved to be very productive, and the frame of its validity is well understood. In this approximation, the particle dynamics are considered in the given field, while the wave evolution is calculated from the energy conservation in the system wave-resonant particles. We should emphasize that numerical solutions of particle equations of motion in the given field used in the present work represent numerical calculations of much simpler type as compared with self-consistent numerical simulations.

The main result of the present work consists in that resonant interaction (for example, of upper hybrid resonance wave with energetic electrons) in the nonlinear stage may lead to a substantial growth of the wave intensity due to energy exchange with energetic particles. While the sign of energy transfer depends on the characteristics of wave and particle distribution function, the energy exchange, in general, is more efficient for frequencies close to cyclotron harmonics.

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